DIFFERENCE SETS AND POLYNOMIALS OF PRIME VARIABLES

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Abstract. Let $\psi(x)$ be a polynomial with rational coefficients. Suppose that ψ has the positive leading coefficient and zero constant term. Let A be a set of positive integers with the positive upper density. Then there exist $x, y \in A$ and a prime p such that $x - y = \psi(p-1)$. Furthermore, if P is a set of primes with the positive relative upper density, then there exist $x, y \in P$ and a prime p such that $x - y = \psi(p-1)$.

1. Introduction

For a set A of positive integers, define

$$\overline{d}(A) = \limsup_{x\to\infty} \frac{|A \cap [1, x]|}{x}.$$

Furstenberg [9, Theorem 1.2] and Sárközy [21] independently confirmed the following conjecture of Lovász:

Theorem 1.1. *Suppose that A is a set of positive integers with $\overline{d}(A) > 0$, then there exist $x, y \in A$ and a positive integer z such that $x - y = z^2$.*

In fact, the z^2 in Theorem 1.1 can be replaced by an arbitrary integral-valued polynomial $f(z)$ with $f(0) = 0$. On the other hand, Sárközy [22] also solved a problem of Erdős:

Theorem 1.2. *Suppose that A is a set of positive integers with $\overline{d}(A) > 0$, then there exist $x, y \in A$ and a prime p such that $x - y = p - 1$.*

For the further developments of Theorems 1.1 and 1.2, the readers may refer to [24], [18], [1], [10], [16], [17], [20]. In the present paper, we shall give a common generalization of Theorems 1.1 and 1.2. Define

$$\Lambda_{b,W} = \{x : Wx + b \text{ is prime}\}$$

for $1 \leq b \leq W$ with $(b, W) = 1$.

Theorem 1.3. *Let $\psi(x)$ be a polynomial with integral coefficients and zero constant term. Suppose that $A \subseteq \mathbb{Z}^+$ satisfies $\overline{d}(A) > 0$. Then there exist $x, y \in A$ and $z \in \Lambda_{1,W}$ such that $x - y = \psi(z)$.*

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Corollary 1.1. *Let $\psi(x)$ be a polynomial with rational coefficients and zero constant term. Suppose that $A \subseteq \mathbb{Z}^+$ satisfies $\overline{d}(A) > 0$. Then there exist $x, y \in A$ and a prime p such that $x - y = \psi(p-1)$.*

Proof. Let W be the least common multiple of the denominators of the coefficients of ψ. Then the coefficients of $\psi^*(x) = \psi(Wx)$ are all integers. Then by Theorem 1.3, there exist $x, y \in A$ and $z \in \Lambda_{1,W}$ such that

$$x - y = \psi^*(z) = \psi(p-1)$$

where $p = Wz + 1$. □

Quite recently, about one month after the first version of this paper was open in the arXiv server, in [3] Bergelson and Lesigne proved that the set

$$\{(\psi_1(p-1), \ldots, \psi_m(p-1)) : p \text{ prime}\}$$

is an enhanced van der Corput set $\mathbb{Z}^m$, where $\psi_1, \ldots, \psi_m$ are polynomials with integral coefficients and zero constant term. Of course, their result can be extended to the set $\{(\psi_1(z), \ldots, \psi_m(z)) : z \in \Lambda_{1,W}\}$ without any special difficulty. On the other hand, Kamae and Mendés France [15] proved that any van der Corput set is also a set of 1-recurrence. Hence Bergelson and Lesigne's result also implies our Theorem 1.3 and Corollary 1.1. In fact, they showed that the set $\{\psi(p-1) : p \text{ prime}\}$ is not only a set of 1-recurrence, but also a set of strong 1-recurrence.

For two sets A, X of positive integers, define

$$\overline{d}_X(A) = \limsup_{x\to\infty} \frac{|A \cap X \cap [1,x]|}{|X \cap [1,x]|}.$$

Let $\mathcal{P}$ denote the set of all primes. In [11], Green established a Roth's-type extension of a result of van der Corput [6] on 3-term arithmetic progressions in primes:

Let P be a set of primes with $\overline{d}_{\mathcal{P}}(P) > 0$, then there exists a non-trivial 3-term arithmetic progressions contained in P.

The key of Green's proof is a transference principle, which transfers a subset $P \subseteq \mathcal{P}$ to a subset $A \subseteq \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ with $|A|/N \geq \overline{d}_{\mathcal{P}}(P)/64$, where N is a large prime. Using Green's ingredients, now we can show that:

Theorem 1.4. *Let $\psi(x)$ be a polynomial with integral coefficients and zero constant term. Suppose that $P \subseteq \mathcal{P}$ satisfies $\overline{d}_{\mathcal{P}}(P) > 0$. Then there exist $x, y \in P$ and $z \in \Lambda_{1,W}$ such that $x - y = \psi(z)$.*

Similarly, we have

Corollary 1.2. *Let $\psi(x)$ be a polynomial with rational coefficients and zero constant term. Suppose that $P \subseteq \mathcal{P}$ satisfies $\overline{d}_{\mathcal{P}}(P) > 0$. Then there exist $x, y \in P$ and a prime p such that $x - y = \psi(p-1)$.*

On the other hand, the well-known Szemerédi theorem [23] asserts that for any set A of positive integers with $\overline{d}(A) > 0$, there exist arbitrarily long arithmetic progressions contained in A. In [2], Bergelson and Leibman extended Theorem 1.1 and Szemerédi's theorem:

Let $\psi_1(x),\dots,\psi_m(x)$ be arbitrary integral-valued polynomials with $\psi_1(0)=\cdots=\psi_m(0)=0$. Then for any set A of positive integers with $\overline{d}(A)>0$, there exist $x\in A$ and a integer z such that $x+\psi_1(z),\dots,x+\psi_m(z)$ are all contained in A.

Recently, Tao and Ziegler [26] proved that:

Let $\psi_1(x),\dots,\psi_m(x)$ be arbitrary integral-valued polynomials with $\psi_1(0)=\cdots=\psi_m(0)=0$. Then for any set P of primes with $\overline{d}_{\mathcal{P}}(P)>0$, there exist $x\in P$ and a integer z such that $x+\psi_1(z),\dots,x+\psi_m(z)$ are all contained in P.

This is a generalization of Green and Tao's celebrated result [12] that the primes contain arbitrarily long arithmetic progressions. Furthermore, with the help of a very deep result due to Green and Tao [13] on the Gowers norms [14], Frantzikinakis, Host and Kra [8] proved that if $\overline{d}(A)>0$ then A contains a 3-term arithmetic progression with the difference $p-1$, where p is a prime. In fact, using the methods of Green and Tao in [13], it is not difficult to replace A by P with $\overline{d}_{\mathcal{P}}(P)>0$ in the result of Frantzikinakis, Host and Kra.

Motivated by the above results, here we propose two conjectures:

Conjecture 1.1. *Let $\psi_1(x),\dots,\psi_m(x)$ be arbitrary polynomials with rational coefficients and zero constant terms. Then for any set A of positive integers with $\overline{d}(A)>0$, there exist $x\in A$ and a prime p such that $x+\psi_1(p-1),\dots,x+\psi_m(p-1)$ are all contained in A.*

Conjecture 1.2. *Let $\psi_1(x),\dots,\psi_m(x)$ be arbitrary polynomials with rational coefficients and zero constant terms. Then for any set P of primes with $\overline{d}_{\mathcal{P}}(P)>0$, there exist $x\in P$ and a prime p such that $x+\psi_1(p-1),\dots,x+\psi_m(p-1)$ are all contained in P.*

The proofs of Theorems 1.3 and 1.4 will be given in section 3 and section 4. Throughout this paper, without the additional mentions, the constants implied by $\ll$, $\gg$ and $O(\cdot)$ will only depend on the degree of ψ.

2. Some Necessary Lemmas on Exponential Sums

Let $\mathbb{T}$ denote the torus $\mathbb{R}/\mathbb{Z}$. For any function f over $\mathbb{Z}$, define $f^{\Delta}(x)=f(x+1)-f(x)$. Also, we abbreviate $e^{2\pi\sqrt{-1}x}$ to $e(x)$. Let $\psi(x)=a_1x^k+\cdots+a_kx$ be a polynomial with integral coefficients. In this section, we always assume that $W,|a_1|,\dots,|a_k|\le\log N$.

Lemma 2.1. *Suppose that $h(x)$ is an arbitrary polynomial and $0<\nu<1$. Then for any $\alpha\in\mathbb{T}$*

$$\sum_{x=1}^{N}h(x)e(\alpha\psi(x))=\frac{1}{q}\sum_{r=1}^{q}e(a\psi(r)/q)\sum_{x=1}^{N}h(x)e((\alpha-a/q)\psi(x))+O_{\deg h}(h(N)N^{\nu})$$

provided that $|\alpha q-a|\le N^{\nu}/\psi(N)$ with $1\le a\le q\le N^{\nu}$.

Proof. Let $\theta = \alpha - a/q$. Then by a partial summation, we have

$$\begin{aligned}&\sum_{x=1}^{N} h(x)e(a\psi(x)/q)e(\theta\psi(x))\\ =&h(N)e(\theta\psi(N))F_N(a/q) - \sum_{y=1}^{N-1}(h(y+1)e(\theta\psi(y+1)) - h(y)e(\theta\psi(y)))F_y(a/q),\end{aligned}$$

where

$$F_y(a/q) := \sum_{x=1}^{y} e(a\psi(x)/q) = \frac{y}{q}\sum_{r=1}^{q} e(a\psi(r)/q) + O(q).$$

Clearly

$$\begin{aligned}&h(y+1)e(\theta\psi(y+1)) - h(y)e(\theta\psi(y))\\ =&(h(y+1) - h(y))e(\theta\psi(y+1)) + h(y)e(\theta\psi(y))(e(\theta\psi^{\Delta}(y)) - 1)\\ =&O(h^{\Delta}(y)) + O(h(y)\theta\psi^{\Delta}(y)).\end{aligned}$$

This concludes that

$$\begin{aligned}&\sum_{x=1}^{N} h(x)e(a\psi(x)/q)e(\theta\psi(x))\\ =&\frac{1}{q}\sum_{r=1}^{q} e(a\psi(r)/q)\sum_{x=1}^{N} h(x)e(\theta\psi(x)) + O(\theta qN\psi^{\Delta}(N)h(N)) + O(qh^{\Delta}(N)N).\end{aligned}$$

□

Define

$$\lambda_{b,W}(x) = \begin{cases}\frac{\phi(W)}{W}\log(Wx+b) & \text{if } Wx+b \text{ is prime},\\ 0 & \text{otherwise},\end{cases}$$

where ϕ is the Euler totient function.

Lemma 2.2. *Suppose that $h(x)$ is an arbitrary polynomial and $B > 1$. Then for any $\alpha \in \mathbb{T}$*

$$\begin{aligned}&\sum_{x=1}^{N} h(x)\lambda_{b,W}(x)e(\alpha\psi(x))\\ =&\frac{\phi(W)}{\phi(Wq)}\sum_{\substack{1\le r\le q\\ (Wr+b,q)=1}} e(a\psi(r)/q)\sum_{x=1}^{N} h(x)e((\alpha - a/q)\psi(x)) + O_{\deg h}(h(N)Ne^{-c\sqrt{\log N}})\end{aligned}$$

provided that $|\alpha q - a| \le (\log N)^B/\psi(N)$ with $1 \le a \le q \le (\log N)^B$, where c is a positive constant.

Proof. Let

$$F_y(a/q) = \sum_{x=1}^{y} \lambda_{b,W}(x)e(a\psi(x)/q)$$
$$= \sum_{\substack{1\le r\le Wq\\(r,q)=1\\ r\equiv b \pmod W}} e(a\psi((r-b)/W)/q) \sum_{\substack{x\in\Lambda_{r,Wq}\\ Wqx+r\le Wy+b}} \frac{\phi(W)q}{\phi(Wq)}\lambda_{r,Wq}(x).$$

The well-known Siegel-Walfisz theorem (cf. [7]) asserts that

$$\sum_{\substack{p\le y \text{ is prime}\\ p\equiv b \pmod q}} \log p = \frac{y}{\phi(q)} + O(ye^{-c'\sqrt{\log y}})$$

provided that $q \le \log^{c_1} y$, where c_1, c' are positive constants. Hence

$$\sum_{\substack{x\in\Lambda_{r,Wq}\\ Wqx+r\le Wy+b}} \lambda_{r,Wq}(x) = \frac{y}{q} + O(Wye^{-c'\sqrt{\log(Wy)}}).$$

It follows that

$$F_y(a/q) = \frac{\phi(W)y}{\phi(Wq)} \sum_{\substack{1\le r\le q\\ (Wr+b,q)=1}} e(a\psi(r)/q) + O(ye^{-c'\sqrt{\log y}/2}).$$

Let $\theta = \alpha - a/q$. Then

$$\sum_{x=1}^{N} h(x)\lambda_{b,W}(x)e(\alpha\psi(x))$$
$$=h(N)e(\theta\psi(N))F_N(a/q) - \sum_{y=1}^{N-1}(h(y+1)e(\theta\psi(y+1)) - h(y)e(\theta\psi(y)))F_y(a/q)$$
$$=\frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1\le r\le q\\ (Wr+b,q)=1}} e(a\psi(r)/q)\sum_{y=1}^{N} h(y)e(\theta\psi(y)) + O(h(N)Ne^{-c'\sqrt{\log N}/3})$$

by noting that

$$h(y+1)e(\theta\psi(y+1)) - h(y)e(\theta\psi(y)) = O(h^{\Delta}(y)) + O(h(y)\theta\psi^{\Delta}(y+1)).$$

□

Lemma 2.3. *For any $\theta \in \mathbb{T}$,*

$$\sum_{x=1}^{N} \psi^{\Delta}(x-1)e(\theta\psi(x)) = \sum_{x=1}^{\psi(N)} e(\theta x) + O(\theta\psi(N)\psi^{\Delta}(N)).$$

Proof. Clearly

$$\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\theta\psi(x))-\sum_{x=1}^{\psi(N)}e(\theta x)=\sum_{x=1}^{N}e(\theta\psi(x))\sum_{y=0}^{\psi^{\Delta}(x-1)-1}(1-e(-\theta y))$$
$$=O\bigg(\sum_{x=1}^{N}\sum_{y=0}^{\psi^{\Delta}(x-1)-1}\theta y\bigg)$$
$$=O(\theta\psi(N)\psi^{\Delta}(N)).$$

□

Lemma 2.4. *For any $\epsilon>0$,*

$$\sum_{x=1}^{N}e(\alpha\psi(x))\ll_{\epsilon}N^{1+\epsilon}\bigg(\frac{a_1}{q}+\frac{a_1}{N}+\frac{q}{N^k}\bigg)^{2^{1-k}}$$

provided that $|\alpha-a/q|\le q^{-2}$.

Proof. We left the proof of Lemma 2.4 as an exercise for the readers, since it is just a little modification of the proof of Wely's inequality [27, Lemma 2.4]. □

Lemma 2.5 (Hua)**.** *Suppose that $(q,a_1,\dots,a_k)=1$. Then*

$$\sum_{r=1}^{q}e(\psi(r)/q)\ll_{\epsilon}q^{1-\frac{1}{k}+\epsilon}$$

for any $\epsilon>0$.

Proof. See [27, Theorem 7.1]. □

Lemma 2.6.

$$\int_{\mathbb{T}}\bigg|\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\alpha\psi(x))\bigg|^{\rho}d\alpha\ll_{\rho}\gcd(\psi)\psi(N)^{\rho-1}$$

for $\rho\ge k2^{k+2}$, where $\gcd(\psi)$ denotes the greatest common divisor of $a_1,\dots,a_k$.

Proof. Notice that

$$\int_0^1\bigg|\sum_{x=1}^{N}(a\psi)^{\Delta}(x-1)e(\alpha a\psi(x))\bigg|^{\rho}d\alpha=a^{\rho-1}\int_0^a\bigg|\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\alpha\psi(x))\bigg|^{\rho}d\alpha$$
$$=a^{\rho}\int_0^1\bigg|\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\alpha\psi(x))\bigg|^{\rho}d\alpha.$$

So without loss of generality, we may assume that $\gcd(\psi)=1$. Let $\nu=1/5$ and $\epsilon=2^{-k}\nu-\frac{k}{2\rho}$. Let

$$\mathfrak{M}_{a,q}=\{\alpha\in\mathbb{T}:|\alpha q-a|\le N^{\nu}/\psi(N)\},\quad\mathfrak{M}=\bigcup_{\substack{1\le a\le q\le N^{\nu}\\(a,q)=1}}\mathfrak{M}_{a,q}$$

and $\mathfrak{m} = \mathbb{T} \setminus \mathfrak{M}$. Clearly $\mathrm{mes}(\mathfrak{M}) \leq N^{3\nu}/\psi(N)$, where $\mathrm{mes}(\mathfrak{M})$ denotes the Lebesgue measure of $\mathfrak{M}$.

If $\alpha \in \mathfrak{m}$, then by Lemma 2.4 we have

$$\begin{aligned}&\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\alpha\psi(x))\\ =&\psi^{\Delta}(N-1)\sum_{x=1}^{N}e(\alpha\psi(x))-\sum_{y=1}^{N-1}(\psi^{\Delta}(y)-\psi^{\Delta}(y-1))\sum_{x=1}^{y}e(\alpha\psi(x))\\ \ll_{\epsilon}&\psi^{\Delta}(N)N^{1+\epsilon-2^{1-k}\nu}.\end{aligned}$$

Hence

$$\int_{\mathfrak{m}}\bigg|\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\alpha\psi(x))\bigg|^{\rho}d\alpha \ll_{\epsilon} \psi(N)^{\rho}N^{\rho(\epsilon-2^{1-k}\nu)}=o(\psi(N)^{\rho-1}).$$

On the other hand, when $\alpha \in \mathfrak{M}$, by Lemmas 2.1 and 2.3,

$$\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\alpha\psi(x))=\frac{1}{q}\sum_{r=1}^{q}e(a\psi(r)/q)\sum_{x=1}^{\psi(N)}e((\alpha-a/q)x)+O(\psi^{\Delta}(N)N^{\nu}).$$

Let $L = \lfloor \rho/2 \rfloor$. Obviously

$$\int_{\mathfrak{M}}\bigg|\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\alpha\psi(x))\bigg|^{\rho}d\alpha \leq \psi(N)^{\rho-2L}\int_{\mathfrak{M}}\bigg|\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\alpha\psi(x))\bigg|^{2L}d\alpha.$$

So it suffices to show that

$$\int_{\mathfrak{M}}\bigg|\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\alpha\psi(x))\bigg|^{2L}d\alpha \ll_{L} \psi(N)^{2L-1}.$$

Now

$$\begin{aligned}&\bigg|\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\alpha\psi(x))\bigg|^{2L}\\ =&\bigg|\frac{1}{q}\sum_{r=1}^{q}e(a\psi(r)/q)\sum_{x=1}^{\psi(N)}e((\alpha-a/q)x)\bigg|^{2L}+O(\psi(N)^{2L-1}\psi^{\Delta}(N)N^{\nu}).\end{aligned}$$

Hence

$$\begin{aligned}&\int_{\mathfrak{M}}\bigg|\sum_{x=1}^{N}\psi^{\Delta}(x-1)e(\alpha\psi(x))\bigg|^{2L}d\alpha\\ =&\sum_{\substack{1\leq a\leq q\leq N^{\nu}\\ (a,q)=1}}\int_{\mathfrak{M}_{a,q}}\bigg|\frac{1}{q}\sum_{r=1}^{q}e(a\psi(r)/q)\sum_{x=1}^{\psi(N)}e((\alpha-a/q)x)\bigg|^{2L}d\alpha\\ &+O(\psi(N)^{2L-1}\psi^{\Delta}(N)N^{\nu}\mathrm{mes}(\mathfrak{M})).\end{aligned}$$

Clearly

$$\int_{\mathfrak{M}_{a,q}}\left|\sum_{x=1}^{\psi(N)}e((\alpha-a/q)x)\right|^{2L}d\alpha\le\int_{\mathbb{T}}\left|\sum_{x=1}^{\psi(N)}e((\alpha-a/q)x)\right|^{2L}d\alpha$$
$$=\sum_{\substack{1\le x_1,\dots,x_{2L}\le\psi(N)\\ x_1+\cdots+x_L=x_{L+1}+\cdots+x_{2L}}}1$$
$$\le\psi(N)^{2L-1}.$$

And by Lemma 2.5,

$$\sum_{\substack{1\le a\le q\le N^\nu\\(a,q)=1}}\left|\frac{1}{q}\sum_{r=1}^{q}e(a\psi(r)/q)\right|^{2L}\ll_\epsilon\sum_{\substack{1\le a\le q\le N^\nu\\(a,q)=1}}q^{-2L(\frac{1}{k}-\epsilon)}\le\sum_{1\le q\le N^\nu}q^{1-2L(\frac{1}{k}-\epsilon)}=O_L(1)$$

since $L>(\frac{1}{k}-\epsilon)^{-1}$. We are done. □

Lemma 2.7.

$$\sum_{\substack{1\le r\le q\\(Wr+b,q)=1}}e(a\psi(r)/q)\ll_\epsilon\gcd(\psi)q^{1-\frac{1}{k(k+1)}+\epsilon}.$$

Proof. Clearly

$$\sum_{\substack{1\le r\le q\\(Wr+b,q)=1}}e(a\psi(r)/q)=\sum_{r=1}^{q}e(a\psi(r)/q)\sum_{d|(Wr+b,q)}\mu(d)$$

where μ is the Möbius function. Note that $d\mid(Wr+b)\implies(d,W)=1$ since $(W,b)=1$. Hence

$$\sum_{\substack{1\le r\le q\\(Wr+b,q)=1}}e(a\psi(r)/q)=\sum_{\substack{d|q\\ b_d\text{ exists}}}\mu(d)\sum_{\substack{1\le r\le q\\ r\equiv b_d\pmod d}}e(a\psi(r)/q),$$

where $1\le b_d\le d$ is the integer such that $Wb_d+b\equiv 0\pmod d$.

For those $d\le q^{\frac{1}{k(k+1)}}$ with b_d exists, we have

$$\sum_{\substack{1\le r\le q\\ r\equiv b_d\pmod d}}e(a\psi(r)/q)=\sum_{r=0}^{q/d-1}e(a\psi(dr+b_d)/q).$$

Write

$$\psi(dr+b_d)=\sum_{i=1}^{k}a_{k-i+1}\sum_{j=0}^{i}\binom{i}{j}d^jr^jb_d^{i-j}$$
$$=\sum_{j=0}^{k}d^jr^j\sum_{i=j}^{k}\binom{i}{j}a_{k-i+1}b_d^{i-j}$$
$$=a_1'r^k+a_2'r^{k-1}+\cdots+a_k'r+a_{k+1}'.$$

Notice that

$$(q, a'_1, \ldots, a'_k) = (q, d^k a_1, a'_2, \ldots, a'_k) \le d^k (q, a_1, a'_2, \ldots, a'_k).$$

Also

$$a'_2 = d^{k-1}(a_2 + k a_1 b_d).$$

Therefore

$$(q, a_1, a'_2, \ldots, a'_k) = (q, a_1, d^{k-1} a_2, \ldots, a'_k) \le d^{k-1}(q, a_1, a_2, \ldots, a'_k).$$

Similarly, we obtain that

$$(q, a'_1, \ldots, a'_k) \le d^{\frac{k(k+1)}{2}} (q, a_1, \ldots, a_k).$$

Thus by Lemma 2.5,

$$\begin{aligned}\sum_{r=0}^{q/d-1} e(a\psi(dr+b_d)/q) \ll_\epsilon & (q/d, a'_1, \ldots, a'_k) \left(\frac{q/d}{(q/d, a'_1, \ldots, a'_k)} \right)^{1-\frac{1}{k}+\frac{\epsilon}{k}} \\ \le & (q, a'_1, \ldots, a'_k)^{\frac{1-\epsilon}{k}} d^{\frac{1-\epsilon}{k}-1} q^{1-\frac{1-\epsilon}{k}} \\ \le & (a_1, \ldots, a_k)^{\frac{1-\epsilon}{k}} d^{(\frac{k+1}{2}+\frac{1}{k})(1-\epsilon)-1} q^{1-\frac{1-\epsilon}{k}}.\end{aligned}$$

On the other hand, clearly

$$\Big| \sum_{r=0}^{q/d-1} e(a\psi(dr+b_d)/q) \Big| \le \frac{q}{d} < q^{1-\frac{1}{k(k+1)}}$$

when $d > q^{\frac{1}{k(k+1)}}$. Thus

$$\begin{aligned}&\Big| \sum_{\substack{1\le r\le q \\ (Wr+b,q)=1}} e(a\psi(r)/q) \Big| \\ \le & \sum_{\substack{d|q,\ d\le q^{\frac{1}{k(k+1)}} \\ \text{and } b_d \text{ exists}}} \Big| \sum_{\substack{1\le r\le q \\ r\equiv b_d \pmod{d}}} e(a\psi(r)/q) \Big| + \sum_{\substack{d|q,\ d> q^{\frac{1}{k(k+1)}} \\ \text{and } b_d \text{ exists}}} \Big| \sum_{\substack{1\le r\le q \\ r\equiv b_d \pmod{d}}} e(a\psi(r)/q) \Big| \\ \ll_\epsilon & d(q) (\gcd(\psi))^{\frac{1-\epsilon}{k}} q^{1-\frac{1-\epsilon}{k}+\frac{1-\epsilon}{2(k+1)}} + q^{1-\frac{1}{k(k+1)}}) \\ \ll_\epsilon & \gcd(\psi) q^{1-\frac{1}{k(k+1)}+\epsilon},\end{aligned}$$

where $d(q)$ is the divisor function. □

Lemma 2.8. *For any $A > 0$, there is a $B = B(A, k) > 0$ such that,*

$$\sum_{x=1}^{N} \lambda_{b,W}(x) e(\alpha\psi(x)) \ll_B N(\log N)^{-A}$$

provided that $|\alpha - a/q| \le q^{-2}$ with $1 \le a \le q$, $(a, q) = 1$ and $(\log N)^B \le q \le \psi(N)(\log N)^{-B}$.

Proof. At least Vinogradov had dealt with the case $\psi(x) = x^k$ and $W = 1$ in [28]. The proof of this Lemma is very standard but too long, so we give the detailed proof as an appendix. □

Lemma 2.9.

$$\int_{\mathbb{T}} \left|\sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\right|^{\rho} d\alpha \ll_{\rho} \gcd(\psi)\psi(N)^{\rho-1}$$

for $\rho \geq k2^{k+2}+1$.

Proof. Without loss of generality, we assume that $\gcd(\psi) = 1$. Let $B > 2\rho$ be a sufficiently large integer satisfying the requirement of Lemma 2.8 for $A = 2\rho$. Let

$$\mathfrak{M}_{a,q} = \{\alpha \in \mathbb{T} : |\alpha q - a| \leq (\log N)^{2B}/\psi(N)\}, \quad \mathfrak{M} = \bigcup_{\substack{1\leq a\leq q\leq (\log N)^{2B}\\ (a,q)=1}} \mathfrak{M}_{a,q}$$

and $\mathfrak{m} = \mathbb{T} \setminus \mathfrak{M}$.

If $\alpha \in \mathfrak{m}$, then there exist $(\log N)^{2B} \leq q \leq \psi(N)(\log N)^{-2B}$ and $1 \leq a \leq q$ with $(a, q) = 1$ such that $|\alpha - a/q| \leq q^{-2}$. By Lemma 2.8,

$$\sum_{x=1}^{y} \lambda_{b,W}(x)e(\alpha\psi(x)) \ll_B y(\log y)^{-2\rho}.$$

for $N(\log N)^{-\frac{B}{k}} \leq y \leq N$. Therefore

$$\begin{aligned}
&\left|\sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\right| \\
=&\left|\psi^{\Delta}(N-1)\sum_{x=1}^{N} e(\alpha\psi(x))\lambda_{b,W}(x) - \sum_{y=1}^{N-1}(\psi^{\Delta})^{\Delta}(y-1)\sum_{x=1}^{y} e(\alpha\psi(x))\lambda_{b,W}(x)\right| \\
\leq&\psi^{\Delta}(N-1)\left|\sum_{x=1}^{N} e(\alpha\psi(x))\lambda_{b,W}(x)\right| + \sum_{1\leq y<N(\log N)^{-\frac{B}{k}}} |(\psi^{\Delta})^{\Delta}(y-1)y| \\
&+ \sum_{N(\log N)^{-\frac{B}{k}}\leq y<N} (\psi^{\Delta})^{\Delta}(y-1)\left|\sum_{x=1}^{y} e(\alpha\psi(x))\lambda_{b,W}(x)\right| \\
\ll_B&\psi(N)(\log N)^{-2\rho}.
\end{aligned}$$

Let $L = \lfloor (\rho-1)/2 \rfloor$, then we have

$$\begin{aligned}
&\int_{\mathfrak{m}} \Big| \sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{\rho} d\alpha \\
\ll_B &(\psi(N)(\log N)^{-2\rho})^{\rho-2L} \int_{\mathfrak{m}} \Big| \sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{2L} d\alpha \\
\ll_L &\psi(N)^{\rho-2L}(\log N)^{-2\rho} \int_{\mathbb{T}} \Big| \sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{2L} d\alpha
\end{aligned}$$

Noting that

$$\begin{aligned}
&\int_{\mathbb{T}} \Big| \sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{2L} d\alpha \\
= &\sum_{\substack{1\le x_1,\dots,x_{2L}\le N \\ \psi(x_1)+\cdots+\psi(x_L)=\psi(x_{L+1})+\cdots+\psi(x_{2L})}} \prod_{j=1}^{2L} \psi^{\Delta}(x_j-1)\lambda_{b,W}(x_j) \\
\le &(\log(WN+b))^{2L} \sum_{\substack{1\le x_1,\dots,x_{2L}\le N \\ \psi(x_1)+\cdots+\psi(x_L)=\psi(x_{L+1})+\cdots+\psi(x_{2L})}} \prod_{j=1}^{2L} \psi^{\Delta}(x_j-1) \\
\ll_L &(\log N)^{2L} \int_{\mathbb{T}} \Big| \sum_{x\le N} \psi^{\Delta}(x-1)e(\alpha\psi(x))\Big|^{2L} d\alpha,
\end{aligned}$$

so using Lemma 2.6 we have

$$\int_{\mathfrak{m}} \Big| \sum_{x\le N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{\rho} d\alpha \ll_L \psi(N)^{\rho-1}(\log N)^{-\rho}.$$

If $\alpha \in \mathfrak{M}_{a,q}$, then by Lemma 2.2

$$\begin{aligned}
&\Big| \sum_{x\le N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\Big|^{\rho} \\
= &\Big| \frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1\le r\le q \\ (Wr+b,q)=1}} e(a\psi(r)/q) \sum_{x\le N} \psi^{\Delta}(x-1)e((\alpha-a/q)\psi(x))\Big|^{\rho} \\
&+ O(\psi(N)^{\rho}(\log N)^{-7B}).
\end{aligned}$$

In view of Lemma 2.7, letting $\epsilon = (k+2)^{-4}$,

$$\sum_{\substack{1\le a\le q\le (\log N)^B \\ (a,q)=1}} \Big| \frac{\phi(W)}{\phi(Wq)} \sum_{\substack{1\le r\le q \\ (Wr+b,q)=1}} e(a\psi(r)/q)\Big|^{\rho} \ll_{\epsilon} \sum_{1\le q\le(\log N)^B} q^{1-\rho(\frac{1}{k(k+1)}-2\epsilon)} = O_{\rho,\epsilon}(1).$$

Applying Lemma 2.6, we concludes that

$$\begin{aligned}
&\int_{\mathfrak{M}}\bigg|\sum_{x\le N}\psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x))\bigg|^{\rho}d\alpha\\
=&\sum_{\substack{1\le a\le q\le(\log N)^B\\(a,q)=1}}\bigg|\frac{\phi(W)}{\phi(Wq)}\sum_{\substack{1\le r\le q\\(Wr+b,q)=1}}e(a\psi(r)/q)\bigg|^{\rho}\\
&\cdot\int_{\mathfrak{M}_{a,q}}\bigg|\sum_{x\le N}\psi^{\Delta}(x-1)e((\alpha-a/q)\psi(x))\bigg|^{\rho}d\alpha+O(\mathrm{mes}(\mathfrak{M})\psi(N)^{\rho}(\log N)^{-7B})\\
\le&\bigg(\sum_{\substack{1\le a\le q\le(\log N)^B\\(a,q)=1}}\bigg|\frac{\phi(W)}{\phi(Wq)}\sum_{\substack{1\le r\le q\\(Wr+b,q)=1}}e(a\psi(r)/q)\bigg|^{\rho}\bigg)\\
&\cdot\int_{\mathbb{T}}\bigg|\sum_{x\le N}\psi^{\Delta}(x-1)e(\alpha\psi(x))\bigg|^{\rho}d\alpha+O(\psi(N)^{\rho-1}(\log N)^{-B})\\
\ll_{\rho,\epsilon}&\psi(N)^{\rho-1}.
\end{aligned}$$

□

Lemma 2.10. *Suppose that ψ is positive and strictly increasing on $[1,N]$. Let $p\ge\psi(N)$ be a prime. Then*

$$\frac{1}{p}\sum_{r=1}^{p}\bigg|\sum_{z=1}^{N}\psi^{\Delta}(z-1)\lambda_{b,W}(z)e(-r\psi(z)/p)\bigg|^{\rho}\ll_{\rho}\gcd(\psi)\psi(N)^{\rho-1}$$

for $\rho\ge k2^{k+2}+1$.

Proof. We require a well-known result of Marcinkiewicz and Zygmund (cf. [11, Lemma 6.5]):

$$\sum_{r\in\mathbb{Z}_p}\bigg|\sum_{x=1}^{p}f(x)e(-xr/p)\bigg|^{\rho}\ll_{\rho}p\int_{\mathbb{T}}|\hat{f}(\theta)|^{\rho}d\theta$$

for arbitrary function $f:\mathbb{Z}_p=\mathbb{Z}/p\mathbb{Z}\to\mathbb{C}$, where

$$\hat{f}(\theta)=\sum_{x=1}^{p}f(x)e(-\theta x).$$

Define

$$f(x)=\begin{cases}\psi^{\Delta}(z-1)\lambda_{b,W}(z) & \text{if } x=\psi(z) \text{ where } 1\le z\le N,\\ 0 & \text{otherwise.}\end{cases}$$

Then

$$\begin{aligned}
&\sum_{r\in\mathbb{Z}_p}\left|\sum_{z=1}^{N}\psi^{\Delta}(z-1)\lambda_{b,W}(z)e(-\psi(z)r/p)\right|^{\rho}\\
=&\sum_{r\in\mathbb{Z}_p}\left|\sum_{x=1}^{p}f(x)e(-xr/p)\right|^{\rho}\\
\ll_{\rho}&p\int_{\mathbb{T}}\left|\sum_{x=1}^{p}f(x)e(-x\theta)\right|^{\rho}d\theta\\
=&p\int_{\mathbb{T}}\left|\sum_{z=1}^{N}\psi^{\Delta}(z-1)\lambda_{b,W}(z)e(-\psi(z)\theta)\right|^{\rho}d\theta\\
\ll_{\rho}&\operatorname{gcd}(\psi)p\psi(N)^{\rho-1},
\end{aligned}$$

where Lemma 2.9 is applied in the last inequality. □

3. Proof of Theorem 1.3

Clearly Theorem 1.3 is a consequence of the following theorem:

Theorem 3.1. *Suppose that $k \geq t \geq 1$ are integers, a_{k-t+1} is a non-zero integer and $0 < \delta \leq 1$. Let $\psi(x) = a_1x^k + a_2x^{k-1} + \cdots + a_{k-t+1}x^t$ be an arbitrary polynomial with integral coefficients and positive leading coefficient. Then for any positive integer W, there exist $N(\delta, W, \psi)$ and $c(\delta, a_{k-t+1}) > 0$ satisfying that*

$$\min_{\substack{A\subseteq\{1,2,\ldots,n\}\\|A|\geq\delta n}}|\{(x,y,z):\,x,y\in A,z\in\Lambda_{1,W},x-y=\psi(z)\}|\geq c(\delta,a_{k-t+1})\frac{Wn^{1+\frac{1}{k}}a_1^{-\frac{1}{k}}}{\phi(W)\log n}$$

if $n \geq N(\delta, W, \psi)$.

Remark. We emphasize that in Theorem 3.1 the constant $c(\delta, a_{k-t+1})$ only depends on k, δ, a_{k-t+1}. As we will see later, this fact is important in the proof of Theorem 1.4.

Proof. Similarly as Tao's arguments [25] on Roth's theorem [19], we shall make an induction on δ. Suppose that $P(\delta)$ is a proposition on $0 < \delta \leq 1$. Assume that $P(\delta)$ satisfies the following conditions:

(i) There exists $0 < \delta_0 < 1$ such that $P(\delta)$ holds for any $\delta_0 \leq \delta \leq 1$.

(ii) There exists a continuous function $\epsilon(\delta) > 0$ such that $\delta + \epsilon(\delta) \leq 1$ for any $0 < \delta \leq \delta_0$ and $P(\delta + \epsilon(\delta))$ holds implies $P(\delta)$ also holds.

(iii) If $0 < \delta' < \delta \leq 1$, then $P(\delta')$ holds implies that $P(\delta)$ also holds.

Then we claim that $P(\delta)$ holds for any $0 < \delta \leq 1$. In fact, assume on the contrary that there exists $0 < \delta \leq 1$ such that $P(\delta)$ doesn't hold. Let

$$\delta^* = \limsup_{\substack{0<\delta\leq 1\\ P(\delta)\text{ doesn't hold}}} \delta.$$

From the condition (i), we know that $\delta^* \le \delta_0$. Since $\delta + \epsilon(\delta)$ is continuous, there exists $0 < \delta_1 < \delta^*$ such that

$$|(\delta^* + \epsilon(\delta^*)) - (\delta_1 + \epsilon(\delta_1))| < \frac{1}{2}\epsilon(\delta^*),$$

i.e., $0 < \delta_1 < \delta^* < \delta_1 + \epsilon(\delta_1) \le 1$. Hence $P(\delta_1 + \epsilon(\delta_1))$ holds but $P(\delta_1)$ doesn't hold by the definition of δ^*. This is obviously leads to a contradiction with the conditions (ii) and (iii).

Suppose that A is subset of $\{1, 2, \dots, n\}$ with $|A| \ge \delta n$. Firstly, we shall show that Theorem 3.1 holds for $\delta \ge 3/4$. Define

$$r_{W,\psi}(A) = |\{(x, y, z) : x, y \in A, z \in \Lambda_{1,W}, x - y = \psi(z)\}|.$$

Clearly

$$|\{z \in \Lambda_{1,W} : 1 \le \psi(z) \le n/3\}| \ge \frac{1}{4k}\frac{Wn^{\frac{1}{k}}a_1^{-\frac{1}{k}}}{\phi(W)\log n},$$

whenever n is sufficiently large (depending on the coefficients of ψ). And for any $1 \le z \le n/3$,

$$\begin{aligned}|\{(x, y) : x, y \in A, x - y = z\}| =&|A \cap (z + A)|\\ =&2|A| - |A \cup (z + A)|\\ \ge&\frac{2 \cdot 3n}{4} - \frac{4n}{3} = \frac{n}{6}.\end{aligned}$$

Hence

$$r_{W,\psi}(A) \ge \frac{1}{24k}\frac{Wn^{1+\frac{1}{k}}a_1^{-\frac{1}{k}}}{\phi(W)\log n}.$$

Now we assume that $\delta < 3/4$. Let $\epsilon = \epsilon(\delta, a_{k-t+1})$ be a small positive real number and $Q = Q(\delta, a_{k-t+1})$ be a large integer to be chosen later. We shall show that if Theorem 3.1 holds for $\delta + \epsilon$, it also holds for δ. Define

$$\psi_q(x) = \psi(qx)/q^t = a_1q^{k-t}x^k + \cdots + a_{k-t+1}x^t.$$

By the induction hypothesis on $\delta + \epsilon$, for any $1 \le q \le Q$

$$\min_{\substack{A\subseteq\{1,2,\dots,n\}\\ |A|\ge(\delta+\epsilon)n}} r_{Wq,\psi_q}(A) \ge \frac{c(\delta + \epsilon, a_{k-t+1})}{2}\frac{Wq}{\phi(Wq)}\frac{n^{1+\frac{1}{k}}(a_1q^{k-t})^{-\frac{1}{k}}}{\log n}$$

provided that

$$n \ge \max_{1\le q\le Q} N(\delta + \epsilon, Wq, \psi_q).$$

Let $\mathbb{A}_m(b, d)$ denote the arithmetic progression $\{b, b+d, \dots, b+(m-1)d\}$. Suppose that

$$n \ge \max\{e^{k(|a_1|+\cdots+|a_{k-t+1}|)Q^{k-t}}, 10^4\epsilon^{-1}Q^t \max_{1\le q\le Q} N(\delta + \epsilon, Wq, \psi_q)\}$$

and $A \subseteq \{1, 2, \dots, n\}$ with $|A| = \delta n$. Let $m = \lfloor 10^{-2}\epsilon Q^{-t}n\rfloor$. Observe that $|\{b : x, y \in \mathbb{A}_m(b, q^t)\}| \le m$ for every pair (x, y). Let

$$A_{b,q^t} = \{1 + (x - b)/q^t\ , x \in A \cap \mathbb{A}_m(b, q^t)\} \subseteq \{1, 2, \dots, m\}.$$

Clearly if $x', y' \in A_{b,q^t}$ and $z' \in \Lambda_{1,Wq}$ satisfy that $x' - y' = \psi_q(z')$, then

$$x = b + (x'-1)q^t,\ y = b + (y'-1)q^t \in A,\ z = z'q \in \Lambda_{1,W}$$

and $x - y = \psi(z)$. So if there exists $1 \le q \le Q$ such that

$$|\{1 \le b \le n - mq^t : |A_{b,q^t}| \ge (\delta+\epsilon)m\}| \ge \epsilon n,$$

then

$$\begin{aligned} r_{W,\psi}(A) \ge& \frac{1}{m}\sum_{1\le b\le n-mq^t} r_{Wq,\psi_q}(A_{b,q^t}) \\ \ge& \epsilon n \frac{c(\delta+\epsilon, a_{k-t+1})}{2}\frac{Wq}{\phi(Wq)}\frac{m^{\frac{1}{k}}(a_1q^{k-t})^{-\frac{1}{k}}}{\log m} \\ \ge& \frac{c(\delta+\epsilon, a_{k-t+1})\epsilon^{1+\frac{1}{k}}}{400Q}\cdot\frac{Wn^{1+\frac{1}{k}}a_1^{-\frac{1}{k}}}{\phi(W)\log n}. \end{aligned}$$

So we may assume that

$$|\{1 \le b \le n - mq^t : |A \cap \mathbb{A}_m(b,q^t)| \ge (\delta+\epsilon)m\}| < \epsilon n \tag{3.1}$$

for each $1 \le q \le Q$. Let

$$M = \max\{x \in \mathbb{Z} : \psi(x) \le n\}.$$

Clearly $M = n^{\frac{1}{k}}a_1^{-\frac{1}{k}}(1+o(1))$. We shall show that

$$\int_{\mathbb{T}}\bigg(\bigg|\sum_{x\le n}\mathbf{1}_A e(\alpha x)\bigg|^2 - \delta^2\bigg|\sum_{x\le n}e(\alpha x)\bigg|^2\bigg)\bigg(\sum_{z\le M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\bigg)d\alpha$$

is relatively small.

For $1 \le q \le Q$, define

$$\mathfrak{M}_{a,q} = \{\alpha : |\alpha - a/q| \le \frac{1}{2}q^{-t}m^{-1}\}.$$

Let

$$\mathfrak{M} = \bigcup_{\substack{1\le a\le q\le Q\\(a,q)=1}}\mathfrak{M}_{a,q},$$

and let $\mathfrak{m} = \mathbb{T}\setminus\mathfrak{M}$. Let B be a sufficiently large integer. For $1 \le q \le (\log M)^B$, define

$$\mathfrak{M}^*_{a,q} = \{\alpha : |\alpha q - a| \le (\log M)^B/\psi(M)\}.$$

Let

$$\mathfrak{M}^* = \bigcup_{\substack{1\le a\le q\le(\log M)^B\\(a,q)=1}}\mathfrak{M}^*_{a,q}$$

and let $\mathfrak{m}^* = \mathbb{T}\setminus\mathfrak{M}^*$.

Suppose that $\alpha \in \mathfrak{m}$. We know $|\alpha q - a| \le (\log M)^B/\psi(M)$ for some $1 \le a \le q < \psi(M)(\log M)^{-B}$ with $(a,q) = 1$. If $\alpha \in \mathfrak{m}^*$, i.e., $q \ge (\log M)^B$, then $|\alpha - a/q| \le q^{-2}$

and $(\log y)^{\frac{B}{2}} \le \psi(y)(\log y)^{-\frac{B}{2}}$ for any $M(\log M)^{-\frac{B}{2k}} \le y \le M$. So applying Lemma 2.8 and a partial summation, we have

$$\sum_{z\le M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \ll_B \psi(M)(\log M)^{-1} \le n(\log M)^{-1},$$

whenever B is sufficiently large.

Now suppose that $q < (\log M)^B$, i.e., $\alpha \in \mathfrak{M}^*$. Applying Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\sum_{z\le M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\\
=&\frac{\phi(W)}{\phi(Wq)}\sum_{\substack{1\le r\le q\\(Wr+1,q)=1}} e(a\psi(r)/q)\sum_{z\le M}\psi^{\Delta}(z-1)e((\alpha-a/q)\psi(z))\\
&+O(\psi^{\Delta}(M)M(\log M)^{-4B})\\
=&\frac{\phi(W)}{\phi(Wq)}\sum_{\substack{1\le r\le q\\(Wr+1,q)=1}} e(a\psi(r)/q)\sum_{z\le n} e((\alpha-a/q)z)+O(\psi^{\Delta}(M)M(\log M)^{-4B}).
\end{aligned}$$

Since $\alpha \in \mathfrak{m}$, either $q > Q$ or $|\alpha - a/q| > \frac{1}{2}q^{-t}m^{-1}$.

If $q > Q$, then in light of Lemma 2.7

$$\left|\frac{\phi(W)}{\phi(Wq)}\sum_{\substack{1\le r\le q\\(Wr+1,q)=1}} e(a\psi(r)/q)\right| \le \left|\frac{1}{\phi(q)}\sum_{\substack{1\le r\le q\\(Wr+1,q)=1}} e(a\psi(r)/q)\right| \le C_1|a_{k-t+1}|q^{-\frac{1}{k(k+2)}}.$$

And if $|\alpha - a/q| > \frac{1}{2}q^{-t}m^{-1}$, then

$$\left|\sum_{z=1}^{n} e((\alpha-a/q)z)\right| = \left|\frac{1-e((\alpha-a/q)n)}{1-e(\alpha-a/q)}\right| \le 4\pi q^t m.$$

Hence for $\alpha \in \mathfrak{m}$

$$\sum_{z\le M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \le C_1|a_{k-t+1}|Q^{-\frac{1}{k(k+2)}}n + 4\pi mQ^t + O(n(\log n)^{-1}).$$

Suppose that $\alpha \in \mathfrak{M}$. Let $\tau = \mathbf{1}_A - \delta$ where $\mathbf{1}_A(x) = 1$ or 0 according whether $x \in A$ or not. Let

$$S(\alpha) = \sum_{c=0}^{m-1} e(\alpha c)$$

and

$$T(\alpha) = \sum_{b=1}^{n} \tau(b)e(\alpha b).$$

Then

$$\begin{aligned} S(\alpha q^t)T(\alpha) &= \sum_{b=1}^{n} \tau(b) \sum_{c=0}^{m-1} e(\alpha(b+cq^t)) \\ &= \sum_{b=1}^{n-mq^t} e(\alpha(b+(m-1)q^t)) \sum_{c=0}^{m-1} \tau(b+cq^t) + R(\alpha), \end{aligned}$$

where $|R(\alpha)| \le 2m^2q^t$. When $|\alpha q^t - aq^{t-1}| \le \frac{1}{2}m^{-1}$,

$$|S(\alpha q^t)| = |S(\alpha q^t - aq^{t-1})| = \left| \frac{1-e(m(\alpha q^t - aq^{t-1}))}{1-e(\alpha q^t - aq^{t-1})} \right| \ge \frac{m}{\pi}.$$

Hence for $\alpha \in \mathfrak{M}_{a,q}$,

$$m|T(\alpha)| \le \pi |S(\alpha q^t)T(\alpha)| \le \pi \left| \sum_{b=1}^{n-mq^t} e(\alpha(b+(m-1)q^t)) \sum_{c=0}^{m-1} \tau(b+cq^t) \right| + \pi |R(\alpha)|.$$

Notice that

$$|\{1 \le b \le n - mq^t : x \in \mathbb{A}_m(b, q^t)\}| \le m,$$

and the equality holds if $1 + (m-1)q^t \le x \le n - mq^t$. It follows that

$$m|A| \ge \sum_{b=1}^{n-mq^t} |A \cap \mathbb{A}_m(b, q^t)| = \sum_{x \in A} \sum_{b=1}^{n-mq^t} \mathbf{1}_{\mathbb{A}_m(b,q^t)}(x) \ge m|A| - 2m^2q^t,$$

whence

$$\left| \sum_{b=1}^{n-mq^t} (|A \cap \mathbb{A}_m(b, q^t)| - (\delta+\epsilon)m) \right| \le \epsilon nm + (2+\delta)m^2q^t.$$

By the assumption (3.1), we have

$$\sum_{\substack{1 \le b \le n-mq^t \\ |A \cap \mathbb{A}_m(b,q^t)| \ge (\delta+\epsilon)m}} (|A \cap \mathbb{A}_m(b, q^t)| - (\delta+\epsilon)m) \le \epsilon n(1-\delta)m.$$

It follows that

$$\begin{aligned} \sum_{b=1}^{n-mq^t} ||A \cap \mathbb{A}_m(b, q^t)| - \delta m| \le& \sum_{b=1}^{n-mq^t} ||A \cap \mathbb{A}_m(b, q^t)| - (\delta+\epsilon)m| + \epsilon nm \\ \le& 2 \sum_{\substack{1 \le b \le n-mq^t \\ |A \cap \mathbb{A}_m(b,q^t)| \ge (\delta+\epsilon)m}} (|A \cap \mathbb{A}_m(b, q^t)| - (\delta+\epsilon)m) \\ &+ \left| \sum_{b=1}^{n-mq^t} (|A \cap \mathbb{A}_m(b, q^t)| - (\delta+\epsilon)m) \right| + \epsilon nm \\ \le& 4\epsilon nm + 4m^2q^t. \end{aligned}$$

Thus for any $\alpha \in \mathfrak{M}$.

$$\begin{aligned}|T(\alpha)| \le& \frac{\pi}{m}\bigg(\bigg|\sum_{b=1}^{n-mq^t} e(\alpha(b+(m-1)q^t))\sum_{c=0}^{m-1}\tau(b+cq^t)\bigg| + 2m^2q^t\bigg)\\ \le& \frac{\pi}{m}\bigg(\sum_{b=1}^{n-mq^t} ||A\cap\mathbb{A}_m(b,q^t)| - \delta m| + 2m^2q^t\bigg)\\ \le& 4\pi\epsilon n + 6\pi m Q^t,\end{aligned}$$

i.e.,

$$\bigg|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x) - \delta\sum_{x=1}^{n}e(\alpha x)\bigg| \le 16\epsilon n.$$

It is easy to see that

$$||x|^2-|y|^2| \le ||x|-|y||^{\frac{2}{\rho}}(|x|+|y|)^{2-\frac{2}{\rho}} \le 4|x-y|^{\frac{2}{\rho}}(|x|^{2-\frac{2}{\rho}}+|y|^{2-\frac{2}{\rho}})$$

for any $\rho \ge 2$. Let $\rho = k2^{k+3}$. Then

$$\begin{aligned}&\bigg|\int_{\mathfrak{M}}\bigg(\bigg|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\bigg|^2 - \delta^2\bigg|\sum_{x=1}^{n}e(\alpha x)\bigg|^2\bigg)\bigg(\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\bigg)d\alpha\bigg|\\ \le& 4(16\epsilon n)^{\frac{2}{\rho}}\int_{\mathfrak{M}}\bigg(\bigg|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\bigg|^{2-\frac{2}{\rho}} + \delta^{2-\frac{2}{\rho}}\bigg|\sum_{x=1}^{n}e(\alpha x)\bigg|^{2-\frac{2}{\rho}}\bigg)\\ &\cdot\bigg|\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\bigg|d\alpha.\end{aligned}$$

By the Hölder inequality,

$$\begin{aligned}&\int_{\mathfrak{M}}\bigg|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\bigg|^{2-\frac{2}{\rho}}\bigg|\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\bigg|d\alpha\\ \le&\bigg(\int_{\mathbb{T}}\bigg|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\bigg|^2 d\alpha\bigg)^{1-\frac{1}{\rho}}\bigg(\int_{\mathbb{T}}\bigg|\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\bigg|^{\rho}d\alpha\bigg)^{\frac{1}{\rho}}.\end{aligned}$$

Applying Lemma 2.9,

$$\int_{\mathbb{T}}\bigg|\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\bigg|^{\rho}d\alpha \le C_2|a_{k-t+1}|\psi(M)^{\rho-1}.$$

Therefore

$$\begin{aligned}&\int_{\mathfrak{M}}\bigg|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\bigg|^{2-\frac{2}{\rho}}\bigg|\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\bigg|d\alpha\\ \le& C_2^{\frac{1}{\rho}}|a_{k-t+1}|^{\frac{1}{\rho}}|(\delta n)^{1-\frac{1}{\rho}}n^{1-\frac{1}{\rho}}.\end{aligned}$$

Similarly,

$$\int_{\mathfrak{m}}\Big|\sum_{x=1}^{n}e(\alpha x)\Big|^{2-\frac{2}{\rho}}\Big|\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big|d\alpha\le C_2^{\frac{1}{\rho}}|a_{k-t+1}|^{\frac{1}{\rho}}n^{2-\frac{2}{\rho}}.$$

It is concluded that

$$\Big|\int_{\mathfrak{m}}\Big(\Big|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\Big|^2-\delta^2\Big|\sum_{x=1}^{n}e(\alpha x)\Big|^2\Big)\Big(\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)d\alpha\Big|$$
$$\le 8C_2^{\frac{1}{\rho}}|a_{k-t+1}|^{\frac{1}{\rho}}\epsilon^{\frac{2}{\rho}}(\delta^{1-\frac{1}{\rho}}+\delta^{2-\frac{2}{\rho}})n^2.$$

Now we have shown that

$$\Big|\int_{\mathbb{T}}\Big(\Big|\sum_{x\le n}\mathbf{1}_Ae(\alpha x)\Big|^2-\delta^2\Big|\sum_{x\le n}e(\alpha x)\Big|^2\Big)\Big(\sum_{z\le M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)d\alpha\Big|$$
$$\le(2C_1|a_{k-t+1}|Q^{-\frac{1}{k(k+2)}}n+5\pi mQ^t)\int_{\mathbb{T}}\Big(\Big|\sum_{x\le n}\mathbf{1}_Ae(\alpha x)\Big|^2+\delta^2\Big|\sum_{x\le n}e(\alpha x)\Big|^2\Big)d\alpha$$
$$+8C_2^{\frac{1}{\rho}}|a_{k-t+1}|^{\frac{1}{\rho}}\epsilon^{\frac{2}{\rho}}(\delta^{1-\frac{1}{\rho}}+\delta^{2-\frac{2}{\rho}})n^2$$
$$\le 4C_1|a_{k-t+1}|Q^{-\frac{1}{k(k+2)}}\delta n^2+\epsilon\delta n^2+16C_2^{\frac{1}{\rho}}|a_{k-t+1}|^{\frac{1}{\rho}}\epsilon^{\frac{2}{\rho}}\delta^{1-\frac{1}{\rho}}n^2.$$

On the other hand, we have

$$\int_{\mathbb{T}}\Big|\sum_{x=1}^{n}e(\alpha x)\Big|^2\Big(\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)d\alpha$$
$$=\sum_{\substack{1\le x,y\le n\\1\le z\le M\\x-y=\psi(z)}}\psi^{\Delta}(z-1)\lambda_{1,W}(z)$$
$$\ge\sum_{\substack{1\le x,y\le n\\M/4+1\le z\le M/2\\x-y=\psi(z)}}\psi^{\Delta}(z-1)\lambda_{1,W}(z)$$
$$\ge\frac{M}{8}(n-\psi(M/2))\psi^{\Delta}(M/4).$$

It follows that

$$\int_{\mathbb{T}}\Big|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\Big|^2\Big(\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)d\alpha$$
$$\ge\delta^2\int_{\mathbb{T}}\Big|\sum_{x=1}^{n}e(\alpha x)\Big|^2\Big(\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\Big)d\alpha$$
$$-4C_1|a_{k-t+1}|Q^{-\frac{1}{k(k+2)}}\delta n^2-\epsilon\delta n^2-16C_2^{\frac{1}{\rho}}|a_{k-t+1}|^{\frac{1}{\rho}}\epsilon^{\frac{2}{\rho}}\delta^{1-\frac{1}{\rho}}n^2$$
$$\ge\frac{k\delta^2n^2}{4^{k+1}}-4C_1|a_{k-t+1}|Q^{-\frac{1}{k(k+2)}}\delta n^2-\epsilon\delta n^2-16C_2^{\frac{1}{\rho}}|a_{k-t+1}|^{\frac{1}{\rho}}\epsilon^{\frac{2}{\rho}}\delta^{1-\frac{1}{\rho}}n^2.$$

Let $\epsilon = 4^{-(k+2)\rho}\delta^{\frac{\rho+1}{2}}C_2^{-\frac{1}{2}}|a_{k-t+1}|^{-\frac{1}{2}}$ and

$$Q = 4^{(k+1)^4}\delta^{-2k(k+2)}C_1^{k(k+2)}|a_{k-t+1}|^{k(k+2)}.$$

Therefore

$$\begin{aligned}&|\{(x,y,z):\, x,y\in A, z\in\Lambda_{1,W}, x-y=\psi(z)\}|\\ \geq&\frac{W/\phi(W)}{\psi^{\Delta}(M)\log(WM+1)}\int_{\mathbb{T}}\bigg|\sum_{x=1}^{n}\mathbf{1}_A(x)e(\alpha x)\bigg|^2\bigg(\sum_{z=1}^{M}\psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z))\bigg)d\alpha\\ \geq&\frac{W\delta^2}{4^{k+2}k\phi(W)}\cdot\frac{n^{1+\frac{1}{k}}a_1^{-\frac{1}{k}}}{\log n}.\end{aligned}$$

This concludes our desired result. □

Finally, let us briefly discuss the bound in Theorem 1.3. Let $R_{W,\psi}(\delta)$ be the least integer n such that for any $A\subseteq\{1,2,\ldots,n\}$, there exist $x,y\in A$ and $z\in\Lambda_{1,W}$ satisfying $x-y=\psi(z)$. In our proof, we choose $\epsilon=\epsilon(\delta)=O_{|a_{k-t}|}(\delta^{O_k(1)})$ and $Q=Q(\delta)=O_{|a_{k-t}|}(\delta^{-O_k(1)})$. So the iteration process $\delta\to\delta+\epsilon(\delta)$ will end after $O_{|a_{k-t}|}(\delta^{-O_k(1)})$ steps. Also, clearly for $\delta>3/4$,

$$R_{W,\psi}(\delta)\ll(|a_1|+\cdots+|a_{k-t}|)(\min\{p:\, p\in\Lambda_{1,W}\})^k.$$

Notice that when the iteration process ends, W will become $WQ^{O_{|a_{k-t}|}(\delta^{-O_k(1)})}$ and a_i will become $a_iQ^{O_{|a_{k-t}|}(\delta^{-O_k(1)})}$. Hence we have

$$R_{W,\psi}(\delta)\leq\exp(O_{W,a_1,\ldots,a_{k-t}}(\delta^{-O_{|a_{k-t}|}(\delta^{-O_k(1)})})),$$

since $\min\{p:\, p\in\Lambda_{1,W}\}\leq e^{O(W)}$. In other words, if a subset $A\subseteq\{1,2,\ldots,n\}$ satisfies $|A|\geq O_{W,a_1,\ldots,a_{k-t}}(n/\log\log\log n)$, then there exist $x,y\in A$ and $z\in\Lambda_{1,W}$ such that $x-y=\psi(z)$. Of course, this bound is very rough. And we believe that it could be improved using some more refined estimations (e.g. [18], [1], [16], [17], [20]).

4. Proof of Theorem 1.4

Write $\psi(x)=a_1x^k+a_2x^{k-1}+\cdots+a_{k-t+1}x^t$ where $a_{k-t+1}\neq 0$. Let $\delta=\overline{d}_{\mathcal{P}}(P)$. Since $\overline{d}_{\mathcal{P}}(P)>0$, there exist infinitely many n such that

$$|P\cap[1,n]|\geq\frac{4\delta}{5}\cdot\frac{n}{\log n}.$$

Define

$$w(n)=\max\{w\leq\log\log\log n:\, n\geq 16\mathcal{W}(w)N(\delta,\mathcal{W}(w),\psi_{\mathcal{W}(w)})\},$$

where $N(\delta,W,\psi)$ is same as the one defined in Theorem 3.1 and $\mathcal{W}(w)=\prod_{\substack{p\leq w\\ p\text{ prime}}}p$. Clearly $\lim_{n\to\infty}w(n)=\infty$. Let $w=w(n)$ and $\mathcal{W}=\mathcal{W}(w)$. Then

$$\sum_{\substack{x\in P\cap[1,n]\\(x,\mathcal{W})=1}}\log x\geq\sum_{x\in P\cap[n^{\frac{2}{3}},n]}\log x\geq\frac{2\log n}{3}(|P\cap[1,n]|-n^{\frac{2}{3}})\geq\frac{\delta}{2}\cdot n.$$

Hence there exists $1 \le b \le \mathcal{W}^t$ with $(b, \mathcal{W}) = 1$ such that

$$\sum_{\substack{x \in P \cap [1,n] \\ x \equiv b \pmod{\mathcal{W}^t}}} \log x \ge \frac{\delta}{2\phi(\mathcal{W}^t)} \cdot n.$$

Let

$$A = \{(x-b)/\mathcal{W}^t : x \in P \cap [1,n],\ x \equiv b \pmod{\mathcal{W}^t}\}.$$

Let N be a prime in the interval $(2n/\mathcal{W}^t, 4n/\mathcal{W}^t]$. Define $\lambda_{b,\mathcal{W}^t,N} = \lambda_{b,\mathcal{W}^t}/N$ and $a = \mathbf{1}_A \lambda_{b,\mathcal{W}^t,N}$. Then

$$\sum_x a(x) \ge \frac{\phi(\mathcal{W}^t)}{\mathcal{W}^t N} \cdot \frac{\delta n}{2\phi(\mathcal{W}^t)} \ge \frac{\delta}{8}.$$

Let

$$\psi_{\mathcal{W}}(x) = \psi(\mathcal{W}x)/\mathcal{W}^t = a_1 \mathcal{W}^{k-t} x^k + \cdots + a_{k-t+1} x^t.$$

Clearly $\psi_{\mathcal{W}}(z)$ is positive and strictly increasing for $1 \le z \le M$, whenever $\mathcal{W}$ is sufficiently large.

Below we consider A as a subset of $\mathbb{Z}_N$. Let $M = \max\{z \in \mathbb{N} : \psi_{\mathcal{W}}(z) < N/2\}$. If $x, y \in A$ and $1 \le z \le M$ satisfy $x - y = \psi_{\mathcal{W}}(z)$ in $\mathbb{Z}_N$, then we also have $x - y = \psi_{\mathcal{W}}(z)$ in $\mathbb{Z}$. In fact, since $1 \le x, y < N/2$ and $1 \le z \le M$, it is impossible that $x - y = \psi_{\mathcal{W}}(z) - N$ in $\mathbb{Z}$. For a function $f : \mathbb{Z}_N \to \mathbb{C}$, define

$$\tilde{f}(r) = \sum_{x \in \mathbb{Z}_N} f(x) e(-xr/N).$$

Lemma 4.1 (Bourgain [4], [5] and Green [11])**.** *Suppose that $\rho > 2$. Then*

$$\sum_r |\tilde{a}(r)|^\rho \le C(\rho),$$

where $C(\rho)$ is a constant only depending on ρ.

Proof. See [11, Lemma 6.6]. □

Lemma 4.2.

$$\sum_{r \in \mathbb{Z}_N} \left| \sum_{z=1}^{M} \psi_{\mathcal{W}}^{\Delta}(z-1) \lambda_{1,\mathcal{W}W}(z) e(-\psi_{\mathcal{W}}(z) r/N) \right|^\rho \le C'(\rho) |a_{k-t+1}| N^\rho.$$

provided that $\rho \ge k2^{k+3}$, where $C'(\rho)$ is a constant only depending on ρ.

Proof. This is an immediate consequence of Lemma 2.10 since $\gcd(\psi_{\mathcal{W}}) \le |a_{k-t+1}|$. □

Let η and ϵ be two positive real numbers to be chosen later. Let

$$R = \{r \in \mathbb{Z}_N : \tilde{a}(r) \ge \eta\}$$

and

$$B = \{r \in \mathbb{Z}_N : \|xr/N\| \le \epsilon \text{ for all } r \in R\}.$$

Define $\beta = \mathbf{1}_B/|B|$ and $a' = a * \beta * \beta$, where

$$f * g(x) = \sum_{y\in\mathbb{Z}_N} f(y)g(x-y).$$

Let $\varrho = k2^{k+3}$.

Lemma 4.3.

$$\sum_{\substack{x,y\in\mathbb{Z}_N\\ 1\le z\le M\\ x-y=\psi_{\mathcal{W}}(z)}} (a'(x)a'(y) - a(x)a(y))\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z) \le C(\epsilon^2\eta^{-\frac{5}{2}} + \eta^{\frac{1}{\varrho}}).$$

Proof. It is not difficult to check that

$$\sum_{\substack{x,y\in\mathbb{Z}_N\\ 1\le z\le M\\ x-y=\psi_{\mathcal{W}}(z)}} a(x)a(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)$$
$$=\frac{1}{N}\sum_{r\in\mathbb{Z}_N} \tilde{a}(r)\tilde{a}(-r)\bigg(\sum_{z=1}^{M} \psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)e(-\psi_{\mathcal{W}}(z)r/N)\bigg).$$

Also, it is easy to see that $(f*g)\tilde{}= \tilde{f}\tilde{g}$. Then

$$\sum_{\substack{x,y\in\mathbb{Z}_N\\ 1\le z\le M\\ x-y=\psi_{\mathcal{W}}(z)}} a'(x)a'(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z) - \sum_{\substack{x,y\in\mathbb{Z}_N\\ 1\le z\le M\\ x-y=\psi_{\mathcal{W}}(z)}} a(x)a(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)$$
$$=\frac{1}{N}\sum_{r\in\mathbb{Z}_N} \tilde{a}(r)\tilde{a}(-r)(\tilde{\beta}(r)^2\tilde{\beta}(-r)^2 - 1)\bigg(\sum_{z=1}^{M} \psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)e(-\psi_{\mathcal{W}}(z)r/N)\bigg).$$

If $r \in R$, then by the proof of Lemma 6.7 of [11], we know that

$$|\tilde{\beta}(r)^2\tilde{\beta}(-r)^2 - 1| \le 2^{16}\epsilon^2.$$

And applying Lemma 2.2 with $\alpha = a = q = 1$,

$$\sum_{z=1}^{M} \psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z) = \sum_{z=1}^{M} \psi_{\mathcal{W}}^{\Delta}(z-1) + O(\psi_{\mathcal{W}}^{\Delta}(M)Me^{-c\sqrt{\log M}}) \le 2\psi_{\mathcal{W}}(M).$$

Therefore

$$\bigg|\sum_{r\in R} \tilde{a}(r)\tilde{a}(-r)(\tilde{\beta}(r)^2\tilde{\beta}(-r)^2 - 1)\bigg(\sum_{z=1}^{M} \psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)e(-\psi_{\mathcal{W}}(z)r/N)\bigg)\bigg|$$
$$\le 2^{16}\epsilon^2\sum_{r\in R} |\tilde{a}(r)|^2 \cdot \bigg|\sum_{z=1}^{M} \psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)e(-\psi_{\mathcal{W}}(z)r/N)\bigg|$$
$$\le 2^{17}\epsilon^2\psi_{\mathcal{W}}(M)|R|.$$

In view of Lemma 4.1 with $\rho = 5/2$, we have $|R| \le C''\eta^{-\frac{5}{2}}$. On the other hand, by Hölder inequality,

$$\begin{aligned}&\Bigg|\sum_{r\notin R}\tilde{a}(r)\tilde{a}(-r)(\tilde{\beta}(r)^2\tilde{\beta}(-r)^2-1)\bigg(\sum_{z=1}^{M}\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)e(-\psi_{\mathcal{W}}(z)r/N)\bigg)\Bigg|\\ \le& 2\sup_{r\notin R}|\tilde{a}(r)|^{\frac{1}{\varrho}}\bigg(\sum_{r\notin R}|\tilde{a}(r)|^{\frac{2\varrho-1}{\varrho-1}}\bigg)^{\frac{\varrho-1}{\varrho}}\bigg(\sum_{r\notin R}\bigg|\sum_{z=1}^{M}\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)e(-\psi_{\mathcal{W}}(z)r/N)\bigg|^{\varrho}\bigg)^{\frac{1}{\varrho}}\\ \le& 2\eta^{\frac{1}{\varrho}}\cdot C((2\varrho-1)/(\varrho-1))^{1-\frac{1}{\varrho}}\cdot(|a_{k-t+1}|C'(\varrho))^{\frac{1}{\varrho}}N,\end{aligned}$$

where in the last step we apply Lemma 4.1 with $\rho = (2\varrho-1)/(\varrho-1)$ and Lemma 4.2 with $\rho = \varrho$. All are done. □

Lemma 4.4. *If $\epsilon^{|R|} \ge 2\log\log w/w$, then $|a'(x)| \le 2/N$ for any $x \in \mathbb{Z}_N$.*

Proof. See [11, Lemma 6.3]. □

Let $A' = \{x \in \mathbb{Z}_N : a'(x) \ge \frac{1}{16}\delta N^{-1}\}$. Then

$$\frac{2}{N}|A'| + \frac{\delta}{16N}(N-|A'|) \ge \sum_{x\in\mathbb{Z}_N} a'(x) = \sum_{x\in\mathbb{Z}_N} a(x) \ge \frac{\delta}{8},$$

whence $|A'|/N \ge \delta/32$. Let $A_1' = A' \cap [1, (N-1)/2]$ and

$$A_2' = \{x - (N-1)/2 : x \in A' \cap [(N+1)/2, N-1]\}.$$

Clearly there exists $i \in \{1, 2\}$ such that $|A_i'|/N \ge \delta/64$. Without loss of generality, we may assume that $|A_1'|/N \ge \delta/64$. Applying Theorem 3.1, we know that

$$\begin{aligned}&|\{(x,y,z) : x, y \in A_1', z \in \Lambda_{1,\mathcal{W}W} \cap [1, M], x - y = \psi_{\mathcal{W}}(z)\}|\\ \ge& c(\delta/64, a_{k-t+1})\frac{\mathcal{W}W(N/2)^{1+\frac{1}{k}}(a_1\mathcal{W}^{k-t})^{-\frac{1}{k}}}{\phi(\mathcal{W}W)\log N}.\end{aligned}$$

Let $c' = \frac{1}{16k}c(\delta/64, a_{k-t+1})$. Clearly

$$|\{(x,y,z) : x, y \in A_1', z \in \Lambda_{1,\mathcal{W}W} \cap [1, c'M], x - y = \psi_{\mathcal{W}}(z)\}| \le \frac{\mathcal{W}W(c'M)}{\phi(\mathcal{W}W)\log M}N.$$

Therefore

$$\begin{aligned}&|\{(x,y,z) : x, y \in A_1', z \in \Lambda_{1,\mathcal{W}^tW} \cap (c'M, M], x - y = \psi_{\mathcal{W}}(z)\}|\\ \ge& \frac{c(\delta/64, a_{k-t+1})}{8}\frac{\mathcal{W}WN^{1+\frac{1}{k}}(a_1\mathcal{W}^{k-t})^{-\frac{1}{k}}}{\phi(\mathcal{W}W)\log N}.\end{aligned}$$

It follows that

$$\begin{aligned}&\sum_{\substack{x,y\in A_1'\\1\le z\le M\\x-y=\psi_{\mathcal{W}}(z)}}\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)\\ \ge&\frac{c(\delta/64,a_{k-t+1})}{8}\frac{\mathcal{W}WN^{1+\frac{1}{k}}(a_1\mathcal{W}^{k-t})^{-\frac{1}{k}}}{\phi(\mathcal{W}W)\log N}\cdot\frac{\psi_{\mathcal{W}}^{\Delta}(c'M)\phi(\mathcal{W}W)\log M}{2\mathcal{W}W}\\ \ge&\frac{c(\delta/64,a_{k-t+1})c'^{k-1}}{64}N^2.\end{aligned}$$

So

$$\begin{aligned}&\sum_{\substack{x,y\in\mathbb{Z}_N\\1\le z\le M\\x-y=\psi_{\mathcal{W}}(z)}}a(x)a(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)\\ \ge&\sum_{\substack{x,y\in\mathbb{Z}_N\\1\le z\le M\\x-y=\psi_{\mathcal{W}}(z)}}a'(x)a'(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)-C(\epsilon^2\eta^{-\frac{5}{2}}+\eta^{\frac{1}{\varrho}})\\ \ge&\frac{\delta^2}{2^8N^2}\sum_{\substack{x,y\in A_1'\\1\le z\le M\\x-y=\psi_{\mathcal{W}}(z)}}\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)-C(\epsilon^2\eta^{-\frac{5}{2}}+\eta^{\frac{1}{\varrho}})\\ \ge&c''(\delta,a_{k-t+1})-C(\epsilon^2\eta^{-\frac{5}{2}}+\eta^{\frac{1}{\varrho}}).\end{aligned}$$

Finally, we may choose $\eta,\epsilon>0$ satisfying $\epsilon^{C''\eta^{-5/2}}\ge 2\log\log w/w$ such that $C(\epsilon^2\eta^{-\frac{5}{2}}+\eta^{\frac{1}{\varrho}})<c''(\delta,a_{k-t+1})/2$, whenever w is sufficiently large. Hence

$$\sum_{\substack{x,y\in\mathbb{Z}_N\\1\le z\le M\\x-y=\psi_{\mathcal{W}}(z)}}a(x)a(y)\psi_{\mathcal{W}}^{\Delta}(z-1)\lambda_{1,\mathcal{W}W}(z)\ge\frac{c''(\delta,a_{k-t+1})}{2}>0$$

for sufficiently large N. □

Appendix: Exponential Sums on Polynomials of Prime Variables

Lemma 4.5.

$$\sum_{x=1}^{N}d_k^2(x)\ll N(\log N)^{k^2-1},\tag{4.1}$$

where

$$d_k(x)=|\{(a_1,\dots,a_k):\ a_1,\dots,a_k\in\mathbb{Z}^+,a_1\cdots a_k=x\}|.$$

Let $K=2^{k-1}$.

Lemma 4.6. *Let* $\psi(x) = a_1x^k + a_2x^{k-1} + \cdots + a_kx$ *be a polynomial with real coefficients and* $a_1 \in \mathbb{Z}^+$. *Then*

$$\sum_{1\le x\le V} e(\alpha\psi(x)) \ll V^{1-\frac{k}{K}}\left(V^{k-1} + V^{\frac{k}{2}}(\log V)^{\frac{k^2-2k}{2}}\left(\sum_{y=1}^{V^{k-1}} \min\{V, \|\alpha k! a_1 y\|^{-1}\}\right)^{\frac{1}{2}}\right)^{\frac{1}{K}} \tag{4.2}$$

for any real α. *In particular, if* $|\alpha - a/q| \le q^{-2}$ *with* $(a,q) = 1$, *then*

$$\sum_{1\le x\le V} e(\alpha\psi(x)) \ll V\left(V^{-\frac{1}{K}} + a_1^{\frac{1}{2K}}(\log(a_1qV))^{\frac{(k-1)^2}{2K}}\left(\frac{1}{q} + \frac{1}{V} + \frac{q}{a_1V^k}\right)^{\frac{1}{2K}}\right). \tag{4.3}$$

Proof. Define the intervals $I_j(V;h_1,\dots,h_j)$ by $I_1(V;h_1) = [1,V] \cap [1-h_1, V-h_1]$ and

$$I_{j+1}(V;h_1,\dots,h_{j+1}) = I_j(V;h_1,\dots,h_j) \cap \{x : x + h_{j+1} \in I_j(V;h_1,\dots,h_j)\}.$$

For $j \ge 1$, we know (cf.[27][Lemma 2.3])that

$$\Bigg|\sum_{1\le x\le V} e(\alpha\psi(x))\Bigg|^{2^j} \le (2V)^{2^j-j-1} \sum_{-V<h_1,\dots,h_j<V} T_j(V;h_1,\dots,h_j),$$

where

$$T_j(V;h_1,\dots,h_j) = \sum_{x\in I_j(V;h_1,\dots,h_j)} e(\Delta_j(\alpha\psi(x);h_1,\dots,h_j)).$$

In particular,

$$T_{k-1}(V;h_1,\dots,h_{k-1}) = \sum_{x\in I_{k-1}(V;h_1,\dots,h_{k-1})} e(\alpha h_1\cdots h_{k-1} g_{k-1}(x;h_1,\dots,h_{k-1})),$$

where

$$g_{k-1}(x;h_1,\dots,h_{k-1}) = k! a_1(x + (h_1 + \cdots + h_{k-1})/2) + (k-1)! a_2.$$

Therefore

$$\begin{aligned}
&\sum_{1\le x\le V} e(\alpha\psi(x))\\
\ll&V^{1-\frac{k}{K}}\bigg(\sum_{-V<h_1,\dots,h_{k-1}<V}\bigg|\sum_{1\le x\le V} e(\alpha k!a_1h_1\cdots h_{k-1}x)\bigg|\bigg)^{\frac{1}{K}}\\
\ll&V^{1-\frac{k}{K}}\bigg(V^{k-1}+\sum_{\substack{-V<h_1,\dots,h_{k-1}<V\\ h_1\cdots h_{k-1}\neq 0}}\min\{V,\|\alpha k!a_1h_1\cdots h_{k-1}\|^{-1}\}\bigg)^{\frac{1}{K}}\\
\le&V^{1-\frac{k}{K}}\bigg(V^{k-1}+\sum_{1\le y\le V^{k-1}} d_{k-1}(y)\min\{V,\|\alpha k!a_1y\|^{-1}\}\bigg)^{\frac{1}{K}}\\
\le&V^{1-\frac{k}{K}}\bigg(V^{k-1}+\bigg(\sum_{1\le y\le V^{k-1}} d_{k-1}(y)^2\bigg)^{\frac{1}{2}}\bigg(\sum_{1\le y\le V^{k-1}}\min\{V,\|\alpha k!a_1y\|^{-1}\}^2\bigg)^{\frac{1}{2}}\bigg)^{\frac{1}{K}}\\
\ll&V^{1-\frac{k}{K}}\bigg(V^{k-1}+V^{\frac{k}{2}}(\log V)^{\frac{k^2-2k}{2}}\bigg(\sum_{1\le y\le V^{k-1}}\min\{V,\|\alpha k!a_1y\|^{-1}\}\bigg)^{\frac{1}{2}}\bigg)^{\frac{1}{K}}.
\end{aligned}$$

Finally, if $|\alpha-a/q|\le q^{-2}$ with $(a,q)=1$, then by Lemma 2.2 of [27], we have

$$\begin{aligned}
\sum_{1\le y\le V^{k-1}}\min\{V,\|\alpha k!a_1y\|^{-1}\}\le&\sum_{y=1}^{k!a_1V^{k-1}}\min\{k!a_1V^k/y,\|\alpha y\|^{-1}\}\\
\ll&k!a_1V^k\log(2k!a_1V^kq)\bigg(\frac{1}{q}+\frac{1}{V}+\frac{q}{k!a_1V^k}\bigg).
\end{aligned}$$

We are done. □

Lemma 4.7. *Suppose that $\psi(x,y)=\sum_{1\le i,j\le k+1} a_{ij}x^{k-i+1}y^{k-j+1}$ is a polynomial with real coefficients. Suppose that $a_{11}\in\mathbb{Z}^+$ and $a_{12}=0$. Then*

$$\begin{aligned}
&\sum_{1\le x\le U}\bigg|\sum_{1\le y\le V} e(\alpha\psi(x,y))\bigg|\\
\ll&UV\bigg(U^{-\frac{1}{K^2}}+V^{-\frac{1}{K^2}}+a_{11}^{\frac{1}{4K^2}}(\log(a_{11}qUV))^{\frac{3k^2-2k+1}{4K^2}}\bigg(\frac{1}{q}+\frac{1}{U}+\frac{q}{a_{11}U^kV^k}\bigg)^{\frac{1}{4K^2}}\bigg)
\end{aligned}\tag{4.4}$$

provided that $|\alpha-a/q|\le q^{-2}$ with $(a,q)=1$.

Proof. Write $\psi(x,y)=\sum_{j=1}^{k+1}\psi_j(x)y^{k-j+1}$. Then by Hölder inequality we have

$$\begin{aligned}
&\sum_{1\le x\le U}\Big|\sum_{1\le y\le V}e(\alpha\psi(x,y))\Big|\\
\le& U^{1-\frac{1}{K}}\bigg(\sum_{1\le x\le U}\Big|\sum_{1\le y\le V}e(\alpha\psi(x,y))\Big|^K\bigg)^{\frac{1}{K}}\\
\ll& U^{1-\frac{1}{K}}V^{1-\frac{k}{K}}\bigg(\sum_{1\le x\le U}\sum_{\substack{|h_1|,\dots,|h_{k-1}|\le V\\ y\in I_{k-1}(V;h_1,\dots,h_{k-1})}}e(\alpha h_1\cdots h_{k-1}g_{k-1}(x,y;h_1,\dots,h_{k-1}))\bigg)^{\frac{1}{K}}\\
\le& U^{1-\frac{1}{K}}V^{1-\frac{k}{K^2}}\bigg(\sum_{\substack{|h_1|,\dots,|h_{k-1}|\le V\\ y\in I_{k-1}(V;h_1,\dots,h_{k-1})}}\Big|\sum_{1\le x\le U}e(\alpha h_1\cdots h_{k-1}g_{k-1}(x,y;h_1,\dots,h_{k-1}))\Big|^K\bigg)^{\frac{1}{K^2}},
\end{aligned}$$

where

$$\begin{aligned}
&g_{k-1}(x,y;h_1,\dots,h_{k-1})\\
=&k!\psi_1(x)(y+(h_1+\cdots+h_{k-1})/2)+(k-1)!\psi_2(x).
\end{aligned}$$

Note that $\deg\psi_2\le k-1$ since $a_{12}=0$. Thus applying Lemma 4.6,

$$\begin{aligned}
&\Big|\sum_{1\le x\le U}e(\alpha h_1\cdots h_{k-1}g_{k-1}(x,y;h_1,\dots,h_{k-1}))\Big|^K\\
\ll& U^{K-k}\bigg(U^{k-1}+U^{\frac{k}{2}}(\log U)^{\frac{k^2-2k}{2}}\Big(\sum_{1\le z\le U^{k-1}}\min\{U,\|\alpha(k!)^2a_{11}h_1\cdots h_{k-1}yz\|^{-1}\}\Big)^{\frac{1}{2}}\bigg)
\end{aligned}$$

provided that $h_1\cdots h_{k-1}\ne 0$. So

$$\begin{aligned}
&\sum_{\substack{|h_1|,\dots,|h_{k-1}|\le V\\ y\in I_{k-1}(V;h_1,\dots,h_{k-1})}}\Big|\sum_{1\le x\le U}e(\alpha h_1\cdots h_{k-1}g_{k-1}(x,y;h_1,\dots,h_{k-1}))\Big|^K\\
\ll& U^KV^{k-1}+\sum_{\substack{|h_1|,\dots,|h_{k-1}|\le V\\ h_1\cdots h_{k-1}\ne 0\\ y\in I_{k-1}(V;h_1,\dots,h_{k-1})}}\Big|\sum_{1\le x\le U}e(\alpha h_1\cdots h_{k-1}g_{k-1}(x,y;h_1,\dots,h_{k-1}))\Big|^K\\
\ll& (\log U)^{\frac{k^2-2k}{2}}U^{K-\frac{k}{2}}\sum_{\substack{|h_1|,\dots,|h_{k-1}|\le V\\ h_1\cdots h_{k-1}\ne 0\\ y\in I_{k-1}(V;h_1,\dots,h_{k-1})}}\Big(\sum_{1\le z\le U^{k-1}}\min\{U,\|\alpha(k!)^2a_{11}h_1\cdots h_{k-1}yz\|^{-1}\}\Big)^{\frac{1}{2}}\\
&+U^{K-1}V^k+U^KV^{k-1}.
\end{aligned}$$

Furthermore,

$$\sum_{\substack{|h_1|,\ldots,|h_{k-1}|\le V\\ h_1\cdots h_{k-1}\neq 0\\ y\in I_{k-1}(V;h_1,\ldots,h_{k-1})}}\bigg(\sum_{1\le z\le U^{k-1}}\min\{U,\|\alpha(k!)^2a_{11}h_1\cdots h_{k-1}yz\|^{-1}\}\bigg)^{\frac12}$$

$$\ll V^{\frac{k}{2}}\bigg(\sum_{\substack{|h_1|,\ldots,|h_{k-1}|\le V\\ h_1\cdots h_{k-1}\neq 0\\ 1\le y\le V}}\sum_{1\le z\le U^{k-1}}\min\{U,\|\alpha(k!)^2a_{11}h_1\cdots h_{k-1}yz\|^{-1}\}\bigg)^{\frac12}$$

$$\ll V^{\frac{k}{2}}\bigg(\sum_{1\le z\le U^{k-1}V^k}d_{k+1}(z)\min\{U,\|\alpha(k!)^2a_{11}z\|^{-1}\}\bigg)^{\frac12}$$

$$\ll V^{\frac{k}{2}}\bigg(\sum_{1\le z\le U^{k-1}V^k}d_{k+1}(z)^2\bigg)^{\frac14}\bigg(\sum_{1\le z\le U^{k-1}V^k}\min\{U,\|\alpha(k!)^2a_{11}z\|^{-1}\}^2\bigg)^{\frac14}$$

$$\ll V^{\frac{3k}{4}}U^{\frac{k}{4}}(\log(UV))^{\frac{k^2+2k}{4}}\bigg(\sum_{1\le z\le (k!)^2a_{11}U^{k-1}V^k}\min\{(k!)^2a_{11}U^kV^k/z,\|\alpha z\|^{-1}\}\bigg)^{\frac14}$$

$$\ll V^{\frac{3k}{4}}U^{\frac{k}{4}}(\log(UV))^{\frac{k^2+2k}{4}}\cdot a_{11}^{\frac14}U^{\frac{k}{4}}V^{\frac{k}{4}}(\log(a_{11}qUV))^{\frac14}\bigg(\frac{1}{q}+\frac{1}{U}+\frac{q}{(k!)^2a_{11}U^kV^k}\bigg)^{\frac14}.$$

All are done. □

Lemma 4.8. *Suppose that $\psi(x,y)=\sum_{1\le i,j\le k+1}a_{ij}x^{k-i+1}y^{k-j+1}$ is a polynomial with real coefficients. Suppose that $a_{11}\in\mathbb{Z}^+$ and $a_{12}=0$. Then*

$$\sum_{U\le x\le 2U}\bigg|\sum_{1\le y\le V/x}e(\alpha\psi(x,y))\bigg|$$
$$\ll V\bigg(U^{\frac1K}V^{-\frac1K}+U^{-\frac{1}{K^2}}++a_{11}^{\frac{1}{4K^2}}(\log(a_{11}qUV))^{\frac{3k^2-2k+1}{4K^2}}\bigg(\frac1q+\frac1U+\frac{q}{a_{11}V^k}\bigg)^{\frac{1}{4K^2}}\bigg) \tag{4.5}$$

provided that $|\alpha-a/q|\le q^{-2}$ with $(a,q)=1$.

Proof. Write $\psi(x,y)=\sum_{j=1}^{k+1}\psi_j(x)y^{k-j+1}$. And let

$$T_{k-1}(x,Q;h_1,\ldots,h_{k-1})=\sum_{y\in I_{k-1}(Q;h_1,\ldots,h_{k-1})}e(\alpha h_1\cdots h_{k-1}g_{k-1}(x,y;h_1,\ldots,h_{k-1}))$$

where

$$g_{k-1}(x,y;h_1,\ldots,h_{k-1})$$
$$=k!\psi_1(x)(y+(h_1+\cdots+h_{k-1})/2)+(k-1)!\psi_2(x).$$

Then

$$\begin{aligned}
&\sum_{U\le x\le 2U}\Big|\sum_{1\le y\le V/x} e(\alpha\psi(x,y))\Big|\\
\le& U^{1-\frac{1}{K}}\Bigg(\sum_{U\le x\le 2U}(V/x)^{K-k}\sum_{|h_1|,\dots,|h_{k-1}|\le V/x} T_{k-1}(x,\lfloor V/x\rfloor;h_1,\dots,h_{k-1})\Bigg)^{\frac{1}{K}}\\
\le& U^{1-\frac{1}{K}}V^{1-\frac{k}{K}}\Bigg(\sum_{|h_1|,\dots,|h_{k-1}|\le V/U}\ \sum_{\substack{U\le x\le 2U\\ x\le\min_{1\le i<k}V/|h_i|}} x^{k-K}T_{k-1}(x,\lfloor V/x\rfloor;h_1,\dots,h_{k-1})\Bigg)^{\frac{1}{K}}.
\end{aligned}$$

By an induction on j, it is not difficult to prove that $I_j(Q_1;h_1,\dots,h_j)\subseteq I_j(Q_2;h_1,\dots,h_j)$ if $Q_1\le Q_2$. Hence for any y, the set

$$\bar I(y;h_1,\dots,h_j)=\{x:\ y\in I_j(\lfloor V/x\rfloor;h_1,\dots,h_j)\}$$

is exactly an interval. Then

$$\begin{aligned}
&\sum_{\substack{U\le x\le 2U\\ 1\le x\le\min_{1\le i<k}V/|h_i|}} x^{k-K}T_{k-1}(x;h_1,\dots,h_{k-1})\\
=&\sum_{y\in I_{k-1}(V/U;h_1,\dots,h_{k-1})}\ \sum_{\substack{U\le x\le 2U\\ 1\le x\le\min_{1\le i<k}V/|h_i|\\ x\in\bar I(y;h_1,\dots,h_{k-1})}} x^{k-K}e(\alpha h_1\cdots h_{k-1}g_{k-1}(x,y;h_1,\dots,h_{k-1})).
\end{aligned}$$

By Lemma 4.6 we know that

$$\begin{aligned}
&\sum_{x=Q_1}^{Q_2} x^{k-K}e(\alpha h_1\cdots h_{k-1}g_{k-1}(x,y;h_1,\dots,h_{k-1}))\\
=&Q_2^{k-K}\sum_{x=1}^{Q_2} e(\alpha h_1\cdots h_{k-1}g_{k-1}(x,y;h_1,\dots,h_{k-1}))\\
&-Q_1^{k-K}\sum_{x=1}^{Q_1-1} e(\alpha h_1\cdots h_{k-1}g_{k-1}(x,y;h_1,\dots,h_{k-1}))\\
&-\sum_{X=Q_1}^{Q_2-1}((X+1)^{k-K}-X^{k-K})\sum_{x=1}^{X} e(\alpha h_1\cdots h_{k-1}g_{k-1}(x,y;h_1,\dots,h_{k-1}))\\
\ll& Q_1^{k-K-1}Q_2^{2-\frac{k}{K}}\Bigg(Q_2^{k-1}+Q_2^{\frac{k}{2}}(\log Q_2)^{\frac{k^2-2k}{2}}\\
&\cdot\Bigg(\sum_{z=1}^{Q_2^{k-1}}\min\{Q_2,\|\alpha(k!)^2a_{11}h_1\cdots h_{k-1}yz\|^{-1}\}\Bigg)^{\frac{1}{2}}\Bigg)^{\frac{1}{K}}.
\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{|h_1|,\dots,|h_{k-1}|\le V/U}\ \sum_{\substack{U\le x\le 2U\\1\le x\le\min_{1\le i<k}V/|h_i|}} x^{k-K}T_{k-1}(x;h_1,\dots,h_{k-1})\\
\ll&V^{k-1}U^{2-K}+U^{k-K+1-\frac{k}{K}}\sum_{\substack{|h_1|,\dots,|h_{k-1}|\le V/U\\h_1\cdots h_{k-1}\ne0}}\ \sum_{1\le y\le V/U}\Bigg(U^{k-1}\\
&+U^{\frac{k}{2}}(\log U)^{\frac{k^2-2k}{2}}\Bigg(\sum_{1\le z\le(2U)^{k-1}}\min\{2U,\|\alpha(k!)^2a_{11}h_1\cdots h_{k-1}yz\|^{-1}\}\Bigg)^{\frac12}\Bigg)^{\frac1K}\\
\le&V^{k-1}U^{2-K}+U^{1-K}V^{k-\frac{k}{K}}\Bigg(V^kU^{-1}+U^{\frac{k}{2}}(\log U)^{\frac{k^2-2k}{2}}\\
&\cdot\sum_{\substack{|h_1|,\dots,|h_{k-1}|\le V/U\\h_1\cdots h_{k-1}\ne0}}\ \sum_{1\le y\le V/U}\Bigg(\sum_{1\le z\le(2U)^{k-1}}\min\{2U,\|\alpha(k!)^2a_{11}h_1\cdots h_{k-1}yz\|^{-1}\}\Bigg)^{\frac12}\Bigg)^{\frac1K}.\end{aligned}$$

Also,

$$\begin{aligned}&\sum_{\substack{|h_1|,\dots,|h_{k-1}|\le V/U\\h_1\cdots h_{k-1}\ne0}}\ \sum_{1\le y\le V/U}\Bigg(\sum_{1\le z\le(2U)^{k-1}}\min\{2U,\|\alpha(k!)^2a_{11}h_1\cdots h_{k-1}yz\|^{-1}\}\Bigg)^{\frac12}\\
\ll&V^{\frac{k}{2}}U^{-\frac{k}{2}}\Bigg(\sum_{\substack{|h_1|,\dots,|h_{k-1}|\le V/U\\h_1\cdots h_{k-1}\ne0}}\ \sum_{1\le y\le V/U}\ \sum_{1\le z\le(2U)^{k-1}}\min\{2U,\|\alpha(k!)^2a_{11}h_1\cdots h_{k-1}yz\|^{-1}\}\Bigg)^{\frac12}\\
\le&V^{\frac{k}{2}}U^{-\frac{k}{2}}\Bigg(\sum_{z=1}^{2^{k-1}V^kU^{-1}}d_{k+1}(z)\min\{2U,\|\alpha(k!)^2a_{11}z\|^{-1}\}\Bigg)^{\frac12}\\
\ll&V^{\frac{k}{2}}U^{-\frac{k}{2}}\cdot V^{\frac{k}{4}}(\log(V^kU^{-1}))^{\frac{k^2+2k}{4}}\Bigg(\sum_{1\le z\le2^{k-1}V^kU^{-1}}\min\{2U,\|\alpha(k!)^2a_{11}z\|^{-1}\}\Bigg)^{\frac14}\\
\ll&V^{\frac{3k}{4}}U^{-\frac{k}{2}}(\log(VU^{-1}))^{\frac{k^2+2k}{4}}\cdot a_{11}^{\frac14}V^{\frac{k}{4}}(\log(a_{11}qV))^{\frac14}\left(\frac1q+\frac1U+\frac{q}{a_{11}V^k}\right)^{\frac14}.\end{aligned}$$

It follows that

$$\sum_{U\le x\le 2U}\Big|\sum_{1\le y\le V/x} e(\alpha\psi(x,y))\Big|$$
$$\ll U^{1-\frac{1}{K}}V^{1-\frac{k}{K}}\Bigg(V^{k-1}U^{2-K}+U^{1-K}V^{k-\frac{k}{K}}\Bigg(V^kU^{-1}$$
$$+a_{11}^{\frac14}V^k(\log(a_{11}qUV))^{\frac{3k^2-2k+1}{4}}\left(\frac1q+\frac1U+\frac{q}{a_{11}V^k}\right)^{\frac14}\Bigg)^{\frac1K}\Bigg)^{\frac1K}$$
$$\ll V\left(U^{\frac1K}V^{-\frac1K}+U^{-\frac{1}{K^2}}++a_{11}^{\frac{1}{4K^2}}(\log(a_{11}qUV))^{\frac{3k^2-2k+1}{4K^2}}\left(\frac1q+\frac1U+\frac{q}{a_{11}V^k}\right)^{\frac{1}{4K^2}}\right).$$

□

From Lemma 4.8, it is easily derived that

$$\sum_{U_1\le x\le U_2}\sum_{y\le V/x} e(\alpha\psi(x,y))$$
$$\ll V\log U_2\left(U_2^{\frac1K}V^{-\frac1K}+U_1^{-\frac{1}{K^2}}+a_{11}^{\frac{1}{4K^2}}(\log(a_{11}qU_2V))^{\frac{3k^2-2k+1}{4K^2}}\left(\frac1q+\frac{1}{U_1}+\frac{q}{a_{11}V^k}\right)^{\frac{1}{4K^2}}\right). \tag{4.6}$$

Lemma 4.9. *Suppose that $\psi(x,y)=\sum_{1\le i,j\le k+1}a_{ij}x^{k-i+1}y^{k-j+1}$ is a polynomial with real coefficients and $a_{11},a_{21},\ldots,a_{k+1,1}\in\mathbb{Z}$. If $a_{11}x^k+a_{21}x^{k-1}+\cdots+a_{k+1,1}\ne 0$ for each $1\le x\le U$, then*

$$\sum_{1\le x\le U}\Big|\sum_{1\le y\le V} e(\alpha\psi(x,y))\Big|$$
$$\ll_\epsilon UV\left(V^{-\frac1K}U^{\frac1K}+a_*^{\frac{1}{2K}}U^{\frac{k}{2K}}(\log(a_*qU^kV))^{\frac{(k-1)^2}{2K}}\left(\frac1q+\frac UV+\frac{q}{V^k}\right)^{\frac{1}{2K}}\right) \tag{4.7}$$

provided that $|\alpha-a/q|\le q^{-2}$ with $(a,q)=1$, where $a_*=|a_{11}|+|a_{21}|+\cdots+|a_{k+1,1}|$.

Proof. Write $\psi(x,y)=\sum_{j=1}^{k+1}\psi_j(x)y^{k-j+1}$. Then by Lemma 4.6 we have

$$\sum_{1\le x\le U}\Big|\sum_{1\le y\le V} e(\alpha\psi(x,y))\Big|$$
$$\ll\sum_{1\le x\le U}V\left(V^{-\frac1K}+|\psi_1(x)|^{\frac{1}{2K}}(\log(|\psi_1(x)|qV))^{\frac{(k-1)^2}{2K}}\left(\frac1q+\frac1V+\frac{q}{|\psi_1(x)|V^k}\right)^{\frac{1}{2K}}\right)$$
$$\ll UV\left(V^{-\frac1K}+a_*^{\frac{1}{2K}}U^{\frac{k}{2K}}(\log(a_*qU^kV))^{\frac{(k-1)^2}{2K}}\left(\frac1q+\frac1V+\frac{q}{V^k}\right)^{\frac{1}{2K}}\right).$$

□

Lemma 4.10. *Suppose that* $\psi(x,y)=\sum_{1\le i,j\le k+1}a_{ij}x^{k-i+1}y^{k-j+1}$ *is a polynomial with real coefficients and* $a_{11},a_{21},\ldots,a_{k+1,1}\in\mathbb{Z}$. *If* $a_{11}x^k+a_{21}x^{k-1}+\cdots+a_{k+1,1}\neq 0$ *for each* $1\le x\le U$, *then*

$$\sum_{1\le x\le U}\bigg|\sum_{1\le y\le V/x}e(\alpha\psi(x,y))\bigg|$$

$$\ll V(\log U)\bigg(V^{-\frac{1}{K}}U^{\frac{1}{K}}+a_*^{\frac{1}{2K}}U^{\frac{k}{2K}}(\log(a_*qU^kV))^{\frac{(k-1)^2}{2K}}\bigg(\frac{1}{q}+\frac{U}{V}+\frac{q}{V^k}\bigg)^{\frac{1}{2K}}\bigg) \quad (4.8)$$

provided that $|\alpha-a/q|\le q^{-2}$ *with* $(a,q)=1$, *where* $a_*=|a_{11}|+|a_{21}|+\cdots+|a_{k+1,1}|$.

Proof. Write $\psi(x,y)=\sum_{j=1}^{k+1}\psi_j(x)y^{k-j+1}$. Now $\psi_1(x)$ is a polynomial with integral coefficients. Hence by Lemma 4.6 we have

$$\sum_{1\le x\le U}\bigg|\sum_{1\le y\le V/x}e(\alpha\psi(x,y))\bigg|$$

$$\ll\sum_{1\le x\le U}Vx^{-1}\bigg(V^{-\frac{1}{K}}x^{\frac{1}{K}}+|\psi_1(x)|^{\frac{1}{2K}}(\log(|\psi_1(x)|qV))^{\frac{(k-1)^2}{2K}}\bigg(\frac{1}{q}+\frac{x}{V}+\frac{qx^k}{|\psi_1(x)|V^k}\bigg)^{\frac{1}{2K}}\bigg)$$

$$\ll V(\log U)\bigg(V^{-\frac{1}{K}}U^{\frac{1}{K}}+a_*^{\frac{1}{2K}}U^{\frac{k}{2K}}(\log(a_*qU^kV))^{\frac{(k-1)^2}{2K}}\bigg(\frac{1}{q}+\frac{U}{V}+\frac{q}{V^k}\bigg)^{\frac{1}{2K}}\bigg).$$

□

Lemma 4.11. *Let* $\psi(x)=a_1x^k+a_2x^{k-1}\cdots+a_kx$ *be a polynomial with integral coefficients and* $a_1\in\mathbb{Z}^+$. *Let* $A\ge 1$ *and* $B>32k^2(k^2+K^2)A$. *Suppose that* $1\le W, a_1\le(\log V)^A$ *and* $1\le U\le V^{1-\delta}$ *for some* $\delta>0$. *Then for any integer* b *and* $1\le c,c'\le W$ *with* $cc'\equiv b \pmod W$, *we have*

$$\sum_{\substack{1\le x\le U\\ x\equiv c\ (\mathrm{mod}\ W)}}\bigg|\sum_{\substack{1\le y\le V/x\\ y\equiv c'\ (\mathrm{mod}\ W)}}e(\alpha\psi((xy-b)/W))\bigg|\ll_{A,B}V(\log V)^{-\frac{B}{16k^2K^2}} \quad (4.9)$$

provided that $|\alpha-a/q|\le q^{-2}$ *with* $(\log V)^B\le q\le\psi(V)(\log V)^{-B}$ *and* $(a,q)=1$.

Proof. Let

$$U_*=\min\{q^{\frac{1}{2k^2}},(2V^k/W^kq)^{\frac{1}{2k^2}},U\}.$$

Apparently $U_*\ge\min\{(\log V)^{\frac{B-(k+1)A}{2k^2}},U\}$ and

$$U_*\le(q\cdot 2V^k/W^kq)^{\frac{1}{4k^2}}\ll V^{\frac{1}{4k}}.$$

Rewrite

$$\sum_{\substack{1\le x\le U\\ x\equiv c\ (\mathrm{mod}\ W)}}\bigg|\sum_{\substack{1\le y\le V/x\\ y\equiv c'\ (\mathrm{mod}\ W)}}e(\alpha\psi((xy-b)/W))\bigg|$$

$$=\bigg(\sum_{\substack{1\le x\le U_*\\ x\equiv c\ (\mathrm{mod}\ W)}}+\sum_{\substack{U_*<x\le U\\ x\equiv c\ (\mathrm{mod}\ W)}}\bigg)\bigg|\sum_{\substack{1\le y\le V/x\\ y\equiv c'\ (\mathrm{mod}\ W)}}e(\alpha\psi((xy-b)/W))\bigg|.$$

Clearly

$$\begin{aligned}&\sum_{\substack{1\le x\le U_*\\ x\equiv c \pmod W}}\Big|\sum_{\substack{1\le y\le V/x\\ y\equiv c' \pmod W}} e(\alpha\psi((xy-b)/W))\Big|\\ =&\frac{1}{W}\sum_{j=1}^{W}\sum_{1\le x\le U_*} e(j(x-c)/W)\Big|\sum_{1\le y\le V/Wx} e(\alpha\psi((x(Wy+c')-b)/W))+O(1)\Big|\\ \le&\sum_{1\le x\le U_*}\Big|\sum_{1\le y\le (V/W)/x} e(\alpha\psi(xy+(xc'-b)/W))\Big|+O(U_*).\end{aligned}$$

By Lemma 4.10,

$$\begin{aligned}&\sum_{1\le x\le U_*}\Big|\sum_{1\le y\le (V/W)/x} e(\alpha\psi(xy+(xc'-b)/W))\Big|\\ \ll& V\log U_*\bigg(V^{-\frac{1}{K}}W^{\frac{1}{K}}U_*^{\frac{1}{K}}+a_1^{\frac{1}{2K}}U_*^{\frac{k}{2K}}(\log(a_1qVW^{-1}U_*))^{\frac{(k-1)^2}{2K}}\Big(\frac{1}{q}+\frac{U_*}{V}+\frac{qW^k}{V^k}\Big)^{\frac{1}{2K}}\bigg).\end{aligned}$$

If $U_*=U$, then $U\le(\log V)^{\frac{B-(k+1)A}{2k^2}}$ and

$$\begin{aligned}&V\log U_*\bigg(V^{-\frac{1}{K}}W^{\frac{1}{K}}U_*^{\frac{1}{K}}+a_1^{\frac{1}{2K}}U_*^{\frac{k}{2K}}(\log(a_1qVW^{-1}U_*))^{\frac{(k-1)^2}{2K}}\Big(\frac{1}{q}+\frac{U_*}{V}+\frac{qW^k}{V^k}\Big)^{\frac{1}{2K}}\bigg)\\ \ll& V\log U(V^{-\frac{1}{K}}W^{\frac{1}{K}}U^{\frac{1}{K}}+a_1^{\frac{1}{2K}}(\log V)^{\frac{(k-1)^2}{2K}}(U^{\frac{k}{2K}}q^{-\frac{1}{2K}}+U^{\frac{k+1}{2K}}V^{-\frac{1}{2K}}+(\log V)^{\frac{(2k^2+k)A-(2k-1)B}{4kK}}))\\ \ll_{A,B}& V(\log V)^{1+\frac{(k-1)^2+(k+2)A}{2K}-\frac{(2k-1)B}{4kK}}.\end{aligned}$$

Below we assume that $U_*<U$, then $(\log V)^{\frac{B-(k+1)A}{2k^2}}\ll U_*\ll V^{\frac{1}{4k}}$ and

$$\begin{aligned}&V\log U_*\bigg(V^{-\frac{1}{K}}W^{\frac{1}{K}}U_*^{\frac{1}{K}}+a_1^{\frac{1}{2K}}U_*^{\frac{k}{2K}}(\log(a_1qVW^{-1}U_*))^{\frac{(k-1)^2}{2K}}\Big(\frac{1}{q}+\frac{U_*}{V}+\frac{qW^k}{V^k}\Big)^{\frac{1}{2K}}\bigg)\\ \ll& V\log V(V^{-\frac{1}{K}}W^{\frac{1}{K}}U_*^{\frac{1}{K}}+a_1^{\frac{1}{2K}}(\log V)^{\frac{(k-1)^2}{2K}}(U_*^{\frac{k}{2K}}q^{-\frac{1}{2K}}+U_*^{\frac{k+1}{2K}}V^{-\frac{1}{2K}}+U_*^{\frac{k-2k^2}{2K}}))\\ \ll_{A,B}& V(\log V)^{1+\frac{(k-1)^2+(k+2)A}{2K}-\frac{(2k-1)B}{4kK}}.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\sum_{\substack{U_*<x\le U\\ x\equiv c \pmod W}}\bigg|\sum_{\substack{1\le y\le V/x\\ y\equiv c' \pmod W}} e(\alpha\psi((xy-b)/W))\bigg|\\
=&\sum_{(U_*-c)/W<x\le(U-c)/W}\bigg|\sum_{\substack{1\le y\le V/(Wx+c)\\ y\equiv c' \pmod W}} e(\alpha\psi(xy+(yc-b)/W))\bigg|\\
=&\sum_{(U_*-c)/W<x\le(U-c)/W}\bigg|\sum_{\substack{1\le y\le V/Wx\\ y\equiv c' \pmod W}} e(\alpha\psi(xy+(yc-b)/W))+O(V/W^2x^2)\bigg|\\
=&\sum_{(U_*-c)/W<x\le(U-c)/W}\bigg|\sum_{\substack{1\le y\le V/Wx\\ y\equiv c' \pmod W}} e(\alpha\psi(xy+(yc-b)/W))\bigg|+O(V/WU_*).\end{aligned}$$

Notice that

$$\begin{aligned}&\sum_{(U_*-c)/W<x\le(U-c)/W}\bigg|\sum_{\substack{1\le y\le V/Wx\\ y\equiv c' \pmod W}} e(\alpha\psi(xy+(yc-b)/W))\bigg|\\
=&\frac{1}{W}\sum_{(U_*-c)/W<x\le(U-c)/W}\bigg|\sum_{1\le y\le V/Wx} e(\alpha\psi(xy+(yc-b)/W))\sum_{j=1}^{W}e((y-c')j/W)\bigg|\\
\le&\max_{1\le j\le W}\sum_{(U_*-c)/W<x\le(U-c)/W}\bigg|\sum_{1\le y\le V/Wx} e(\alpha\psi(xy+(yc-b)/W+(y-c')j/W)\bigg|.\end{aligned}$$

Hence by Lemma 4.8,

$$\begin{aligned}&\sum_{(U_*-c)/W<x\le(U-c)/W}\bigg|\sum_{1\le y\le (V/W)/x} e(\alpha\psi(xy+(yc-b)/W+(y-c')j/W)\bigg|\\
\ll&VW^{-1}\log(UW^{-1})\Big(U^{\frac{1}{K}}V^{-\frac{1}{K}}+U_*^{-\frac{1}{K^2}}W^{\frac{1}{K^2}}+a_1^{\frac{1}{4K^2}}(\log V)^{\frac{3k^2-2k+1}{4K^2}}U_*^{-\frac{1}{4K^2}}W^{\frac{1}{4K^2}}\Big)\\
\ll_{A,B}&V(\log V)^{1+\frac{3k^2-2k+1+3A}{4K^2}-\frac{B}{8k^2K^2}}.\end{aligned}$$

□

Lemma 4.12. *Let $\psi(x)=a_1x^k+a_2x^{k_1}\cdots+a_kx$ be a polynomial with integral coefficients and $a_1\in\mathbb{Z}^+$. Let $A\ge 1$ and $B>16k^3A$. Suppose that $1\le W,a_1\le(\log(UV))^A$. Let $g(x)$ be a polynomial with the degree at most k satisfying that the coefficient of x^k in $g(Wx)$ is an integer. Then for any integer b and $1\le c,c'\le W$ with $cc'\equiv b \pmod W$, we have*

$$\sum_{\substack{1\le x\le U\\ x\equiv c \pmod W}}\bigg|\sum_{\substack{1\le y\le V\\ y\equiv c' \pmod W}} e(\alpha(\psi((xy-b)/W)+g(y)))\bigg|\ll_{A,B}UV(\log(UV))^{-\frac{B}{16kK^2}} \tag{4.10}$$

provided that $|\alpha - a/q| \le q^{-2}$ *with* $(\log V)^B \le q \le \psi(V)(\log V)^{-B}$ *and* $(a,q)=1$.

Proof. Suppose that $U \ge (\log V)^{\frac{B}{2k}}$. Then by Lemma 4.7,

$$\sum_{\substack{1\le x\le U\\ x\equiv c \pmod W}} \Bigg| \sum_{\substack{1\le y\le V\\ y\equiv c' \pmod W}} e(\alpha(\psi((xy-b)/W)+g(y))) \Bigg|$$

$$= \sum_{1\le x\le (U-c)/W+1} \frac{1}{W}\Bigg|\sum_{j=1}^{W}\sum_{1\le y\le V} e(\alpha(\psi(xy-(W-c)y/W-b/W)+g(y))+j(y-c')/W)\Bigg|$$

$$\le \frac{1}{W}\sum_{j=1}^{W}\sum_{1\le x\le (U-c)/W+1}\Bigg|\sum_{1\le y\le V} e(\alpha(\psi(xy-(W-c)y/W-b/W)+g(y))+j(y-c')/W)\Bigg|$$

$$\ll UV\left(U^{-\frac{1}{K^2}}W^{\frac{1}{K^2}}+V^{-\frac{1}{K^2}}+a_{11}^{\frac{1}{4K^2}}(\log(a_{11}qUV))^{\frac{3k^2-2k+1}{4K^2}}\left(\frac{1}{q}+\frac{W}{U}+\frac{qW^k}{a_{11}U^kV^k}\right)^{\frac{1}{4K^2}}\right)$$

$$\ll_{A,B} UV(\log(UV))^{\frac{3k^2-2k+1+(k+1)A}{4K^2}-\frac{B}{8kK^2}}.$$

Also, if $U \le (\log V)^{\frac{B}{2k}}$, then by Lemma 4.9,

$$\sum_{\substack{1\le x\le U\\ x\equiv c \pmod W}} \Bigg| \sum_{\substack{1\le y\le V\\ y\equiv c' \pmod W}} e(\alpha(\psi((xy-b)/W)+g(y))) \Bigg|$$

$$\le \sum_{1\le x\le U}\Bigg|\sum_{1\le y\le (V-c')/W+1} e(\alpha(\psi(xy-(W-c')x/W-b/W)+g(Wy-W+c')))\Bigg|$$

$$\ll UV\left(V^{-\frac{1}{K}}W^{\frac{1}{K}}+a_*^{\frac{1}{2K}}U^{\frac{k}{2K}}(\log(a_*qU^kV))^{\frac{(k-1)^2}{2K}}\left(\frac{1}{q}+\frac{W}{V}+\frac{qW^k}{V^k}\right)^{\frac{1}{2K}}\right)$$

$$\ll_{A,B} UV(\log(UV))^{\frac{(k-1)^2+(k+2)A}{2K}-\frac{B}{4K}}.$$

□

Theorem 4.1. *Let* $\psi(x)=a_1x^k+a_2x^{k-1}\cdots+a_kx$ *be a polynomial with integral coefficients and* $a_1\in\mathbb{Z}^+$. *Let* $A\ge 1$ *and* $B>64k^2(k^2+K^2)A$. *Suppose that* $1\le W, a_1\le(\log N)^A$. *Then we have*

$$\sum_{\substack{1\le x\le N\\ Wx+b\,\text{is prime}}} \log(Wx+b)e(\alpha\psi(x)) \ll_{A,B} N(\log N)^{-\frac{B}{64k^2K^2}} \tag{4.11}$$

provided that $|\alpha-a/q|\le q^{-2}$ *with* $(\log N)^{B+1}\le q\le\psi(N)(\log N)^{-B-1}$ *and* $(a,q)=1$.

Proof. For a proposition P, define $\mathbf{1}_P=1$ or 0 according to whether P holds. Let $F(x)=e(\alpha\psi((x-b)/W))\mathbf{1}_{x\equiv b \pmod W}$. Let $V=WN+b$ and $X=V^{2/5}$. Clearly

$$(\log V)^B \le (\log N)^{B+1}\le q\le \psi(N)(\log N)^{-B-1}\le\psi(V)(\log V)^{-B}.$$

By Vaughan's identity we have,

$$\sum_{X<x\le V}\Lambda(x)F(x)=S_1-S_2-S_3,$$

where

$$S_1=\sum_{1\le d\le X}\mu(d)\sum_{1\le z\le V/d}\sum_{x\le V/dz}\Lambda(x)F(xdz),$$

$$S_2=\sum_{1\le d\le X}\mu(d)\sum_{1\le z\le V/d}\sum_{x\le\min\{X,V/dz\}}\Lambda(x)F(xdz),$$

and

$$S_3=\sum_{X<u\le V}\sum_{\substack{1\le d\le X\\ d|u}}\mu(d)\sum_{X<x\le V/u}\Lambda(x)F(xu).$$

In fact, letting $\tau_u=\sum_{1\le d|u,d\le X}\mu(d)$, we have

$$\sum_{1\le u\le V}\tau_u\sum_{X<x\le V/u}\Lambda(x)F(xu)=\sum_{X<u\le V}\tau_u\sum_{X<x\le V/u}\Lambda(x)F(xu)+\sum_{X<x\le V}\Lambda(x)F(x),$$

since $\tau_1=1$ and $\tau_u=0$ for $1<u\le X$. On the other hand,

$$\begin{aligned}\sum_{1\le u\le V}\tau_u\sum_{X<x\le V/u}\Lambda(x)F(xu)&=\sum_{1\le u\le V}\sum_{d|u,1\le d\le X}\mu(d)\sum_{X<x\le V/u}\Lambda(x)F(xu)\\&=\sum_{1\le d\le X}\mu(d)\sum_{1\le z\le V/d}\sum_{X<x\le V/dz}\Lambda(x)F(xdz).\end{aligned}$$

First, we compute

$$\begin{aligned}|S_1|=&\Big|\sum_{d\le X}\mu(d)\sum_{xz\le V/d}\Lambda(x)e(\alpha\psi((dxz-b)/W))\mathbf{1}_{dxz\equiv b\pmod W}\Big|\\=&\Big|\sum_{1\le d\le X}\mu(d)\sum_{1\le u\le V/d}e(\alpha\psi((du-b)/W))\mathbf{1}_{du\equiv b\pmod W}\sum_{x|u}\Lambda(x)\Big|\\\le&\sum_{1\le d\le X}\Big|\sum_{1\le u\le V/d}e(\alpha\psi((du-b)/W))\mathbf{1}_{du\equiv b\pmod W}\log u\Big|\\\le&\sum_{1\le d\le X}\Big|\sum_{1\le u\le V/d}e(\alpha\psi((du-b)/W))\mathbf{1}_{du\equiv b\pmod W}\int_1^u\frac{dt}{t}\Big|\\\le&\int_1^V\sum_{1\le d\le X}\Big|\sum_{t\le u\le V/d}e(\alpha\psi((du-b)/W))\mathbf{1}_{du\equiv b\pmod W}\Big|\frac{dt}{t}.\end{aligned}$$

Clearly

$$\begin{aligned}
&\sum_{1\le d\le X}\left|\sum_{t\le u\le V/d} e(\alpha\psi((du-b)/W))\mathbf{1}_{du\equiv b \pmod W}\right| \\
&= \sum_{1\le d\le \min\{X,V/t\}}\left|\sum_{t\le u\le V/d} e(\alpha\psi((du-b)/W))\mathbf{1}_{du\equiv b \pmod W}\right| \\
&= \sum_{\substack{1\le c\le W\\ (c,W)=1}}\sum_{\substack{1\le d\le \min\{X,V/t\}\\ d\equiv c \pmod W}}\left|\sum_{\substack{t\le u\le V/d\\ uc\equiv b \pmod W}} e(\alpha\psi((du-b)/W))\right|.
\end{aligned}$$

So it suffices to estimate

$$\sum_{\substack{1\le d\le \min\{X,V/t\}\\ d\equiv c \pmod W}}\left|\sum_{\substack{1\le u<t\\ uc\equiv b \pmod W}} e(\alpha\psi((du-b)/W))\right|$$

and

$$\sum_{\substack{1\le d\le \min\{X,V/t\}\\ d\equiv c \pmod W}}\left|\sum_{\substack{1\le u\le V/d\\ uc\equiv b \pmod W}} e(\alpha\psi((du-b)/W))\right|.$$

Applying Lemma 4.11,

$$\sum_{\substack{1\le d\le \min\{X,V/t\}\\ d\equiv c \pmod W}}\left|\sum_{\substack{1\le u\le V/d\\ uc\equiv b \pmod W}} e(\alpha\psi((du-b)/W))\right| \ll V(\log V)^{-\frac{B}{16k^2K^2}}.$$

Since

$$\sum_{\substack{1\le d\le \min\{X,V/t\}\\ d\equiv c \pmod W}}\left|\sum_{\substack{1\le u<t\\ uc\equiv b \pmod W}} e(\alpha\psi((du-b)/W))\right| \le Xt,$$

we may assume that $t\ge V^{\frac{1}{2}}$. Then by Lemma 4.12,

$$\sum_{\substack{1\le d\le \min\{X,V/t\}\\ d\equiv c \pmod W}}\left|\sum_{\substack{1\le u<t\\ uc\equiv b \pmod W}} e(\alpha\psi((du-b)/W))\right| \ll V(\log t)^{-\frac{B}{16kK^2}}.$$

Similarly,

$$\begin{aligned}|S_2| = &\sum_{1\le d\le X}\mu(d)\sum_{1\le z\le V/d}\sum_{1\le x\le\min\{X,V/dz\}}\Lambda(x)e(\alpha\psi((dxz-b)/W))\mathbf{1}_{dxz\equiv b\ (\mathrm{mod}\ W)}\\ \le&\sum_{1\le d\le X}\sum_{1\le x\le X}\Lambda(x)\bigg|\sum_{1\le z\le V/dx}e(\alpha\psi((dxz-b)/W))\mathbf{1}_{dxz\equiv b\ (\mathrm{mod}\ W)}\bigg|\\ \le&\sum_{1\le y\le X^2}\sum_{\substack{1\le x\le X\\ x|y}}\Lambda(x)\bigg|\sum_{1\le z\le V/y}e(\alpha\psi((yz-b)/W))\mathbf{1}_{yz\equiv b\ (\mathrm{mod}\ W)}\bigg|\\ \le&\log V\sum_{1\le y\le X^2}\bigg|\sum_{1\le z\le V/y}e(\alpha\psi((yz-b)/W))\mathbf{1}_{yz\equiv b\ (\mathrm{mod}\ W)}\bigg|\\ \ll&V(\log V)^{1-\frac{B}{16k^2K^2}},\end{aligned}$$

where Lemma 4.11 is used in the last step.

Finally, let

$$S_3(U_1,U_2)=\sum_{U_1\le u\le U_2}\tau_u\sum_{X<x\le V/u}\Lambda(x)e(\alpha\psi((xu-b)/W))\mathbf{1}_{xu\equiv b\ (\mathrm{mod}\ W)}$$

with $X\le U_1\le U_2\le 2U_1$, where $\tau_u=\sum_{1\le d\le X,d|u}\mu(d)$. Clearly $S_3(U_1,U_2)\neq 0$ only if $X<V/U_1$. Since $|\tau_u|\le d(u)$, we have

$$\begin{aligned}&|S_3(U_1,U_2)|\\ \le&\bigg(\sum_{U_1\le u\le U_2}|\tau_u|^2\bigg)^{\frac12}\bigg(\sum_{u=U_1}^{U_2}\bigg|\sum_{X<x\le V/u}\Lambda(x)e(\alpha\psi((xu-b)/W))\mathbf{1}_{xu\equiv b\ (\mathrm{mod}\ W)}\bigg|^2\bigg)^{\frac12}\\ \le&U_2^{\frac12}(\log U_2)^{\frac32}\bigg(\sum_{U_1\le u\le U_2}\sum_{\substack{X<x,y\le V/u\\ xu\equiv b\ (\mathrm{mod}\ W)\\ yu\equiv b\ (\mathrm{mod}\ W)}}\Lambda(x)\Lambda(y)e(\alpha(\psi((xu-b)/W)-\psi((yu-b)/W)))\bigg)^{\frac12}.\end{aligned}$$

Now for $1\le c,c'\le W$ with $cc'\equiv b\ (\mathrm{mod}\ W)$,

$$\begin{aligned}&\sum_{\substack{U_1\le u\le U_2\\ u\equiv c\ (\mathrm{mod}\ W)}}\sum_{\substack{X<x,y\le V/u\\ x\equiv c'\ (\mathrm{mod}\ W)\\ y\equiv c'\ (\mathrm{mod}\ W)}}\Lambda(x)\Lambda(y)e(\alpha(\psi((xu-b)/W)-\psi((yu-b)/W)))\\ =&\sum_{\substack{X<x,y\le V/U_1\\ x\equiv c'\ (\mathrm{mod}\ W)\\ y\equiv c'\ (\mathrm{mod}\ W)}}\Lambda(x)\Lambda(y)\sum_{\substack{U_1\le u\le\min\{U_2,V/x,V/y\}\\ u\equiv c\ (\mathrm{mod}\ W)}}e(\alpha(\psi((xu-b)/W)-\psi((yu-b)/W)))\\ =&2\sum_{\substack{X<x<y\le V/U_1\\ x\equiv c'\ (\mathrm{mod}\ W)\\ y\equiv c'\ (\mathrm{mod}\ W)}}\Lambda(x)\Lambda(y)\sum_{\substack{U_1\le u\le\min\{U_2,V/y\}\\ u\equiv c\ (\mathrm{mod}\ W)}}e(\alpha(\psi((xu-b)/W)-\psi((yu-b)/W)))\\ &+O((V/U_1-X)/W).\end{aligned}$$

And

$$\sum_{\substack{X<x<y\le V/U_1\\ x\equiv c' \pmod W\\ y\equiv c' \pmod W}} \Lambda(x)\Lambda(y) \sum_{\substack{U_1\le u\le \min\{U_2,V/y\}\\ u\equiv c \pmod W}} e(\alpha(\psi((xu-b)/W)-\psi((yu-b)/W)))$$

$$= \sum_{\substack{V/U_2<y\le V/U_1\\ y\equiv c' \pmod W}} \Lambda(y) \sum_{\substack{X<x<y\\ x\equiv c' \pmod W}} \Lambda(x) \sum_{\substack{U_1\le u\le V/y\\ u\equiv c \pmod W}} e(\alpha(\psi((xu-b)/W)-\psi((yu-b)/W)))$$

$$+ \sum_{\substack{X<y\le V/U_2\\ y\equiv c' \pmod W}} \Lambda(y) \sum_{\substack{X<x<y\\ x\equiv c' \pmod W}} \Lambda(x) \sum_{\substack{U_1\le u\le U_2\\ u\equiv c \pmod W}} e(\alpha(\psi((xu-b)/W)-\psi((yu-b)/W))).$$

If $V/U_2 < y$, then by Lemma 4.12,

$$\sum_{\substack{X<x<y\\ x\equiv c' \pmod W}} \Bigg| \sum_{\substack{U_1\le u\le V/y\\ u\equiv c \pmod W}} e(\alpha(\psi((xu-b)/W)-\psi((yu-b)/W)))\Bigg|$$

$$\ll (X+y)U_1(\log U_1)^{-\frac{B}{16kK^2}} + (X+y)(V/y)(\log(V/y))^{-\frac{B}{16kK^2}}$$

$$\ll (X+y)(U_1+V/y)(\log V)^{-\frac{B}{16kK^2}}.$$

Also, if $y \le V/U_2$, then by Lemma 4.12,

$$\sum_{\substack{X<x<y\\ x\equiv c' \pmod W}} \sum_{\substack{U_1\le u\le U_2\\ u\equiv c \pmod W}} e(\alpha(\psi((xu-b)/W)-\psi((yu-b)/W)))$$

$$\ll (X+y)U_1(\log U_1)^{-\frac{B}{16kK^2}} + (X+y)U_2(\log U_2)^{-\frac{B}{16kK^2}}$$

$$\ll (X+y)(U_1+U_2)(\log V)^{-\frac{B}{16kK^2}}.$$

Hence

$$\sum_{\substack{X<x<y\le V/U_1\\ x\equiv c' \pmod W\\ y\equiv c' \pmod W}} \Lambda(x)\Lambda(y) \sum_{\substack{U_1\le u\le \min\{U_2,V/y\}\\ u\equiv c \pmod W}} e(\alpha(\psi((xu-b)/W)-\psi((yu-b)/W)))$$

$$\ll (\log V)^{2-\frac{B}{16kK^2}} \Bigg(\sum_{\substack{V/U_2<y\le V/U_1\\ y\equiv c' \pmod W}} (X+y)(U_1+V/y) + \sum_{\substack{X<y\le V/U_2\\ y\equiv c' \pmod W}} (X+y)(U_1+U_2)\Bigg)$$

$$\ll V^2U_1^{-1}(\log V)^{2-\frac{B}{16kK^2}}.$$

It follows that

$$S_3(U_1,U_2) \ll U_2^{\frac12}(\log U_2)^{\frac32}(V^2U_1^{-1}(\log V)^{2-\frac{B}{16kK^2}})^{\frac12} \ll V(\log V)^{3-\frac{B}{32kK^2}},$$

and

$$S_3 \ll V(\log V)^{4-\frac{B}{32kK^2}}.$$

All are done. □

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