Spacelike Willmore surfaces in 4-dimensional Lorentzian space forms

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Abstract

Spacelike Willmore surfaces in 4-dimensional Lorentzian space forms, a topic in Lorentzian conformal geometry which parallels the theory of Willmore surfaces in S^4, are studied in this paper. We define two kinds of transforms for such a surface, which produce the so-called left/right polar surfaces and the adjoint surfaces. These new surfaces are again conformal Willmore surfaces. For them holds interesting duality theorem. As an application spacelike Willmore 2-spheres are classified. Finally we construct a family of homogeneous spacelike Willmore tori.

Keywords: Spacelike Willmore surfaces; adjoint transforms; polar surfaces; duality theorem

1 Introduction

Willmore surfaces are the critical surfaces with respect to the conformally invariant Willmore functional. Many interesting results related to them have been obtained (see [2,4,10,16]), and now they are recognized as one of the most important surface classes in Möbius geometry.

For Lorentzian space forms there is also a parallel theory of conformal geometry. Thus it is natural to generalize the notion of Willmore surfaces to such a context. This idea was first followed by Alias and Palmer in [1]. They considered the codim-1 case and established such a theory as Bryant did in [2]: the conformal Gauss map was introduced; the Willmore functional was defined as the area with respect to the metric induced from this map; a surface is Willmore if, and only if, its conformal Gauss map is harmonic. Later Deng and Wang [8] treated timelike Willmore surfaces in Lorentzian 3-space; Nie [17] established a theory of conformal geometry about hypersurfaces in Lorentzian space forms and computed the first variation of Willmore functional.

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In this paper we take the next step to study spacelike Willmore surfaces in Q^4_1, the conformal compactification of the 4-dimensional Lorentzian space forms R^4_1, S^4_1 and H^4_1. In many aspects the theory is almost the same as in Möbius geometry, except that we have a distinctive construction as below.

For a spacelike surface $[Y]$ immersed into Q^4_1, the normal plane is Lorentzian at each point. The null lines $[L], [R]$ in this plane define two conformal maps into Q^4_1, called *the left and the right polar surface*, while these transforms are called $(-)$transform and $(+)$transform, respectively. Conversely, Y is also the right polar surface of $[L]$, and the left polar surface of $[R]$ (when $[L]$ and $[R]$ are immersions). That means $(-)$transform and $(+)$transform are mutual inverses to each other (this is true even without the Willmore condition). Applying these transforms successively, we obtain a sequence of conformal surfaces as described by the following diagram:

$$\begin{tikzcd}
 & {[\hat{Y}]} \arrow[dl, "-"'] \arrow[dr, bend left, "+"] & & {[Y]} \arrow[dl, bend left, "-"] \arrow[dr, bend left, "+"] & & {[\tilde{Y}]} \arrow[dl, bend left, "-"] \arrow[dr, "+"] & \\
\cdots & & {[L]} \arrow[ul, bend left, "-"] \arrow[ur, bend left, "+"] & & {[R]} \arrow[ul, bend left, "-"] \arrow[ur, bend left, "+"] & & \cdots
\end{tikzcd}$$

Our main result says that they are all Willmore surfaces if $[Y]$ is assumed to be so.

It is interesting to notice that the two-step transforms $[\hat{Y}]$ and $[\tilde{Y}]$ are located on *the central sphere* of the original Willmore surface $[Y]$ at corresponding point, which mimics the property of the adjoint transforms in S^n as introduced by the first author [14] (indeed they could be introduced in the same manner). In the special case that $[\hat{Y}] = [\tilde{Y}]$, this yields a Willmore surface sharing the same central sphere congruence as $[Y]$. It generalizes the duality theorem of Bryant [2] and Ejiri [10], and such surfaces will still be called *S-Willmore* surfaces as in [10,13,14]. In particular, there is a surprising analogy between our transforms and the so-called *forward and backward Bäcklund transforms* defined by Burstall et al. for Willmore surfaces in S^4 [4].

When the underlying surface M is compact, an important problem is to classify all Willmore immersions of M and to find the values of their Willmore functionals (i.e. to determine the critical values and critical points of the Willmore functional). For Willmore 2-spheres in S^3 and S^4 this question was perfectly answered by Bryant [2] and Montiel [14], respectively. Precisely speaking, any Willmore 2-spheres in S^4 is the conformal compactification of a complete minimal surface in R^4, or the twistor projection of a complex curve in the twistor space $\mathbb{C}P^3$. This follows from the duality theorem and the vanishing theorem about holomorphic forms on S^2. By the same method we could obtain similar characterization result in the Lorentzian space.

Theorem. *Any spacelike Willmore 2-sphere in Q^4_1 is either the conformal compactification of a complete spacelike stationary surface (i.e. H=0) in R^4_1, or a polar surface of such a surface (in the latter case the surface is the twistor projection of a holomorphic curve in the twistor space of Q^4_1). For a surface of the second type, its Willmore functional always equals zero.*

This paper is organized as follows. In Section 2, we describe the Lorentzian conformal space Q^4_1 as well as round 2-spheres in it. The general theory about spacelike surfaces and the characterization of Willmore surfaces are given in Section 3 and Section 4. Then we study the transforms of spacelike Willmore surfaces in Section 5. These transforms are utilized to classify spacelike Willmore 2-spheres in Section 6. Finally we discuss some special examples in Section 7 and construct a family of homogeneous spacelike Willmore tori which are not S-Willmore.

In the sequel $y : M \to Q^4_1$ will always denote a smooth spacelike immersion from an oriented surface M unless it is explicitly claimed otherwise.

2 Lorentzian conformal geometry of Q^4_1

Let $\mathbb{R}^n_s$ be the space $\mathbb{R}^n$ equipped with the quadric form

$$\langle x, x\rangle = \sum_{1}^{n-s} x_i^2 - \sum_{n-s+1}^{n} x_i^2.$$

In this paper we will mainly work with $\mathbb{R}^6_2$ whose light cone is denoted as C^5. The quadric

$$Q^4_1 = \{\ [x] \in \mathbb{R}P^5 \mid x \in C^5 \setminus \{0\}\}$$

is exactly the projectived light cone. The standard projection $\pi : C^5\setminus\{0\} \to Q^4_1$ is a fiber bundle with fiber $\mathbb{R} \setminus \{0\}$. It is easy to see that Q^4_1 is equipped with a Lorentzian metric induced from projection $S^3 \times S^1 \to Q^4_1$. Here

$$S^3 \times S^1 = \{x \in \mathbb{R}^6_2 \mid \sum_{i=1}^{4} x_i^2 = x_5^2 + x_6^2 = 1\} \subset C^5 \setminus \{0\} \tag{1}$$

is endowed with the Lorentzian metric $g(S^3) \oplus (-g(S^1))$, where $g(S^3)$ and $g(S^1)$ are standard metrics on S^3 and S^1. So there is a conformal structure of Lorentzian metric $[h]$ on Q^4_1. By a theorem of Cahen and Kerbrat [6], we know that the conformal group of $(Q^4_1, [h])$ is exactly the orthogonal group $O(4, 2)/\{\pm 1\}$, which keeps the inner product of $\mathbb{R}^6_2$ invariant and acts on Q^4_1 by

$$T([x]) = [xT],\ T \in O(4, 2). \tag{2}$$

As in the Riemannian case, there are three 4-dimensional Lorentzian space forms, each with constant sectional curvature $c = 0, +1, -1$, respectively. They are defined by

$$\begin{aligned}
&R^4_1,\ c = 0;\\
&S^4_1 := \{x \in \mathbb{R}^5_1 \mid \langle x, x\rangle = 1\}, c = 1;\\
&H^4_1 := \{x \in \mathbb{R}^5_2 \mid \langle x, x\rangle = -1\}, c = -1.
\end{aligned}$$

Each of them could be embedded as a proper subset of Q_1^4:

$$\begin{array}{ll} \varphi_0 : R_1^4 \to Q_1^4, & \varphi_0(x) = [(\frac{-1+\langle x,x\rangle}{2}, x, \frac{1+\langle x,x\rangle}{2})]; \\ \varphi_+ : S_1^4 \to Q_1^4, & \varphi_+(x) = [(x,1)]; \\ \varphi_- : H_1^4 \to Q_1^4, & \varphi_-(x) = [(1,x)]. \end{array} \tag{3}$$

It is easy to verify that these maps are conformal embeddings. In particular, the Lorentzian conformal space Q_1^4 could be viewed as the conformal compactification of R_1^4 by attaching the light-cone at infinity to it, i.e.

$$Q_1^4 = \varphi_0(R_1^4) \cup C_\infty,$$

where $C_\infty = \{(a,u,a) \in \mathbb{R}P^5 \mid \langle u,u\rangle = 0, a \in \mathbb{R}\}$. Thus Q_1^4 is the proper space to study the conformal geometry of these Lorentzian space forms.

We note that the description above is valid in n-dimensional space. The whole theory parallels Möbius geometry, and Lorentzian space forms are viewed as conic sections of Q_1^n.

Lorentzian conformal geometry is also analogous to Möbius geometry in that we have round spheres as the most important conformally invariant objects. For our purpose here we only discuss *round 2-spheres* (they were named *conformal 2-spheres* in [1]). Each of them could be identified with a 4-dim Lorentzian subspace in $\mathbb{R}_2^6$. Given such a 4-space V, the round 2-sphere is given by

$$S^2(V) := \{[v] \in Q_1^4 \mid v \in V\}.$$

Such spheres share the same properties as the round 2-spheres in Möbius geometry: they are not only topological 2-spheres, but also geodesic 2-spheres when viewed as subsets of some Lorentzian space form; they are totally umbilic spacelike surfaces. In our terms the moduli space Σ of all round 2-spheres in Q_1^4 can be identified with the Grassmannian manifold

$$G_{3,1}(\mathbb{R}_2^6) := \{\text{4-dim Lorentzian subspaces of } \mathbb{R}_2^6\}.$$

3 Basic equations for a surface in Q_1^4

For a surface $y : M \to Q_1^4$ and any open subset $U \subset M$, a local lift of y is just a map $Y : U \to C^5 \setminus \{0\}$ such that $\pi \circ Y = y$. Two different local lifts differ by a scaling, so the metric induced from them are conformal to each other.

Let M be a Riemann surface. An immersion $y : M \to Q_1^4$ is called a conformal spacelike surface, if $\langle Y_z, Y_z\rangle = 0$ and $\langle Y_z, Y_{\bar z}\rangle > 0$ for any local lift Y and any complex coordinate z on M. (Here $Y_z = \frac{1}{2}(Y_u - iY_v)$ is the complex tangent vector for $z = u+iv$, and $Y_{\bar z}$ its complex conjugate.) For such a surface there is a decomposition $M \times \mathbb{R}_2^6 = V \oplus V^\perp$, where

$$V = \mathrm{Span}\{Y, \mathrm{d}Y, Y_{z\bar z}\} \tag{4}$$

is a Lorentzian rank-4 subbundle independent to the choice of Y and z. The orthogonal complement $V^\perp$ is also a Lorentzian subbundle, which might be identified with the normal bundle of y in Q^4_1. Their complexifications are denoted separately as $V_{\mathbb{C}}$ and $V_{\mathbb{C}}^{\perp}$.

Fix a local coordinate z. There is a local lift Y satisfying $|\mathrm{d}Y|^2=|\mathrm{d}z|^2$, called the canonical lift (with respect to z). Choose a frame $\{Y,Y_z,Y_{\bar z},N\}$ of $V_{\mathbb{C}}$, where $N\in\Gamma(V)$ is uniquely determined by

$$\langle N,Y_z\rangle=\langle N,Y_{\bar z}\rangle=\langle N,N\rangle=0,\langle N,Y\rangle=-1. \tag{5}$$

For $V^{\perp}$ which is a Lorentzian plane at every point of M, a natural frame is $\{L,R\}$ such that

$$\langle L,L\rangle=\langle R,R\rangle=0,\langle L,R\rangle=-1. \tag{6}$$

So L and R span the two null lines in $V^{\perp}$ separately. They are determined up to a real factor around each point.

Given frames as above, it is straightforward to write down the structure equations of Y. First note that Y_{zz} is orthogonal to Y, Y_z and $Y_{\bar z}$. So there must be a complex function s and a section $\kappa\in\Gamma(V_{\mathbb{C}}^{\perp})$ such that

$$Y_{zz}=-\frac{s}{2}Y+\kappa. \tag{7}$$

This defines two basic invariants κ and s depending on coordinates z. Similar to the case in Möbius geometry, κ and s are interpreted as *the conformal Hopf differential* and *the Schwarzian* of y, separately (see [5][14]). Decompose κ as

$$\kappa=\lambda_1L+\lambda_2R. \tag{8}$$

Let D denote the normal connection, i.e. the connection in the bundle $V^{\perp}$. We have

$$D_zL=\alpha L,\quad D_zR=-\alpha R$$

for the connection 1-form $\alpha\mathrm{d}z$. Denote

$$\langle\kappa,\bar\kappa\rangle=-\beta,\quad D_{\bar z}\kappa=\gamma_1L+\gamma_2R, \tag{9}$$

where

$$\begin{cases}\beta=\lambda_1\bar\lambda_2+\lambda_2\bar\lambda_1,\\ \gamma_1=\lambda_{1\bar z}+\lambda_1\bar\alpha,\\ \gamma_2=\lambda_{2\bar z}-\lambda_2\bar\alpha.\end{cases} \tag{10}$$

The structure equations are given as follows:

$$\begin{cases}Y_{zz}=-\frac{s}{2}Y+\lambda_1L+\lambda_2R,\\ Y_{z\bar z}=\beta Y+\frac12 N,\\ N_z=2\beta Y_z-sY_{\bar z}+2\gamma_1L+2\gamma_2R,\\ L_z=\alpha L-2\gamma_2Y+2\lambda_2Y_{\bar z},\\ R_z=-\alpha R-2\gamma_1Y+2\lambda_1Y_{\bar z},\end{cases} \tag{11}$$

The conformal Gauss, Codazzi and Ricci equations as integrable conditions are:

$$\begin{cases}s_{\bar z}=-2\beta_z-4\lambda_1\bar\gamma_2-4\lambda_2\bar\gamma_1,\\ \mathrm{Im}(\gamma_{1\bar z}+\gamma_1\bar\alpha+\frac{\bar s}{2}\lambda_1)=0,\\ \mathrm{Im}(\gamma_{2\bar z}-\gamma_2\bar\alpha+\frac{\bar s}{2}\lambda_2)=0,\\ D_{\bar z}D_zL-D_zD_{\bar z}L=2(\lambda_2\bar\lambda_1-\bar\lambda_2\lambda_1)L,\\ D_{\bar z}D_zR-D_zD_{\bar z}R=-2(\lambda_2\bar\lambda_1-\bar\lambda_2\lambda_1)R.\end{cases} \tag{12}$$

These are quite similar to the theory in [5]. In particular, the second and the third equation above could be combined and written as a single conformal Codazzi equation:

$$\mathrm{Im}(D_{\bar z}D_{\bar z}\kappa + \frac{\bar s}{2}\kappa) = 0. \tag{13}$$

Remark 3.1. Another important fact we will need later is that $\kappa\ (\mathrm{d}z)^{\frac{3}{2}}(\mathrm{d}\bar z)^{-\frac{1}{2}}$ is a globally defined vector-valued complex differential form.

4 Willmore functional and Willmore surfaces

Definition 4.1. *For a conformal spacelike surface $y : M \to Q_1^4$, the 4-dim Lorentzian subspace*

$$V = \mathrm{Span}\{Y, \mathrm{d}Y, Y_{z\bar z}\}$$

at one point $p \in M$ is identified with a round 2-sphere $S^2(V)$ in Q_1^4 as in Section 2. We call it the central sphere *of the surface y at p.*

The notion of central spheres comes from Möbius geometry, where it is of great importance in the study of surfaces (and general submanifolds) [2,5,8,19]. It is also known as *the mean curvature sphere* of the immersed surface y at p, characterized as the unique round 2-sphere y^* tangent to y at p and sharing the same mean curvature vector as y at this point. (The ambient space is endowed with a metric of some space form). In the Lorentzian case this is also true.

Proposition 4.2. *A surface immersed in a lorentzian space form envelops its central sphere congruence and shares the same mean curvature with these round 2-spheres at corresponding points.*

Proof. To prove the conclusion for the flat space R_1^4, consider a surface $y : M \to R_1^4$ and a point $p \in M$. Let $y^* : S^2 \to R_1^4$ be the *mean curvature sphere* associated with y at p as characterized above. It suffices to show that y^* coincides with the central sphere of y at p. Embed surface y into Q_1^4 via φ_0 as given by (3), with lift

$$Y = \Big(\frac{-1+\langle y,y\rangle}{2}, y, \frac{1+\langle y,y\rangle}{2}\Big).$$

Computation shows that the central sphere of y at p, identified with $V = \mathrm{Span}\{Y, \mathrm{Re}(Y_z), \mathrm{Im}(Y_z), Y_{z\bar z}\}$, is determined by the position vector $y(p)$, the tangent plane $\mathrm{d}y(TM_p)$, and the mean curvature vector $H(p)$, which coincide with those of y^* by our assumption. So y, y^* share the same central sphere at p. Yet for the round 2-sphere y^*, its central sphere at any point is exactly itself (they fall into the same 4-dim subspace), which verifies our assertion. For surfaces in S_1^4 or H_1^4 the proof is similar. □

Corollary 4.3. *In particular, if the central sphere congruence of y is a family of planes in R_1^4, this surface must have mean curvature zero at every point, thus be a stationary surface in R_1^4.*

The central sphere congruence is conformally invariant in the sense that for two surfaces $[Y^{'}], [Y]$ differing to each other by the action of $T \in O(4,2)$, their central spheres at corresponding points also differ by this transformation. This tells us that although the mean curvature sphere of a surface at one point is defined in terms of metric geometry, it is indeed a conformal invariant by Proposition 4.2 and the observation above. Viewed as a map from M to Σ, the moduli space of round 2-spheres, it has another name, *the conformal Gauss map* of y [1,2]. In Section 2 we have identified Σ with the Grassmannian $G_{3,1}(\mathbb{R}^6_2)$, which could be further embedded into the space of multi-vectors (of certain type and of length 1) in $\mathbb{R}^6_2$:

$$\Sigma \simeq G_{3,1}(\mathbb{R}^6_2) \hookrightarrow \Lambda_{3,1}(\mathbb{R}^6_2).$$

The latter is endowed with the canonical semi-Riemannian metric as usual. This provides the appropriate framework for the discussion of the geometry of the conformal Gauss map.

Definition 4.4. *For a conformally immersed surface $y : M \to Q^4_1$ with canonical lift Y (with respect to a local coordinate z), define*

$$G := Y \wedge Y_u \wedge Y_v \wedge N = -2i \cdot Y \wedge Y_z \wedge Y_{\bar{z}} \wedge N, \;\; z = u + iv,$$

where $N \equiv 2Y_{z\bar{z}} (\bmod Y)$ is the frame vector determined in (5)*. Note that $\langle G, G \rangle = 1$ and that G is well defined. We call $G : M \to G_{3,1}(\mathbb{R}^6_2)$* the conformal Gauss map *of y. It is noteworthy that $V \in G_{3,1}(\mathbb{R}^6_2)$ determines $V^{\perp} \in G_{1,1}(\mathbb{R}^6_2)$ and vice versa. Hence the geometry of G is equivalent to the geometry of the associated map*

$$G^{\perp} := L \wedge R : M \to G_{1,1}(\mathbb{R}^6_2),$$

where L, R are normal vectors as given in (6)*.*

The conformal Gauss map is important in that it induces a conformally invariant conformal metric. Direct computation using (11) shows

Proposition 4.5. *For a conformal surface $y : M \to Q^4_1$, G induces a metric*

$$g := \frac{1}{4}\langle \mathrm{d}G, \mathrm{d}G \rangle = \langle \kappa, \bar{\kappa} \rangle |dz|^2$$

on M, where $\kappa = \lambda_1 L + \lambda_2 R$ is the conformal Hopf differential. This metric might be positive definite, negative definite, or degenerate according to the sign of $\langle \kappa, \bar{\kappa} \rangle = -\beta = -(\lambda_1 \bar{\lambda}_2 + \lambda_2 \bar{\lambda}_1)$.

Now we can introduce the Willmore functional and Willmore surfaces.

Definition 4.6. The Willmore functional *of y is defined as the area of M with respect to the metric above:*

$$W(y) := \frac{i}{2} \int_M |\kappa|^2 dz \wedge d\bar{z}. \tag{14}$$

An immersed surface $y : M \to Q^4_1$ is called a Willmore surface*, if it is a critical surface of the Willmore functional with respect to any variation of the map $y : M \to Q^4_1$.*

Willmore surfaces can be characterized as follows, which is similar to the conclusions in codim-1 case [1,8] as well as in Möbius geometry [2,5,10,14].

Theorem 4.7. *For a conformal spacelike surface $y : M \to Q_1^4$, the following four conditions are equivalent:*

(i) y is Willmore.

(ii) The conformal Gauss map G is a harmonic map into $G_{3,1}(\mathbb{R}_2^6)$.

(iii) The conformal Hopf differential κ of y satisfies the Willmore condition as below, which is stronger than the conformal Codazzi equation (13)*:*

$$D_{\bar{z}}D_{\bar{z}}\kappa + \frac{\bar{s}}{2}\kappa = 0. \tag{15}$$

(iv) In a Lorentzian space form of sectional curvature c, y satisfies the Euler-Lagrange equation

$$\Delta\vec{H} - 2(|\vec{H}|^2 + K - c)\vec{H} = 0. \tag{16}$$

Here $\Delta, \vec{H}, K$ are the Laplacian of the induced metric, the mean curvature vector, and the Gaussian curvature of y, respectively.

The proof to Theorem 4.7 is completely the same as in Möbius geometry (we refer the reader to [15]). Note that when we take a variation y_t of the immersion y, generally y_t is not conformal to y, hence we have to consider the variation of the Willmore functional with respect to a varied complex structure J_t over M. Yet one can verify that this change of complex structure J contributes nothing to the first variation of the Willmore functional. Then the Willmore condition (15) can be derived easily.

The equivalent condition (16) in this theorem also implies that stationary surfaces (i.e. surfaces with mean curvature $\overrightarrow{H} = 0$) in Lorentzian space forms are Willmore. Indeed they belong to a subclass of Willmore surfaces, the so-called *S-Willmore surfaces*. The latter are exactly those Willmore surfaces with dual surfaces (see the next section). See Ejiri [10] and Ma [14] for the counterpart in Möbius geometry.

Definition 4.8. *A conformal Willmore surface $y : M \to Q_1^4$ is called a* S-Willmore surface *if it satisfies $D_{\bar{z}}\kappa \parallel \kappa$, i.e., $D_{\bar{z}}\kappa = -\frac{\bar{\mu}}{2}\kappa$ for some local function μ when $\kappa \neq 0$.*

Definition 4.9. *Let $y : M \to Q_1^4$ be a spacelike surface. We call y* null-umbilic *if its Hopf differential is isotropic, i.e. $\langle\kappa, \kappa\rangle = 0$ (equivalently, λ_1 or λ_2 vanishes). y is umbilic if $\kappa = 0$ (equivalently, $\lambda_1 = \lambda_2 = 0$).*

So far our notions, constructions and results can all be easily generalized to n-dimensional spaces. Yet in Q_1^4 null-umbilic surfaces have a special meaning. They are equivalent to $\mathcal{O}^+$-holomorphic maps into the twistor space of Q_1^4 according to [12]. Thus they are similar to isotropic surfaces in S^4 (which are also twistor projection of complex curves). Yet there are also important differences. For example, isotropic surfaces in S^4 are always S-Willmore, yet for null-umbilic surfaces this is not necessarily true. (Only under the additional Willmore condition can we show that a null-umbilic surface is S-Willmore.)

5 Transforms of Spacelike Willmore surfaces

In this section, we will define two transforms for surfaces in Q_1^4 and show that the new surfaces derived from them are also Willmore if the original surface is Willmore.

5.1 Right/left polar surfaces; $(+/-)$transforms

For a conformal spacelike surface $y : M \to Q_1^4$ with canonical lift $Y : M \to R_2^6$ with respect to complex coordinate $z = u + iv$, its normal plane at any point is spanned by two lightlike vectors L, R. Suppose that R_2^6 is endowed with a fixed orientation and that

$$\{Y, Y_u, Y_v, N, R, L\}$$

form a positively oriented frame. $\{R, L\}$ might also be viewed as a frame of the normal plane compatible with the orientation of M and that of the ambient space. Since $\langle L, R\rangle = -1$ has been fixed in (6), either one of the null lines $[L]$ ($[R]$) is well-defined.

Definition 5.1. *The two maps*

$$[L], [R] : M \to Q_1^4$$

are named **the left and the right polar surface** *of $y = [Y]$, respectively.*

Remark 5.2. Denote $e_+ = \frac{1}{\sqrt{2}}(R - L)$, $e_- = \frac{1}{\sqrt{2}}(R + L)$. Then $\{e_+, e_-\}$ is a positively oriented orthonormal frame of the normal plane, and L, R could be written as

$$L = \frac{1}{\sqrt{2}}(e_- - e_+), \quad R = \frac{1}{\sqrt{2}}(e_- + e_+).$$

Thus we also call $[L]$ the **$(-)$transform**, and $[R]$ the **$(+)$transform** of $[Y]$. At the same time these names correspond to the directions of these transforms in the diagram below:

$$\begin{tikzcd} & {[Y]} \arrow[dl, "-"'] \arrow[dr, "+"] & \\ {[L]} & & {[R]} \end{tikzcd}$$

The name *polar surfaces* comes from Lawson's similar construction for minimal surfaces in S^3 [11].

Proposition 5.3. *The polar surfaces $[L], [R] : M \to Q_1^4$ are both conformal maps. $[L]$ ($[R]$) is degenerate if, and only if, $\lambda_2 = 0$ ($\lambda_1 = 0$); it is a spacelike immersion otherwise. The original surface $[Y]$ is the left polar surface of $[R]$ (the right polar surface of $[L]$) when $[R]$ ($[L]$) is not degenerate.*

Proof. The first two conclusions for $[L]$ follow directly from

$$L_z = \alpha L - 2\gamma_2 Y + 2\lambda_2 Y_{\bar{z}}$$

by (11). Differentiating this equation once more, by (10)(11) we find

$$\begin{aligned} L_{z\bar z} = \bar\alpha L_z + \alpha L_{\bar z} &+ (\alpha_{\bar z} - \alpha\bar\alpha + 2\lambda_2\bar\lambda_1)L \\ &+ 2\lambda_2\bar\lambda_2\cdot R - 2\left(\gamma_{2\bar z} - \gamma_2\bar\alpha + \frac{\bar s}{2}\lambda_2\right)Y. \end{aligned} \tag{17}$$

When $\lambda_2 \neq 0$, we can verify directly that Y and

$$\hat Y = 2\left|\frac{\gamma_2}{\lambda_2}\right|^2 Y - 2\frac{\bar\gamma_2}{\bar\lambda_2}\,Y_z - 2\frac{\gamma_2}{\lambda_2}\,Y_{\bar z} + N + \frac{\gamma_{2\bar z} - \gamma_2\bar\alpha + \frac{\bar s}{2}\lambda_2}{\lambda_2\bar\lambda_2}\,L \tag{18}$$

are two lightlike vectors in the orthogonal complement of $\mathrm{Span}\{L, L_z, L_{\bar z}, L_{z\bar z}\}$ with $\langle Y, \hat Y\rangle = -1$, and that $\{L, L_z, L_{\bar z}, L_{z\bar z}, Y, \hat Y\}$ is again a positively oriented frame. So for $[L]$, the left polar surface of $[Y]$, its right polar surface is exactly $[Y]$. For $[R]$ the proof is similar. In other words, the $(+)$transform is the inverse to the $(-)$transform and vice versa when all surfaces concerned are immersed. $\square$

Remark 5.4. On the other hand, $[\hat Y]$ might be viewed as the 2-step $(-)$transform of $y = [Y]$. Similarly we have the 2-step $(+)$transform $[\tilde Y]$ as the right polar surface of $[R]$:

$$\begin{tikzcd}
{[\hat Y]} \arrow[dr, bend left, "+"] & & {[Y]} \arrow[dl, bend left, "-"] \arrow[dr, bend left, "+"] & & {[\tilde Y]} \arrow[dl, bend left, "-"] \\
& {[L]} \arrow[ul, bend left, "-"] \arrow[ur, bend left, "+"] & & {[R]} \arrow[ul, bend left, "-"] \arrow[ur, bend left, "+"] &
\end{tikzcd}$$

Note that $[L], [R]$ are also 2-step transforms to each other.

5.2 $(+/-)$transforms preserve Willmore property

Assume $y : M \to Q_1^4$ is an immersed spacelike Willmore surface with canonical lift $Y : M \to R_2^6$ for a given coordinate z locally. We want to show that the $(+)$transform and $(-)$transform again produce Willmore surfaces.

Assume that the left polar surface $[L]$ is an immersion, i.e. $\lambda_2 \neq 0$. Set

$$-\frac{\bar\mu}{2} := \frac{\gamma_2}{\lambda_2}. \tag{19}$$

According to the conclusions of Theorem 4.7 and Proposition 5.3, we need to show that the conformal Gauss map of $[L]$, represented by $Y \wedge \hat Y$, is a harmonic map. The Willmore condition (15) for y amounts to say

$$\begin{cases} \gamma_{1\bar z} + \gamma_1\bar\alpha + \frac{\bar s}{2}\lambda_1 = 0, \\ \gamma_{2\bar z} - \gamma_2\bar\alpha + \frac{\bar s}{2}\lambda_2 = 0. \end{cases} \tag{20}$$

Hence the expression of $\hat Y$ in (18) is simplified to

$$\hat Y := \frac{|\mu|^2}{2}Y + \bar\mu Y_z + \mu Y_{\bar z} + N. \tag{21}$$

The Willmore condition also implies

$$\mu_z - \frac{\mu^2}{2} - s = 0, \tag{22}$$

because one can verify directly that

$$2(\gamma_{2\bar{z}} - \gamma_2\bar{\alpha} + \frac{\lambda_2\bar{s}}{2}) = (-\bar{\mu}_{\bar{z}} + \frac{\bar{\mu}^2}{2} + \bar{s})\lambda_2$$

using the expressions of γ_2 (10) and μ (19).

For convenience of computation, set a new frame

$$\{Y, \hat{Y}, P, \bar{P}, L, R\}, \quad \text{with } P := Y_z + \frac{\mu}{2}Y,$$

so that $\langle Y, P\rangle = \langle \hat{Y}, P\rangle = 0$. Differentiating $\hat{Y}$ and invoking (22), we find

$$\hat{Y}_z = \frac{\mu}{2}\hat{Y} + \rho P + \sigma L, \tag{23}$$

where

$$\begin{cases} \rho = \bar{\mu}_z + 2\lambda_1\bar{\lambda}_2 + 2\lambda_2\bar{\lambda}_1 = \bar{\mu}_z + 2\beta, \\ \sigma = 2\gamma_1 + \lambda_1\bar{\mu}. \end{cases} \tag{24}$$

For the frame $\{Y, \hat{Y}, P, \bar{P}, L, R\}$, the structure equations are

$$\begin{cases} Y_z = -\frac{\mu}{2}Y + P, \\ \hat{Y}_z = \ \ \frac{\mu}{2}\hat{Y} + \rho P + \sigma L, \\ P_z = \ \ \frac{\mu}{2}P + \lambda_1 L + \lambda_2 R, \\ \bar{P}_z = -\frac{\mu}{2}\bar{P} + \frac{\rho}{2}Y + \frac{1}{2}\hat{Y}, \\ L_z = \ \ \alpha L + 2\lambda_2\bar{P}, \\ R_z = -\alpha R + 2\lambda_1\bar{P} - \sigma Y. \end{cases} \tag{25}$$

Now the Willmore condition, (22) and the first one in (20), yields

$$\rho_{\bar{z}} = \bar{\mu}\rho - 2\bar{\lambda}_2\sigma, \tag{26}$$

$$\sigma_{\bar{z}} = (-\bar{\alpha} + \frac{\bar{\mu}}{2})\sigma. \tag{27}$$

The computation is straightforward by the expressions (24)(10) and the first equation in (12) (the conformal Gauss equation).

After these preparations, now we can compute out that

$$\begin{aligned} (Y\wedge\hat{Y})_z &= P\wedge\hat{Y} + \rho Y\wedge P + \sigma Y\wedge L, \\ (Y\wedge\hat{Y})_{z\bar{z}} &= \frac{\rho+\bar{\rho}}{2}Y\wedge\hat{Y} + \sigma\bar{P}\wedge L + \bar{\sigma}P\wedge L + \rho\bar{P}\wedge P + \bar{\rho}P\wedge\bar{P} \\ &\quad + \underbrace{(\rho_{\bar{z}} - \bar{\mu}\rho + 2\sigma\bar{\lambda}_2)}_{=0} Y\wedge P + \underbrace{(\sigma_{\bar{z}} - \tfrac{\bar{\mu}}{2}\sigma + \bar{\alpha}\sigma)}_{=0} Y\wedge L. \end{aligned}$$

Thus $Y\wedge\hat{Y}$ is a (conformal) harmonic map into $G_{1,1}(\mathbb{R}^6_2)$ as desired. This shows that $[L]$ is Willmore. For $[R]$ the proof is similar. Sum together, we have proved

Theorem 5.5. *Let $y : M \to Q_1^4$ be a spacelike Willmore surface. Then its left and right polar surfaces $[L], [R] : M \to Q_1^4$ are also spacelike Willmore surfaces when they are not degenerate.*

Since the $(-)$transform and the $(+)$transform preserve the Willmore property, the same holds true for the 2-step transforms $[\hat{Y}], [\tilde{Y}]$ when they are defined.

Definition 5.6. *When $[Y]$ is Willmore, $[\hat{Y}], [\tilde{Y}]$ are also conformal spacelike Willmore surfaces, called separately* the left adjoint transform *and* the right adjoint transform *of $[Y]$.*

Remark 5.7. Another equivalent way to define adjoint transforms of a given Willmore surface is to follow the idea in [14]. In particular, the adjoint transforms defined at here share many properties as before. Taking $[\hat{Y}]$ for example, we have:

(1) The (left) adjoint transform $[\hat{Y}]$ is conformal to $[Y]$; it locates on the central sphere congruence of $[Y]$ according to (21) and Definition 4.1.

(2) $Y \wedge \hat{Y}$ is a conformal harmonic map into the Grassmannian $G_{1,1}(\mathbb{R}^6_2)$.

(3) When the two adjoint transforms coincide, this surface $[\hat{Y}] = [\tilde{Y}]$ will share the same central sphere congruence with $[Y]$. (See the duality theorem in the next subsection.)

The interested reader may confer [14] for a comparison. Here we derive them from the polar surfaces, which seems more natural in our context. Note that $[L], [R]$ are also adjoint transforms to each other, as visualized below:

$$\begin{tikzcd}
{[\hat{Y}]} \arrow[rr, dotted, leftrightarrow] \arrow[dr, bend left, "+"] & & {[Y]} \arrow[rr, dotted, leftrightarrow] \arrow[dl, bend left, "-"] \arrow[dr, bend left, "+"] & & {[\tilde{Y}]} \arrow[dl, bend left, "-"] \\
& {[L]} \arrow[ul, bend left, "-"] \arrow[ur, bend left, "+"] \arrow[rr, dotted, leftrightarrow] & & {[R]} \arrow[ul, bend left, "-"] \arrow[ur, bend left, "+"] &
\end{tikzcd}$$

The chain of $(-)$transforms and $(+)$transforms also demonstrates a striking similarity with the backward and forward Bäcklund transforms introduced for Willmore surfaces in S^4 [4]. In particular, the 2-step Bäcklund transforms there could also be identified with the adjoint transforms in [14]. An interesting difference is that our $(-/+)$transforms are defined in a conformally invariant way, whereas the 1-step Bäcklund transforms are only properly defined in some affine space R^4.

5.3 Duality theorem of S-Willmore surfaces

In the picture given above, a special case is noteworthy, namely that when $[\hat{Y}] = [\tilde{Y}]$. This might be characterized by the following

Theorem 5.8 (Duality Theorem). *Let $y = [Y] : M \to Q^4_1$ be a spacelike S-Willmore surface with polar surfaces $[L], [R]$ and adjoint transforms $[\hat{Y}], [\tilde{Y}]$. Suppose that both of $[L], [R]$ are not degenerate, i.e., $\lambda_1 \neq 0, \lambda_2 \neq 0$. Then the conditions below are equivalent:*

(1) $[\hat{Y}] = [\tilde{Y}]$, i.e., the two adjoint transforms coincide.

(2) $y = [Y]$ is a S-Willmore surface, i.e. $D_{\bar{z}}\kappa = -\frac{\bar{\mu}}{2}\kappa$ for some μ.

(3) $[\hat{Y}]$ (or $[\tilde{Y}]$) shares the same central sphere congruence with $[Y]$.

Proof. When y is Willmore, its right adjoint transform $[\tilde{Y}]$ might be given in a formula similar to (21) with

$$\tilde{Y} := \frac{|\mu_1|^2}{2} Y + \bar{\mu}_1 Y_z + \mu_1 Y_{\bar{z}} + N, \quad -\frac{\bar{\mu}_1}{2} := \frac{\gamma_1}{\lambda_1}.$$

Thus it is obvious that $[\hat{Y}] = [\tilde{Y}]$ if and only if $-\bar{\mu}/2 = \gamma_1/\lambda_1 = \gamma_2/\lambda_2$, which is equivalent to the S-Willmore condition. This shows "(1) $\Leftrightarrow$ (2)". By (25) we also know that

$$\mathrm{Span}\{\hat{Y}, \hat{Y}_z, \hat{Y}_{\bar{z}}, \hat{Y}_{z\bar{z}}\} = \mathrm{Span}\{Y, Y_z, Y_{\bar{z}}, Y_{z\bar{z}}\}$$

if and only if $\sigma := 2\gamma_1 + \lambda_1\bar{\mu} = 0$, where $-\bar{\mu}/2 := \gamma_2/\lambda_2$. This shows that $[\hat{Y}]$ has the same central sphere congruence as $[Y]$ exactly when $-\bar{\mu}/2 = \gamma_1/\lambda_1 = \gamma_2/\lambda_2$. So "(3) $\Leftrightarrow$ (2)", and the proof is completed. $\square$

Remark 5.9. Condition (3) in this theorem tells us that when $[Y]$ is S-Willmore, $[\hat{Y}]$ must also be S-Willmore. Each of them could be obtained as the unique adjoint transform, or the second envelopping surface of the central sphere congruence, of the other. $[\hat{Y}]$ is called *the dual Willmore surface* of $[Y]$, and vice versa. Note that when $\lambda_2 \equiv 0 (\lambda_1 \equiv 0)$, $[L]([R])$ degenerates to a single point. This happens exactly when y is a null-umbilic surface in Q^4_1. Yet the dual Willmore surface could still be defined if the other $\lambda_i \neq 0$.

Corollary 5.10. *When $y = [Y] : M \to Q^4_1$ is a S-Willmore surface without umbilic points, $[L]$ and $[R]$ are a pair of S-Willmore surfaces being adjoint transform to each other (one of them might be degenerate). In particular, the $(-/+)$transforms preserve the S-Willmore property.*

Proof. Since y has no umbilic points, λ_1, λ_2 could not vanish simultaneously. Without loss of generality, assume $\lambda_2 \neq 0$. Then $[L]$ is an immersion. By the Duality Theorem above, we see that $[\hat{Y}](= [\tilde{Y}])$ is defined. The transform chain appeared in Remark 5.7 then closes up as below:

$$\begin{tikzcd}
 & {[Y]} \arrow[dl, bend right=15, "-"] \arrow[dr, bend left=30, "+"] \arrow[dd, dash, dotted] & \\
{[L]} \arrow[ur, bend left=30, "+"] \arrow[rr, dash, dotted] \arrow[dr, bend left=15, "-"'] & & {[R]} \arrow[ul, bend right=15, "-"'] \arrow[dl, bend left=30, "+"] \\
 & {[\hat{Y}]} \arrow[ul, bend left=30, "+"] \arrow[ur, bend right=15, "-"] &
\end{tikzcd}$$

It tells us that $[L]$ is the 2-step $(-)$transform and the 2-step $(+)$transform of $[R]$ at the same time. Equivalently, that means $[L]$ and $[R]$ are the left and the right adjoint transform of each other. This proves the conclusion by the Duality Theorem above. $\square$

6 Spacelike Willmore 2-spheres in Q^4_1

In this section, we will classify spacelike Willmore 2-spheres in Q^4_1. This is done by constructing globally defined holomorphic forms on S^2; the vanishing of such forms then enables us to draw strong conclusions. The reader will see that our method and result are still similar to the case for Willmore 2-spheres in S^4 [4,16].

Lemma 6.1. *(i) Let $y: M \to Q^4_1$ be a spacelike Willmore surface with conformal Hopf differential κ for a given coordinate z. Then the 6-form*

$$\Theta(\mathrm{d}z)^6 = \left[\langle D_{\bar z}\kappa, \kappa\rangle^2 - \langle \kappa, \kappa\rangle \cdot \langle D_{\bar z}\kappa, D_{\bar z}\kappa\rangle\right](\mathrm{d}z)^6 \tag{28}$$

is a globally defined holomorphic 6-form on M.
(ii) When $M = S^2$, we have $\Theta \equiv 0$ and y is S-Willmore. On the subset $M_0 \subset M$ where y has no umbilic points, let Y be the canonical lift of y, and $\hat Y$ a local lift of its dual Willmore surface satisfying $\langle Y, \hat Y\rangle = -1$. Then

$$\Omega(\mathrm{d}z)^8 = \langle Y_{zz}, Y_{zz}\rangle\langle \hat Y_{zz}, \hat Y_{zz}\rangle(\mathrm{d}z)^8 \tag{29}$$

is a globally defined holomorphic 8-form on S^2. So $\Omega \equiv 0$.

Proof. It is easy to verify that these two differential forms are well-defined (one may use the fact that $\kappa\ (\mathrm{d}z)^{\frac{3}{2}}(\mathrm{d}\bar z)^{-\frac{1}{2}}$ is globally defined). The holomorphicity of $\Theta(\mathrm{d}z)^6$ follows directly from the Willmore condition (15).

For conclusion (ii), by the well-known fact that every holomorpic form on S^2 must vanish, we know $\Theta \equiv 0$. On the other hand, $\Theta = (\lambda_1\gamma_2 - \lambda_2\gamma_1)^2$ by (8)(9). So on S^2 we have $\lambda_1\gamma_2 - \lambda_2\gamma_1 = 0$. It is just the S-Willmore condition. Thus on M_0 where $\kappa \neq 0$, there is $D_{\bar z}\kappa = -\frac{\bar\mu}{2}\kappa$ for some local function μ. Define $\hat Y$ and ρ as in (21) and (24), and compute $\hat Y_{zz}$ using (25). We get $\langle \hat Y_{zz}, \hat Y_{zz}\rangle = -2\rho^2\lambda_1\lambda_2$. Hence

$$\Omega = \langle Y_{zz}, Y_{zz}\rangle\langle \hat Y_{zz}, \hat Y_{zz}\rangle = 4(\rho\lambda_1\lambda_2)^2.$$

Note that in the S-Willmore case

$$\sigma = \bar\mu\lambda_1 + 2\gamma_1 = 0 = \bar\mu\lambda_2 + 2\gamma_2. \tag{30}$$

So $\rho_{\bar z} = \bar\mu\rho$ according to (26). On the other hand,

$$(\lambda_1\lambda_2)_{\bar z} = -\frac{1}{2}\langle \kappa, \kappa\rangle_{\bar z} = -\langle D_{\bar z}\kappa, \kappa\rangle = \frac{\bar\mu}{2}\langle \kappa, \kappa\rangle = -\bar\mu\lambda_1\lambda_2.$$

Combined together, they show that $(\rho\lambda_1\lambda_2)_{\bar z} = 0$ and $\Omega(\mathrm{d}z)^8$ is a holomorphic differential form defined on M_0.

To show $\Omega(\mathrm{d}z)^8$ extends to M as a holomorphic form, note that by (30),

$$\bar{\mu}_z\lambda_1\lambda_2 = (\bar{\mu}\lambda_1\lambda_2)_z - \bar{\mu}(\lambda_1\lambda_2)_z = (-2\gamma_1\lambda_2)_z + 2\gamma_1(\lambda_2)_z + 2\gamma_2(\lambda_1)_z$$

is a smooth function (depending on z). Then for $\rho = \bar{\mu}_z + 2\lambda_1\bar{\lambda}_2 + 2\lambda_2\bar{\lambda}_1$ (24), we see that $(\rho\lambda_1\lambda_2)^2(\mathrm{d}z)^8$ extends smoothly to M as desired. It is holomorphic both on M_0 and in the interior of $M \setminus M_0$ (it vanishes in the latter case). So it is holomorphic on the whole $M = S^2$. This completes the proof. $\square$

Theorem 6.2. *Let $y : S^2 \to Q^4_1$ be a spacelike Willmore 2-sphere. Then it must be a surface among the following two classes:*

(i) it is the conformal compactification of a stationary surface in $\mathbb{R}^4_1$.

(ii) it is one of the polar surfaces of a surface in class (i).

Proof. First we observe that (10)(20) may be re-written as

$$\begin{cases} \lambda_{1\bar{z}} = -\bar{\alpha}\lambda_1 + \gamma_1, \\ \gamma_{1\bar{z}} = -\frac{\bar{s}}{2}\lambda_1 - \bar{\alpha}\gamma_1. \end{cases}$$

By a lemma of Chern (see section 4 in [7]), either λ_1 is identically zero on S^2, or it has only isolated zeroes. The same conclusion holds for λ_2. Now that we have shown $\rho\lambda_1\lambda_2 \equiv 0$, one of $\rho, \lambda_1, \lambda_2$ must vanish identically on S^2.

If $\rho \equiv 0$, $[\hat{Y}]$ degenerates to a single point due to (23) and $\sigma = 0$. Applying a transformation $T \in O(4,2)$ if necessary, we can set $\hat{Y} = (1,0,0,0,0,1)$ and $Y = (\frac{-1+\langle u,u\rangle}{2}, u, \frac{1+\langle u,u\rangle}{2})$ where $u : U \to \mathbb{R}^4_1$. Let z be an arbitrary complex coordinate. Then we have

$$Y_{z\bar{z}} = aY + \langle Y_z, Y_{\bar{z}}\rangle N, \quad \hat{Y} = N + \bar{\mu}Y_z + \mu Y_{\bar{z}} + \langle Y_z, Y_{\bar{z}}\rangle|\mu|^2 Y,$$

where a, μ are two functions. It is easy to see

$$\hat{Y}_z = -\mu\langle Y_z, Y_{\bar{z}}\rangle\hat{Y} + \cdots .$$

So $\mu \equiv 0$ and $Y_{z\bar{z}} = aY + \langle Y_z, Y_{\bar{z}}\rangle\hat{Y}$. Replacing by u leads to

$$(\langle u_{z\bar{z}}, u\rangle, u_{z\bar{z}}, \langle u_{z\bar{z}}, u\rangle) = (\frac{-a + a\langle u,u\rangle}{2}, au, \frac{a + a\langle u,u\rangle}{2}).$$

This implies $a \equiv 0$ and $u_{z\bar{z}} \equiv 0$. So u is a stationary surface in $\mathbb{R}^4_1$, and $y = [Y]$ belongs to class (i). [1]

If $\lambda_1 \equiv 0 (\lambda_2 \equiv 0)$, $[R]([L])$ is a point. Using the conclusion in (i) for surface $[L]([R])$ finishes the proof. $\square$

Remark 6.3. Note that surfaces of class (ii) are exactly spacelike null-umbilic Willmore surfaces. So one has

$$\langle \kappa, \bar{\kappa}\rangle = -\lambda_1\bar{\lambda}_2 - \lambda_2\bar{\lambda}_1 = 0.$$

As a consequence, its induced conformal metric $\langle\kappa,\bar{\kappa}\rangle(\mathrm{d}z)^2$ as well as the Willmore functional is always zero, which is different from the case in S^4. Here a left question is: *For a space-like Willmore 2-sphere in Q^4_1, if its Willmore functional equals zero, must it be of type (ii)?*

[1] An alternative proof is by the meaning of the mean curvature sphere. Since every central sphere of y passing through a fixed point $[\hat{Y}]$ of Q^4_1, which could be viewed as a point at infinity for some affine $\mathbb{R}^4_1$, each sphere is a plane in this $\mathbb{R}^4_1$. Corollary 4.3 implies the conclusion.

7 Examples

First let us see some special Willmore surfaces contained in a 3-dimensional space.

Example 7.1. Embed $\mathbb{R}^3 \subset \mathbb{R}^4_1$ via $u \to (u,1)$, $\mathbb{R}^3_1 \subset \mathbb{R}^4_1$ via $u \to (1,u)$.

(i) Let $u : M^2 \to \mathbb{R}^3$ be a minimal surface. Then $(u,1) : M^2 \to \mathbb{R}^4_1$ is a spacelike stationary surface in $\mathbb{R}^4_1$, and

$$Y = \Big(\frac{\langle u,u\rangle}{2} - 1, u, 1, \frac{\langle u,u\rangle}{2}\Big) : M^2 \to C^5$$

gives a spacelike S-Willmore surface $[Y] : M^2 \to Q^4_1$. (Essentially this comes from the conformal embedding φ_0.) Let $g : M^2 \to S^4_1 \subset R^5_1$ denote the conformal Gauss map of u as in [2] and $e = (-1,0,0,0,-1,1) \in \mathbb{R}^6_2$. It is straightforward to verify that $[e + (g,0)]$ and $[e - (g,0)]$ are the polar surfaces of $[Y]$.

(ii) Let $u : M^2 \to \mathbb{R}^3_1$ be a spacelike maximal surface. Then $(1,u) : M^2 \to \mathbb{R}^4_1$ is a spacelike stationary surface in $\mathbb{R}^4_1$, and

$$Y = \Big(\frac{\langle u,u\rangle}{2}, 1, u, \frac{\langle u,u\rangle}{2} + 1\Big) : M^2 \to C^5$$

gives a spacelike S-Willmore surface $[Y] : M^2 \to Q^4_1$. Let $\tilde{g} : M^2 \to H^4_1 \subset R^5_2$ denote the conformal Gauss map of u as in [1] and $\tilde{e} = (1,1,0,0,0,1) \in \mathbb{R}^6_2$. Then $[\tilde{e} + (0,\tilde{g})]$ and $[\tilde{e} - (0,\tilde{g})]$ are the polar surfaces of $[Y]$.

(iii) Suppose $u : M^2 \to \mathbb{R}^3$ is a Laguerre minimal surface and $n : M^2 \to S^2$ its Gauss map. Its Laguerre lift

$$Y = (n, u\cdot n, -u\cdot n, 1) : M^2 \to C^5$$

gives a spacelike S-Willmore surface $[Y] : M^2 \to Q^4_1$. We denote $g' : M^2 \to R^4_1 \hookrightarrow C^5$ its Laguerre Gauss map (see [18]). Then the point $[(0,0,0,1,-1,0)]$ and $[g']$ are the polar surfaces of $[Y]$.

Example 7.2. Consider a spacelike Willmore surface $y = [Y]$ both of type (i) and type (ii) as in Theorem 6.2. That means either of $[\hat{Y}]$ and $[L]$ is a single point, and $[Y]$ is the conformal compactification of a stationary surface $x : M^2 \to \mathbb{R}^4_1$. Without loss of generality, suppose

$$\hat{Y} = (1,0,0,0,0,1),\ L = (0,1,0,0,1,0),\quad Y = \Big(\frac{-1+\langle x,x\rangle}{2}, x, \frac{1+\langle x,x\rangle}{2}\Big).$$

From $\langle Y, L\rangle = 0$, we see that the surface $x = (x_1,x_2,x_3,x_4)$ must satisfy $x_1 = x_4$, which means that x in fact is a zero mean curvature surface in $\mathbb{R}^3_0 \subset \mathbb{R}^4_1$. For details of such surfaces, see [18].

Among compact surfaces, 2-spheres and tori are simplest and most important. In general, Willmore tori are not necessarily S-Willmore surfaces. Here we give such a class of spacelike Willmore tori which are homogenous.

Example 7.3. Let

$$e_1=(\cos\frac{t\theta}{\sqrt{t^2-1}}\cos\phi,\cos\frac{t\theta}{\sqrt{t^2-1}}\sin\phi,\sin\frac{t\theta}{\sqrt{t^2-1}}\cos\phi,\sin\frac{t\theta}{\sqrt{t^2-1}}\sin\phi),$$

$$e_2=\frac{\partial e_1}{\partial\phi}=e_{1\phi},e_3=\frac{\sqrt{t^2-1}}{t}e_{1\theta},e_4=\frac{\sqrt{t^2-1}}{t}e_{2\theta},$$

where $t>1$. Let

$$\begin{aligned}&Y_t(\theta,\phi):\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}^6_2\\&Y_t(\theta,\phi)=(e_1,\cos\frac{\theta}{\sqrt{t^2-1}},\sin\frac{\theta}{\sqrt{t^2-1}}).\end{aligned}\tag{31}$$

For simplicity, we omit the subscript "$_t$" of Y_t. We have that $y=[Y]:\mathbb{R}\times\mathbb{R}\to Q^4_1$ is a spacelike Willmore torus of Q^4_1 for any rational number $t>1$.

For the lift Y we set $z=\theta+i\phi$. It is easy to verify Y is a canonical lift with respect to z. We have

$$\begin{cases}Y_{z\bar z}=-\dfrac{t^2}{4(t^2-1)}Y+\dfrac{1}{2}N,\\ Y_{zz}=-\dfrac{1}{4(t^2-1)}Y-\dfrac{it}{2\sqrt{2}\sqrt{t^2-1}}(L-R),\\ L_z=-\dfrac{i}{2\sqrt{t^2-1}}L-\dfrac{t}{2\sqrt{2}(t^2-1)}Y+\dfrac{it}{\sqrt{2}\sqrt{t^2-1}}Y_{\bar z},\\ R_z=\dfrac{i}{2\sqrt{t^2-1}}R-\dfrac{t}{2\sqrt{2}(t^2-1)}Y-\dfrac{it}{\sqrt{2}\sqrt{t^2-1}}Y_{\bar z},\\ N_z=-\dfrac{t^2}{2(t^2-1)}Y_z-\dfrac{1}{2(t^2-1)}Y_{\bar z}+\dfrac{t}{2\sqrt{2}(t^2-1)}L+\dfrac{t}{2\sqrt{2}(t^2-1)}R.\end{cases}\tag{32}$$

Here

$$\begin{cases}Y_z=\dfrac{1}{2\sqrt{t^2-1}}(te_3-i\sqrt{t^2-1}e_2,-\sin\dfrac{\theta}{\sqrt{t^2-1}},\cos\dfrac{\theta}{\sqrt{t^2-1}}),\\ N=\dfrac{1}{2}(-e_1,\cos\dfrac{\theta}{\sqrt{t^2-1}},\sin\dfrac{\theta}{\sqrt{t^2-1}}),\\ L=\dfrac{1}{\sqrt{2}\sqrt{t^2-1}}(\sqrt{t^2-1}e_4+e_3,-t\sin\dfrac{\theta}{\sqrt{t^2-1}},t\cos\dfrac{\theta}{\sqrt{t^2-1}}),\\ R=\dfrac{1}{\sqrt{2}\sqrt{t^2-1}}(-\sqrt{t^2-1}e_4+e_3,-t\sin\dfrac{\theta}{\sqrt{t^2-1}},t\cos\dfrac{\theta}{\sqrt{t^2-1}}).\end{cases}\tag{33}$$

So it is easy to see that Y is spacelike Willmore and not S-Willmore.
The adjoint surface of Y with respect to L is

$$\hat Y=\frac{1}{2}\left(\frac{2-t^2}{t^2-1}e_1+\frac{1}{\sqrt{t^2-1}}e_2,\frac{t^2}{t^2-1}\cos\frac{\theta}{\sqrt{t^2-1}},\frac{t^2}{t^2-1}\sin\frac{\theta}{\sqrt{t^2-1}}\right).$$

The adjoint surface of Y with respect to R is

$$\tilde Y=\frac{1}{2}\left(\frac{2-t^2}{t^2-1}e_1-\frac{1}{\sqrt{t^2-1}}e_2,\frac{t^2}{t^2-1}\cos\frac{\theta}{\sqrt{t^2-1}},\frac{t^2}{t^2-1}\sin\frac{\theta}{\sqrt{t^2-1}}\right).$$

We point out that Y is a homogenous torus which is the orbit of the subgroup

$$G = \begin{pmatrix} e_1^T & e_2^T & e_3^T & e_4^T & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\frac{\theta}{\sqrt{t^2-1}} & -\sin\frac{\theta}{\sqrt{t^2-1}} \\ 0 & 0 & 0 & 0 & \sin\frac{\theta}{\sqrt{t^2-1}} & \cos\frac{\theta}{\sqrt{t^2-1}} \end{pmatrix}$$

acting on $(1,0,0,0,1,0)^T$. Here ^{T} denotes transposition.

If y_t is a torus, then t must be some rational number. Suppose $t = \frac{p}{q}$, where $p, q \in \mathbb{N}$. Then the Willmore functional of y_t is

$$W(y_t) = \frac{p^2}{\sqrt{p^2 - q^2}}\pi^2 \tag{34}$$

So the minimum of Willmore functional of y_t is $\frac{4}{\sqrt{3}}\pi^2$.

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