

A parameterized version of the Borsuk-Ulam theorem

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Abstract

We show that for a “continuous” family of Borsuk-Ulam situations, parameterized by points of a compact manifold W , its solution set also depends “continuously” on the parameter space W . “Continuity” here means that the solution set supports a homology class which maps onto the fundamental class of W . When $W \subset \mathbb{R}^{m+1}$ we also show how to construct such a “continuous” family starting from a family depending in the same way continuously on the points of ∂W . This solves a problem related to a conjecture which is relevant for the construction of equilibrium strategies in repeated two-player games with incomplete information.

A new method (of independent interest) used in this context is a canonical symmetric squaring construction in Čech homology with $\mathbb{Z}/2$ -coefficients.

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1 Introduction

The classical Borsuk-Ulam theorem states that for every continuous map $f: S^n \rightarrow \mathbb{R}^n$ there exists $v \in S^n$ such that $f(v) = f(-v)$. We think of $\{v \in S^n \mid f(v) = f(-v)\}$ as the set of solutions to the Borsuk-Ulam equation. The original Borsuk-Ulam theorem states that this set is non-empty.

There are many generalizations of the Borsuk-Ulam theorem. In this paper, we generalize to continuous families of Borsuk-Ulam situations and study the

global structure of the solution set (and its dependence on the parameters). At the same time, we generalize to correspondences, i.e. “multi-valued functions”.

More precisely, our “family version” of the Borsuk-Ulam equation starts with a continuous map $F: W \times S^n \rightarrow \mathbb{R}^n$ and asks about properties of the solution set $B := \{(w, v) \in W \times S^n \mid F(w, v) = F(w, -v)\}$. In particular, we establish a suitable kind of continuous dependence of the solution sets $B_w := \{v \in S^n \mid F(w, v) = F(w, -v)\}$ on $w \in W$. Of course, this has to be made precise in a suitable way. Our main result, Theorem 2.4, implies that the solution set B (as subset of $W \times S^n$) supports a (Čech) homology class which is mapped to the fundamental class if W is a compact manifold. If $W = [0, 1]$ this means essentially that there is a path $s: [0, 1] \rightarrow B$ with $s(0) \in \{0\} \times S^n$ and $s(1) \in \{1\} \times S^n$. The homological statement is the correct generalization if W is an arbitrary manifold. [3, Theorem 3.1] is a similar family Borsuk-Ulam theorem to ours, with the precursor [17, Theorem 4.2]. Note that the assumptions in these works are quite different to ours, and so are the methods.

We show then that a special case of a continuous family of Borsuk-Ulam situations arises as follows. Given is an n -dimensional manifold with boundary, embedded into \mathbb{R}^n and on its boundary a continuous function (or correspondence) with values in \mathbb{R}^{n-1} . Each of its interior points can be considered as a “midpoint”, giving rise to a corresponding Borsuk-Ulam equation (that the function assumes equal values on opposite points of the boundary, “opposite” with respect to the interior point). We establish in Theorem 2.8 that this is indeed a continuous family of Borsuk-Ulam situations. As a consequence (Theorem 2.9), the solution set has the (homological) continuity property alluded to above.

This result is strongly reminiscent to a (more complicated) Borsuk-Ulam type statement (Conjecture 5.5) which is part of a program in game theory. This program would establish the existence of equilibria in general repeated two-player games with incomplete information. We describe the game theoretic context in some detail in Section 5.

The proof of our main Theorem 2.4 relies on a new homological construction which we expect to be of independent interest, described in Section 3. It is a functorial symmetric square: a transformation $a \mapsto a^s: H_k(X; \mathbb{Z}/2) \rightarrow H_{2k}(X^s; \mathbb{Z}/2)$, where $X^s = (X \times X/\tau, \Delta)$, $\tau: X \times X \rightarrow X \times X$ interchanges the factors, and Δ is the image of the diagonal. One should think of a^s as one half of the Künneth product $a \times a$ (defined on “half of $X \times X$ ”). The main property we use (and establish) states that if X is a manifold and a its fundamental class, then a^s is the fundamental class of the relative manifold X^s .

The paper is organized as follows: in Section 2 we state our version of the family Borsuk-Ulam theorem and introduce the relevant notation. In Section 3 we discuss the symmetric squaring transformation. In Section 4.1 we prove our Borsuk-Ulam type theorem and in Section 5 we describe the game theoretic background and motivation.

2 A parameterized Borsuk-Ulam theorem

Below, H denotes the Čech homology functor with $\mathbb{Z}/2$ coefficients; see also Section 3.

2.1 Definition. Let X be a compact space and (W, W_0) a compact pair. Later, (W, W_0) will be a compact manifold with boundary.

- (1) We say that a map $f: X \rightarrow W$ is *H-essential for (W, W_0)* if it induces a surjection $H_d(X, f^{-1}(W_0)) \rightarrow H_d(W, W_0)$ for $d = \dim W$, compare to [1].
- (2) Let $p: E \rightarrow W$ be a fiber bundle, e.g. $E = Y \times W$ for some topological space Y . Assume that $X \subset E$. We say that X has *property \mathcal{S} for (W, W_0)* if the projection $p|_X: X \rightarrow W$ is *H-essential for (W, W_0)* . Here, \mathcal{S} stands for “spanning”, compare with [14, 15].
- (3) In case W is a manifold with boundary ∂W (possibly empty) or W is a compact subset of \mathbb{R}^n with topological boundary $\partial W \neq W$ in \mathbb{R}^n , then in the definitions above we say “for W ” in place of “for $(W, \partial W)$ ”.

2.2 Remark. Suppose W is a connected compact manifold with boundary ∂W . The top-degree homology group of $(W, \partial W)$ is generated by a single element, which we call the *fundamental class* of the manifold W and denote $[W]$. Thus f being *H-essential for $(W, \partial W)$* is equivalent to the existence of a homology class $\alpha \in H(X, f^{-1}(\partial W))$ such that $f_*(\alpha) = [W]$. We say that such a class α *witnesses* the *H-essentiality* of the map f . We may also speak of an analogously defined witnessing of the property \mathcal{S} of X for $(W, \partial W)$.

2.3 Definition. Let $S = S^m$ denote the m -sphere. We assume that W is a compact PL-manifold (boundary admitted), and $Z \subset W \times S \times \mathbb{R}^m$ is compact.

We associate to Z its *Borsuk-Ulam correspondence* $B(Z) \subset W \times \mathbb{R}^m$ defined by

$$(w, e) \in B(Z) \iff \exists x \in S : (w, x, e) \in Z \text{ and } (w, -x, e) \in Z.$$

The first of our two results on correspondences asserts the following:

2.4 Theorem. *If Z has property \mathcal{S} for $W \times S$, then $B(Z)$ has it for W .*

2.5 Remark. (1) Note that, if $W = \{pt\}$ and Z is the graph of a continuous function $S \rightarrow \mathbb{R}^m$, then the Borsuk-Ulam theorem states that $B(Z)$ is non-empty, whence the name chosen for the Borsuk-Ulam correspondence.

- (2) For the same reason, Theorem 2.4 is a parameterized Borsuk-Ulam theorem. Indeed, it yields that theorem when $W = \{pt\}$, and, loosely speaking, it asserts in general the following: for a “continuous” family of Borsuk-Ulam situations, parameterized by a manifold W , its solution set depends continuously on the parameters in the sense that it supports a homology class which hits the fundamental class of W (in particular, it surjects onto W).

With pr_W denoting the projection to W we want to describe the construction of the class $\gamma \in H(B(Z), \text{pr}_W^{-1}(\partial W) \cap B(Z))$ satisfying $(\text{pr}_W)_*(\gamma) = [W]$ more precisely (which will of course also be necessary to establish its properties). To do this, we will use a relative squaring construction in Čech homology. We believe that this construction can be useful in other contexts and therefore deserves independent interest; it will be described in the Section 3.

2.6 Remark. The theorem above remains true when the manifold pair $(W, \partial W)$ is replaced by a *relative PL-manifold*, by which we mean a compact pair (W, W_0) such that $W \setminus W_0$ is a PL-manifold. We will skip the argument, which however will be implicit in the proof of the result below. We get notable examples of such relative manifolds by taking W to be a compact n -dimensional subset of \mathbb{R}^n , $n < \infty$, and W_0 to be its topological boundary in \mathbb{R}^n .

In our second theorem we assume that W is a compact subset of \mathbb{R}^{m+1} of codimension zero, of course with a topological boundary in \mathbb{R}^{m+1} , which we denote ∂W . We keep denoting by $S = S^m$ the m -sphere.

2.7 Definition. The *spherical correspondence* Z associated to a compact set $Y \subset \partial W \times E$, where E is an arbitrary space, is defined by

$$Z := \overline{\{(w, x, e) \in W^\circ \times S \times E \mid (w + \lambda x, e) \in Y \text{ for some } \lambda > 0\}}.$$

Here W° is the interior of the manifold W with boundary, or the topological interior of the subset of \mathbb{R}^n , respectively. The over-line stands, as usual, for the closure, i.e. Z is the closure in $W \times S \times E$ of the set of points listed.

If we think of Y as a multi-valued map from ∂W to E , Z is a canonical “extension” to W using the convex structure of \mathbb{R}^{m+1} and keeping track of the directions needed for the extension. It collects all Borsuk-Ulam equations arising if the points in W are treated as “midpoints” of W .

2.8 Theorem. *If $Y \subset \partial W \times E$ has property \mathcal{S} for $(\partial W, \emptyset)$ then the spherical correspondence $Z \subset W \times S \times E$ associated to Y as above has property \mathcal{S} for $W \times S$.*

As a consequence, we immediately obtain from Theorem 2.8 and Theorem 2.4 the following corollary.

2.9 Corollary. *Let $W \subset \mathbb{R}^n$ be a compact n -dimensional embedded manifold with boundary and assume that $Y \subset \partial W \times \mathbb{R}^{n-1}$ has property \mathcal{S} for $(\partial W, \emptyset)$.*

Then $Z := \{(x, v) \in W \times \mathbb{R}^{n-1} \mid \exists x_1, x_2 \in \partial W : x \in [x_1, x_2], (x_1, v), (x_2, v) \in Y\}$ satisfies property \mathcal{S} for $(W, \partial W)$.

For earlier results related to Corollary 2.9 see [6, 10].

3 Čech homology and symmetric homology squaring

Throughout this note, all manifolds are finite-dimensional, possibly with a nonempty boundary, and all spaces encountered will be subspaces of manifolds. The homology groups we are using will exclusively be *Čech homology* groups. Their properties can be found in [5, Chapters IX, X] and in [4, VIII,13]. We list the most important properties (not all of them will be relevant to us):

- (1) Čech homology is defined for compact pairs.
- (2) Čech homology satisfies excision in a very strong form: if $f: (X, A) \rightarrow (Y, B)$ is a map of compact pairs such that $f|: X \setminus A \rightarrow Y \setminus B$ is a homeomorphism, then $f_*: H(X, A) \rightarrow H(Y, B)$ is an isomorphism of Čech homology groups.

- (3) More generally, the *Vietoris theorem* about maps with acyclic fibers holds: if $f: (X, A := f^{-1}(B)) \rightarrow (Y, B)$ is a surjective map of compact pairs such that the reduced Čech homology groups $\check{H}(f^{-1}(y))$ are trivial for all $y \in Y \setminus B$, then $f_*: H(X, A) \rightarrow H(Y, B)$ is an isomorphism of Čech homology groups [16] (using [5, Theorem 5.4]).
- (4) For a Euclidean neighborhood retract (ENR), e.g. for a topological manifold, Čech homology and singular homology are canonically isomorphic.
- (5) Continuity: The Čech homology functor commutes with taking an inverse limit.
- (6) For Čech homology, there is a natural intersection pairing: If (X, A) and (Y, B) are two compact pairs inside an m -dimensional manifold W (possibly with non-empty boundary), then there is an intersection pairing

$$H_p(X, A) \otimes H_q(Y, B) \ni \alpha \otimes \beta \mapsto \alpha \bullet \beta \in H_{p+q-m}((X, A) \cap (Y, B))$$

which is natural for inclusions of pairs. (Here, $(X, A) \cap (Y, B) := (X \cap Y, A \cap Y \cup X \cap B)$.) The reasoning in [4, p. 342] not only shows that the product $\alpha \bullet \beta$ can be computed in any given neighborhood of $X \cap Y$, but actually that it can be defined whenever a neighborhood of $X \cap Y$ is a manifold (even if W isn't).

- (7) For *compact* subsets of manifolds, and with coefficients in a field (e.g. $\mathbb{Z}/2$), Čech homology is a homology theory, in particular with a long exact sequence of a pair. This does not hold in general for \mathbb{Z} -coefficients. The main reason however for which we work with $\mathbb{Z}/2$ -coefficients is to avoid orientability assumptions.

As in [15] these properties allow to define a certain restriction operator, as follows. Suppose (X, A) and (Y, B) are compact pairs in a space Z . If $X \cap (Y \setminus B)$ is a relatively open subset of $X \setminus A$ then we say that (Y, B) is *admissible* for (X, A) . In this case we have canonical homomorphisms

$$H(X, A) \rightarrow H(X, X \setminus (Y \setminus B)) \rightarrow H(X \cap Y, X \cap B)$$

the first of which is induced by the inclusion and the second is the excision of $X \setminus Y$ from X . The arising composition $H(X, A) \rightarrow H(X \cap Y, X \cap B)$ will be called *restriction* and denoted $\alpha \mapsto \alpha|(Y, B)$. Note that then any other compact pair (Y', B') in Z such that $X \cap (Y', B') = X \cap (Y, B)$ is admissible for (X, A) , too, and $\alpha|(Y', B') = \alpha|(Y, B)$.

From now on we assume that H denotes the graded Čech homology functor with $\mathbb{Z}/2$ coefficients. Our aim is to construct a “symmetric squaring”, similar to the squaring $H_k(X, A) \rightarrow H_{2k}(X \times X, A \times X \cup X \times A)$ obtained from the exterior homology product [4], but which takes values in the homology of the (slightly modified) *symmetric* square of the pair (X, A) . We explain this below.

Let $X \supset A$ be compacta. On $X \times X$ we have the coordinate-switching involution τ . We put

$$(X, A)^s = \left(X \times X / \tau, (\Delta \cup (A \times X) \cup (X \times A)) / \tau \right) \text{ and } X^s = (X, \emptyset)^s$$

where $\Delta = \{(x, x) : x \in X\}$ is the diagonal. Clearly, a map $f: (X, A) \rightarrow (Y, B)$ of compact pairs induces a map $(X, A)^s \rightarrow (Y, B)^s$, denoted f^s . (Here and below the superscript s stands for “symmetric square”.)

3.1 Theorem. *For each k there is an assignment*

$$H_k(X, A) \ni \alpha \mapsto \alpha^s \in H_{2k}((X, A)^s)$$

such that

(1) *It is natural: if $f: (X, A) \rightarrow (Y, B)$ is a map of compact pairs then*

$$(f_*(\alpha))^s = f_*^s(\alpha^s) \text{ for } \alpha \in H_k(X, A)$$

(2) *If $(X, A) = (M, \partial M)$ for some compact PL-manifold M and $\alpha = [M]$ is its fundamental class, then α^s is the fundamental class of the relative PL-manifold $(M, \partial M)^s$.*

Before proving this theorem we need also to explain what we mean by a *fundamental class* of a relative PL-manifold. When Y is a genuine connected manifold and $B = \partial Y$ is its boundary (possibly empty), then the fundamental class $[Y] \in H_m(Y, B)$, $m = \dim Y$, has already been defined above. It is unique, because we use $\mathbb{Z}/2$ as the coefficients. In case Y is disconnected but remains an m -manifold we define $[Y] \in H_m(Y, \partial Y)$ as the sum of the images of the classes $[Y_i]$ under the homomorphisms induced by inclusions $(Y_i, \partial Y_i) \hookrightarrow (Y, \partial Y)$, where Y_i runs over the (finitely many) components of Y . It is clear that when M is a codimension 0 submanifold of Y , contained in the interior of Y , then $[M]$ is the restriction of $[Y]$ to $(M, \partial M)$. Thus in the general case of a relative PL-manifold (Y, B) we may exhaust $Y \setminus B$ by compact m -manifolds $M_i, i \in \mathbb{Z}$, each contained in the interior of the next, and define the fundamental class of (Y, B) as $\alpha = \varprojlim \alpha_i \in H_m(Y, B)$, where $\alpha_i \in H_m(Y, Y \setminus M_i^\circ)$ is the image of $[M_i]$ under the map $H_m(M_i, \partial M_i) \rightarrow H_m(Y, Y \setminus M_i^\circ)$ induced by the inclusion $(M_i, \partial M_i) \hookrightarrow (Y, Y \setminus M_i^\circ)$. This fundamental class α is uniquely characterized by the property that $\alpha|(M, \partial M) \neq 0$, for each compact m -manifold $M \subset Y \setminus B$.

A word should be said about the exhaustion of $Y \setminus A$ by the manifolds M_i . In the presence of a PL-structure on $Y \setminus A$ its existence is obvious. However, for $m \geq 6$ it exists also when $Y \setminus A$ is a topological manifold, see [7, p. 108].

Proof of Theorem 3.1. We assume first that X and A are compact polyhedra, in which case Čech and singular homology of (X, A) coincide. Let U be a neighborhood of the diagonal Δ in $X \times X/\tau$ and let

$$(X, A)_U^s = (X, A)^s \cup (X \times X/\tau, U).$$

Let $\sigma = \sigma_U$ be a representative of $\alpha \in H_k(X, A)$ by a singular chain $\sigma = \sum_{i=1}^n \sigma_i$ such that for all i, j the image of $\sigma_i \times \sigma_j$ under the projection $p: X \times X \rightarrow X \times X/\tau$ is either contained entirely in U or does not meet the diagonal Δ at all. Then we define σ^s to be the chain $\sum_{i < j} p_*(\sigma_i \times \sigma_j)$ in the relative chains of $(X, A)_U^s$. Technically speaking, we have to subdivide $\sigma_i \times \sigma_j$ into simplices. This can be done in an arbitrary fashion —since we are dealing with $\mathbb{Z}/2$ -coefficients, not even the orientations play a role. Note that $p_*(\sigma_i \times \sigma_j) = p_*(\sigma_j \times \sigma_i)$ since we quotient out by τ . Thus

$$\partial(\sigma^s) = \sum_{i < j} p_*(\partial\sigma_i \times \sigma_j) + p_*(\sigma_i \times \partial\sigma_j) = \sum_i p_*(\sigma_i \times \sum_{j \neq i} \partial\sigma_j)$$

By the smallness condition imposed on the σ_j 's each $p_*(\sigma_i \times \partial\sigma_i)$ lies in U . Hence $\partial(\sigma^s) = \sum_i p_*(\sigma_i \times \sum_j \partial\sigma_j) = 0$ as a chain in $(X, A)_U^s$, since $\sum_i \sigma_i$ is a cycle. This shows that σ_U^s is a cycle in $(X, A)_U^s$.

One checks immediately that its homology class (actually, even the cycle itself) does not depend on the ordering of the simplices σ_i chosen above.

Similarly, when $\mu = \mu_U$ is a second “ U -small” representative of α then σ_U^s and μ_U^s turn out to represent the same element of $H_{2k}((X, A)_U^s)$. Indeed, we may take a $(k+1)$ -chain $\sum \tau_k$ such that $\mu = \partial(\sum_n \tau_n) + \sigma$ and assume after subdivision that τ satisfy the same “smallness” condition as σ and μ . Clearly it suffices to treat the case where there is just one simplex τ , and to argue by induction. Assuming that in the representation $\mu = \sum_i \sigma_i + \partial\tau$ all the σ_i 's precede the simplices α_k of $\partial\tau$ we get $\mu^s = \sigma^s + p_*(\sigma \times \partial\tau) + (\partial\tau)^s$, where all the summands $p_*(\alpha_k \times \alpha_l)$ of $(\partial\tau)^s$ lie in U . Since moreover $p_*(\sigma \times \partial\tau) = \partial(p_*(\sigma \times \tau))$ (because $\partial\sigma = 0$) it follows that μ^s and σ^s are homologous in $(X, A)_U^s$.

Now, the classes $[\sigma_U]^s$ form an element of the inverse system $\{H_{2k}((X, A)_U^s)\}$, with homomorphisms induced by the inclusions. (The U 's run over all the neighborhoods of the diagonal.) Since $\varprojlim H((X, A)_U^s) = H((X, A)^s)$ by (6), this system uniquely defines an element $\alpha^s \in H_{2k}((X, A)^s)$.

This concludes the construction in case of polyhedral pairs (X, A) and it is clear that in case X is a PL-manifold and α is its fundamental class then α^s is by construction the fundamental class of $(X, \partial X)^s$.

When (X, A) is an arbitrary compact pair, we may represent it as an inverse limit of compact polyhedral pairs (X_i, A_i) . Given $\alpha \in H_k(X, A)$ let us denote its image in $H_k(X_i, A_i)$ by α_i . Then (α_i^s) is an element of $\varprojlim H_{2k}((X_i, A_i)^s)$, whose limit is the class $\alpha^s \in H_{2k}((X, A)^s)$ we are after. By the very construction, naturality is true even on the level of cycles. \square

3.2 Remark. Above, one can replace the field $\mathbb{Z}/2$ by other coefficients, over which suitable manifolds are orientable. More details about “invariant homology squaring” and about the parameterized Borsuk-Ulam theorem can be found in the Göttingen diploma thesis of Denise Krempasky (maiden name Nakiboglu) [8]. Moreover, [8] contains a generalization of the construction to homology with integer coefficients in even dimensions. It is an interesting problem to lift the invariant squaring construction to the generalized homology theories “non-oriented bordism” or even to oriented bordism. This is work in progress [9].

Finally, we need to establish that restriction is compatible with some other operations.

3.3 Lemma. (1) Let (X, A) and (X', A') be compact pairs in spaces Z and Z' , respectively, and let a map $f : Z' \rightarrow Z$ satisfy $f(X') \subset X$ and $f(A') \subset A$. Then for any compact pair (Y, B) in Z , admissible for (X, A) , one has

$$g_*(\alpha)|(Y, B) = h_*(\alpha|(f^{-1}(Y), f^{-1}(B))) \text{ for } \alpha \in H(X', A')$$

where $g : (X', A') \rightarrow (X, A)$ and $h : X' \cap (f^{-1}(Y), f^{-1}(B)) \rightarrow X \cap (Y, B)$ are the restrictions of f .

(2) Let (X, A) and (Y, B) be compact pairs in a space Z . If (Y, B) is admissible for (X, A) , then $(Y, B)^s$ is admissible for $(X, A)^s$ and $(\alpha|(Y, B))^s = \alpha^s|(Y, B)^s$ for $\alpha \in H(X, A)$.

- (3) Let $(X, A), (X', A')$ and (Y, B) be compact pairs in a relative manifold (Z, Z_0) . If $X \cap X' \cap Z_0 = \emptyset$ and (Y, B) is admissible for both pairs (X, A) and $(X_0, A_0) := (X, A) \cap (X', A')$, then $(\alpha' \bullet \alpha)|(Y, B) = \alpha' \bullet (\alpha|(Y, B))$ for $\alpha \in H(X, A)$ and $\alpha' \in H(X', A')$.

Proof. The proofs of the three parts are similar. For (1), the conclusion follows from the definition of the restriction and the commutativity of the diagram induced on the homology level by the following one:

$$\begin{array}{ccccc} (X', A') & \longrightarrow & (X', X' \setminus (f^{-1}(Y) \setminus f^{-1}(B))) & \longleftarrow & X' \cap (f^{-1}(Y), f^{-1}(B)) \\ \downarrow & & \downarrow & & \downarrow \\ (X, A) & \longrightarrow & (X, X \setminus (Y \setminus B)) & \longleftarrow & X \cap (Y, B) \end{array}$$

where horizontal maps are inclusions and the vertical ones are induced by f .

For (2), we consider a similar diagram

$$\begin{array}{ccccc} H(X, A) & \longrightarrow & H(X, X \setminus (Y \setminus B)) & \longleftarrow & H(X \cap (Y, B)) \\ \downarrow & & \downarrow & & \downarrow \\ H((X, A)^s) & \longrightarrow & H((X, X \setminus (Y \setminus B))^s) & \longleftarrow & H((X \cap (Y, B))^s) \end{array}$$

where horizontal arrows are inclusion-induced and vertical ones are given by the assignment $\alpha \mapsto \alpha^s$. By (1) in Theorem 3.1 the diagram is commutative, whence again the conclusion follows from the definition of the restriction.

Finally, (3) follows in the same manner using the commutative diagram

$$\begin{array}{ccccc} H(X, A) & \longrightarrow & H(X, X \setminus (Y \setminus B)) & \xleftarrow{\cong} & H(X \cap (Y, B)) \\ \downarrow & & \downarrow & & \downarrow \\ H(X_0, A_0) & \longrightarrow & H(X_0, X_0 \setminus (Y \setminus B)) & \xleftarrow{\cong} & H(X_0 \cap (Y, B)) \end{array}$$

where the vertical arrows are \bullet -multiplications by a fixed $\alpha' \in H(X', A')$. \square

4 Proof of the main results

4.1 Proof of Theorem 2.4; the naturality of the witnessing class.

We use the notation of Section 2 concerning Theorem 2.4 and denote \mathbb{R}^m by E . We consider E as being embedded in the sphere $\tilde{E} = E \cup \{\infty\}$. Hence $Z \subset W \times S \times \tilde{E}$ and $Z^s \subset (W \times S \times \tilde{E})^s$. The last space also contains the ‘‘anti-diagonal’’ $A = \{[(w, x, e), (w, -x, e)] : w \in W, x \in S, e \in \tilde{E}\}$, with well-defined projections $A \rightarrow W \times \tilde{E}$ and $p_W: A \rightarrow W$. Note that there is an obvious mapping $A \cap Z^s \rightarrow B(Z)$ which preserves the projections to W . (Here and below we identify the pair $A \cap Z^s = (A \cap (Z \times Z/\tau), \emptyset)$ with the space $A \cap (Z \times Z/\tau)$.) Hence it suffices to show that the projection $p_W|_{A \cap Z^s}$ is H -essential, i.e. sends some homology class $\beta \in H_k((A, p_W^{-1}(\partial W)) \cap Z^s)$ to $[W]$.

To this end let $k = \dim W$ and $m = \dim S = \dim E$ and choose $\alpha \in H_{k+m}(Z, (\partial W \times S \times \tilde{E}) \cap Z)$ which projects to $[W \times S] \in H_{k+m}((W, \partial W) \times S)$.

Note that A is over $W \times \tilde{E}$ homeomorphic to the $(2m+k)$ -manifold $W \times P \times \tilde{E}$, where P is the real projective m -space, and $(A, \partial A) \subset ((W, \partial W) \times S \times \tilde{E})^s$. We claim that one can take $\beta := [A] \bullet \alpha^s$, where $[A] \in H_{2m+k}(A, \partial A)$ is a fundamental class and \bullet is taken in the relative $2(2m+k)$ -manifold $(W \times S \times \tilde{E})^s$. Note that the product $[A] \bullet \alpha^s$ makes sense, because the anti-diagonal A misses the singular set of this relative manifold.

Indeed, Z is a compact set in $W \times S \times E$ and, by its choice, α is equal to $[W \times S \times \{0\}]$ in $H_{k+m}((W, \partial W) \times S \times \tilde{E})$, by which we mean that the images of these homology classes under the inclusion-induced homomorphisms coincide. Similarly, $[W \times S \times \{0\}]$ is in $H_{k+m}((W, \partial W) \times S \times \tilde{E})$ equal to the fundamental class of the manifold $M := W \times \{(x, u(x))\}_{x \in S}$, where for $u: S \subset \mathbb{R}^{m+1} \rightarrow E = \mathbb{R}^m$ we choose the projection onto the first m coordinates. (The advantage of this u will be seen below.) Thus $\alpha^s = [M]^s$ in $H_{2(k+m)}(((W, \partial W) \times S \times \tilde{E})^s)$ and $[A] \bullet \alpha^s = [A] \bullet [M]^s$ in $H_j(A, \partial A)$, with $j = 2(k+m) + (2m+k) - 2(2m+k) = k$. But, by Theorem 3.1, $[M]^s$ is the fundamental class of the relative manifold $(M, \partial M)^s$. The latter intersects $(A, \partial A)$ transversally along $(N, \partial N)$, where the manifold $N := \{(w, e_{m+1}, 0), (w, -e_{m+1}, 0)\}: w \in W\}$ is disjoint from the singular set Δ of $M \times M/\tau$. Hence $\beta = [A] \bullet [M]^s = [N]$ in $H_k(A, \partial A)$ and $(p_W)_*(\beta) = (p_W)_*([N])$ in $H_k(W, \partial W)$. Our claim follows, for p_W maps N homeomorphically onto W . \square

4.1 Remark. In fact we established the following: if α witnesses property \mathcal{S} of Z for $W \times S$, then an explicit class γ_α witnessing property \mathcal{S} of $B(Z)$ for W may be defined by $\gamma_\alpha = p_*(\beta) = p_*([A] \bullet \alpha^s)$, where $p: (A, p_W^{-1}(\partial W)) \cap Z^s \rightarrow (B(Z), (\partial W \times E) \cap B(Z))$ is induced by the projection $q: A \rightarrow W \times \tilde{E}$.

Below we show that the above dependence between the witnesses γ_α and α satisfies a certain naturality property.

4.2 Proposition. *If V is a top-dimensional compact submanifold of W then the class $\gamma_{\alpha|(V, \partial V) \times S \times \tilde{E}}$ is the restriction of γ_α to $(V, \partial V) \times \tilde{E}$.*

Proof. Implicit in the formulation above is that $\alpha|(V, \partial V) \times S \times \tilde{E}$ is a witness of the property \mathcal{S} of $Z \cap p_W^{-1}(V)$ for V ; compare to [15, Lemma 2a)]. The latter follows also from part (1) of Lemma 3.3, for $[V]$ is the restriction of $[W]$ to $(V, \partial V)$.

For the same reason, to prove the desired equality it suffices to show that

$$([A] \bullet \alpha^s)|(A_V, \partial A_V) = [A_V] \bullet (\alpha|(V, \partial V) \times S \times \tilde{E})^s$$

where we write A_V for $q^{-1}(V \times \tilde{E})$.

But as noted after the definition of the restriction in Section 2, the pair $(A_V, \partial A_V)$ may be replaced by $((V, \partial V) \times S \times \tilde{E})^s$. By parts (2) and (3) of Lemma 3.3 the left hand side is thus equal to $([A]|(A_V, \partial A_V)) \bullet (\alpha|(V, \partial V) \times S \times \tilde{E})^s$. The result follows, for $[A]|(A_V, \partial A_V) = [A_V]$. \square

4.2 Transport of the spanning property: proof of Theorem 2.8

We now use the notation of Theorem 2.8. For the proof of this theorem we consider a number of intermediate correspondences. First, by the Künneth

formula, it follows that the correspondence $Y \times W \subset \partial W \times E \times W$ has property \mathcal{S} for $\partial W \times W$.

Next, we define the correspondence $\tilde{Z} \subset W \times S \times Y$ by

$$\tilde{Z} := \overline{\{(v, x, (w, e)) \in W \times S \times Y : v \neq w, x = (w - v)/|w - v|\}}.$$

and set $Z_0 := \tilde{Z} \cap (\partial W \times S \times Y)$. The map $\tilde{Z} \rightarrow W \times Y$, induced by the projection, restricts to a homeomorphism $\tilde{Z} \setminus Z_0 \rightarrow (W \setminus \partial W) \times Y$ and therefore induces a Čech homology isomorphism

$$H(\tilde{Z}, Z_0) \rightarrow H((W, \partial W) \times Y).$$

Define $V \subset W \times S \times \partial W$ as

$$V := \overline{\{(v, x, w) \in W \times S \times \partial W : v \neq w, x = (w - v)/|w - v|\}}$$

and set $V_0 := V \cap (\partial W \times S \times \partial W)$. As above, the projection

$$p: V \rightarrow W \times \partial W$$

restricts to a homeomorphism $V \setminus V_0 \rightarrow (W \setminus \partial W) \times \partial W$ and therefore induces a Čech homology isomorphism $H(V, V_0) \rightarrow H((W, \partial W) \times \partial W)$.

Now consider the commutative diagram

$$\begin{array}{ccc} (W, \partial W) \times Y & \longrightarrow & (W, \partial W) \times \partial W \\ \uparrow & & \uparrow p \\ (\tilde{Z}, Z_0) & \longrightarrow & (V, V_0) \\ \downarrow & & \downarrow q \\ (Z, \partial Z) & \longrightarrow & (W, \partial W) \times S. \end{array}$$

where the lower vertical maps are induced by the projection along ∂W . We already know that the upper vertical maps induce Čech homology isomorphisms and that the upper horizontal map induces an epimorphism on H_{2m+1} . Hence so does the middle horizontal map and to prove this for the lowest one it remains to show the following

Claim. *With the above notation, $H_{2m+1}(q)$ is an epimorphism.*

Proof. Without loss of generality we can assume that W° is connected.

Fix v in the interior of W and let D_v be a small disc centered at v contained in the interior of W , and with boundary the sphere S_v . Consider the commutative diagram

$$\begin{array}{ccc} H_{2m+1}((W, \partial W) \times \partial W) & \longrightarrow & H_{2m+1}((D_v, S_v) \times \partial W) \\ \uparrow & & \uparrow \\ H_{2m+1}(V, V_0) & \longrightarrow & H_{2m+1}(p^{-1}((D_v, S_v) \times \partial W)) \\ \downarrow q_* & & \downarrow f_* \\ H_{2m+1}((W, \partial W) \times S) & \longrightarrow & H_{2m+1}((D_v, S_v) \times S) \end{array}$$

where the horizontal homomorphisms are restrictions, the upper vertical ones are induced by p , and f is the restriction of q . The upper vertical arrows are isomorphisms. So are the horizontal ones, the reason for the upper and lower ones being that restriction takes the generator $[W]$ of $H_{m+1}(W, \partial W)$ to a generator of $H_{m+1}(D_v, S_v)$.

Therefore, to prove the claim it suffices to show that the map

$$g: (D_v, S_v) \times \partial W \rightarrow (D_v, S_v) \times S; (u, w) \mapsto (u, \frac{w - u}{|w - u|})$$

induces an epimorphism on H_{2m+1} . (We are using the fact that f is the composition of g with p restricted to the set $p^{-1}(D_v \times \partial W)$, on which it is a homeomorphism.) However, f is homotopic to the map

$$g': (D_v, S_v) \times \partial W \rightarrow (D_v, S_v) \times S; (u, w) \mapsto (u, \frac{w - v}{|w - v|}).$$

By Borsuk's separation criterion [5, p.302] the map $\partial W \rightarrow S, w \mapsto \frac{w-v}{|w-v|}$, induces an epimorphism in Čech homology H_m . Thus, by the Künneth formula, $H_{2m+1}(g')$ is an epimorphism. Hence the same follows for $H_{2m+1}(g)$, as desired. \square

5 Relations to game theory

The inspiration for this paper came originally from games of incomplete information. Infinitely repeated two-person, non-zero-sum games of incomplete information on one side were introduced by Aumann, Maschler, Stearns [2]. There is a finite set K of states of nature and two players. Nature chooses a state $k \in K$ according to a commonly known probability distribution p_0 on K . The first player, but not the second player, is informed of nature's choice. The finite sets of moves for the players are the same for all states. The chosen state remains constant throughout the play. Although the chosen state k , along with the moves of the players, determines the stage payoffs, during the play the second player learns nothing about his payoff, as this could give him information about the state of nature.

The first player and second player have the finite sets I and J of moves, respectively.

For any finite set S let $\Delta(S)$ be the probability simplex $\{\lambda \in [0, 1]^S \mid \sum_{s \in S} \lambda^s = 1\}$.

We assume that there are two sets of finite signals R and S , received by the first and second players, respectively, and a stochastic signaling function, $\Lambda: K \times I \times J \rightarrow \Delta(R \times S)$. After each stage in which $i \in I$ and $j \in J$ were played, and k is the state of nature, a member of R and a member of S is determined by $\Lambda(k, i, j)$ and communicated to Player One and Player Two, respectively. The only knowledge the players have of the moves of their opponents is through their observations of the sets R and S , and the only knowledge Player Two has of the state of nature is from the initial probability p_0 and the received sequence of signals in S .

An equilibrium of the game is a pair of strategies such that for every state k there are limits a^k and b^k (as the number n of stages goes to infinity) for the

accumulated averages up to the stage n of the expected payoffs of Players One and Two, respectively, and neither player can obtain a higher limit superior of this same average payoff (for Player Two as weighted by the initial probability distribution on K) by choosing a different strategy.

5.1 Conjecture. *For all signalling functions Λ the corresponding infinitely repeated game of incomplete information on one side has an equilibrium.*

Obviously, this is the principal conjecture in this field. In full generality, it is still open. Special cases of increasing generality have been established by Simon, Spieź, and Toruńczyk [14], Renault [11] and again Simon, Spieź, and Toruńczyk [15]. By a suitable construction, to the most general game considered in the present paper one can assign a game as treated by [15] with an essential preservation of the payoff structure. Therefore, for those only interested in economic applications the full generality of Conjecture 5.1 will be unnecessary.

How do equilibria in the non-zero-sum games relate to topological theorems resembling the Borsuk-Ulam Theorem?

The general approach is the construction of rather simple strategies, depending on a small set of parameters. Then, conditions for such a strategy to be an equilibrium are derived. Finally, one has to show that among the strategies considered there is at least one which satisfies these conditions. Such constructions have already been considered in [2, 11, 13, 14, 15].

Given that the two players know each other's strategies (an assumption that can be made of an equilibrium), the critical information used by the second player in the special strategies considered is the conditional probability on the set K of states that is induced by the long term observable behavior of the first player. This conditional probability defines a multi-dimensional martingale (as many dimensions as are states in K) starting at the initial probability p_0 . We restrict ourselves to strategies where this martingale is controlled entirely by the first player and that it converges after only a few initial stages to some point in a finite subset $V \subset \Delta(K)$ of probabilities for the state of nature. Two conditions together imply that the (constructed) strategy is an equilibrium. The first, "individual rationality" for each $v \in V$, is classically under control [2] and we don't discuss it here.

The second condition is called "incentive compatibility". Incentive compatibility requires that the first player has no incentive to infer through her behavior one of the $v \in V$ over another member of V . If the vector $y \in \mathbb{R}^k$ represents the expected payoff to the first player starting from the initial probability p_0 and y_v the expected payoff to the first player after v is chosen, this means that $y_v^k = y^k$ if $v^k > 0$ (with some modification if $v^k = 0$ which leads to the introduction of slack variables and the requirement of saturation below). Hence we are describing a set V whose convex hull includes the initial probability p_0 with corresponding vectors $(y_v \mid v \in V)$ that are all equal. With some form of continuity of these payoff vectors we recognize a relation to the Borsuk-Ulam Theorem. This relation becomes stronger if there are restrictions on how we can convexify members of V , for example that the initial p_0 must be in the convex combination of at most two members of V .

Continuing with the game context, a family \mathcal{L} of subsets of K is introduced. A member L of \mathcal{L} is a set of states for which the first player can confirm that the chosen state belongs to L without revealing anything more about the chosen

state. A choice of an L in \mathcal{L} (which must contain the chosen state) by the first player determines a kind of sub-game, which motivates the mathematics described below.

Without going into further detail, we present two statements (of Borsuk-Ulam type): Assertion 5.3 and Conjecture 5.5. Together, they imply Conjecture 5.1, as asserted in [13, p. 40], compare [12]. This follows rather easily using the methods of [15]. To state them, we first make a definition.

5.2 Definition. (1) If A is a subset of an affine space, $\text{co}(A)$ stands for its convex hull.

(2) If $F \subset X \times Y$ and $y \in Y$, then $F^{-1}(y) := \{x \in X \mid (x, y) \in F\}$.

(3) If $F \subset \Delta(L) \times Y$ we define $cF \subset \Delta(L) \times Y$ by $cF := \{(x, y) \mid x \in \text{co}(F^{-1}(y))\}$.

(4) If $F \subset \Delta(L) \times Y$ and $Y \subset \mathbb{R}^L$ then is $F^+ \subset \Delta(L) \times Y$ defined by

$$F^+ := \{(p, y) \mid \exists (p, x) \in F : x^l \leq y^l \forall l \in L \text{ and } x^l = y^l \text{ if } p^l > 0\}.$$

We call F^+ the Y -saturation of F . If $F = F^+$ then F is called saturated.

(5) A constant correspondence $\Delta(K) \ni p \mapsto U$ is denoted U^{cst} (that is, $U^{cst} := \Delta(K) \times U$).

In both the Assertion 5.3 and Conjecture 5.5 below we assume that $I = [a, b] \subset \mathbb{R}$ is a non-trivial closed segment, K is a finite set and \mathcal{L} is a family of non-empty proper subsets of K which covers K . Each simplex $\Delta(L)$, $L \in \mathcal{L}$, is considered as a subset of $\Delta(K)$.

5.3 Assertion. Suppose for every $L \in \mathcal{L} \cup \{K\}$ there is given a saturated correspondence $F_L \subset \Delta(L) \times I^L$ with property \mathcal{S} for $\Delta(L)$ and a closed convex set $U_L \subset I^K$ containing the point $v_+ = (b, \dots, b)$. Set

$$\widetilde{F}_L = F_L \times I^{K \setminus L} \subset \Delta(K) \times I^K \text{ for } L \in \mathcal{L} \quad (5.4)$$

Also let $U := \bigcap_{L \in \mathcal{L} \cup \{K\}} U_L$, $F := U^{cst} \cap F_K$ and $G := U^{cst} \cap \bigcup_{L \in \mathcal{L}} \widetilde{F}_L$, and define a correspondence $\Gamma \subset \Delta(K) \times I^K$ so that for $y \in I^K$

$$\Gamma^{-1}(y) = \text{co}(G^{-1}(y)) \cup \bigcup_{x \in F^{-1}(y)} \text{co}(\{x\} \cup G^{-1}(y))$$

The assertion is that if $\widetilde{F}_L \subset U_L^{cst}$ for each $L \in \mathcal{L} \cup \{K\}$, then Γ has property \mathcal{S} for $\Delta(K)$.

The key property in the definition of Γ is that for every $y \in \Delta(K)$ we convexify sets in $F^{-1}(y) \cup G^{-1}(y)$ that have at most one member x whose projection to $\Delta(K)$ belongs to $\Delta(K) \setminus \bigcup_{L \in \mathcal{L}} \Delta(L)$.

5.5 Conjecture. This time suppose the saturated correspondences $F_L \subset \Delta(L) \times I^L$ with property \mathcal{S} for $\Delta(L)$ are given merely for those sets $L \in \mathcal{L}$ which are maximal with respect to inclusion. For all $L \in \mathcal{L}$ let

$$F_L := \bigcup_{L \subset J \in \max \mathcal{L}} F_J \cap (\Delta(L) \times I^L) \subset \Delta(L) \times I^L$$

and define \tilde{F}_L as in (5.4) and $\Gamma \subset \Delta(K) \times I^K$ so that for $y \in I^K$

$$\Gamma^{-1}(y) := \bigcup \{ \text{co}(x_{L_1}, \dots, x_{L_m}) \mid x_{L_i} \in \tilde{F}_{L_i}^{-1}(y), L_i = L_j \in \max \mathcal{L} \Rightarrow i = j \}.$$

The conjecture is that if $L_1 \cap L_2 \in \mathcal{L}$ for all $L_1, L_2 \in \mathcal{L}$, then Γ has property \mathcal{S} for $\Delta(K)$.

Because—in a way similar to Assertion 5.3—in Conjecture 5.5 we have a restriction on how we convexify sets to obtain Γ , its character is close to the original Borsuk-Ulam Theorem (where one convexifies using opposite points). It suggests the statement of Corollary 2.9, proved above. Indeed, this relation to Conjecture 5.5 was one of the main motivations for the work presented in this paper.

We believe that we can prove Assertion 5.3. We plan to publish this in a forthcoming paper. Conjecture 5.5 is still open. Provided we can establish it, as well, we will eventually put all the details together and present a proof of the game theoretic Conjecture 5.1.

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