RELATIVE SINGULARITY CATEGORIES AND GORENSTEIN-PROJECTIVE MODULES

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Abstract. We introduce the notion of relative singularity category with respect to any self-orthogonal subcategory ω of an abelian category. We introduce the Frobenius category of ω-Cohen-Macaulay objects, and under some reasonable conditions, we show that the stable category of ω-Cohen-Macaulay objects is triangle-equivalent to the relative singularity category. As applications, we relate the stable category of (unnecessarily finitely-generated) Gorenstein-projective modules with singularity categories of rings. We prove that for a Gorenstein ring, the stable category of Gorenstein-projective modules is compactly generated and its compact objects coincide with finitely-generated Gorenstein-projective modules up to direct summands.

1. Introduction

1.1. Throughout, $\mathcal{A}$ is an abelian category, $\omega \subseteq \mathcal{A}$ its full additive subcategory. Denote by $C^b(\mathcal{A})$, $K^b(\mathcal{A})$ and $D^b(\mathcal{A})$ the category of bounded complexes, the bounded homotopy category and the bounded derived category of $\mathcal{A}$, respectively, both of whose shift functors will be denoted by [1]. Recall that for any $X, Y \in \mathcal{A}$, the n-th extension group $\mathrm{Ext}^n_{\mathcal{A}}(X, Y)$ is defined to be $\mathrm{Hom}_{D^b(\mathcal{A})}(X, Y[n])$, $n \geq 0$ (see [14], p.62). The subcategory ω is said to be *self-orthogonal* if for any $X, Y \in \omega$, $n \geq 1$, $\mathrm{Ext}^n_{\mathcal{A}}(X, Y) = 0$. Consider the following composite of functors

$$K^b(\omega) \longrightarrow K^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A}),$$

where the first is the inclusion functor, and the second the quotient functor. By [12], Chapter II, Lemma 3.4 (or Chapter III, Lemma 2.1), the composite functor is fully-faithful if and only if ω is self-orthogonal.

Let $\omega \subseteq \mathcal{A}$ be a self-orthogonal additive subcategory. By the argument above, we may view $K^b(\omega)$ as a triangulated subcategory of $D^b(\mathcal{A})$. Define the *relative singularity category*

Supported in part by the National Nature Science Foundation of China (Grant No. 10501041, 10601052).
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$D_\omega(\mathcal{A})$ of $\mathcal{A}$ with respect to ω to be the Verdier quotient category

$$D_\omega(\mathcal{A}) := D^b(\mathcal{A})/K^b(\omega).$$

The motivation of introducing relative singularity category is twofold: (1) A special case of relative singularity category is of particular interest: let $\mathcal{P}$ denote the subcategory consisting of projective objects, which is clearly self-orthogonal, then the relative singularity category with respect to $\mathcal{P}$ is called the *singularity category* of $\mathcal{A}$. Denote it by $D_{\rm sg}(\mathcal{A})$ (compare [23]). This terminology is justified by the fact that: the singularity category $D_{\rm sg}(\mathcal{A})$ vanishes if and only if the category $\mathcal{A}$ has enough projectives and every object is of finite projective dimension. (2) Singularity categories may be described as relative singularity categories via tilting subcategories, and this viewpoint allows us to describe singularity category by various tilting subcategories. Precisely, let $\mathcal{A}$ have enough projectives, a *tilting subcategory* $\mathcal{T}$ is a self-orthogonal subcategory such that $K^b(\mathcal{T}) = K^b(\mathcal{P})$ inside $D^b(\mathcal{A})$. Then we have $D_{\rm sg}(\mathcal{A}) = D_{\mathcal{T}}(\mathcal{A})$ for any tilting subcategory $\mathcal{T}$.

1.2. The paper is organized as follows: In section 2, we study the relative singularity category $D_\omega(\mathcal{A})$ and various related subcategories of the abelian category $\mathcal{A}$, in particular, the category of ω-Cohen-Macaulay objects. As the main theorem, we prove that there is a full exact embedding of the stable category of ω-Cohen-Macaulay objects into the relative singularity category, and further under some reasonable conditions, the embedding is an equivalence. In section 3, we apply the result to the module category of rings, and we rediscover the result of Buchweitz-Happel which says that for a Gorenstein ring, the singularity category is triangle-equivalent to the stable category of finitely-generated Gorenstein-projective modules, and we also find a similar result holds in the unnecessarily finitely-generated case. We relate the stable category of T-Cohen-Macaulay objects to the stable category of finitely-generated Gorenstein-projective modules over the endomorphism ring $\mathrm{End}_{\mathcal{A}}(T)$, where T is any self-orthogonal object in $\mathcal{A}$. In section 4, we show that for a Gorenstein ring, the stable category of Gorenstein-projective modules is compactly generated and its subcategory of compact objects is the stable category of finitely-generated Gorenstein-projective modules up to direct summands.

For triangulated categories, we refer to [12, 14, 25]. We abuse the notions of triangle-functors and exact functors between triangulated categories. For Gorenstein rings and Gorenstein-projective modules, we refer to [11, 9, 13, 8].

2. Relative singularity category and ω-Cohen-Macaulay objects

2.1. In this subsection, we will introduce some subcategories of the abelian category $\mathcal{A}$ (compare [3, 10]). At this moment, $\omega \subseteq \mathcal{A}$ is an arbitrary additive subcategory. Consider the following full subcategories:

$$\begin{aligned}\widehat{\omega} := \{X \in \mathcal{A} \mid &\text{ there exists an exact sequence}\\ &0 \to T^{-n} \to T^{1-n} \to \cdots \to T^0 \to X \to 0, \text{ each } T^{-i} \in \omega,\ n \geq 0\};\\ \omega^{\perp} := \{X \in \mathcal{A} \mid &\operatorname{Ext}^i_{\mathcal{A}}(T, X) = 0, \text{ for all } T \in \omega,\ i \geq 1\};\\ {}_{\omega}\mathcal{X} := \{X \in \mathcal{A} \mid &\text{ there exists an exact sequence}\\ &\cdots \to T^{-n} \overset{d^{-n}}{\to} T^{1-n} \to \cdots \to T^0 \overset{d^0}{\to} X \to 0, \text{ each } T^{-i} \in \omega,\ \operatorname{Ker} d^i \in \omega^{\perp}\}.\end{aligned}$$

If ω is self-orthogonal, using the dimension-shift technique in homological algebra, we infer that $\widehat{\omega} \subseteq \omega^{\perp}$ and ${}_{\omega}\mathcal{X} \subseteq \omega^{\perp}$, and thus we get $\widehat{\omega} \subseteq {}_{\omega}\mathcal{X}$. Consequently, if ω is self-orthogonal, we obtain that

$$\omega \subseteq \widehat{\omega} \subseteq {}_{\omega}\mathcal{X} \subseteq \omega^{\perp}.$$

Dually, we have the following three full subcategories:

$$\begin{aligned}\overset{\vee}{\omega} := \{X \in \mathcal{A} \mid &\text{ there exists an exact sequence}\\ &0 \to X \to T^0 \to \cdots \to T^{n-1} \to T^n \to 0, \text{ each } T^i \in \omega,\ n \geq 0\};\\ {}^{\perp}\omega := \{X \in \mathcal{A} \mid &\operatorname{Ext}^i_{\mathcal{A}}(X, T) = 0, \text{ for all } T \in \omega,\ i \geq 1\};\\ \mathcal{X}_{\omega} := \{X \in \mathcal{A} \mid &\text{ there exists an exact sequence}\\ &0 \to X \overset{d^{-1}}{\to} T^0 \overset{d^0}{\to} T^1 \to \cdots \to T^{n-1} \overset{d^{n-1}}{\to} T^n \to \cdots, \text{ each } T^i \in \omega,\ \operatorname{Coker} d^i \in {}^{\perp}\omega\}.\end{aligned}$$

Similarly, if ω is self-orthogonal, we have $\omega \subseteq \overset{\vee}{\omega} \subseteq \mathcal{X}_{\omega} \subseteq {}^{\perp}\omega$.

Let ω be a self-orthogonal subcategory. We define the *category of ω-Cohen-Macaulay objects* to be the full subcategory $\alpha(\omega) := \mathcal{X}_{\omega} \cap {}_{\omega}\mathcal{X}$. By [3], Proposition 5.1, the full subcategories ${}_{\omega}\mathcal{X}$ and $\mathcal{X}_{\omega}$ are closed under extensions, and therefore so is $\alpha(\omega)$. Hence, $\alpha(\omega)$ becomes an exact category whose conflations are just short exact sequences with terms in $\alpha(\omega)$ (for terminology, see [17]). Observe that objects in ω are (relative) projective and injective, and then it is not hard to see that $\alpha(\omega)$ is a Frobenius category, whose projective-injective objects are precisely contained in the additive closure add ω of ω. Consider the stable category $\underline{\alpha(\omega)}$ of $\alpha(\omega)$ modulo ω (or equivalently modulo add ω). Then by [12], $\underline{\alpha(\omega)}$ is a triangulated category.

For each $X \in \alpha(\omega)$, from the definition (and the dimension-shift technique if needed), we have an exact sequence in $K(\omega)$

$$T^{\bullet} = \quad \to \cdots \to T^{-n} \to T^{-n+1} \to \cdots \to T^{-1} \to T^0 \to T^1 \to \cdots \to T^n \to T^{n+1} \to \cdots$$

such that each of its cocycles $Z^i(T^{\bullet})$ lies in ${}^{\perp}\omega \cap \omega^{\perp}$, and $X = Z^0(T^{\bullet})$. Such a complex $T^{\bullet}$ will be called an *ω-complete resolution* for X. It is worthy observing that an exact complex

$T^\bullet \in K(\omega)$ is an ω-complete resolution if and only if for each $T \in \omega$, the Hom complexes $\mathrm{Hom}(T, T^\bullet)$ and $\mathrm{Hom}(T^\bullet, T)$ are exact. One may compare [5], Definition 5.5.

2.2. Consider the following composite of natural functors

$$F : \alpha(\omega) \longrightarrow \mathcal{A} \xrightarrow{i_{\mathcal{A}}} D^b(\mathcal{A}) \xrightarrow{Q_\omega} D_\omega(\mathcal{A}),$$

where the first functor is the inclusion, the second is the full embedding which sends objects in $\mathcal{A}$ to the stalk complexes concentrated at degree 0, and the last is the quotient functor $Q_\omega : D^b(\mathcal{A}) \longrightarrow D_\omega(\mathcal{A})$. Note that $F(\omega) = 0$, and thus F induces a unique functor $\underline{F}$ from $\underline{\alpha(\omega)}$ to $D_\omega(\mathcal{A})$.

Our main result is

Theorem 2.1. *Let $\omega \subseteq \mathcal{A}$ be a self-orthogonal additive subcategory. Then the natural functor $\underline{F} : \underline{\alpha(\omega)} \longrightarrow D_\omega(\mathcal{A})$ is a fully-faithful triangle-functor.*

Assume further that $\widehat{\mathcal{X}_\omega} = \mathcal{A} = {}_\omega\overset{\vee}{\mathcal{X}}$. Then $\underline{F}$ is an equivalence, thus a triangle-equivalence.

Note that the subcategories $\widehat{\mathcal{X}_\omega}$ and ${}_\omega\overset{\vee}{\mathcal{X}}$ are defined as in **2.1**, by replacing ω by $\mathcal{X}_\omega$ and ${}_\omega\mathcal{X}$, respectively.

2.3. We will divide the proof of Theorem 2.1 into proving several propositions. Note that we will always view $\mathcal{A}$ as the full subcategory of $D^b(\mathcal{A})$ consisting of stalk complexes concentrated at degree zero.

We need some notation. A complex $X^\bullet \in C^b(\omega)$ is said to *negative* if $X^n = 0$ for all $n \geq 0$. Denote by $D^{<0}(\omega)$ to be full subcategory of $K^b(\omega)$ whose objects are isomorphic t o some negative complexes in $C^b(\omega)$. Similarly, we have the subcategory $D^{>0}(\omega)$.

Lemma 2.2. *(1). For $M \in {}^\perp\omega$ and $X^\bullet \in D^{<0}(\omega)$, we have $\mathrm{Hom}_{D^b(\mathcal{A})}(M, X^\bullet) = 0$.*
(2). For $N \in \omega^\perp$ and $Y^\bullet \in D^{>0}(\omega)$, we have $\mathrm{Hom}_{D^b(\mathcal{A})}(Y^\bullet, N) = 0$.

Proof. We only show (1). Consider $\mathcal{L} := \{Z^\bullet \in D^b(\mathcal{A}) \mid \mathrm{Hom}_{D^b(\mathcal{A})}(M, Z^\bullet) = 0\}$. By the self-orthogonal property of ω, we have $\omega[i] \in \mathcal{L}$ for all $i > 0$. Observe that $\mathcal{L}$ is closed under extensions, and complexes in $D^{<0}(\omega)$ are obtained by iterated extensions from objects in $\bigcup_{i>0} \omega[i]$, thus we infer that $D^{<0}(\omega) \subseteq \mathcal{L}$. ■

In what follows, morphisms in $D^b(\mathcal{A})$ will be denoted by arrows, and those whose cones lie in $K^b(\omega)$ will be denoted by doubled arrows; morphisms in $D_\omega(\mathcal{A})$ will be denoted by right fractions (for the definition, see [25]).

Let $M, N \in \mathcal{A}$. We consider the natural map

$$\theta_{M,N} : \mathrm{Hom}_{\mathcal{A}}(M, N) \longrightarrow \mathrm{Hom}_{D_\omega(\mathcal{A})}(Q_\omega(M), Q_\omega(N)), \quad f \longmapsto f/\mathrm{Id}_M.$$

Set $\omega(M,N) = \{f \in \mathrm{Hom}_{\mathcal{A}}(M,N) \mid f$ factors through objects in $\omega\}$. Then $\theta_{M,N}$ vanishes on $\omega(M,N)$ because $Q_\omega(\omega) = 0$.

The following observation is crucial in our proof, compare [10], Lemma 2.1 and [23], Proposition 1.21.

Lemma 2.3. *In the following two cases: (1) $M \in \mathcal{X}_\omega$ and $N \in \omega^\perp$; (2) $M \in {}^\perp\omega$ and $N \in {}_\omega\mathcal{X}$, the morphism $\theta_{M,N}$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(M,N)/\omega(M,N) \simeq \mathrm{Hom}_{D_\omega(\mathcal{A})}(M,N).$$

Proof. We only show case (1). First, we show that $\theta_{M,N}$ is surjective. For this, consider any morphism $a/s : M \overset{s}{\Longleftarrow} Z^\bullet \overset{a}{\longrightarrow} N$ in $D_\omega(\mathcal{A})$, where $Z^\bullet$ is a complex, both a and s are morphisms in $D^b(\mathcal{A})$, and the cone of s, $C^\bullet = \mathrm{Con}(s)$, lies in $K^b(\omega)$. Hence we have a distinguished triangle in $D^b(\mathcal{A})$

$$Z^\bullet \overset{s}{\Longrightarrow} M \longrightarrow C^\bullet \longrightarrow Z^\bullet[1]. \tag{2.1}$$

Since $M \in \mathcal{X}_\omega$, we have a long exact sequence

$$0 \longrightarrow M \overset{\varepsilon}{\longrightarrow} T^0 \overset{d^0}{\longrightarrow} T^1 \longrightarrow \cdots \longrightarrow T^n \overset{d^n}{\longrightarrow} T^{n+1} \longrightarrow \cdots$$

where each $T^i \in \omega$ and $\mathrm{Ker} d^i \in {}^\perp\omega$. Hence in $D^b(\mathcal{A})$, M is isomorphic to the following complex

$$T^\bullet := \quad 0 \longrightarrow T^0 \overset{d^0}{\longrightarrow} T^1 \longrightarrow \cdots \longrightarrow T^n \overset{d^n}{\longrightarrow} T^{n+1} \longrightarrow \cdots,$$

and furthermore, M is isomorphic to the good truncation $\tau^{\leq l}T^\bullet$ for any $l \geq 0$. Note the following natural triangle in $K^b(\mathcal{A})$

$$(\sigma^{<l}T^\bullet)[-1] \longrightarrow \mathrm{Ker} d^l[-l] \overset{s''}{\Longrightarrow} \tau^{\leq l}T^\bullet \longrightarrow \sigma^{<l}T^\bullet, \tag{2.2}$$

where $\sigma^{<l}T^\bullet$ is the brutal truncation. Take s' to be the following composite in $D^b(\mathcal{A})$

$$\mathrm{Ker} d^l[-l] \overset{s''}{\Longrightarrow} \tau^{\leq l}T^\bullet \longrightarrow T^\bullet \overset{\varepsilon}{\Longleftarrow} M.$$

Thus from the triangle (2.2), we get a triangle in $D^b(\mathcal{A})$

$$(\sigma^{<l}T^\bullet)[-1] \longrightarrow \mathrm{Ker} d^l[-l] \overset{s'}{\Longrightarrow} M \overset{\varepsilon}{\longrightarrow} \sigma^{<l}T^\bullet. \tag{2.3}$$

Since $C^\bullet \in K^b(\omega)$, we may assume that

$$C^\bullet = \quad \cdots \longrightarrow 0 \longrightarrow W^{-t'} \longrightarrow \cdots \longrightarrow W^{t-1} \longrightarrow W^t \longrightarrow 0 \longrightarrow \cdots,$$

where $W^i \in \omega$, $t, t' \geq 0$. Set $l_0 = t+1$, $E = \mathrm{Ker} d^{l_0}$. Note that $E \in {}^\perp\omega$ and $C^\bullet[l_0] \in D^{<0}(\omega)$, by Lemma 2.2(1), we get

$$\mathrm{Hom}_{D^b(\mathcal{A})}(E[-l_0], C^\bullet) = \mathrm{Hom}_{D^b(\mathcal{A})}(E, C^\bullet[l_0]) = 0.$$

Hence, the morphism $E[-l_0] \overset{s'}{\Longrightarrow} M \longrightarrow C^\bullet$ is 0. By the triangle (2.1), we infer that there exists $h : E[-l_0] \longrightarrow Z^\bullet$ such that $s' = s \circ h$, and thus $a/s = (a \circ h)/s'$.

Note that $N \in \omega^\perp$ and $(\sigma^{<l_0} T^\bullet)[-1] \in D^{>0}(\omega)$, by Lemma 2.2(2), we have

$$\mathrm{Hom}_{D^b(\mathcal{A})}((\sigma^{<l_0} T^\bullet)[-1], N) = 0.$$

Applying the cohomological functor $\mathrm{Hom}_{D^b(\mathcal{A})}(-, N)$ to the triangle (2.3), we obtain the following exact sequence (here, take $l = l_0$)

$$\mathrm{Hom}_{D^b(\mathcal{A})}(M, N) \overset{\mathrm{Hom}_{D^b(\mathcal{A})}(s', N)}{\longrightarrow} \mathrm{Hom}_{D^b(\mathcal{A})}(E[-l_0], N) \longrightarrow \mathrm{Hom}_{D^b(\mathcal{A})}((\sigma^{<l_0} T^\bullet)[-1], N).$$

Thus, there exists $f : M \longrightarrow N$ such that $f \circ s' = a \circ h$. Hence, we have

$$a/s = (a \circ h)/s' = (f \circ s')/s' = \theta_{M,N}(f),$$

proving that $\theta_{M,N}$ is surjective.

Next, we will show $\mathrm{Ker}\theta_{M,N} = \omega(M, N)$, then we are done. It is already known that $\omega(M, N) \subseteq \mathrm{Ker}\theta_{M,N}$. Conversely, consider $f : M \longrightarrow N$ such that $\theta_{M,N}(f) = 0$. Hence there exists $s : Z^\bullet \Longrightarrow M$ such that $f \circ s = 0$, where s is a morphism in $D^b(\mathcal{A})$ whose cone $C^\bullet \in K^b(\omega)$. Using the notation above, we obtain that $s' = s \circ h$. Thus $f \circ s' = 0$. By the triangle (2.3), we infer that there exists $f' : \sigma^{<l_0} T^\bullet \longrightarrow N$ such that $f' \circ \varepsilon = f$.

Consider the following natural triangle

$$T^0[-1] \longrightarrow \sigma^{>0}(\sigma^{<l_0} T^\bullet) \Longrightarrow \sigma^{<l_0} T^\bullet \overset{\pi}{\longrightarrow} T^0. \tag{2.4}$$

Since $N \in \omega^\perp$ and $\sigma^{>0}(\sigma^{<l_0} T^\bullet) \in D^{>0}(\omega)$, by Lemma 2.2(2), we have

$$\mathrm{Hom}_{D^b(\mathcal{A})}(\sigma^{>0}(\sigma^{<l_0} T^\bullet), N) = 0.$$

Thus the composite morphism $\sigma^{>0}(\sigma^{<l_0} T^\bullet) \Longrightarrow \sigma^{<l_0} T^\bullet \overset{f'}{\longrightarrow} N$ is 0, and furthermore, by the triangle (2.4), we infer that there exists $g : T^0 \longrightarrow N$ such that $g \circ \pi = f'$. So we get $f = g \circ (\pi \circ \varepsilon)$, which proves that f factors through ω inside $D^b(\mathcal{A})$. Note again that $i_\mathcal{A} : \mathcal{A} \longrightarrow D^b(\mathcal{A})$ is fully-faithful, and we can obtain that f factors through ω in $\mathcal{A}$, i.e., $f \in \omega(M, N)$. This finishes the proof. ■

Recall the notion of ∂-functor, compare [18], section 1. Let $(\mathfrak{a}, \mathcal{E})$ be an exact category, $\mathcal{C}$ a triangulated category. An additive functor $F : \mathfrak{a} \longrightarrow \mathcal{C}$ is said to be a ∂-functor, if for each conflation $(i, d) : X \overset{i}{\longrightarrow} Y \overset{d}{\longrightarrow} Z \in \mathcal{E}$, there exists a morphism $w_{(i,d)} : F(Z) \longrightarrow F(X)[1]$ such that the triangle is distinguished

$$F(X) \overset{F(i)}{\longrightarrow} F(Y) \overset{F(d)}{\longrightarrow} F(Z) \overset{w_{(i,d)}}{\longrightarrow} F(X)[1],$$

moreover, the morphisms w are natural in the sense that given a morphism between two conflations

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xrightarrow{d} & Z \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{i'} & Y' & \xrightarrow{d'} & Z', \end{array}$$

then we have a morphism of triangles

$$\begin{array}{ccccccc} F(X) & \xrightarrow{F(i)} & F(Y) & \xrightarrow{F(d)} & F(Z) & \xrightarrow{w_{(i,\, d)}} & F(X)[1] \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle F(g)} & & \downarrow{\scriptstyle F(h)} & & \downarrow{\scriptstyle F(f)[1]} \\ F(X') & \xrightarrow{F(i')} & F(Y') & \xrightarrow{F(d')} & F(Z') & \xrightarrow{w_{(i',\, d')}} & F(X')[1]. \end{array}$$

We will need the following fact, which is direct from definition.

Lemma 2.4. *Let $F : \mathfrak{a} \longrightarrow \mathcal{C}$ be a ∂-functor. Assume $j : \mathfrak{b} \longrightarrow \mathfrak{a}$ is an exact functor between two exact categories, $\pi : \mathcal{C} \longrightarrow \mathcal{D}$ a triangle-functor between two triangulated categories. Then the composite functor $\pi F j : \mathfrak{b} \longrightarrow \mathcal{D}$ is a ∂-functor.*

Next fact is very useful, and well-known, compare [12], p.23.

Lemma 2.5. *Let $(\mathfrak{a},\ \mathcal{E})$ be a Frobenius category, $\underline{\mathfrak{a}}$ its stable category modulo projectives. Assume $F : \mathfrak{a} \longrightarrow \mathcal{C}$ is a ∂-functor, which vanishes on projective objects. The induced functor $\underline{F} : \underline{\mathfrak{a}} \longrightarrow \mathcal{C}$ is a triangle-functor.*

Proof. Since F vanishes on projective objects, then the functor $\underline{F}$ is defined. Recall that the translation functor S on $\underline{\mathfrak{a}}$ is defined such that for each X, we have a fixed conflation $X \xrightarrow{i_X} I(X) \xrightarrow{d_X} S(X)$, where $I(X)$ is injective (for details, see [12]). By assumption, we have the distinguished triangle in $\mathcal{C}$

$$F(X) \xrightarrow{F(i_X)} F(I(X)) \xrightarrow{F(d_X)} F(S(X)) \xrightarrow{w_{(i_X, d_X)}} F(X)[1].$$

Since $F(I(X)) \simeq 0$, we infer that $w_{(i_X, d_X)}$ is an isomorphism. Set $\eta_X := w_{(i_X, d_X)}$. In fact, by the naturalness of w, we can obtain that η_X is natural in X, in other words, $\eta : FS \longrightarrow [1]F$ is a natural isomorphism. Recall that all the distinguished triangles in $\underline{\mathfrak{a}}$ arise from conflations in $\mathfrak{a}$ ([10], Lemma 2.1), then one may show that $(\underline{F}, \eta)$ is a triangle-functor easily. We omit the details. ■

Proof of Theorem 2.1: By Lemma 2.3, we know that $\underline{F}$ is fully-faithful. It is classical that $i_{\mathcal{A}} : \mathcal{A} \longrightarrow D^b(\mathcal{A})$ is a ∂-functor (by [14], p.63, Remark). Then by Lemma 2.4, we know that the composite functor F is also a ∂-functor. Now by Lemma 2.5, we deduce that $\underline{F}$ is a triangle-functor.

Now assume that $\widehat{\mathcal{X}_\omega} = \mathcal{A} = {}_\omega\overset{\vee}{\mathcal{X}}$. It suffices to show that $\underline{F}$ is dense, that is, the image $\mathrm{Im}\underline{F} = D_\omega(\mathcal{A})$. By above, we know that $\mathrm{Im}\underline{F}$ is a triangulated subcategory, and it is direct to see that $D_\omega(\mathcal{A})$ is generated by the image $Q_\omega(\mathcal{A})$ of $\mathcal{A}$ in the sense of [12], p.71. Hence it is enough to show that $Q_\omega(\mathcal{A})$ lies in $\mathrm{Im}\underline{F}$.

Assume $X \in \mathcal{A}$. Since ω cogenerates $\mathcal{X}_\omega$ and $X \in \widehat{\mathcal{X}_\omega} = \mathcal{A}$, by Auslander-Buchweitz decomposition theorem ([2], Theorem 1.1), we have an exact sequence

$$0 \longrightarrow Y \longrightarrow X' \longrightarrow X \longrightarrow 0,$$

where $Y \in \widehat{\omega}$, and $X' \in \mathcal{X}_\omega$. Since $Y \in \widehat{\omega}$, then inside $D^b(\mathcal{A})$ we have $Y \in K^b(\omega)$. Consequently, $Q_\omega(Y) \simeq 0$. Note that the above exact sequence induces a distinguished triangle in $D^b(\mathcal{A})$ ([14], p.63), and thus we have the induced distinguished triangle in $D_\omega(\mathcal{A})$

$$Q_\omega(Y) \longrightarrow Q_\omega(X') \longrightarrow Q_\omega(X) \longrightarrow Q_\omega(Y)[1].$$

Now since $Q_\omega(Y) \simeq 0$, we deduce that $Q_\omega(X') \simeq Q_\omega(X)$. On the other hand, ω generates ${}_\omega\mathcal{X}$ and $X' \in {}_\omega\overset{\vee}{\mathcal{X}} = \mathcal{A}$, by the dual of Auslander-Buchweitz decomposition theorem, we have an exact sequence

$$0 \longrightarrow X' \longrightarrow X'' \longrightarrow Y' \longrightarrow 0,$$

where $Y' \in \overset{\vee}{\omega}$, and $X'' \in {}_\omega\mathcal{X}$. By the same argument as above, we deduce that $Q_\omega(X') \simeq Q_\omega(X'')$, and consequently, $Q_\omega(X) \simeq Q_\omega(X'')$. As we noted in **2.1** that $\overset{\vee}{\omega} \subseteq \mathcal{X}_\omega$, and in the exact sequence above, both Y and X' lie in $\mathcal{X}_\omega$, and by Proposition 5.1 in [3], $\mathcal{X}_\omega$ is closed under extensions, we infer that $X'' \in \mathcal{X}_\omega$, and thus $X'' \in \alpha(\omega)$. Hence $Q_\omega(X'') = F(X'')$, and we see that $Q_\omega(X)$ lies in the image of $\underline{F}$. This completes the proof. ■

3. Gorenstein-projective modules and singularity categories

3.1. Let R be a ring with unit. Denote by R-Mod the category of left R-modules, and R-Proj its full subcategory of projective modules. A complex $P^\bullet = (P^n, d^n)$ in $C(R\text{-Proj})$ is said to be *totally-acyclic* ([21], section 7), if for each projective module Q, the Hom complexes $\mathrm{Hom}_R(Q, P^\bullet)$ and $\mathrm{Hom}_R(P^\bullet, Q)$ are exact. Hence a complex $P^\bullet$ is totally-acyclic if and only if it is acyclic (= exact) and for each n, the cocycle $\mathrm{Ker}d^n$ lies in ${}^\perp R$-Proj. A module M is said to be *Gorenstein-projective*, if there exists a totally-acyclic complex $P^\bullet$ such that its zeroth cocycle is M. In this case, $P^\bullet$ is said to be a *complete resolution* of M. Denote by R-GProj the full subcategory consisting of Gorenstein-projective modules.

Observe that a module M is Gorenstein-projective if and only if there exists an exact sequence $0 \longrightarrow M \xrightarrow{\varepsilon} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \longrightarrow \cdots$ such that each cocycle $\mathrm{Ker}d^i \in {}^\perp R$-Proj. Set $\mathcal{A} = R$-Mod, $\omega = R$-Proj. Thus ${}_\omega\mathcal{X} = \mathcal{A}$ and $\alpha(\omega) = \mathcal{X}_\omega$. By the above observation, we have $\alpha(\omega) = R\text{-GProj}$. In this case, the relative singularity category is the (big) singularity category of R (compare [23])

$$D'_{\rm sg}(R) = D^b(R\text{-Mod})/K^b(R\text{-Proj}).$$

Note that $D'_{\rm sg}(R)$ vanishes if and only if every module has finite projective dimension, and then it is equivalent to that the ring R has finite left global dimension.

The following result can be read from the general theory developed in section 2.

Proposition 3.1. *(1) The category* R-GProj *is a Frobenius category with projective-injective objects exactly contained in* R-Proj.
(2) The natural functor $\underline{F} : R\text{-}\underline{\rm GProj} \longrightarrow D'_{\rm sg}(R)$ *is fully-faithful and exact.*

A sufficient condition making $\underline{F}$ an equivalence is that the ring R is Gorenstien. This was first observed by Buchweitz [9]. Recall that a ring R is said to be *Gorenstein*, if R is two-sided noetherian, and the regular module R has finite injective dimension both as a left and right module.

We need the following fact, which is known to experts.

Lemma 3.2. *Let R be a Gorenstein ring. Then we have* $R\text{-GProj} = {}^\perp R\text{-Proj}$.

Proof. Note that $R\text{-GProj} \subseteq {}^\perp R\text{-Proj}$. For the converse, denote $\mathcal{L}$ the full subcategory of R-Mod, consisting of modules of finite injective dimension. By [11], Lemma 10.2.13, $\mathcal{L}$ is preenveloping (= covariantly-finite), i.e., for any module M, there exists a morphism $g_M : M \longrightarrow C_M$ such that $C_M \in \mathcal{L}$ and any morphism from M to a module in $\mathcal{L}$ factors through g_M (such a morphism g_M is called an $\mathcal{L}$-preenvelop (= right $\mathcal{L}$-approximation)). We note that the morphism g_M is mono, by noting that the injective hull of M factors through g_M.

Now assume $M \in {}^\perp R\text{-Proj}$. Take an exact sequence

$$0 \longrightarrow K \longrightarrow P^0 \xrightarrow{\theta} C_M \longrightarrow 0, \tag{3.1}$$

where P^0 is projective. Since C_M has finite injective dimension, by [11], Proposition 9.1.7, it also has finite projective dimension. Thus we infer that K has finite projective dimension. Note that $M \in {}^\perp R\text{-Proj}$, and by the dimension-shift argument, we have $\mathrm{Ext}^1_R(M, K) = 0$. Applying the functor $\mathrm{Hom}_R(M, -)$ to (3.1), we obtain a long exact sequence, and from which, we read a surjective map $\mathrm{Hom}_R(M, \theta) : \mathrm{Hom}_R(M, P) \longrightarrow \mathrm{Hom}_R(M, C_M)$. In particular, the morphism g_M factor through θ, and thus we get a morphism $h : M \longrightarrow P^0$ such that $g_M = \theta \circ h$. Since g_M is an $\mathcal{L}$-preenvelop, and g_M factors through h (note $P^0 \in \mathcal{L}$), and we deduce that h is also an $\mathcal{L}$-preenvelop. In particular, h is mono. Consider the exact sequence

$$0 \longrightarrow M \xrightarrow{h} P^0 \longrightarrow M' \longrightarrow 0. \tag{3.2}$$

For any projective module Q, applying the functor $\mathrm{Hom}_R(-,Q)$, and we obtain a long exact sequence, from which we read that $\mathrm{Ext}^i_R(M,Q)=0$ for $i\geq 1$ (for $i=1$, we need the fact that h is an $\mathcal{L}$*-preenvelop*). Thus $M'\in {}^{\perp}R\text{-Proj}$. Applying the same argument to M', we may get an exact sequence $0\longrightarrow M'\longrightarrow P^1\longrightarrow M''\longrightarrow 0$ with P projective and $M''\in {}^{\perp}R\text{-Proj}$. Continue this process, and we obtain a long exact sequence $0\longrightarrow M\longrightarrow P^0\longrightarrow P^1\longrightarrow P^2\longrightarrow\cdots$ with cocycles in ${}^{\perp}R$-Proj, that is, $M\in R$-GProj. Thus we are done. ■

Now we have the following variant of Buchweitz-Happel's theorem (compare [9], Theorem 4.4.1 and [13], Theorem 4.6, also see [5], Theorem 6.9).

Theorem 3.3. *Let R be a Gorenstein ring. Then the natural functor*

$$\underline{F}: \underline{R\text{-GProj}}\longrightarrow D'_{\mathrm{sg}}(R)$$

is a triangle-equivalence.

Proof. We have noted the following fact: set $\mathcal{A}=R$-Mod, $\omega=R$-Proj, then ${}_{\omega}\mathcal{X}=\mathcal{A}$ and $\alpha(\omega)=R$-GProj. Hence by Theorem 2.1, we know that to obtain the result, it suffices to show that $\widehat{R\text{-GProj}}=R$-Mod. Assume inj.dim ${}_RR=d$. Then every projective module has injective dimension at most d. Let X be any R-module. Take an exact sequence

$$0\longrightarrow M\longrightarrow P^{d-1}\longrightarrow P^{d-2}\longrightarrow\cdots\longrightarrow P^1\longrightarrow P^0\longrightarrow X\longrightarrow 0,$$

where each P^i is projective. By dimension-shift technique, we have, for each projective module Q, $\mathrm{Ext}^i_R(M,Q)\simeq \mathrm{Ext}^{i+d}_R(X,Q)=0$, $i\geq 1$. Hence $M\in {}^{\perp}R$-Proj, and by Lemma 3.2, $M\in R$-GProj. Hence, $X\in\widehat{R\text{-GProj}}$. Thus we are done. ■

3.2. In this subsection, we consider another self-orthogonal subcategory $\omega'=R$-proj, the full subcategory of finite-generated projective modules, of the category $\mathcal{A}=R$-Mod. From the definition in **2.1**, it is not hard to see that

$$\begin{aligned}{}_{\omega'}\mathcal{X}=\{M\in R\text{-Mod}\mid &\text{ there exists an exact sequence}\\ &\cdots\longrightarrow P^n\longrightarrow P^{n-1}\longrightarrow\cdots\longrightarrow P^1\longrightarrow P^0\longrightarrow M\longrightarrow 0,\ \text{each } P^n\in R\text{-proj}\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{X}_{\omega'}=\{M\in R\text{-Mod}\mid &\text{ there exists an exact sequence}\\ 0\longrightarrow M\longrightarrow P^0\xrightarrow{d^0}P^1\longrightarrow\cdots\longrightarrow P^n\xrightarrow{d^n}P^{n+1}\longrightarrow\cdots, &\text{ each } P^n\in R\text{-proj},\ \mathrm{Coker}d^n\in {}^{\perp}R\text{-proj}\}.\end{aligned}$$

Set $\alpha(\omega')=R$-Gproj. Hence R-Gproj is a Frobenius category, whose projective-injective objects are exactly contained in R-proj. Observe that $R\text{-Gproj}\subseteq R\text{-GProj}$, and we have an induced inclusion of triangulated categories $\underline{R\text{-Gproj}}\hookrightarrow\underline{R\text{-GProj}}$.

Denote by R-mod the full subcategory consisting of finitely-presented modules. Let R be a left-coherent ring. Observe that in this case, R-mod is an abelian subcategory of R-Mod, and $R\text{-mod} = {}_{\omega'}\mathcal{X}$ (compare [1], p.41). Therefore, if R is left-coherent, we have

$$R\text{-Gproj} = \{M \in R\text{-mod} \mid \text{there exists an exact sequence}$$

$$0 \longrightarrow M \longrightarrow P^0 \stackrel{d^0}{\longrightarrow} P^1 \longrightarrow \cdots \longrightarrow P^n \stackrel{d^n}{\longrightarrow} P^{n+1} \longrightarrow \cdots, \text{ each } P^n \in R\text{-proj}, \ \mathrm{Coker} d^n \in {}^{\perp}R\text{-proj}\}.$$

The following observation is interesting.

Lemma 3.4. *Let R be a left-coherent ring. Then we have* $R\text{-GProj} \cap R\text{-mod} = R\text{-Gproj}$.

Proof. Let $M \in R$-Gproj. Then we have an exact sequence $0 \longrightarrow M \longrightarrow P^0 \stackrel{d^0}{\longrightarrow} P^1 \longrightarrow \cdots \longrightarrow P^n \stackrel{d^n}{\longrightarrow} P^{n+1} \longrightarrow \cdots$, where $P^i \in R$-proj and each $\mathrm{Coker} d^i \in {}^{\perp}R$-proj. Since each module $\mathrm{Coker} d^i$ is finitely-generated, and thus $\mathrm{Coker} d^i \in {}^{\perp}R$-proj implies that $\mathrm{Coker} d^i \in {}^{\perp}R$-Proj immediately. Thus we have $M \in R$-GProj. Hence $R\text{-Gproj} \subseteq R\text{-GProj} \cap R\text{-mod}$.

Conversely, assume that $M \in R\text{-GProj} \cap R\text{-mod}$. Then there exists an exact sequence $0 \longrightarrow M \stackrel{\varepsilon}{\longrightarrow} P \longrightarrow X \longrightarrow 0$, where $P \in R$-Proj and $X \in R$-GProj. By adding proper projective modules to P and X, we may assume that P is free. Since M is finitely-generated, we may decompose $P = P^0 \oplus P'^0$ such that P^0 is finitely-generated and $\mathrm{Im}\varepsilon \subseteq P^0$. Consider the exact sequence $0 \longrightarrow M \stackrel{\varepsilon}{\longrightarrow} P^0 \longrightarrow M' \longrightarrow 0$. We have $M' \oplus P'^0 \simeq X$, and note that $R\text{-GProj} \subseteq R\text{-Mod}$ is closed under taking direct summands (by Proposition 5.1 in [3], or [11]), we deduce that $M' \in R$-GProj. Observe that $M' \in R$-mod, and we have $M' \in R\text{-GProj} \cap R\text{-mod}$. Applying the same argument to M', we can find an exact sequence $0 \longrightarrow M' \longrightarrow P^1 \longrightarrow M'' \longrightarrow 0$ such that P^1 is finitely-generated projective, and $M'' \in R\text{-GProj} \cap R\text{-mod}$. Continue this process, we can derive a long exact sequence $0 \longrightarrow M \longrightarrow P^0 \longrightarrow P^1 \longrightarrow \cdots$. This is the required sequence proving $M \in R$-Gproj. ■

Let R be left-coherent. Set $\mathcal{A}' = R$-mod. The relative singularity of $\mathcal{A}'$ with respect to ω' is the usual singularity category of the ring R ([23])

$$D_{\rm sg}(R) = D^b(R\text{-mod})/K^b(R\text{-proj}).$$

The following is read directly from Theorem 2.1.

Proposition 3.5. *Let R be a left-coherent ring. The natural functor* $\underline{F} : \underline{R\text{-Gproj}} \longrightarrow D_{\rm sg}(R)$ *is a fully-faithful triangle-functor.*

Remark 3.6. Consider the natural embedding $D^b(R\text{-mod}) \hookrightarrow D^b(R\text{-Mod})$, and observe that $K^b(R\text{-Proj}) \cap D^b(R\text{-mod}) = K^b(R\text{-proj})$, and for any $P^\bullet \in K^b(R\text{-Proj})$, $X^\bullet \in D^b(R\text{-mod})$, then any morphism (inside $D^b(R\text{-Mod})$) from $P^\bullet$ to $X^\bullet$ factors through an object of $K^b(R\text{-proj})$ (just take a projective resolution $Q^\bullet \in K^{-,b}(R\text{-proj})$ of $X^\bullet$, then the brutally truncated complex $\sigma^{\geq -n}Q^\bullet$, for large n, is the required object). Now, It follows that the natural induced functor $D_{\mathrm{sg}}(R) \longrightarrow D'_{\mathrm{sg}}(R)$ is a full embedding (by [25], 4-2 Theorem). Finally, we have a commutative diagram of fully-faithful triangle-functors

$$\begin{array}{ccc} R\underline{\mathrm{Gproj}} & \overset{\underline{F}}{\hookrightarrow} & D_{\mathrm{sg}}(R) \\ \downarrow & & \downarrow \\ R\text{-}\underline{\mathrm{GProj}} & \overset{\underline{F}}{\hookrightarrow} & D'_{\mathrm{sg}}(R). \end{array}$$

A sufficient condition that the functor $\underline{F}$ in Proposition 3.5 is an equivalence is also that the ring R is Gorenstein. We need the following result.

Lemma 3.7. *Let R be a Gorenstein ring. Then we have*

$$R\text{-Gproj} = \{M \in R\text{-mod} \mid \mathrm{Ext}^i_R(M,P) = 0,\ P \in R\text{-proj},\ i \geq 1\}.$$

Proof. Just note that the left hand side is equal to R-mod $\cap\, {}^{\perp}R$-Proj. Then the result follows from Lemma 3.2 and Lemma 3.4 directly. Let us remark that the lemma can be also proved by the cotilting theory. ■

Using Proposition 3.5 and Lemma 3.7 and applying a similar argument as Theorem 3.3, we have the following result. Note that the result was first shown by Buchweitz [9] and its dual version was shown independently by Happel in the finite-dimensional case [13] (compare [5], Corollary 4.13 or [10], Theorem 2.5). A special case of the result was given by Rickard ([24], Theorem 2.1) which says that the singularity category of a self-injective algebra is triangle-equivalent to its stable module category (compare Keller-Vossieck [20]).

Theorem 3.8. (Buchweitz-Happel) *Let R be a Gorenstein ring. Then the natural functor*

$$\underline{F} : R\text{-}\underline{\mathrm{Gproj}} \longrightarrow D_{\mathrm{sg}}(R)$$

is a triangle-equivalence.

3.3. Let T be a self-orthogonal object in any abelian category $\mathcal{A}$. Set $\alpha(T) = \alpha(\mathrm{add}\ T)$. We will relate $\alpha(T)$ to the category of Gorenstein-projective modules over the endomorphism ring.

Theorem 3.9. *Let T be a self-orthogonal object, and let $R = \mathrm{End}_{\mathcal{A}}(T)^{\mathrm{op}}$. Then the functor* $\mathrm{Hom}_{\mathcal{A}}(T,-) : \alpha(T) \longrightarrow R$-Gproj *is fully-faithful, and it induce a full embedding of triangulated categories $\underline{\alpha(T)} \longrightarrow R\text{-}\underline{\mathrm{Gproj}}$.*

Part of the theorem follows from an observation of Xi ([26], Proposition 5.1), which we will recall. Let $T \in \mathcal{A}$ be any object, $R = \mathrm{End}_R(T)^{\mathrm{op}}$. Then we have the functor

$$\mathrm{Hom}_{\mathcal{A}}(T,-) :\ \mathcal{A} \longrightarrow R\text{-Mod}.$$

In general, it is not fully-faithful. However, it is well-known that it induces an equivalence

$$\mathrm{add}\ T \simeq R\text{-proj},$$

in particular, the restriction of $\mathrm{Hom}_{\mathcal{A}}(T,-)$ to add T is fully-faithful. Actually, we can define a larger subcategory, on which $\mathrm{Hom}_{\mathcal{A}}(T,-)$ is fully-faithful. For this, recall that a morphism $g:\ T_0 \longrightarrow M$ with $T_0 \in \mathrm{add}\ T$ is a *T-precover (= right T-approximation)* of M, if any morphism from T to M factors through g. Consider the following full subcategory

$$\begin{aligned}\mathrm{App}(T) := \{M \in \mathcal{A} \mid \text{there exists an exact sequence } T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0,\\ T_i \in \mathrm{add}\ T,\ f_0 \text{ is a } T\text{-precovers},\ f_1 : T_1 \longrightarrow \mathrm{Ker} f_0 \text{ is a } T\text{-precover}\}.\end{aligned}$$

For $M \in \mathrm{App}(T)$, such a sequence $T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0$ will be called a *T-presentation* of M.

The following result is contained in [26] in slightly different form.

Lemma 3.10. *The functor* $\mathrm{Hom}_{\mathcal{A}}(T,-)$ *induces a full embedding of* $\mathrm{App}(T)$ *into* R-mod.

Proof. The proof resembles the argument in [4], p.102. Let $M \in \mathrm{App}(T)$ with T-presentation $T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \longrightarrow 0$. Since f_0 and $f_1 : T_1 \longrightarrow \mathrm{Ker} f_0$ are T-precovers, we have the following exact sequence of R-modules

$$\mathrm{Hom}_{\mathcal{A}}(T,T_1) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(T,f_1)} \mathrm{Hom}_{\mathcal{A}}(T,T_0) \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(T,f_0)} \mathrm{Hom}_{\mathcal{A}}(T,M) \longrightarrow 0.$$

Recall the equivalence $\mathrm{Hom}_{\mathcal{A}}(T,-) : \mathrm{add}\ T \simeq R\text{-proj}$. Thus the left-hand side two terms in the sequence above are finite-generated R-modules, and we infer that $\mathrm{Hom}_{\mathcal{A}}(T,M)$ is a finite-presented R-module. Let $M' \in \mathrm{App}(T)$ with T-presentation $T_1' \xrightarrow{f_1'} T_0' \xrightarrow{f_0'} M' \longrightarrow 0$. Given any homomorphism of R-modules $\theta : \mathrm{Hom}_{\mathcal{A}}(T,M) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(T,M')$. Thus by a similar argument as the comparison theorem in homological algebra, we have the following diagram in R-mod

$$\begin{array}{ccccccc}
\mathrm{Hom}_{\mathcal{A}}(T,T_1) & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(T,f_1)} & \mathrm{Hom}_{\mathcal{A}}(T,T_0) & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(T,f_0)} & \mathrm{Hom}_{\mathcal{A}}(T,M) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta} & & \\
\mathrm{Hom}_{\mathcal{A}}(T,T_1') & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(T,f_1')} & \mathrm{Hom}_{\mathcal{A}}(T,T_0') & \xrightarrow{\mathrm{Hom}_{\mathcal{A}}(T,f_0)} & \mathrm{Hom}_{\mathcal{A}}(T,M') & \longrightarrow & 0.
\end{array}$$

Using the equivalence add $T \simeq R$-proj again, we have morphisms $g_i : T_i \longrightarrow T_i'$ such that $\mathrm{Hom}_{\mathcal{A}}(T,g_i) = \theta_i,\ i = 0,1$. Thus $g_0 \circ f_1 = f_1' \circ g_1$. Thus we have a unique morphism $g : M \longrightarrow M'$ making the diagram commute

$$\begin{array}{ccccccc}
T_1 & \xrightarrow{f_1} & T_0 & \xrightarrow{f_0} & M & \longrightarrow & 0\\
\downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g} & & \\
T_1' & \xrightarrow{f_1'} & T_0' & \xrightarrow{f_0'} & M' & \longrightarrow & 0.
\end{array}$$

Now it is not hard to see that $\mathrm{Hom}_{\mathcal{A}}(T, g) = \theta$, i.e., $\mathrm{Hom}_{\mathcal{A}}(T, -) : \mathrm{App}(T) \longrightarrow R\text{-mod}$ is full. We will omit the proof of faithfulness, which is somehow the inverse of the above proof. ■

Proof of Theorem 3.9: Set $\omega = \mathrm{add}\ T$. First note that any epimorphism $f : T_0 \longrightarrow M$ with $T_0 \in \mathrm{add}\ T$ and $\mathrm{Ker} f \in T^{\perp}$, is a T-precover. This can be seen from the long exact sequence obtained by applying $\mathrm{Hom}_{\mathcal{A}}(T, -)$ to the sequence $0 \longrightarrow \mathrm{Ker} f \longrightarrow T_0 \xrightarrow{f} M \longrightarrow 0$. Thus we infer that ${}_{\omega}\mathcal{X} \subseteq \mathrm{App}(T)$, and then $\alpha(T) \subseteq \mathrm{App}(T)$. Thus $\mathrm{Hom}_{\mathcal{A}}(T, -)$ is fully-faithful on $\alpha(T)$. What is left to show is that for each $M \in \alpha(T)$, $\mathrm{Hom}_{\mathcal{A}}(T, M) \in R\text{-Gproj}$. Take a complete T-resolution $T^{\bullet} = (T^n, d^n)$ for M. Then the complex $P^{\bullet} = \mathrm{Hom}_{\mathcal{A}}(T, T^{\bullet})$ is exact with its 0-cocycle $\mathrm{Hom}_{\mathcal{A}}(T, M)$. Note that $P^{\bullet}$ is a complex of finitely-generated projective R-modules. Note that we have an isomorphism of Hom complexes $\mathrm{Hom}_{\mathcal{A}}(T^{\bullet}, T) \simeq \mathrm{Hom}_R(P^{\bullet}, R)$, using the equivalence $\mathrm{Hom}_{\mathcal{A}}(T, -) : \mathrm{add}\ T \simeq R\text{-proj}$ and noting that $\mathrm{Hom}_{\mathcal{A}}(T, T) = R$. However $\mathrm{Hom}_{\mathcal{A}}(T^{\bullet}, T)$ is exact, hence we infer that $P^{\bullet}$ is a complete resolution for $\mathrm{Hom}_{\mathcal{A}}(T, M)$. Thus $\mathrm{Hom}_{\mathcal{A}}(T, M) \in R\text{-Gproj}$.

Note that $\mathrm{Hom}_{\mathcal{A}}(T, -)$ preserves short exact sequences in $\alpha(T)$, and thus the composite $\alpha(T) \longrightarrow R\text{-Gproj} \longrightarrow R\text{-}\underline{\mathrm{Gproj}}$ is a ∂-functor, which sends $\mathrm{add}\ T$ to zero. By Lemma 2.5, the induced functor $\underline{\alpha(T)} \longrightarrow R\text{-}\underline{\mathrm{Gproj}}$ is a triangle-functor, the fully-faithfulness of which follows directly from the one of $\mathrm{Hom}_{\mathcal{A}}(T, -) : \alpha(T) \longrightarrow R\text{-Gproj}$. ■

4. Compact generators for Gorenstein-projective modules

4.1. Let us begin with some notions. Let $\mathcal{C}$ be a triangulated category with arbitrary (small) coproducts. An object $C \in \mathcal{C}$ is said to be *compact*, if the functor $\mathrm{Hom}_{\mathcal{C}}(C, -)$ commutes with coproducts. Denote by $\mathcal{C}^c$ the full subcategory of $\mathcal{C}$ consisting of compact objects, which is easily seen to be a thick triangulated subcategory. The triangulated category $\mathcal{C}$ is said to be *compactly generated*, if there is a set S of compact objects such that there is no proper triangulated category containing S and closed under coproducts [22].

Let R be a ring. It is easy to see that the triangulated category $R\text{-}\underline{\mathrm{GProj}}$ has arbitrary coproducts, and the natural embedding $R\text{-}\underline{\mathrm{Gproj}} \longrightarrow R\text{-}\underline{\mathrm{GProj}}$ gives that $R\text{-}\underline{\mathrm{Gproj}} \subseteq (R\text{-}\underline{\mathrm{Gproj}})^c$.

We have our main result. Note that similar results were obtained by Beligiannis ([6], Theorem 6.7 and [7], Theorem 6.6), and Iyengar-Krause ([15], Theorem 5.4 (2)) using different methods in different setups. We would like to thank Beligiannis to remark that one might find another proof of the following result using Gorenstein-injective modules, and a suitable combination of results and arguments in [7] and [8].

Theorem 4.1. *Let R be a Gorenstein ring. Then the triangulated category $R\text{-}\underline{\mathrm{GProj}}$ is compactly generated, and its subcategory of compact objects $(R\text{-}\underline{\mathrm{GProj}})^c$ is the additive closure of $R\text{-}\underline{\mathrm{Gproj}}$.*

Before proving Theorem 4.1, we need to recall some well-known facts on the homotopy category of projective modules. Denote by $K_{\mathrm{proj}}(R)$ the smallest triangulated category of $K(R\text{-Proj})$ containing R and closed under coproducts. Denote by $K^{\mathrm{ex}}(R\text{-Proj})$ the full subcategory of $K(R\text{-Proj})$ consisting of exact complexes. For each complex $P^\bullet \in K(R\text{-Proj})$, there is a unique triangle

$$\mathbf{p}(P^\bullet) \longrightarrow P^\bullet \longrightarrow \mathbf{a}(P^\bullet) \longrightarrow \mathbf{p}(P^\bullet)[1]$$

with $\mathbf{p}(P^\bullet) \in K_{\mathrm{proj}}(R)$ and $\mathbf{a}(P^\bullet) \in K^{\mathrm{ex}}(R\text{-Proj})$. Thus we have an exact functor $\mathbf{a}: K(R\text{-Proj}) \longrightarrow K^{\mathrm{ex}}(R\text{-Proj})$. Moreover, we have an exact sequence of triangulated categories

$$0 \longrightarrow K_{\mathrm{proj}}(R) \stackrel{\mathrm{inc}}{\longrightarrow} K(R\text{-Proj}) \stackrel{\mathbf{a}}{\longrightarrow} K^{\mathrm{ex}}(R\text{-Proj}) \longrightarrow 0,$$

where "inc" denotes the inclusion functor (for details, see [19] and compare [21], Corollary 3.9).

The following result is essentially due to Jørgensen [16].

Lemma 4.2. *Let R be a Gorenstein ring. Then the homotopy category $K(R\text{-Proj})$ is compactly generated, and its subcategory of compact object is $K^{+,b}(R\text{-proj})$.*

Proof. To see the lemma, we need the results of Jørgensen: let R be a ring, recall the duality $* = \mathrm{Hom}_R(-, R): R\text{-proj} \longrightarrow R^{\mathrm{op}}\text{-proj}$, which can be extended to another duality $*: K^-(R\text{-proj}) \longrightarrow K^+(R^{\mathrm{op}}\text{-proj})$. By [16], Theorem 2.4, if the ring R is coherent and every flat R-module has finite projective dimension, then the homotopy category $K(R\text{-Proj})$ is compactly-generated, and then by [16], Theorem 3.2 (and its proof), the subcategory of compact objects is $K(R\text{-Proj})^c = \{P^\bullet \in K^+(R\text{-proj}) \mid \text{the complex } (P^\bullet)^* \in K^{-,b}(R^{\mathrm{op}}\text{-proj})\}$. Note the following two facts: (1) for a Gorenstein ring R, every flat module has finite projective dimension by [11], Chapter 9, section 1; (2) for a Gorenstein ring R, we have an induced duality $*: K^{-,b}(R\text{-proj}) \longrightarrow K^{+,b}(R^{\mathrm{op}}\text{-proj})$, which is because that the regular module has finite injective dimension. Combining the above two facts and Jørgensen's results, we have the lemma. ■

Next result is also known, compare [9], Theorem 4.4.1.

Lemma 4.3. *Let R be a Gorenstein ring. The following composite functor*

$$K^{\rm ex}(R\text{-Proj}) \stackrel{Z^0}{\longrightarrow} R\text{-}\underline{\rm GProj}$$

is a triangle-equivalence, where Z^0 is the functor of taking the zeroth cocycyles.

Proof. Note that since ${}_RR$ has finite injective dimension, we infer that, by the dimension-shift technique, every complex $P^\bullet \in K^{\rm ex}(R\text{-Proj})$, its cocycles Z^i lie in ${}^\perp R$-Proj, and furthur Z^i are Gorenstein-projective. Hence the above functor is well-defined. Note that the functor is induced by the corresponding functor of taking the zeroth cocycles $Z^0 : C^{\rm ex}(R\text{-Proj}) \longrightarrow R\text{-}\underline{\rm GProj}$, and note that Z^0 is an exact functor between two exact categories, preserving projective-injective objects. Hence the induced functor Z^0 is a triangle-functor by [12], p.23. The proof of fully-faithfulness and denseness of Z^0 is same as the argument in [10], Appendix (compare [5], Theorem 3.11). Or, we observe that each exact complex $P^\bullet \in K(R\text{-Proj})$ is a complete resolution (= totally-acyclic complex in [21], section 7), and the result follows directly from the dual of [21], Proposition 7.2. ■

Proof of Theorem 4.1: We will see that the result follows from the following result of Thomason-Trobaugh-Yao-Neeman [22]: let $\mathcal{C}$ be a compactly generated and S a subset of compact objects, $\mathcal{R}$ the smallest triangulated subcategory which contains S and closed under coproducts, then the quotient category $\mathcal{C}/\mathcal{R}$ is compactly generated, and every compact objects in $\mathcal{C}/\mathcal{R}$ is a direct summand of $\pi(C)$ for some compact object C in $\mathcal{C}$, where $\pi : \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{R}$ is the quotient functor. To apply the theorem in our situation, by Lemma 4.2 we may put $\mathcal{C} = K(R\text{-Proj})$, and $S = \{R\}$, and then $\mathcal{R} = K_{\rm proj}(R)$. Via the functor $\mathbf{a}$ and the functor Z^0 in Lemma 4.3, we identify the quotient category $\mathcal{C}/\mathcal{R}$ with $R\text{-}\underline{\rm GProj}$. Hence the triangulated category $R\text{-}\underline{\rm GProj}$ is compactly generated, every object G in $(R\text{-}\underline{\rm GProj})^c$ is a direct summand of the image of the compact object in $K(R\text{-Proj})$, and thus by Lemma 4.2 again, there exists $P^\bullet \in K^{+,b}(R\text{-proj})$ such that G is a direct summand of $Z^0(\mathbf{a}(P^\bullet))$.

Assume that $P^\bullet = (P^n, d^n)$, and take a positive number n_0 such that $H^n(P^\bullet) = 0$, $n \geq n_0$. Consider the natural distinguished triangle

$$\sigma^{\geq n_0} P^\bullet \longrightarrow P^\bullet \longrightarrow \sigma^{<n_0} P^\bullet \longrightarrow (\sigma^{\geq n_0} P^\bullet)[1],$$

where σ is the brutal truncation. Since $\sigma^{<n_0} P^\bullet \in K^b(R\text{-proj}) \subseteq K_{\rm proj}(R)$, we get $\mathbf{a}(\sigma^{<n_0} P^\bullet) = 0$. Thus by applying the exact functor $\mathbf{a}$ to the above triangle, we have $\mathbf{a}(P^\bullet) \simeq \mathbf{a}(\sigma^{\geq n_0} P^\bullet)$. Applying the dimension-shift technique to the following exact sequence and noting that the injective dimension of ${}_RR$ is finite

$$0 \longrightarrow Z^{n_0}(P^\bullet) \longrightarrow P^{n_0} \stackrel{d^{n_0}}{\longrightarrow} P^{n_0+1} \longrightarrow \cdots \longrightarrow P^n \stackrel{d^n}{\longrightarrow} P^{n+1} \longrightarrow \cdots,$$

we infer that $Z^{n_0}(P^\bullet)$ lies in ${}^\perp R$-proj, and by Lemma 3.7, we have $Z^{n_0}(P^\bullet) \in R$-Gproj, and thus it is not hard to see that $\mathbf{a}(\sigma^{\geq n_0} P^\bullet)$ is a shifted complete resolution of $Z^{n_0}(P^\bullet)$

(and in this case, $\mathbf{p}(\sigma^{\geq n_0}P^\bullet)$ is the truncated projective resolution of $Z^{n_0}(P^\bullet)$). Therefore $Z^0(\mathbf{a}(\sigma^{\geq n_0}P^\bullet))$ is the n_0-th syzygy of $Z^{n_0}(P^\bullet)$, and thus it lies in R-Gproj. Hence G is a direct summand of a module in R-Gproj. This completes the proof. ■

Acknowledgement Some results in this paper appeared in the second chapter of my Ph.D thesis, and I would like to thank Prof. Pu Zhang for his supervision. I also would like to thank Prof. Apostolos Beligiannis and Prof. Henning Krause very much for their helpful remarks.

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