Cohomology of $\mathfrak{osp}(1|2)$ acting on linear differential operators on the supercircle $S^{1|1}$

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Abstract

We compute the first cohomology spaces $H^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$ $(\lambda, \mu \in \mathbb{R})$ of the Lie superalgebra $\mathfrak{osp}(1|2)$ with coefficients in the superspace $\mathfrak{D}_{\lambda,\mu}$ of linear differential operators acting on weighted densities on the supercircle $S^{1|1}$. The structure of these spaces was conjectured in [4]. In fact, we prove here that the situation is a little bit more complicated. (To appear in LMP.)

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1 Introduction

The space of weighted densities with weight λ (or λ-densities) on S^1, denoted by:

$$\mathcal{F}_\lambda = \left\{ f(dx)^\lambda, \ f \in C^\infty(S^1) \right\}, \quad (\lambda \in \mathbb{R}),$$

is the space of sections of the line bundle $(T^*S^1)^{\otimes\lambda}$. Let $\mathrm{Vect}(S^1)$ be the Lie algebra of all vector fields $F\frac{d}{dx}$ on S^1, $(F \in C^\infty(S^1))$. With the *Lie derivative*, $\mathcal{F}_\lambda$ is a $\mathrm{Vect}(S^1)$-module. Alternatively, the $\mathrm{Vect}(S^1)$ action can be written as follows:

$$L^\lambda_{F\frac{d}{dx}}(f(dx)^\lambda) = (Ff' + \lambda fF')(dx)^\lambda, \tag{1.1}$$

where f', F' are $\frac{df}{dx}$, $\frac{dF}{dx}$.

Let A be a differential operator on S^1. We see A as the linear mapping $f(dx)^\lambda \mapsto (Af)(dx)^\mu$ from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ (λ, μ in $\mathbb{R}$). Thus the space of differential operators is a $\mathrm{Vect}(S^1)$ module, denoted $\mathcal{D}_{\lambda,\mu}$. The $\mathrm{Vect}(S^1)$ action is:

$$L^{\lambda,\mu}_X(A) = L^\mu_X \circ A - A \circ L^\lambda_X. \tag{1.2}$$

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If we restrict ourselves to the Lie subalgebra of $\mathrm{Vect}(S^1)$ generated by $\left\{\frac{d}{dx},\, x\frac{d}{dx},\, x^2\frac{d}{dx}\right\}$, isomorphic to $\mathfrak{sl}(2)$, we get a family of infinite dimensional $\mathfrak{sl}(2)$ modules, still denoted $\mathcal{D}_{\lambda,\mu}$.

P. Lecomte, in [5], found the cohomology spaces $\mathrm{H}^1(\mathfrak{sl}(2);\mathcal{D}_{\lambda,\mu})$ and $\mathrm{H}^2(\mathfrak{sl}(2);\mathcal{D}_{\lambda,\mu})$. These spaces appear naturally in the problem of describing the deformations of the $\mathfrak{sl}(2)$-module $\mathcal{D}$ of the differential operators acting on $\mathcal{S}^n=\bigoplus_{k=-n}^{n}\mathcal{F}_{\frac{1+k}{2}}$. More precisely, the first cohomology space $\mathrm{H}^1(\mathfrak{sl}(2);V)$ classifies the infinitesimal deformations of a $\mathfrak{sl}(2)$ module V and the obstructions to integrability of a given infinitesimal deformation of V are elements of $\mathrm{H}^2(\mathfrak{sl}(2);V)$. Thus, for instance, the infinitesimal deformations of the $\mathfrak{sl}(2)$ module $\mathcal{D}$ are classified by:

$$\mathrm{H}^1(\mathfrak{sl}(2);\mathcal{D})=\oplus_{k=0}^{n}\mathrm{H}^1\left(\mathfrak{sl}(2);\mathcal{D}_{\frac{1-k}{2},\frac{1+k}{2}}\right)\oplus\oplus_{k=-n}^{n}\mathrm{H}^1\left(\mathfrak{sl}(2);\mathcal{D}_{\frac{1+k}{2},\frac{1+k}{2}}\right).$$

In this paper we are interested to the study of the corresponding super structures. More precisely, we consider here the superspace $S^{1|1}$ equipped with its standard *contact structure 1-form* α, and introduce the superspace $\mathfrak{F}_\lambda$ of λ-densities on the supercircle $S^{1|1}$.

Let $\mathcal{K}(1)$ be the Lie superalgebra of contact vector fields, $\mathfrak{F}_\lambda$ is naturally a $\mathcal{K}(1)$- module. For each λ, μ in $\mathbb{R}$, any differential operator on $S^{1|1}$ becomes a linear mapping from $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$, thus the space of differential operators becomes a $\mathcal{K}(1)$-module denoted $\mathfrak{D}_{\lambda,\mu}$.

To the symplectic Lie algebra $\mathfrak{sl}(2)$ corresponds the ortosymplectic Lie superalgebra $\mathfrak{osp}(1|2)$ which is naturally realized as a subalgebra of $\mathcal{K}(1)$. Restricting our $\mathcal{K}(1)$-modules to $\mathfrak{osp}(1|2)$, we get $\mathfrak{osp}(1|2)$-modules still denoted $\mathfrak{F}_\lambda$, $\mathfrak{D}_{\lambda,\mu}$.

We compute here the first cohomology spaces $\mathrm{H}^1(\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu})$, ($\lambda$, μ in $\mathbb{R}$), getting a result very close to the classical spaces $\mathrm{H}^1(\mathfrak{sl}(2);\mathcal{D}_{\lambda,\mu})$. Especially, these spaces have the same dimension. Moreover, we give explicit formulae for all the non trivial 1-cocycles.

These spaces arise in the classification of infinitesimal deformations of the $\mathfrak{osp}(1|2)$-module of the differential operators acting on $\mathfrak{S}^n=\bigoplus_{k=1-n}^{n}\mathfrak{F}_{\frac{k}{2}}$. We hope to be able to describe in the future all the deformations of this module.

2 Definitions and Notations

2.1 The Lie superalgebra of contact vector fields on $S^{1|1}$

We define the supercircle $S^{1|1}$ through its space of functions, $C^\infty(S^{1|1})$. A $C^\infty(S^{1|1})$ has the form:

$$F(x,\theta)=f_0(x)+\theta f_1(x),$$

where x is the even variable and θ the odd variable: we have $\theta^2=0$. Even elements in $C^\infty(S^{1|1})$ are the functions $F(x,\theta)=f_0(x)$, the functions $F(x,\theta)=\theta f_1(x)$ are odd elements. Note $p(F)$ the parity of a homogeneous function F.

Let $\mathrm{Vect}(S^{1|1})$ be the superspace of vector fields on $S^{1|1}$:

$$\mathrm{Vect}(S^{1|1})=\left\{F_0\partial_x+F_1\partial_\theta\quad F_i\in C^\infty(S^{1|1})\right\},$$

where ∂_θ and ∂_x stand for $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial_x}$. The vector fields $f(x)\partial_x$, and $\theta f(x)\partial_\theta$ are even, the vector fields $\theta f(x)\partial_x$, and $f(x)\partial_\theta$ are odd. The superbracket of two vector fields is bilinear and defined for two homogeneous vector fields by:

$$[X, Y] = X \circ Y - (-1)^{p(X)p(Y)} Y \circ X.$$

Denote $\mathfrak{L}_X$ the Lie derivative of a vector field, acting on the space of functions, forms, vector fields,...

The supercircle $S^{1|1}$ is equipped with the standard contact structure given by the following even 1-form:

$$\alpha = dx + \theta d\theta.$$

We consider the Lie superalgebra $\mathcal{K}(1)$ of contact vector fields on $S^{1|1}$. That is, $\mathcal{K}(1)$ is the superspace of conformal vector fields on $S^{1|1}$ with respect to the 1-form α:

$$\mathcal{K}(1) = \big\{X \in \mathrm{Vect}(S^{1|1}) \mid \text{there exists } F \in C^\infty(S^{1|1}) \text{ such that } \mathfrak{L}_X(\alpha) = F\alpha\big\}.$$

Let us define the vector fields η and $\overline{\eta}$ by

$$\eta = \partial_\theta + \theta\partial_x, \quad \overline{\eta} = \partial_\theta - \theta\partial_x.$$

Then any contact vector field on $S^{1|1}$ can be written in the following explicit form:

$$X_F = F\partial_x + \frac{1}{2}\eta(F)(\partial_\theta - \theta\partial_x) = -F\overline{\eta}^2 + \frac{1}{2}\eta(F)\overline{\eta}, \text{ where } F \in C^\infty(S^{1|1}).$$

Of course, $\mathcal{K}(1)$ is a subalgebra of $\mathrm{Vect}(S^{1|1})$, and $\mathcal{K}(1)$ acts on $C^\infty(S^{1|1})$ through:

$$\mathfrak{L}_{X_F}(G) = FG' + \frac{1}{2}(-1)^{(p(F)+1)p(G)}\overline{\eta}(F) \cdot \overline{\eta}(G). \tag{2.3}$$

Let us define the contact bracket on $C^\infty(S^{1|1})$ as the bilinear mapping such that, for a couple of homogenous functions F, G,

$$\{F, G\} = FG' - F'G + \frac{1}{2}(-1)^{p(F)+1}\overline{\eta}(F) \cdot \overline{\eta}(G), \tag{2.4}$$

Then the bracket of $\mathcal{K}(1)$ can be written as:

$$[X_F,\ X_G] = X_{\{F,G\}}.$$

2.2 The superalgebra $\mathfrak{osp}(1|2)$

Recall the Lie algebra $\mathfrak{sl}(2)$ is isomorphic to the Lie subalgebra of $\mathrm{Vect}(S^1)$ generated by

$$\left\{\frac{d}{dx},\ x\frac{d}{dx},\ x^2\frac{d}{dx}\right\}.$$

Similarly, we now consider the orthosymplectic Lie superalgebra as a subalgebra of $\mathcal{K}(1)$:

$$\mathfrak{osp}(1|2) = \text{Span}(X_1,\, X_x,\, X_{x^2},\, X_{x\theta},\, X_\theta).$$

The space of even elements is isomorphic to $\mathfrak{sl}(2)$:

$$(\mathfrak{osp}(1|2))_0 = \text{Span}(X_1,\, X_x,\, X_{x^2}) = \mathfrak{sl}(2).$$

The space of odd elements is two dimensional:

$$(\mathfrak{osp}(1|2))_1 = \text{Span}(X_{x\theta},\, X_\theta).$$

The new commutation relations are

$$\begin{aligned}
&[X_{x^2}, X_\theta] = -X_{x\theta}, \qquad && [X_x, X_\theta] = -\frac{1}{2}X_\theta, \quad && [X_1, X_\theta] = 0,\\
&[X_{x^2}, X_{x\theta}] = 0, && [X_x, X_{x\theta}] = \frac{1}{2}X_{x\theta}, && [X_1, X_{x\theta}] = X_\theta,\\
&[X_{x\theta}, X_\theta] = \frac{1}{2}X_x.
\end{aligned}$$

2.3 The space of weighted densities on $S^{1|1}$

In the super setting, by replacing dx by the 1-form α, we get analogous definition for weighted densities *i.e.* we define the space of λ-densities as

$$\mathfrak{F}_\lambda = \left\{ \phi = F(x,\theta)\alpha^\lambda \;\;|\;\; F(x,\theta) \in C^\infty(S^{1|1}) \right\}. \tag{2.5}$$

As a vector space, $\mathfrak{F}_\lambda$ is isomorphic to $C^\infty(S^{1|1})$, but the Lie derivative of the density $G\alpha^\lambda$ along the vector field X_F in $\mathcal{K}(1)$ is now:

$$\mathfrak{L}_{X_F}(G\alpha^\lambda) = \mathfrak{L}^\lambda_{X_F}(G)\alpha^\lambda, \quad \text{with} \;\; \mathfrak{L}^\lambda_{X_F}(G) = \mathfrak{L}_{X_F}(G) + \lambda F'G. \tag{2.6}$$

Or, if we put $F = a(x) + b(x)\theta$, $G = g_0(x) + g_1(x)\theta$,

$$\mathfrak{L}^\lambda_{X_F}(G) = L^\lambda_{a\partial_x}(g_0) + \frac{1}{2}\, bg_1 + \left(L^{\lambda+\frac{1}{2}}_{a\partial_x}(g_1) + \lambda g_0 b' + \frac{1}{2} g_0' b \right)\theta. \tag{2.7}$$

Especially, we have

$$\begin{cases} \mathfrak{L}^\lambda_{X_a}(g_0) = L^\lambda_{a\partial_x}(g_0), \\ \\ \mathfrak{L}^\lambda_{X_{b\theta}}(g_0) = (\lambda g_0 b' + \frac{1}{2}g_0'b)\theta \end{cases} \quad \text{and} \quad \begin{array}{l} \mathfrak{L}^\lambda_{X_a}(g_1\theta) = \theta L^{\lambda+\frac{1}{2}}_{a\partial_x}(g_1), \\ \\ \mathfrak{L}^\lambda_{X_{b\theta}}(g_1\theta) = \frac{1}{2} b g_1. \end{array}$$

Of course, for all λ, $\mathfrak{F}_\lambda$ is a $\mathcal{K}(1)$-module:

$$[\mathfrak{L}^\lambda_{X_F}, \mathfrak{L}^\lambda_{X_G}] = \mathfrak{L}^\lambda_{[X_F,\, X_G]}.$$

We thus obtain a one-parameter family of $\mathcal{K}(1)$-modules on $C^\infty(S^{1|1})$ still denoted by $\mathfrak{F}_\lambda$.

2.4 Differential Operators on Weighted Densities

A differential operator on $S^{1|1}$ is an operator on $C^\infty(S^{1|1})$ of t he following form:

$$A=\sum_{i=0}^{\ell}\widetilde{a}_i(x,\theta)\partial_x^i+\sum_{i=0}^{\ell}\widetilde{b}_i(x,\theta)\partial_x^i\partial_\theta.$$

In [4], it is proved that any local operator A on $S^{1|1}$ is in fact a differential operator.

Of course, any differential operator defines a linear mapping from $\mathfrak{F}_\lambda$ to $\mathfrak{F}_\mu$ for any λ, $\mu\in\mathbb{R}$, thus the space of differential operators becomes a family of $\mathcal{K}(1)$ and $\mathfrak{osp}(1|2)$ modules denoted $\mathfrak{D}_{\lambda,\mu}$, for the natural action:

$$\mathfrak{L}_{X_F}^{\lambda,\mu}(A)=\mathfrak{L}_{X_F}^{\mu}\circ A-(-1)^{p(A)p(F)}A\circ\mathfrak{L}_{X_F}^{\lambda}. \tag{2.8}$$

3 The space $H^1(\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu})$

3.1 Lie superalgebra cohomology (see [2])

Let $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ be a Lie superalgebra and $A=A_0\oplus A_1$ a $\mathfrak{g}$ module. We define the *cochain complex* associated to the module as an exact sequence:

$$0\longrightarrow C^0(\mathfrak{g},\ \mathrm{A})\longrightarrow\cdots\longrightarrow \mathrm{C}^{\mathrm{q}-1}(\mathfrak{g},\ \mathrm{A})\stackrel{\delta^{\mathrm{q}-1}}{\longrightarrow}\mathrm{C}^{\mathrm{q}}(\mathfrak{g},\ \mathrm{A})\cdots.$$

The spaces $C^q(\mathfrak{g},\ \mathrm{A})$ are the spaces of super skew-symmetric q linear mappings:

$$C^0(\mathfrak{g},\ \mathrm{A})=\mathrm{A},\quad \mathrm{C}^{\mathrm{q}}(\mathfrak{g},\ \mathrm{A})=\bigoplus_{\mathrm{q_0+q_1=q}}\mathrm{Hom}(\bigwedge^{\mathrm{q_0}}\mathfrak{g}_0\otimes\mathrm{S}^{\mathrm{q_1}}\mathfrak{g}_1,\ \mathrm{A}).$$

Elements of $C^q(\mathfrak{g},\ \mathrm{A})$ are called *cochains*. The spaces $C^q(\mathfrak{g},\ \mathrm{A})$ is $\mathbb{Z}_2$ graded:

$$C^q(\mathfrak{g},\ \mathrm{A})=\mathrm{C}_0^{\mathrm{q}}(\mathfrak{g},\ \mathrm{A})+\mathrm{C}_1^{\mathrm{q}}(\mathfrak{g},\ \mathrm{A}),\ \ \text{with}\ \ \mathrm{C}_{\mathrm{p}}^{\mathrm{q}}(\mathfrak{g},\ \mathrm{A})=\bigoplus_{\substack{q_0+q_1=q\\ q_1+r=p\quad mod2}}\mathrm{Hom}(\bigwedge^{\mathrm{q_0}}\mathfrak{g}_0\otimes\mathrm{S}^{\mathrm{q_1}}\mathfrak{g}_1,\mathrm{A_r}).$$

The linear mapping δ^q (or, briefly δ) is called the *coboundary operator*. This operator is a generalization of the usual Chevalley coboundary operator for Lie algebra to the case of Lie superalgebra. Explicitly, it is defined as follows. Take a cochain $c\in C^q(\mathfrak{g},\ \mathrm{A})$, then for q_0, q_1 with $q_0+q_1=q+1$, $\delta^q c$ is:

$$
\begin{aligned}
&\delta^q c(g_1,\ \dots,\ g_{q_0},\ h_1,\ \dots,\ h_{q_1})\\
&\qquad = \sum_{1\le s<t\le q_0} (-1)^{s+t-1} c([g_s,g_t], g_1,\dots,\hat{g}_s,\dots,\hat{g}_t,\dots,g_{q_0},\ h_1,\dots,h_{q_1})\\
&\qquad\quad + \sum_{s=1}^{q_0}\sum_{t=1}^{q_1}(-1)^{s-1} c(g_1,\dots,\hat{g}_s,\dots,g_{q_0},\ [g_s,h_t], h_1,\dots,\hat{h}_t,\dots,h_{q_1})\\
&\qquad\quad + \sum_{1\le s<t\le q_1} c([h_s,h_t], g_1,\dots,g_{q_0},\ h_1,\dots,\hat{h}_s,\dots,\hat{h}_t,\dots,h_{q_1})\\
&\qquad\quad + \sum_{s=1}^{q_0}(-1)^s g_s c(g_1,\dots,\hat{g}_s,\dots,g_{q_0},\ h_1,\dots,h_{q_1})\\
&\qquad\quad + (-1)^{q_0-1}\sum_{s=1}^{q_1} h_s c(g_1,\dots,g_{q_0},\ h_1,\dots,\hat{h}_s,\dots,h_{q_1}).
\end{aligned}
$$

where $g_1,\dots,g_{q_0}$ are in $\mathfrak{g}_0$ and $h_1,\dots,h_{q_1}$ in $\mathfrak{g}_1$.

The relation $\delta^q\circ\delta^{q-1}=0$ holds. The kernel of δ^q, denoted $Z^q(\mathfrak{g},\ \mathrm{A})$, is the space of q *cocycles*, among them, the elements in the range of δ^{q-1} are called q *coboundaries.* We note $B^q(\mathfrak{g},\ \mathrm{A})$ the space of q coboundaries.

By definition, the q^{th} cohomolgy space is the quotient space

$$H^q(\mathfrak{g},\ \mathrm{A}) = \mathrm{Z^q}(\mathfrak{g},\ \mathrm{A})/\mathrm{B^q}(\mathfrak{g},\ \mathrm{A}).$$

One can check that $\delta^q(C^q_p(\mathfrak{g},\ \mathrm{A}))\subset \mathrm{C^{q+1}_p}(\mathfrak{g},\ \mathrm{A})$ and then we get the following sequences

$$0\longrightarrow C^0_p(\mathfrak{g},\ \mathrm{A})\longrightarrow\cdots\longrightarrow \mathrm{C^{q-1}_p}(\mathfrak{g},\ \mathrm{A})\stackrel{\delta^{q-1}}{\longrightarrow}\mathrm{C^q_p}(\mathfrak{g},\ \mathrm{A})\cdots,$$

where $p=0$ or 1. The cohomology spaces are thus graded by

$$H^q_p(\mathfrak{g},\ \mathrm{A}) = \mathrm{Ker}\delta^{\mathrm{q}}|_{\mathrm{C^q_p}(\mathfrak{g},\ \mathrm{A})}/\delta^{\mathrm{q-1}}(\mathrm{C^{q-1}_p}(\mathfrak{g},\ \mathrm{A})).$$

3.2 The main theorem

The main result in this paper is the following:

Theorem 3.1. *The cohomolgy spaces* $\mathrm{H}^1_p(\mathfrak{g},\mathfrak{D}_{\lambda,\mu})$ *are finite dimensional. An explicit description of these spaces is the following:*

1)The space $\mathrm{H}^1_0(\mathfrak{osp}(1|2),\mathfrak{D}_{\lambda,\mu})$ *is*

$$\mathrm{H}^1_0(\mathfrak{osp}(1|2),\mathfrak{D}_{\lambda,\mu})\simeq\begin{cases}\mathbb{R} \ \textit{if} \quad \lambda=\mu,\\ 0 \ \textit{otherwise.}\end{cases}\tag{3.9}$$

A base for the space $\mathrm{H}^1_0(\mathfrak{osp}(1|2),\mathfrak{D}_{\lambda,\lambda})$ *is given by the cohomology class of the* 1*-cocycle:*

$$\Upsilon_{\lambda,\lambda}(X_F)=F'. \tag{3.10}$$

2)The space $\mathrm{H}^1_1(\mathfrak{osp}(1|2),\mathfrak{D}_{\lambda,\mu})$ *is*

$$\mathrm{H}^1_1(\mathfrak{osp}(1|2),\mathfrak{D}_{\lambda,\mu})\simeq\begin{cases}\mathbb{R}^2 & \textit{if}\quad \lambda=\frac{1-k}{2},\ \mu=\frac{k}{2},\\ 0 & \textit{otherwise.}\end{cases} \tag{3.11}$$

A base for the space $\mathrm{H}^1_1(\mathfrak{osp}(1|2),\mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}})$ *is given by the cohomology classes of the* 1*-cocycles:*

$$\begin{aligned}\Upsilon_{\frac{1-k}{2},\frac{k}{2}}(X_F)&=\overline{\eta}^2(F)\overline{\eta}^{2k-1},\\ \widetilde{\Upsilon}_{\frac{1-k}{2},\frac{k}{2}}(X_F)&=(k-1)\eta^4(F)\overline{\eta}^{2k-3}+\eta^3(F)\overline{\eta}^{2k-2}.\end{aligned} \tag{3.12}$$

Note that the 1-cocycle $\widetilde{\Upsilon}_{\frac{1-k}{2},\frac{k}{2}}$ coincides with the 1-cocycle γ_{2k-1} given by Gargoubi et al. in [4]. The proof of Theorem 3.1 will be the subject of subsection 3.4.

3.3 Relationship between $\mathrm{H}^1(\mathfrak{osp}(1|2),\mathfrak{D}_{\lambda,\mu})$ and $\mathrm{H}^1(\mathfrak{sl}(2),\mathcal{D}_{\lambda,\mu})$

Before proving the theorem 3.1 we present here some results illustrating the analogy between the cohomlogy spaces in super and classical settings.

First, note that:

1) As a $\mathfrak{sl}(2)$-module, we have $\mathfrak{F}_\lambda\simeq\mathcal{F}_\lambda\oplus\Pi(\mathcal{F}_{\lambda+\frac{1}{2}})$ and $\mathfrak{osp}(1|2)\simeq\mathfrak{sl}(2)\oplus\Pi(\mathfrak{h})$, where $\mathfrak{h}$ is the subspace of $\mathcal{F}_{-\frac{1}{2}}$ spanned by $\{dx^{-\frac{1}{2}},xdx^{-\frac{1}{2}}\}$ and Π is the change of parity.

2) As a $\mathfrak{sl}(2)$-module, we have for the homogeneous components of $\mathfrak{D}_{\lambda,\mu}$:

$$(\mathfrak{D}_{\lambda,\mu})_0\simeq\mathcal{D}_{\lambda,\mu}\oplus\mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}\quad\text{and}\quad(\mathfrak{D}_{\lambda,\mu})_1\simeq\Pi(\mathcal{D}_{\lambda+\frac{1}{2},\mu}\oplus\mathcal{D}_{\lambda,\mu+\frac{1}{2}}).$$

Proposition 3.1. *Any 1-cocycle* $\Upsilon\in Z^1(\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu})$*, is decomposed into* (Υ',Υ'') *in* $Hom(\mathfrak{sl}(2);\mathfrak{D}_{\lambda,\mu})\oplus Hom(\mathfrak{h};\mathfrak{D}_{\lambda,\mu})$*.* Υ' *and* Υ'' *are solutions of the following equations:*

$$\Upsilon'([X_{g_1},X_{g_2}])-\mathfrak{L}^{\lambda,\mu}_{X_{g_1}}\Upsilon'(X_{g_2})+\mathfrak{L}^{\lambda,\mu}_{X_{g_2}}\Upsilon'(X_{g_1})=0, \tag{3.13}$$

$$\Upsilon''([X_g,X_{h\theta}])-\mathfrak{L}^{\lambda,\mu}_{X_g}\Upsilon''(X_{h\theta})+\mathfrak{L}^{\lambda,\mu}_{X_{h\theta}}\Upsilon'(X_g)=0, \tag{3.14}$$

$$\Upsilon'([X_{h_1\theta},X_{h_2\theta}])-\mathfrak{L}^{\lambda,\mu}_{X_{h_1\theta}}\Upsilon''(X_{h_2\theta})-\mathfrak{L}^{\lambda,\mu}_{X_{h_2\theta}}\Upsilon''(X_{h_1\theta})=0, \tag{3.15}$$

here, g*,* g_1*,* g_2 *are polynomials in the variable* x*, with degree at most 2, and* h*,* h_1*,* h_2 *are affine functions in the variable* x*.*

Proof. The equations (3.13), (3.14) and (3.15) are equivalent to the fact that Υ is a 1-cocycle. For any $X_F,\, X_G \in \mathfrak{osp}(1|2)$,

$$\delta\Upsilon(X_F, X_G) := \Upsilon([X_F, X_G]) - \mathfrak{L}^{\lambda,\mu}_{X_F}\Upsilon(X_G) + (-1)^{p(F)p(G)}\mathfrak{L}^{\lambda,\mu}_{X_G}\Upsilon(X_F) = 0.$$

□

According to the $\mathbb{Z}_2$-grading, the even component Υ_0 and the odd component Υ_1 of any 1-cocycle Υ can be decomposed as $\Upsilon_0 = (\Upsilon_{000}, \Upsilon_{00\frac{1}{2}}, \Upsilon_{110}, \Upsilon_{11\frac{1}{2}})$ and $\Upsilon_1 = (\Upsilon_{010}, \Upsilon_{01\frac{1}{2}}, \Upsilon_{100}, \Upsilon_{10\frac{1}{2}})$, where

$$\begin{cases} \Upsilon_{000}: & \mathfrak{sl}(2) & \to & \mathcal{D}_{\lambda,\mu}, \\ \Upsilon_{00\frac{1}{2}}: & \mathfrak{sl}(2) & \to & \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}, \\ \Upsilon_{110}: & \mathfrak{h} & \to & \mathcal{D}_{\lambda,\mu+\frac{1}{2}}, \\ \Upsilon_{11\frac{1}{2}}: & \mathfrak{h} & \to & \mathcal{D}_{\lambda+\frac{1}{2},\mu} \end{cases} \quad \text{and} \quad \begin{cases} \Upsilon_{010}: & \mathfrak{sl}(2) & \to & \mathcal{D}_{\lambda,\mu+\frac{1}{2}}, \\ \Upsilon_{01\frac{1}{2}}: & \mathfrak{sl}(2) & \to & \mathcal{D}_{\lambda+\frac{1}{2},\mu} \\ \Upsilon_{100}: & \mathfrak{h} & \to & \mathcal{D}_{\lambda,\mu}, \\ \Upsilon_{10\frac{1}{2}}: & \mathfrak{h} & \to & \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}. \end{cases}$$

The decomposition $\Upsilon = (\Upsilon', \Upsilon'')$ given in proposition 3.1 corresponds to

$$\Upsilon' = (\Upsilon_{000}, \Upsilon_{00\frac{1}{2}}, \Upsilon_{010}, \Upsilon_{01\frac{1}{2}}) \quad \text{and} \quad \Upsilon'' = (\Upsilon_{110}, \Upsilon_{11\frac{1}{2}}, \Upsilon_{100}, \Upsilon_{10\frac{1}{2}}).$$

By considering the equation (3.13), we can see the components Υ_{000}, $\Upsilon_{00\frac{1}{2}}$, Υ_{010} and $\Upsilon_{01\frac{1}{2}}$ as 1-cocycles on $\mathfrak{sl}(2)$ with coefficients respectively in $\mathcal{D}_{\lambda,\mu}$, $\mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}$, $\mathcal{D}_{\lambda,\mu+\frac{1}{2}}$, and $\mathcal{D}_{\lambda+\frac{1}{2},\mu}$.

The first cohomology space $\mathrm{H}^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu})$ was computed by Gargoubi and Lecomte [3, 5]. The result is the following:

$$\mathrm{H}^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda,\mu}) \simeq \begin{cases} \mathbb{R} & \text{if} \quad \lambda = \mu \\ \mathbb{R}^2 & \text{if} \quad (\lambda,\mu) = (\frac{1-k}{2}, \frac{1+k}{2}) \quad \text{where} \quad k \in \mathbb{N}\setminus\{0\} \\ 0 & \quad\quad \text{otherwise.} \end{cases} \tag{3.16}$$

The space $\mathrm{H}^1(\mathfrak{sl}(2); \mathcal{D}_{\lambda,\lambda})$ is generated by the cohomology class of the 1-cocycle

$$C'_\lambda(F\frac{d}{dx})(fdx^\lambda) = F'fdx^\lambda. \tag{3.17}$$

For $k \in \mathbb{N}\setminus\{0\}$, the space $\mathrm{H}^1(\mathfrak{sl}(2); \mathcal{D}_{\frac{1-k}{2},\frac{1+k}{2}})$ is generated by the cohomology classes of the 1-cocycles, C_k and $\tilde{C}_k$ defined by

$$C_k(F\frac{d}{dx})(fdx^{\frac{1-k}{2}}) = F'f^{(k)}dx^{\frac{1+k}{2}} \quad \text{and} \quad \widetilde{C}_k(F\frac{d}{dx})(fdx^{\frac{1-k}{2}}) = F''f^{(k-1)}dx^{\frac{1+k}{2}}. \tag{3.18}$$

We shall need the following description of $\mathfrak{sl}(2)$ invariant mappings.

Lemma 3.2. *Let*

$$A : \mathfrak{h} \times \mathcal{F}_\lambda \to \mathcal{F}_\mu, \qquad (hdx^{-\frac{1}{2}}, fdx^\lambda) \mapsto A(h,f)dx^\mu$$

be a bilinear differential operator. If A is $\mathfrak{sl}(2)$-invariant then

$$\mu = \lambda - \frac{1}{2} + k, \quad \textit{where} \quad k \in \mathbb{N}$$

and the following relation holds

$$A_k(h,f) = a_k(hf^{(k)} + k(2\lambda+k-1)h'f^{(k-1)}), \quad \textit{where} \quad k(k-1)(2\lambda+k-1)(2\lambda+k-2)a_k = 0.$$

Proof. A straightforward computation.

□

Now, let us study the relationship between these 1-cocycles and their analogues in the super setting. We know that any element $\Upsilon \in Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})$ is decomposed into $\Upsilon = \Upsilon' + \Upsilon''$ where $\Upsilon' \in Hom(\mathfrak{sl}(2), \mathfrak{D}_{\lambda,\mu})$ and $\Upsilon'' \in Hom(\mathfrak{h}, \mathfrak{D}_{\lambda,\mu})$. The following lemma shows the close relationship between the cohomolgy spaces $H^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})$ and $H^1(\mathfrak{sl}(2), \mathcal{D}_{\lambda,\mu})$.

Lemma 3.3. *The 1-cocycle Υ is a coboundary for $\mathfrak{osp}(1|2)$ if and only if Υ' is a coboundary for $\mathfrak{sl}(2)$.*

Proof. It is easy to see that if Υ is a coboundary for $\mathfrak{osp}(1|2)$ then Υ' is a coboundary over $\mathfrak{sl}(2)$. Now, assume that Υ' is a coboundary for $\mathfrak{sl}(2)$, that is, there exists $\widetilde{A} \in \mathfrak{D}_{\lambda,\mu}$ such that for all g polynomial in the variable x with degree at most 2

$$\Upsilon'(X_g) = \mathfrak{L}^{\lambda,\mu}_{X_g}\widetilde{A}.$$

By replacing Υ by $\Upsilon - \delta\widetilde{A}$, we can suppose that $\Upsilon' = 0$. But, in this case, the map Υ'' must satisfy, for all h, h_1, h_2 polynomial with degree 0 or 1 and g polynomial with degree 0,1 or 2, the following equations

$$\mathfrak{L}^{\lambda,\mu}_{X_g}\Upsilon''(X_{h\theta}) - \Upsilon''([X_g, X_{h\theta}]) = 0, \tag{3.19}$$

$$\mathfrak{L}^{\lambda,\mu}_{X_{h_1\theta}}\Upsilon''(X_{h_2\theta}) + \mathfrak{L}^{\lambda,\mu}_{X_{h_2\theta}}\Upsilon''(X_{h_1\theta}) = 0. \tag{3.20}$$

1) If Υ is an even 1-cocycle then Υ'' is decomposed into $\Upsilon''_{00} : \mathfrak{h} \otimes \mathcal{F}_{\lambda+\frac{1}{2}} \to \mathcal{F}_\mu$ and $\Upsilon''_{01} : \mathfrak{h} \otimes \mathcal{F}_\lambda \to \mathcal{F}_{\mu+\frac{1}{2}}$. The equation (3.19) tell us that Υ''_{00} and Υ''_{01} are $\mathfrak{sl}(2)$ invariant bilinear maps. Therefore, the expressions of Υ''_{00} and Υ''_{01} are given by Lemma 3.2. So, we must have $\mu = \lambda + k = (\lambda + \frac{1}{2}) - \frac{1}{2} + k$ (and then $\mu + \frac{1}{2} = \lambda - \frac{1}{2} + k + 1$). More precisely, using the equation (3.20), we get up to a factor:

$$\Upsilon = \begin{cases} 0 & \text{if } \; k(k-1)(2\lambda+k)(2\lambda+k-1) \neq 0 \quad \text{or} \quad k=1 \text{ and } \lambda \notin \{0, -\dfrac{1}{2}\}, \\ \delta(\theta\partial_\theta\partial_x^k) & \text{if } \; (\lambda,\mu) = (\dfrac{-k}{2}, \dfrac{k}{2}), \\ \delta(\partial_x^k - \theta\partial_\theta\partial_x^k) & \text{if } \; (\lambda,\mu) = (\dfrac{1-k}{2}, \dfrac{1+k}{2}) \quad \text{or} \quad \lambda = \mu. \end{cases}$$

2) If Υ is an odd 1-cocycle then Υ'' is decomposed into $\Upsilon''_{00} : \mathfrak{h} \otimes \mathcal{F}_\lambda \to \mathcal{F}_\mu$ and $\Upsilon''_{01} : \mathfrak{h} \otimes \mathcal{F}_{\lambda+\frac{1}{2}} \to \mathcal{F}_{\mu+\frac{1}{2}}$. As in the previous case, the expressions of Υ''_{00} and Υ''_{01} are given by Lemma 3.2. So, we must have $\mu = \lambda - \frac{1}{2} + k$ (and then $\mu + \frac{1}{2} = (\lambda + \frac{1}{2}) - \frac{1}{2} + k$.) More precisely, using the equation (3.20), we get:

$$\Upsilon = \begin{cases} 0 & \text{if} \quad k(k-1)(2\lambda + k - 1) \neq 0 \\ \delta(\theta) & \text{if} \quad \mu = \lambda - \frac{1}{2}, \\ \delta(\partial_\theta) & \text{if} \quad \mu = \lambda + \frac{1}{2}, \\ \delta(\theta\partial_x^k) & \text{if} \quad (\lambda, \mu) = (\frac{1-k}{2}, \frac{k}{2}). \end{cases}$$

□

Now, the space $Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})$ of 1-cocycles is $\mathbb{Z}_2$-graded:

$$Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu}) = Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})_0 \oplus Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})_1. \tag{3.21}$$

Therefore, any element $\Upsilon \in Z^1(\mathfrak{osp}(1|2), \mathfrak{D}_{\lambda,\mu})$ is decomposed into an even part Υ_0 and odd part Υ_1. Each of Υ_0 and Υ_1 is decomposed into two components: $\Upsilon_0 = (\Upsilon_{00}, \Upsilon_{11})$ and $\Upsilon_1 = (\Upsilon_{01}, \Upsilon_{10})$, where

$$\begin{cases} \Upsilon_{00} : \mathfrak{sl}(2) & \to (\mathfrak{D}_{\lambda,\mu})_0, \\ \Upsilon_{11} : \mathfrak{h} & \to (\mathfrak{D}_{\lambda,\mu})_1, \end{cases} \quad \text{and} \quad \begin{cases} \Upsilon_{01} : \mathfrak{sl}(2) & \to (\mathfrak{D}_{\lambda,\mu})_1, \\ \Upsilon_{10} : \mathfrak{h} & \to (\mathfrak{D}_{\lambda,\mu})_0. \end{cases}$$

The components Υ_{11} and Υ_{10} of Υ_0 and Υ_1 are also decomposed as follows: $\Upsilon_{11} = \Upsilon_{110} + \Upsilon_{11\frac{1}{2}}$ and $\Upsilon_{10} = \Upsilon_{100} + \Upsilon_{10\frac{1}{2}}$, where $\Upsilon_{110} \in Hom\left(\mathfrak{h}, \mathcal{D}_{\lambda,\mu+\frac{1}{2}}\right)$, $\Upsilon_{11\frac{1}{2}} \in Hom\left(\mathfrak{h}, \mathcal{D}_{\lambda+\frac{1}{2},\mu}\right)$, $\Upsilon_{100} \in Hom\left(\mathfrak{h}, \mathcal{D}_{\lambda,\mu}\right)$, $\Upsilon_{10\frac{1}{2}} \in Hom\left(\mathfrak{h}, \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}\right)$.

As in [1], the following lemma gives the general form of each of Υ_{110} and $\Upsilon_{11\frac{1}{2}}$.

Lemma 3.4. *Up to a coboundary, the maps Υ_{110}, $\Upsilon_{11\frac{1}{2}}$, Υ_{100} and $\Upsilon_{10\frac{1}{2}}$ are given by*

$$\begin{aligned} \Upsilon_{110}(X_{h\theta}) &= a_0 h\theta\partial_x^k + a_1 h'\theta\partial_x^{k-1} \quad \textit{and} \quad \Upsilon_{11\frac{1}{2}}(X_{h\theta}) &= b_0 h\partial_\theta\partial_x^k + b_1 h'\partial_\theta\partial_x^{k-1}, \\ \Upsilon_{110}(X_{h\theta}) &= c_0 h\theta\partial_x^k + c_1 h'\theta\partial_x^{k-1} \quad \textit{and} \quad \Upsilon_{11\frac{1}{2}}(X_{h\theta}) &= d_0 h\partial_\theta\partial_x^k + d_1 h'\partial_\theta\partial_x^{k-1}, \end{aligned}$$

where the coefficients a_i, b_i, c_i, and d_i are constants.

Proof. The coefficients a_i, b_i, c_i, and d_i a priori are some functions of x, but we shall now prove $\partial_x a_i = \partial_x b_i = 0$ (and similarly $\partial_x c_i = \partial_x d_i = 0$). To do that, we shall simply show that $\mathfrak{L}^{\lambda,\mu}_{\partial_x}(\Upsilon_{11}) = 0$.

First, for all h polynomial with degree 0 or 1, we have

$$(\mathfrak{L}^{\lambda,\mu}_{\partial_x}\Upsilon_{11})(X_{h\theta}) = \mathfrak{L}^{\lambda,\mu}_{\partial_x}(\Upsilon_{11}(X_{h\theta})) - \Upsilon_{11}([\partial_x, X_{h\theta}]). \tag{3.22}$$

On the other hand, from Lemma 3.3, it follows that, up to a coboundary, Υ_{00} is a linear combination of some 1-cocycles for sl(2) given by (3.17) and (3.18). So, we have $\Upsilon_{00}(\partial_x) = 0$ and then

$$\mathfrak{L}^{\lambda,\mu}_{X_{h\theta}}(\Upsilon_{00}(\partial_x)) = 0.$$

Therefore, the equation (3.22) becomes, for all h,

$$-(\mathfrak{L}^{\lambda,\mu}_{\partial_x}\Upsilon_{11})(X_{h\theta})=\Upsilon_{11}([\partial_x,X_{h\theta}])-\mathfrak{L}^{\lambda,\mu}_{\partial_x}(\Upsilon_{11}(X_{h\theta}))+\mathfrak{L}^{\lambda,\mu}_{X_{h\theta}}(\Upsilon_{00}(\partial_x)). \tag{3.23}$$

The right-hand side of (3.23) is nothing but $\delta\Upsilon_0(\partial_x,X_{h\theta})$. But, Υ_0 is a 1-cocycle, then $\mathfrak{L}^{\lambda,\mu}_{\partial_x}(\Upsilon_{11})=0$. Lemma 3.4 is proved.

□

3.4 Proof of Theorem 3.1

The first cohomology space $\mathrm{H}^1(\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu})$ inherits the $\mathbb{Z}_2$-grading from (3.21) and is decomposed into odd and an even subspaces:

$$\mathrm{H}^1(\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu})=\mathrm{H}^1_0(\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu})\oplus\mathrm{H}^1_1(\mathfrak{osp}(1|2);\mathfrak{D}_{\lambda,\mu}).$$

We compute each part separetly.

1) Let Υ_0 be a non trivial even 1-cocycle for $\mathfrak{osp}(1|2)$ in $\mathfrak{D}_{\lambda,\mu}$. According to the $\mathbb{Z}_2$-grading, Υ_0 should retain the following general form: $\Upsilon_0=\Upsilon_{000}+\Upsilon_{00\frac{1}{2}}+\Upsilon_{110}+\Upsilon_{11\frac{1}{2}}$ such that

$$\begin{cases}\Upsilon_{000} & : & \mathfrak{sl}(2) & \rightarrow & \mathcal{D}_{\lambda,\mu},\\ \Upsilon_{00\frac{1}{2}} & : & \mathfrak{sl}(2) & \rightarrow & \mathcal{D}_{\lambda\frac{1}{2}f,\mu+\frac{1}{2}},\\ \Upsilon_{110} & : & \mathfrak{h} & \rightarrow & \mathcal{D}_{\lambda,\mu+\frac{1}{2}},\\ \Upsilon_{11\frac{1}{2}} & : & \mathfrak{h} & \rightarrow & \mathcal{D}_{\lambda+\frac{1}{2},\mu}.\end{cases} \tag{3.24}$$

Then, by using Lemma 3.3, we deduce that, up to coboundary, Υ_{000} and $\Upsilon_{00\frac{1}{2}}$ can be expressed in terms of C'_λ, C_k and $\widetilde{C}_k$ where $\lambda\in\mathbb{R}$ and $k\in\mathbb{N}\setminus\{0\}$. We thus consider three cases:

i) $\lambda=\mu$, $\Upsilon_{000}=\alpha C'_\lambda$, and $\Upsilon_{00\frac{1}{2}}=\beta C'_{\lambda\frac{1}{2}f}$.

ii) $(\lambda,\mu)=(\frac{1-k}{2},\frac{1+k}{2})$, $\Upsilon_{000}=\alpha_1C_k+\alpha_2\widetilde{C}_k$, and $\Upsilon_{00\frac{1}{2}}=0$.

iii) $(\lambda,\mu)=(\frac{-k}{2},\frac{k}{2})$, $\Upsilon_{000}=0$ and $\Upsilon_{00\frac{1}{2}}=\alpha_1C_k+\alpha_2\widetilde{C}_k$.

Put $\Upsilon'=\Upsilon_{000}+\Upsilon_{00\frac{1}{2}}$ and $\Upsilon''=\Upsilon_{110}+\Upsilon_{11\frac{1}{2}}$. In each case, the 1-cocycle Υ_0 must satisfy

$$\begin{cases}\Upsilon''[X_g,X_{\theta h}] & = & \mathfrak{L}^{\lambda,\mu}_{X_g}\Upsilon''(X_{\theta h})-\mathfrak{L}^{\lambda,\mu}_{X_{\theta h}}\Upsilon'(X_g),\\ \Upsilon'[X_{\theta h_1},X_{\theta h_2}] & = & \mathfrak{L}^{\lambda,\mu}_{X_{\theta h_1}}\Upsilon''(X_{\theta h_2})+\mathfrak{L}^{\lambda,\mu}_{X_{\theta h_2}}\Upsilon''(X_{\theta h_1}),\end{cases} \tag{3.25}$$

where h, h_1, and h_2 are polynomials of degree 0 or 1, g polynomial of degree 0, 1 or 2.

Now, thanks to Lemma 3.4, we can write

$$\Upsilon_{110}(X_{h\theta})=a_0h\theta\partial_x^k\ +a_1h'\theta\partial_x^{k-1}\ \text{ and }\ \Upsilon_{11\frac{1}{2}}(v_{h\theta})=b_0h\partial_\theta\partial_x^k+b_1h'\partial_\theta\partial_x^{k-1}.$$

Let us now solve the equations (3.25). We obtain $\lambda = \mu$ and $\Upsilon_{\lambda,\lambda}(X_F) = F'$. This completes the proof of part 1).

2) Consider a non trivial odd 1-cocycle Υ_1 for $\mathfrak{osp}(1|2)$ in $\mathfrak{D}_{\lambda,\mu}$ and its decomposition $\Upsilon_1 = \Upsilon_{010} + \Upsilon_{01\frac{1}{2}} + \Upsilon_{100} + \Upsilon_{10\frac{1}{2}}$, where

$$\left\{\begin{array}{lllll} \Upsilon_{010} & : & \mathfrak{sl}(2) & \rightarrow & \mathcal{D}_{\lambda,\mu+\frac{1}{2}}, \\ \Upsilon_{01\frac{1}{2}} & : & \mathfrak{sl}(2) & \rightarrow & \mathcal{D}_{\lambda+\frac{1}{2},\mu}, \\ \Upsilon_{100} & : & \mathfrak{h} & \rightarrow & \mathcal{D}_{\lambda,\mu}, \\ \Upsilon_{10\frac{1}{2}} & : & \mathfrak{h} & \rightarrow & \mathcal{D}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}. \end{array}\right. \tag{3.26}$$

We must have $(\lambda,\mu) = (\frac{1-k}{2}, \frac{k}{2})$ with $k \in \mathbb{N}\setminus\{0\}$. Moreover $\Upsilon_1 = \Upsilon_{000} + \Upsilon_{00\frac{1}{2}} + \Upsilon_{110} + \Upsilon_{11\frac{1}{2}}$ is a 1-cocycle for $\mathcal{K}(1)$ if and only if

$$\left\{\begin{array}{lll} \Upsilon_{010} & = & \alpha_1 C_k + \alpha_2 \widetilde{C}_k \\ \Upsilon_{01\frac{1}{2}} & = & \beta_1 C_{k-1} + \beta_2 \widetilde{C}_{k-1} \\ \Upsilon''[X_g,\, X_{\theta h}] & = & \mathfrak{L}^{\lambda,\mu}_{X_g}\Upsilon''(X_{\theta h}) - \mathfrak{L}^{\lambda,\mu}_{X_{\theta h}}\Upsilon'(X_g), \\ \Upsilon'[X_{\theta h_1},\, X_{\theta h_2}] & = & \mathfrak{L}^{\lambda,\mu}_{X_{\theta h_1}}\Upsilon''(X_{\theta h_2}) + \mathfrak{L}^{\lambda,\mu}_{X_{\theta h_2}}\Upsilon''(X_{\theta h_1}), \end{array}\right. \tag{3.27}$$

where $\Upsilon' = \Upsilon_{010} + \Upsilon_{01\frac{1}{2}}$ and $\Upsilon'' = \Upsilon_{100} + \Upsilon_{10\frac{1}{2}}$.

As above, we then can write

$$\Upsilon_{100}(X_{h\theta}) = a_0 h\theta\partial_x^k \ + a_1 h'\theta\partial_x^{k-1} \quad \text{and} \quad \Upsilon_{10\frac{1}{2}}(v_{h\theta}) = b_0 h\partial_\theta\partial_x^k + b_1 h'\partial_\theta\partial_x^{k-1}.$$

According to Lemma 3.3, the map Υ_1 is a non trivial 1-cocycle if and only if at least one of the maps Υ_{010} and $\Upsilon_{01\frac{1}{2}}$ is a non trivial 1-cocycle for $\mathfrak{sl}(2)$, that means $(\alpha_1,\alpha_2,\beta_1,\beta_2) \neq (0,0,0,0)$. Let us determine the linear maps Υ_{100} and $\Upsilon_{10\frac{1}{2}}$. Up to factor, we get:

$$\Upsilon_1 = \alpha_1 \Upsilon_{\frac{1-k}{2},\frac{k}{2}} + \alpha_2 \widetilde{\Upsilon}_{\frac{1-k}{2},\frac{k}{2}} + a_0\delta(2\theta\partial_x^k).$$

Thus, the cohomology classes of $\Upsilon_{\frac{1-k}{2},\frac{k}{2}}$ and $\widetilde{\Upsilon}_{\frac{1-k}{2},\frac{k}{2}}$ generate $H^1_1(\mathfrak{osp}(1|2), \mathfrak{D}_{\frac{1-k}{2},\frac{k}{2}})$. The proof is now complete.

References

[1] I. Basdouri, M. Ben Ammar, N. Ben Fraj, M. Boujelbene and K. Kammoun *Cohomology of the Lie Superalgebra of Contact Vector Fields on $\mathbb{R}^{1|1}$ and Deformations of the Superspace of Symbols*, math.RT/0702645.

[2] Fuchs D B, *Cohomology of infinite-dimensional Lie algebras*, Plenum Publ. New York, 1986.

[3] H. Gargoubi, *Sur la géométrie de l'espace des opérateurs différentiels lineaires sur $\mathbb{R}$*, Bull. Soc. Roy. Sci. Liège. Vol. 69, 1, 2000, 2147.

[4] H. Gargoubi, N. Mellouli and V. Ovsienko *Differential Operators on Supercircle: Conformally Equivariant Quantization and Symbol Calculus*, Letters in Mathematical Physics (2007) **79**: 5165.

[5] P. B. A. Lecomte, *On the cohomology of* $\mathfrak{sl}(n+1;\mathbb{R})$ *acting on differential operators and* $\mathfrak{sl}(n+1;\mathbb{R})$*-equivariant symbols*, Indag. Math. NS. 11 (1), (2000), 95 114.

[6] A. Nijenuis, R. W. Richardson Jr., *Deformations of homomorphisms of Lie groups and Lie algebras*, Bull. Amer. Math. Soc. **73** (1967), 175–179.