

SYMMETRIZED CUT-JOIN EQUATION OF MARINO-VAFA FORMULA

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ABSTRACT. In this note, we symmetrized the cut-join equation from the proof of Marino-Vafa formula. One can derive more recursion formulas of Hodge integrals out of this polynomial equations. We also give some applications.

1. INTRODUCTION

The Marino-Vafa formula(Liu-Liu-Zhou's theorem, cf [14]) gives a close formula for certain Hodge integrals with three λ classes. One of its specialization is the famous ESLV formula[3]. By applying a transcendental changing of variable, Goulden, Jackson and Vainshtein get a symmetrized cut-join equation[5], which is a polynomial identities with Hodge integral numbers with one λ class as coefficients. Comparing the lowest degree terms, Goulen, Jackson and Vakil[6] were able to give a short proof of λ_g conjecture, which was first proved by Okounkov and Pandharipande[4]. On the other hand, by using the result of [6], Chen, Li and Liu [1] gave a short proof of Witten conjecture Kontsevich theorem.

In this paper, we study another transcendental changing of variable formula, apply it to the Marino-Vafa formula itself, and get a symmetrized cut-join equation, which is again a polynomial identity, but with Hodge integrals with three λ classes as coefficients. We expect more Hodge integrals can be computed from our symmetrized cut-join equation. As an example, we illustrate how to get the Witten conjecture Kontsevich theorem from our newly derived symmetrized cut-join equation.

We study the new transcendental change of variable formula in section 2, which is essentially some calculus based on Formal Lagrange Inversion Theorem. In section 3, we symmetrize the cut-join equation satisfied by the generating series of Hodge integral studied in [14]. Applied the change of variable formula developed in section 2 to the symmetrized cut-join equation in section 3, we derived a polynomial cut-join equation, which is the theorem 3 in section 4, and this is the main result of this paper. We illustrate some application of our result in section 5.

2. PRELIMINARY

We first quote a result from the standard text book on combinatorics. For a proof and more about this theorem, we refer the book Enumerative Combinatorics by Richard Stanley[18].

Theorem 1. *Formal Lagrange Inversion Theorem: Let $F[x] = \sum_{i=1}^{+\infty} a_i x^i \in xK[[x]]$ where $a_1 \neq 0$ and K is a field of characteristic 0. Let $k, n \in \mathbb{Z}$, then*

$$(2.1) \quad n[x^n]F^{-1}(x)^k = k[x^{n-k}]\left(\frac{x}{F(x)}\right)^n = k[x^{-k}]F(x)^{-n}$$

Where $F^{-1}(x)$ denote the formal inverse function of $F(x)$ and $[x^n]F(x)$ is the coefficient of x^n in the formal power series $F(x)$.

In particular, take $k = 1$, we have

$$(2.2) \quad n[x^n]F^{-1}(x) = [x^{-1}]F(x)^{-n}$$

The inverse function of $x(1-x)^\tau$ will play a crucial role in this paper. Take $F(x) = x(1-x)^\tau$ in the above theorem for some fixed complex number τ , then

$$(2.3) \quad \frac{1}{F(x)^n} = \frac{1}{x^n} (1-x)^{-n\tau} = \frac{1}{x^n} \sum_{r=0}^{+\infty} \frac{\prod_{a=0}^{r-1} (n\tau + a)}{r!} x^r$$

and

$$(2.4) \quad [x^{-1}] \frac{1}{F(x)^n} = \frac{\prod_{a=0}^{n-2} (n\tau + a)}{(n-1)!}$$

Let $\omega(x)$ denote $F^{-1}(x)$. We will study some basic properties of this function in this section. The formal Lagrange inversion theorem gives the unique formal power series solution of the equation

$$(2.5) \quad \omega(x)(1 - \omega(x))^\tau = x$$

$$(2.6) \quad \omega(x) = F^{-1}(x) = \sum_{n=1}^{+\infty} \frac{\prod_{a=0}^{n-2} (n\tau + a)}{n!} x^n$$

One can compute the derivative

$$(2.7) \quad x\omega(x)' = \frac{\omega(1-\omega)}{1-(1+\tau)\omega}$$

Let $y = \frac{1}{1-(1+\tau)\omega}$, we have

$$(2.8) \quad (1+\tau)x\omega' = (1-\omega)\left(\frac{1}{1-(1+\tau)\omega} - 1\right)$$

and

$$(2.9) \quad (1+\tau)^2 x\omega' = \frac{\tau}{1-(1+\tau)\omega} + 1 - (1+\tau)(1-\omega) = \tau y - \tau + (1+\tau)\omega$$

so

$$(2.10) \quad y = 1 + \left(\frac{1+\tau}{\tau}\right) \sum_{n=1}^{+\infty} \frac{\prod_{a=0}^{n-1} (n\tau + a)}{n!} x^n$$

For a formal power series $f(x)$, if we change the variable to ω , and then to y , we have the following relations:

$$(2.11) \quad x \frac{df}{dx} = \frac{1-\omega}{1-(1+\tau)\omega} \omega \frac{df}{d\omega} = y(y-1) \left(\frac{y\tau+1}{\tau+1}\right) \frac{df}{dy}$$

3. SYMMETRIZATION

In [14], they studied the generating function $\mathcal{C} = \sum_{g \geq 0, n \geq 1} \mathcal{C}_n^g \lambda^{2g-2+n}$, where

$$\begin{aligned} \mathcal{C}_n^g &= \sum_{d \geq 1} \sum_{\mu \vdash d, l(\mu)=n} -\frac{\sqrt{-1}^{d+n}}{|\text{Aut} \mu|} (\tau(1+\tau))^{n-1} \prod_{i=1}^n \frac{\prod_{a=1}^{\mu_i-1} (\mu_i \tau + a)}{(\mu_i - 1)!} \int_{\mathcal{M}_{g,n}} \frac{\Gamma_g(\tau)}{\prod_{i=1}^n (1 - \mu_i \psi_i)} \cdot \mathbf{P}_\mu \\ &= -\frac{\sqrt{-1}^n (\tau(1+\tau))^{n-1}}{n!} \sum_{\mu_1, \mu_2, \dots, \mu_n \geq 1}^{+\infty} \sqrt{-1}^{|\mu|} \prod_{i=1}^n \frac{\prod_{a=1}^{\mu_i-1} (\mu_i \tau + a)}{(\mu_i - 1)!} \\ &\quad \sum_{k=0}^{3g-3} \sum_{b_1+b_2+\dots+b_n=3g-3+n-k} \int_{\mathcal{M}_{g,n}} \Gamma_g^k(\tau) \prod_{i=1}^n \psi_i^{b_i} \prod_{i=1}^n \mu_i^{b_i} \cdot \mathbf{P}_\mu \\ &= -\frac{(\tau(1+\tau))^{n-1}}{n!} \sum_{k=0}^{3g-3} \sum_{b_1+b_2+\dots+b_n=3g-3+n-k} < \tau_{b_1} \cdots \tau_{b_n} \Gamma_g^k(\tau) > \prod_{i=1}^n \phi_{b_i}(\vec{\mathbf{P}}) \end{aligned}$$

Here we denote

$$\phi_i(\vec{\mathbf{p}}) = \sum_{m \geq 1} \sqrt{-1}^{m+1} \mathbf{p}_m \frac{\prod_{a=1}^{m-1} (m\tau + a)}{(m-1)!} m^i = \frac{1}{\tau} \sum_{m \geq 1} \sqrt{-1}^{m+1} \mathbf{p}_m \frac{\prod_{a=0}^{m-1} (m\tau + a)}{m!} m^i$$

for infinitely many formal variables $\vec{\mathbf{p}} = \{\mathbf{p}_1, \mathbf{p}_2, \dots\}$ and

$$\Gamma_g(\tau) = \Lambda_g^\vee(1) \Lambda_g^\vee(\tau) \Lambda_g^\vee(-\tau - 1)$$

This apparently complicated generating function naturally appeared when one computes the open Gromov-Witten invariants of local Calabi-Yau, cf[7]. Motivated by the duality between topological string theory and Chern-Simon theory, Marino-Vafa formula gives a closed expression of the above generating function \mathcal{C} in terms of some combinatorial data associated to representations of symmetric groups. In their proof of the Marino-Vafa formula, Liu-Liu-Zhou show that the generating function \mathcal{C} satisfies a cut-join equation.

$$(3.1) \quad \frac{\partial \mathcal{C}}{\partial \tau} = \frac{\sqrt{-1}\lambda}{2} \sum_{i,j \geq 1} (ij \mathbf{p}_{i+j} \frac{\partial^2 \mathcal{C}}{\partial \mathbf{p}_i \partial \mathbf{p}_j} + ij \mathbf{p}_{i+j} \frac{\partial \mathcal{C}}{\partial \mathbf{p}_i} \frac{\partial \mathcal{C}}{\partial \mathbf{p}_j} + (i+j) \mathbf{p}_i \mathbf{p}_j \frac{\partial \mathcal{C}}{\partial \mathbf{p}_{i+j}})$$

In this section we will symmetrize this cut-join equation and make a transcendental change of the variable, so that the resulting symmetrized cut-join equation become a polynomial one.

Define the symmetrization operator

$$\Xi_n \mathbf{p}_\alpha = (\sqrt{-1})^{-n-|\alpha|} \sum_{\sigma \in S_n} x_{\sigma(1)}^{\alpha_1} \cdots x_{\sigma(n)}^{\alpha_n}$$

for $n \geq 1$ if $l(\alpha) = n$ with $\alpha = (\alpha_1, \dots, \alpha_n)$, and 0 otherwise.

We have

$$(3.2) \quad \Xi_n \mathcal{C}_n^g = -\frac{(\tau(1+\tau))^{n-1}}{n!} \sum_{k=0}^{3g-3} \sum_{b_1+b_2+\dots+b_n=3g-3+n-k} < \tau_{b_1} \cdots \tau_{b_n} \Gamma_g^k(\tau) > \sum_{\sigma \in S_n} \prod_{i=1}^n \phi_{b_i}(x_{\sigma(i)})$$

where

$$(3.3) \quad \phi_i(x) = \frac{1}{\tau} \sum_{m \geq 1} \frac{\prod_{a=0}^{m-1} (m\tau + a)}{m!} m^i x^m = \frac{1}{\tau+1} \left(x \frac{d}{dx}\right)^i (y-1)$$

Let \mathbf{C} denote the change of variable from x to y , one has the following relation

$$\mathbf{C} x \frac{d}{dx} = y(y-1) \left(\frac{y\tau+1}{\tau+1}\right) \frac{d}{dy} \mathbf{C}$$

Apply to $\phi_i(x)$ for $i \geq 0$, we get

$$(3.4) \quad \mathbf{C} \phi_i(x) = \mathbf{C} \frac{1}{\tau+1} \left(x \frac{d}{dx}\right)^i (y-1) = [y(y-1) \left(\frac{y\tau+1}{\tau+1}\right) \frac{d}{dy}]^i \left(\frac{y-1}{\tau+1}\right)$$

Clearly, this is a polynomial in the new variable y of degree $2i+1$.

Set $\Xi^{\{a_1, \dots, a_m\}} \mathbf{p}_\alpha = \Xi \mathbf{p}_\alpha |_{x_i \rightarrow x_{a_i}, i=1, \dots, m}$

The following three lemmas are from the section 4 of [5]. However, one should be careful that in our case there are some extra coefficients appear due to our definition of the symmetrized operator. We ignore the proof, which one can find in [5].

Lemma 3.1. *Let α and β be partitions with $l(\alpha) = k$ and $l(\beta) = m$. Then*

$$\Xi^{\{1, \dots, m+k\}} \mathbf{p}_\alpha \mathbf{p}_\beta = \sum_{(\mathcal{A}, \mathcal{B})} (\Xi^{\mathcal{A}} \mathbf{p}_\alpha) (\Xi^{\mathcal{B}} \mathbf{p}_\beta)$$

where the sum is over all ordered partitions $(\mathcal{A}, \mathcal{B})$ of $\{1, \dots, m+k\}$ with $|\mathcal{A}| = k$ and $|\mathcal{B}| = m$

Lemma 3.2. *Let α be a partition $l(\alpha) = m$, and let $1 \leq l \leq m$, then*

$$\begin{aligned} x_l \frac{\partial}{\partial x_l} \Xi^{\{1, \dots, m\}} \mathbf{p}_\alpha &= \sqrt{-1}^{-m-|\alpha|} \sum_{\sigma \in S_m} \alpha_{\sigma(l)} \prod_{i=1}^m x_i^{\alpha_{\sigma(i)}} \\ &= \sum_{\sigma \in S_m} \alpha_{\sigma(l)} \prod_{i=1}^m [\sqrt{-1}^{-1} (-\sqrt{-1} x_i)^{\alpha_{\sigma(i)}}] \\ &= \sum_{i \geq 1} \sqrt{-1}^{-(i+1)} (x_l)^{i \Xi^{\{1, \dots, m\}} - \{l\}} \frac{\partial \mathbf{p}_\alpha}{\partial \mathbf{p}_i} \end{aligned}$$

Lemma 3.3. *Let α be a partition with $l(\alpha) = m + 1$ and $1 \leq l \leq m$, then*

$$x_l \frac{\partial}{\partial x_l} x_{m+1} \frac{\partial}{\partial x_{m+1}} \Xi^{\{1, \dots, m+1\}} \mathbf{p}_\alpha |_{x_{m+1}=x_l} = \sum_{i, j \geq 1} \sqrt{-1}^{-(i+j+2)} x_l^{i+j} \Xi^{\{1, \dots, m+1\} - \{l, m+1\}} ij \frac{\partial^2 \mathbf{p}_\alpha}{\partial \mathbf{p}_i \partial \mathbf{p}_j}$$

Now we apply the operator $\Xi^{\{1, \dots, m\}}$ to the cut-joint equation to get a symmetrized one. Notice that $\Xi^{\{1, \dots, m\}}$ commutes with taking derivative with respect to τ , the left hand side gives

$$\Xi^{\{1, \dots, m\}} \frac{\partial \mathcal{C}}{\partial \tau} = \frac{\partial}{\partial \tau} (\Xi^{\{1, \dots, m\}} \mathcal{C}) = \sum_{g \geq 0} \lambda^{2g-2+m} \frac{\partial}{\partial \tau} (\Xi^{\{1, \dots, m\}} \mathcal{C}_m^g)$$

Next we study the effect of $\Xi^{\{1, \dots, m\}}$ on the right hand side. By lemma 3.3,

$$\begin{aligned} \Xi^{\{1, \dots, m\}} \sum_{i, j \geq 1} \mathbf{p}_{i+j} ij \frac{\partial^2 \mathbf{p}_\alpha}{\partial \mathbf{p}_i \partial \mathbf{p}_j} &= \sum_{l=1}^m \sum_{i, j \geq 1} \sqrt{-1}^{-(i+j+1)} x_l^{i+j} \Xi^{\{1, \dots, m\} - \{l\}} ij \frac{\partial^2 \mathbf{p}_\alpha}{\partial \mathbf{p}_i \partial \mathbf{p}_j} \\ &= \sqrt{-1} \sum_{l=1}^m x_l \frac{\partial}{\partial x_l} x_{m+1} \frac{\partial}{\partial x_{m+1}} \Xi^{\{1, \dots, m+1\}} \mathbf{p}_\alpha |_{x_{m+1}=x_l} \end{aligned}$$

$$\begin{aligned} \Xi^{\{1, \dots, m\}} \sum_{i, j \geq 1} \mathbf{p}_{i+j} ij \frac{\partial^2 \mathcal{C}}{\partial \mathbf{p}_i \partial \mathbf{p}_j} &= \sum_{g \geq 0} \lambda^{2g-2+m+1} \sum_{i, j \geq 1} \Xi^{\{1, \dots, m\}} \mathbf{p}_{i+j} ij \frac{\partial^2 \mathcal{C}_{m+1}^g}{\partial \mathbf{p}_i \partial \mathbf{p}_j} \\ &= \sqrt{-1} \sum_{g \geq 0} \lambda^{2g-2+m+1} \sum_{l=1}^m x_l \frac{\partial}{\partial x_l} x_{m+1} \frac{\partial}{\partial x_{m+1}} \Xi^{\{1, \dots, m+1\}} \mathcal{C}_{m+1}^g |_{x_{m+1}=x_l} \end{aligned}$$

Let $l(\alpha) = k$ and $l(\beta) = m - k + 1$, and so we have

$$\begin{aligned} \Xi^{\{1, \dots, m\}} \sum_{i, j \geq 1} \mathbf{p}_{i+j} ij \frac{\partial \mathbf{p}_\alpha}{\partial \mathbf{p}_i} \frac{\partial \mathbf{p}_\beta}{\partial \mathbf{p}_j} &= \sum_{l=1}^m \sum_{i, j \geq 1} (\sqrt{-1})^{-(1+i+j)} x_l^{i+j} \Xi^{\{1, \dots, m\} - \{l\}} (i \frac{\partial \mathbf{p}_\alpha}{\partial \mathbf{p}_i}) (j \frac{\partial \mathbf{p}_\beta}{\partial \mathbf{p}_j}) \\ &= \sqrt{-1} \sum_{l=1}^m \sum_{(A, B)} (\sum_{i \geq 1} \sqrt{-1}^{-(i+1)} x_l^i \Xi^{A_i} \frac{\partial \mathbf{p}_\alpha}{\partial \mathbf{p}_i}) (\sum_{j \geq 1} \sqrt{-1}^{-(j+1)} x_l^j \Xi^{B_j} \frac{\partial \mathbf{p}_\beta}{\partial \mathbf{p}_j}) \\ &= \sqrt{-1} \sum_{l=1}^m \sum_{(A, B)} (x_l \frac{\partial}{\partial x_l} \Xi^{A \cup \{l\}} \mathbf{p}_\alpha) (x_l \frac{\partial}{\partial x_l} \Xi^{B \cup \{l\}} \mathbf{p}_\beta) \\ &= \sqrt{-1} \Theta_{k-1} (x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, \dots, k\}} \mathbf{p}_\alpha) (x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, k+1, \dots, m\}} \mathbf{p}_\beta) \end{aligned}$$

$$\begin{aligned}
& \Xi^{\{1, \dots, m\}} \sum_{i, j \geq 1} \mathbf{p}_{i+j} i j \frac{\partial \mathcal{C}}{\partial \mathbf{p}_i} \frac{\partial \mathcal{C}}{\partial \mathbf{p}_j} \\
&= \sum_{g_1, g_2 \geq 0} \sum_{1 \leq k \leq m} \lambda^{2g_1 - 2 + k} \cdot \lambda^{2g_2 - 2 + (m - k + 1)} \cdot \Xi^{\{1, \dots, m\}} \sum_{i, j \geq 1} \mathbf{p}_{i+j} i j \frac{\partial \mathcal{C}_k^{g_1}}{\partial \mathbf{p}_i} \frac{\partial \mathcal{C}_{m-k+1}^{g_2}}{\partial \mathbf{p}_j} \\
&= \sum_{g_1, g_2 \geq 0} \sum_{1 \leq k \leq m} \lambda^{2g_1 + 2g_2 - 3 + m} \sqrt{-1} \Theta_{k-1} \left(x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, \dots, k\}} \mathcal{C}_k^{g_1} \right) \left(x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, k+1, \dots, m\}} \mathcal{C}_{m-k+1}^{g_2} \right) \\
& \Xi^{\{1, \dots, m\}} \sum_{i, j \geq 1} (i+j) \mathbf{p}_i \mathbf{p}_j \frac{\partial \mathbf{p}_\alpha}{\partial \mathbf{p}_{i+j}} = 2 \sum_{1 \leq l < k \leq m} \sum_{i, j \geq 1} \sqrt{-1}^{-(i+j+2)} x_l^i x_k^j \Xi^{\{1, \dots, m\} - \{l, k\}} (i+j) \frac{\partial \mathbf{p}_\alpha}{\partial \mathbf{p}_{i+j}} \\
&= 2 \sum_{1 \leq l < k \leq m} \sum_{r \geq 1} \sqrt{-1}^{-(r+2)} \cdot \frac{x_k^r x_l - x_l^r x_k}{x_k - x_l} \cdot \Xi^{\{1, \dots, m\} - \{l, k\}} \frac{\partial \mathbf{p}_\alpha}{\partial \mathbf{p}_r} \\
&= -2\sqrt{-1} \sum_{1 \leq l \neq k \leq m} \frac{x_k}{x_l - x_k} x_l \frac{\partial}{\partial x_l} \Xi^{\{1, \dots, m\} - \{k\}} \mathbf{p}_\alpha \\
&= -2\sqrt{-1} \Theta_1 \frac{x_2}{x_1 - x_2} \cdot x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, 3, 4, \dots, m\}} \mathbf{p}_\alpha \\
& \Xi^{\{1, \dots, m\}} \sum_{i, j \geq 1} (i+j) \mathbf{p}_i \mathbf{p}_j \frac{\partial \mathcal{C}}{\partial \mathbf{p}_{i+j}} = \sum_{g \geq 0} \lambda^{2g-3+m} \Xi^{\{1, \dots, m\}} \sum_{i, j \geq 1} (i+j) \mathbf{p}_i \mathbf{p}_j \frac{\partial \mathcal{C}_{m-1}^g}{\partial \mathbf{p}_{i+j}} \\
&= \sum_{g \geq 0} -2\sqrt{-1} \lambda^{2g-3+m} \cdot \Theta_1 \frac{x_2}{x_1 - x_2} \cdot x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, 3, 4, \dots, m\}} \mathcal{C}_{m-1}^g
\end{aligned}$$

Collecting all these terms, the following theorem is proved.

Theorem 2. *The symmetrized cut-join equation of Marino-Vafa formula is*

$$\begin{aligned}
& \sum_{g \geq 0} \lambda^{2g-2+m} \frac{\partial}{\partial \tau} \Xi^{\{1, \dots, m\}} \mathcal{C}_m^g \\
&= -\frac{1}{2} \sum_{g \geq 0} \lambda^{2g+m} \left(\sum_{l=1}^m x_l \frac{\partial}{\partial x_l} x_{m+1} \frac{\partial}{\partial x_{m+1}} \Xi^{\{1, \dots, m+1\}} \mathcal{C}_{m+1}^g \right) \Big|_{x_{m+1}=x_l} \\
&\quad - \frac{1}{2} \sum_{g_1, g_2 \geq 0} \sum_{1 \leq k \leq m} \lambda^{2g_1 + 2g_2 + m - 2} \Theta_{k-1} \left(x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, \dots, k\}} \mathcal{C}_k^{g_1} \right) \left(x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, k+1, \dots, m\}} \mathcal{C}_{m-k+1}^{g_2} \right) \\
&\quad + \sum_g \lambda^{2g+m-2} \Theta_1 \frac{x_2}{x_1 - x_2} \cdot x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, 3, 4, \dots, m\}} \mathcal{C}_{m-1}^g
\end{aligned}$$

Comparing the coefficients of λ^{2g-2+m} , we get

$$\begin{aligned}
\frac{\partial}{\partial \tau} \Xi^{\{1, \dots, m\}} \mathcal{C}_m^g &= -\frac{1}{2} \sum_{l=1}^m x_l \frac{\partial}{\partial x_l} x_{m+1} \frac{\partial}{\partial x_{m+1}} \Xi^{\{1, \dots, m+1\}} \mathcal{C}_{m+1}^{g-1} \Big|_{x_{m+1}=x_l} \\
&\quad - \frac{1}{2} \sum_{0 \leq a \leq g} \sum_{1 \leq k \leq m} \Theta_{k-1} \left(x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, \dots, k\}} \mathcal{C}_k^a \right) \left(x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, k+1, \dots, m\}} \mathcal{C}_{m-k+1}^{g-a} \right) \\
&\quad + \Theta_1 \frac{x_2}{x_1 - x_2} \cdot x_1 \frac{\partial}{\partial x_1} \Xi^{\{1, 3, 4, \dots, m\}} \mathcal{C}_{m-1}^g
\end{aligned}$$

4. CHANGING OF VARIABLE

Now we want to make a change of the variable for the equation we obtained in the last section. We first deal with the right hand side. As in [5], to obtain a polynomial expression in the variable y_i , one has to combined all the unstable terms, which are logarithm transcendental.

In the second term, combined the unstable terms $a = 0, k = 1$ and $a = g, k = m$

$$\sum_{l=1}^m (x_l \frac{\partial}{\partial x_l} \Xi^{\{l\}} \mathcal{C}_l^0) (x_l \frac{\partial}{\partial x_l} \Xi^{\{1,2,\dots,m\}} \mathcal{C}_m^g)$$

and recall that (3.2)

$$(4.1) \quad \Xi^{\{l\}} \mathcal{C}_l^0 = \Xi^{\{l\}} \sum_{d=1}^{+\infty} -\sqrt{-1}^{d+1} \frac{\prod_{a=1}^{d-1} (d\tau + a)}{(d-1)!} \cdot d^{-2} \cdot \mathbf{p}_d = - \sum_{d=1}^{+\infty} \frac{\prod_{a=1}^{d-1} (d\tau + a)}{d!} \cdot \frac{x_l^d}{d}$$

$$(4.2) \quad (x_l \frac{\partial}{\partial x_l})^2 \Xi^{\{l\}} \mathcal{C}_l^0 = -\frac{1}{\tau} \sum_{d=1}^{+\infty} \frac{\prod_{a=0}^{d-1} (d\tau + a)}{d!} x_l^d = -\frac{y_l - 1}{\tau + 1} = -\frac{\omega_l}{1 - (\tau + 1)\omega_l}$$

since $x \frac{\partial}{\partial x} = \frac{1-\omega}{1-(\tau+1)\omega} \cdot \omega \frac{\partial}{\partial \omega}$ (2.11), we find the unique expression of

$$(4.3) \quad x_l \frac{\partial}{\partial x_l} \Xi^{\{l\}} \mathcal{C}_l^0 = \ln(1 - \omega_l)$$

The unstable terms $a = 0, k = 2$ and $a = g, k = m - 1$ gives

$$\Theta_1(x_1 \frac{\partial}{\partial x_1} \Xi^{\{1,2\}} \mathcal{C}_2^0) (x_1 \frac{\partial}{\partial x_1} \Xi^{\{1,3,\dots,m\}} \mathcal{C}_{m-1}^g)$$

and

$$(4.4) \quad \begin{aligned} \Xi^{\{1,2\}} \mathcal{C}_2^0 &= -\tau(\tau + 1) \sum_{\mu_1 \geq 1, \mu_2 \geq 1} \frac{x_1^{\mu_1} x_2^{\mu_2}}{\mu_1 + \mu_2} \prod_{i=1,2} \frac{\prod_{a=1}^{\mu_i-1} (\mu_i \tau + a)}{(\mu_i - 1)!} \\ &= -\frac{\tau + 1}{\tau} \sum_{\mu_1 \geq 1, \mu_2 \geq 1} \frac{x_1^{\mu_1} x_2^{\mu_2}}{\mu_1 + \mu_2} \prod_{i=1,2} \frac{\prod_{a=0}^{\mu_i-1} (\mu_i \tau + a)}{\mu_i!} \\ (x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}) \Xi^{\{1,2\}} \mathcal{C}_2^0 &= -\tau(\tau + 1) \cdot \frac{\omega_1 \omega_2}{[1 - (\tau + 1)\omega_1][1 - (\tau + 1)\omega_2]} \end{aligned}$$

One can verify that

$$\Xi^{\{1,2\}} = -\ln\left(\frac{\omega_1 - \omega_2}{x_1 - x_2}\right) - \tau(\ln(1 - \omega_1) + \ln(1 - \omega_2))$$

is the unique solution, and so

$$\begin{aligned} x_1 \frac{\partial}{\partial x_1} \Xi^{\{1,2\}} &= -\frac{\omega_1(1 - \omega_1)}{(\omega_1 - \omega_2)(1 - (\tau + 1)\omega_1)} + \frac{x_1}{x_1 - x_2} + \frac{\tau\omega_1}{1 - (\tau + 1)\omega_1} \\ &= -\frac{\omega_1}{\omega_1 - \omega_2} \left(1 + \frac{\tau\omega_2}{1 - (\tau + 1)\omega_1}\right) + \frac{x_1}{x_1 - x_2} \end{aligned}$$

Remember we have $\omega(1 - \omega)^\tau = x$, ω depends on the parameter τ . Taking partial derivative to τ , we find

$$\frac{\partial \omega}{\partial \tau} \cdot \frac{1 - (\tau + 1)\omega}{\omega(1 - \omega)} + \ln(1 - \omega) = 0$$

Move the terms involve $\ln(1 - \omega_l)$ to the left the symmetrized cut-join equation, since fix $\omega, \partial y(\omega, \tau)/\partial \tau = y^2 \omega$, we have

$$\begin{aligned} \mathbf{C}\left\{\frac{d}{d\tau}\Xi^{\{1,\dots,m\}}\mathcal{C}_m^g - \sum_{l=1}^m \frac{\partial\omega_l}{\partial\tau} \cdot \frac{\partial}{\partial\omega_l}\Xi^{\{1,\dots,m\}}\mathcal{C}_m^g\right\} &= \frac{d}{d\tau}\mathbf{C}\Xi^{\{1,\dots,m\}}\mathcal{C}_m^g(y_1(\omega_1,\tau), \dots, y_m(\omega_m,\tau), \tau) \\ &= \left(\frac{\partial}{\partial\tau} + \sum_{l=1}^m \omega_l y_l^2 \frac{\partial}{\partial y_l}\right)\mathbf{C}\Xi^{\{1,\dots,m\}}\mathcal{C}_m^g(y_1, \dots, y_m, \tau) \end{aligned}$$

Theorem 3. *After the transcendental changing of variables to y , the symmetrized generating series $\mathbf{C}\Xi^{\{1,\dots,m\}}\mathcal{C}_m^g(y_1, \dots, y_m, \tau)$ is a polynomial of the variables y_i 's of total degree $6g - 6 + 3m$, and satisfy the following cut-joint equation:*

$$\begin{aligned} &\left(\frac{\partial}{\partial\tau} + \sum_{l=1}^m \frac{y_l(y_l-1)}{\tau+1} \cdot \frac{\partial}{\partial y_l}\right)\mathbf{C}\Xi^{\{1,\dots,m\}}\mathcal{C}_m^g(y_1, \dots, y_m, \tau) \\ &= -\frac{1}{2}\sum_{l=1}^m y_l(y_l-1)\left(\frac{y_l\tau+1}{\tau+1}\right)\frac{\partial}{\partial y_l} \cdot y_{m+1}(y_{m+1}-1)\left(\frac{y_{m+1}\tau+1}{\tau+1}\right)\frac{\partial}{\partial y_{m+1}}\mathbf{C}\Xi^{\{1,\dots,m+1\}}\mathcal{C}_{m+1}^{g-1}|_{y_{m+1}=y_l} \\ &\quad -\frac{1}{2}\sum_{1\leq a\leq g-1}\sum_{1\leq k\leq m}\Theta_{k-1}(y_1(y_1-1)\left(\frac{y_1\tau+1}{\tau+1}\right)\frac{\partial}{\partial y_1}\mathbf{C}\Xi^{\{1,\dots,k\}}\mathcal{C}_k^a) \\ &\quad \cdot (y_1(y_1-1)\left(\frac{y_1\tau+1}{\tau+1}\right)\frac{\partial}{\partial y_1}\mathbf{C}\Xi^{\{1,k+1,\dots,m\}}\mathcal{C}_{m-k+1}^{g-a}) \\ &\quad -\sum_{k=3}^m \Theta_{k-1}(y_1(y_1-1)\left(\frac{y_1\tau+1}{\tau+1}\right)\frac{\partial}{\partial y_1}\mathbf{C}\Xi^{\{1,\dots,k\}}\mathcal{C}_k^0)(y_1(y_1-1)\left(\frac{y_1\tau+1}{\tau+1}\right)\frac{\partial}{\partial y_1}\mathbf{C}\Xi^{\{1,k+1,\dots,m\}}\mathcal{C}_{m-k+1}^g) \\ &\quad + \Theta_1 \frac{y_1^2(y_1-1)(y_2-1)}{y_1-y_2}\left(\frac{y_1\tau+1}{\tau+1}\right)\frac{\partial}{\partial y_1}\mathbf{C}\Xi^{\{1,3,\dots,m\}}\mathcal{C}_{m-1}^g \end{aligned}$$

5. APPLICATIONS

This cut-join equation is a generalization of the symmetrized cut-join equation of [6]. In their equation, only Hodge integrals with at most one λ class show up, while in ours equation, Hodge integrals have up to three λ classes. This is not surprising, since their starting point ESLV formula, as Liu-Liu-Zhou showed [14] and [15], is the large τ limit of Marino-Vafa formula. Thus we expect taking large τ limit of our symmetrized cut-join equation, one should be able to recover the equation of [5].

To illustrate the application of our symmetrized cut-join equation, we make a similar derivation of the Witten conjecture (Kontevich Theorem) as in [1]. We don't regard this as a new proof.

Both of the two sides are polynomials of m variables y_1, \dots, y_m , of total degree $6g - 5 + 3m$. We compare this leading term. Recall $\Gamma_g(\tau) = \Lambda_g^\vee(1)\Lambda_g^\vee(\tau)\Lambda_g^\vee(-\tau-1)$, and only its constant $(-)^g[\tau(\tau+1)]^g$ has contribution in the leading degree term. Denote \mathbf{F}_d the operator sending a formal power series to its degree d part.

$$\begin{aligned} &\mathbf{F}_{6g-6+3m}\mathbf{C}\Xi^{\{1,\dots,m\}}\mathcal{C}_m^g(y_1, \dots, y_m, \tau) \\ &= (-1)^{g-1}\left(\frac{\tau^2}{1+\tau}\right)^{2g-2+m}\sum_{b_1+\dots+b_m=3g-3+m}\langle \tau_{b_1}, \dots, \tau_{b_m} \rangle \prod_{i=1}^m (2b_i-1)!! y_i^{2b_i+1} \end{aligned}$$

Where in the above equation, we adopt the following abbreviation and the genus g is determined by the restriction $j_1 + \dots + j_n + d = 3g - 3 + n$ if the degree of ω is d .

$$(5.1) \quad \langle \tau_{j_1} \dots \tau_{j_n} \omega \rangle := \int_{\mathcal{M}_{g,n}} \psi_1^{j_1} \dots \psi_n^{j_n} \omega.$$

For the left hand side, only the derivatives of y_i have contributions:

$$\begin{aligned}
& \mathbf{F}_{6g-5+3m} \left(\frac{\partial}{\partial \tau} + \sum_{l=1}^m \frac{y_l(y_l-1)}{\tau+1} \cdot \frac{\partial}{\partial y_l} \right) \mathbf{C}\Xi^{\{1, \dots, m\}} \mathbf{C}_m^g(y_1, \dots, y_m, \tau) \\
&= \frac{1}{\tau+1} \sum_{l=1}^m y_l^2 \frac{\partial}{\partial y_l} \mathbf{F}_{6g-6+3m} \mathbf{C}\Xi^{\{1, \dots, m\}} \mathbf{C}_m^g(y_1, \dots, y_m, \tau) \\
&= \frac{(-1)^{g-1}}{1+\tau} \left(\frac{\tau^2}{1+\tau} \right)^{2g-2+m} \sum_{l=1}^m \sum_{b_1+\dots+b_m=3g-3+m} \langle \tau_{b_1}, \dots, \tau_{b_m} \rangle \cdot (2b_l+1)!! y_l^{2b_l+2} \prod_{i=1, i \neq l}^m (2b_i-1)!! y_i^{2b_i+1}
\end{aligned}$$

Now go to the right hand side, after applying the operator the $\mathbf{F}_{6g-5+3m}$, the first term becomes

$$\begin{aligned}
& \frac{1}{2} \cdot \frac{(-1)^{g-1}}{1+\tau} \left(\frac{\tau^2}{1+\tau} \right)^{2g-2+m} \sum_{l=1}^m \sum_{b_1+\dots+b_{m+1}=3g-5+m} \langle \tau_{b_1}, \dots, \tau_{b_{m+1}} \rangle \\
& \quad \cdot (2b_l+1)!! (2b_{m+1}+1)!! y_l^{2b_l+2b_{m+1}+6} \prod_{i=1, i \neq l}^m (2b_i-1)!! y_i^{2b_i+1}
\end{aligned}$$

and the second term becomes.

$$\begin{aligned}
& \frac{(-1)^{g-1}}{2(1+\tau)} \left(\frac{\tau^2}{1+\tau} \right)^{2g-2+m} \\
& \cdot \sum_{1 \leq a \leq g-1} \sum_{1 \leq k \leq m} \Theta_{k-1} \left[\sum_{b_1+\dots+b_k=3a-3+k} \langle \tau_{b_1}, \dots, \tau_{b_k} \rangle (2b_1+1)!! y_1^{2b_1+3} \prod_{i=2}^k (2b_i-1)!! y_i^{2b_i+1} \right] \\
& \cdot \left[\sum_{b'_1+b_{k+1}+\dots+b_m=3(g-a-1)+(m+1-k)} \langle \tau_{b'_1}, \tau_{b_{k+1}}, \dots, \tau_{b_m} \rangle (2b'_1+1)!! y_1^{2b'_1+3} \prod_{i=k+1}^m (2b_i-1)!! y_i^{2b_i+1} \right]
\end{aligned}$$

The third term basically is the same as the second, except that the summation range is fixing $a = 0$ and k varies from 3 to m .

$$\begin{aligned}
& \frac{(-1)^{g-1}}{1+\tau} \left(\frac{\tau^2}{1+\tau} \right)^{2g-2+m} \\
& \cdot \sum_{3 \leq k \leq m} \Theta_{k-1} \left[\sum_{b_1+\dots+b_k=k-3} \langle \tau_{b_1}, \dots, \tau_{b_k} \rangle (2b_1+1)!! y_1^{2b_1+3} \prod_{i=2}^k (2b_i-1)!! y_i^{2b_i+1} \right] \\
& \cdot \left[\sum_{b'_1+b_{k+1}+\dots+b_m=3(g-1)+(m+1-k)} \langle \tau_{b'_1}, \tau_{b_{k+1}}, \dots, \tau_{b_m} \rangle (2b'_1+1)!! y_1^{2b'_1+3} \prod_{i=k+1}^m (2b_i-1)!! y_i^{2b_i+1} \right]
\end{aligned}$$

Together with the second term, these gave all the stable cut contributions, and we combine them in the sequel.

The fourth term is

$$\begin{aligned}
& \frac{1}{2} \frac{(-1)^{g-1}}{1+\tau} \left(\frac{\tau^2}{1+\tau} \right)^{2g-2+m} \Theta_1 \sum_{b_1+b_3+\dots+b_m=3g-4+m} \langle \tau_{b_1}, \tau_{b_3}, \dots, \tau_{b_m} \rangle \\
& (2b_1+1)!! y_1 y_2 \left(\frac{y_1^{2b_1+4} - y_2^{2b_1+4}}{y_1 - y_2} \right) \prod_{i=3}^m (2b_i-1)!! y_i^{2b_i+1}
\end{aligned}$$

Collect all these, and comparing the coefficients of $y_l^{2b_l+2} \prod_{i=1, i \neq l}^m y_i^{2b_i+1}$, we get the Dijkgraaf-Verlinde-Verlinde formula, which is equivalent to Witten conjecture. See also [1] and [9] for more detail.

For other more interesting applications, one may take other special values of τ , or consider other degree in terms in theorem 3. For example, the lowest and the next lowest degree terms of theorem 3 may give some relations for Hodge integrals $\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{j_1} \cdots \psi_n^{j_n} \lambda_g \lambda_{g-1} \lambda_{g-3}$, which may be interesting.

REFERENCES

- [1] L. Chen, Y. Li, K. Liu. *Localization, Hurwitz numbers and Witten conjecture*, preprint, 2006
- [2] R. Dijkgraaf. *Intersection Theory, Integrable Hierarchies and Topological Field Theory*, New symmetry principles in quantum field theory (Cargse, 1991), 95-158, NATO Adv. Sci. Inst. Ser. B Phys., 295, Plenum, New York, 1992.
- [3] T. Ekedahl, S. Lando, M. Shapiro, A. Vainshtein. *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146** (2001), 297-327.
- [4] C. Faber, R. Pandharipande. *Hodge integrals, partition matrices, and the λ_g conjecture*, Ann. of math. **157** (2003), 97-124.
- [5] I.P. Goulden, D. M. Jackson and A. Vainshtein. *The number of ramified coverings of the sphere by the torus and surfaces of higher genera*, Ann. Combinatorics 4 (2000) 27-46.
- [6] I.P. Goulden, D. M. Jackson and R. Vakil. *A short proof of λ_g conjecture without Gromov-Witten theory: Hurwitz theory and the moduli of curves*, preprint: math.AG/0604297.
- [7] S. Katz, C.-C. Liu. *Enumerative geometry of stable maps with Lagrangian boundary conditions and multiple covers of the disc*, Adv.Theor.Math.Phys. **5** (2002), 1-49.
- [8] M. Kazarian, S. Lando. *An algebro-geometric proof of Witten's conjecture*, MPIM-preprint, 2005-55.
- [9] Y.-S. Kim, K. Liu. *A simple proof of Witten conjecture through localization*, preprint: math.AG/0508384.
- [10] M. Kontsevich. *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), no. 1,1-23.
- [11] A.M. Li, G. Zhao, Q. Zheng. *The number of ramified coverings of a Riemann surface by Riemann surface*, Comm.Math.Phys. **213** (2000), 685-696.
- [12] J. Li. *Stable Morphisms to singular schemes and relative stable morphisms*, J. Diff. Geom. **57** (2001), 509-578.
- [13] Y. Li. *Some Results of Mariño-Vafa Formula*, math.AG/0601167. (to be published on Math. Res. Lett. Volume 13, Issue 6 (2006))
- [14] C.-C. Liu, K. Liu, J. Zhou. *A proof of a conjecture of Mariño-Vafa on Hodge Integrals*, J. Differential Geom. **65** (2003), 289-340.
- [15] _____. *Mariño-Vafa formula and Hodge integral identities*, J.Algebraic Geom. **15** (2006), 379-398.
- [16] M. Mirzakhani. *Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces*, preprint, 2003.
- [17] A. Okounkov, R. Pandharipande. *Gromov-Witten theory, Hurwitz numbers, and Matrix models, I*, preprint: math.AG/0101147
- [18] R. Stanley. *Enumerative Combinatorics*, Cambridge University Press, 1997.
- [19] E. Witten. *Two-dimensional gravity and intersection theory on moduli space*, Surveys in differential geometry (Cambridge, MA, 1990), 243-310, Lehigh Univ., Bethlehem, PA, 1991.

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