ON THE LOWER CENTRAL SERIES OF AN ASSOCIATIVE ALGEBRA

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(with an appendix by Pavel Etingof)

Abstract. For an associative algebra A, define its lower central series $L_0(A) = A$, $L_i(A) = [A, L_{i-1}(A)]$, and the corresponding quotients $B_i(A) = L_i(A)/L_{i+1}(A)$. In this paper, we study the structure of $B_i(A_n)$ for a free algebra A_n. We construct a basis for $B_2(A_n)$ and determine the structure of $B_3(A_2)$ and $B_4(A_2)$. In the appendix, we study the structure of $B_2(A)$ for any associative algebra A over $\mathbb{C}$.

1. Introduction

Let A be an associative algebra. Let us regard it as a Lie algebra with commutator $[a, b] = ab - ba$. Then one can inductively define the lower central series filtration of A: $L_1(A) = A$, $L_i(A) = [A, L_{i-1}(A)]$, and the corresponding quotients $B_i(A) = L_i(A)/L_{i+1}(A)$. It is an interesting problem to understand the structure of the spaces $B_i(A)$ for a given algebra A.

The study of $B_i(A)$ was initiated in the paper by B. Feigin and B. Shoikhet [FS], who considered the case when $A = A_n$ is the free associative algebra in n generators over $\mathbb{C}$. Their main results are that $B_i(A_n)$ for $i > 1$ are representations of the Lie algebra W_n of polynomial vector fields in n variables, and that $B_2(A_n)$ is isomorphic, as a W_n-module, to the space of closed (or equivalently, exact) polynomial differential forms on $\mathbb{C}^n$ of positive even degree.

The goal of this paper is to continue the study of the structure of $B_i(A_n)$, and more generally, of $B_i(A)$ for any associative algebra A. More specifically, in Section 2 we give a new simple proof of the result of Feigin and Shoikhet on the structure of $B_2(A_n)$ for $n = 2, 3$, by constructing an explicit basis of this space. In Section 3, we generalize this basis to the case $n > 3$, and use it to determine the structure of the space $B_2(A_n^R)$, where A_n^R is the quotient of A_n by the relations $x_i^{m_i} = 0$, where m_i are positive integers. In Sections 4,5 we obtain some information about the structure of $B_m(A_2)$ as a W_2-module. In Section 6 we determine the structures of $B_3(A_2)$ and $B_4(A_2)$, thus confirming conjectures from [FS]. Finally, in the appendix, the structure of $B_2(A)$ is studied for any associative algebra A over $\mathbb{C}$.

2. The structure of $B_{2,2}$ and $B_{3,2}$

2.1. **Some notations.** Let A_n be the free algebra over $\mathbb{C}$ in n generators $x_1, \ldots, x_n$. Let $m_1, \ldots, m_n$ be positive integers, and R be the set of relations $x_i^{m_i} = 0$, $i = 1, \ldots, n$. Let $A_n^R = A_n/(R)$. From now on, we denote $B_i(A_n)$ by $B_{n,i}$, and $B_i(A_n^R)$ by $B_{n,i}^R$.

Let each generator x_i have degree 1. We say that $w \in B_{n,k}$ has multidegree $(i_1, \ldots, i_n)$ if every x_s occurs i_s times in every monomial of w. The set of all multidegree $\mathbf{i} = (i_1, \ldots, i_n)$ elements in $B_{n,k}$ is denoted by $B_{n,k}[\mathbf{i}]$. Note that not all $w \in B_{n,k}$ will have a multi-degree. However, monomials and brackets of monomials in $B_{n,k}$ will have a multi-degree. Let $l = i_1 + \cdots + i_n$, and call l the degree of w. We denote $B_{n,k}[l]$ to be the set of all degree l elements in $B_{n,k}$.

2.2. **Basis for** $B_{2,2}[l]$**.** In this section, we find a basis for $B_{2,2}[l]$.

Proposition 2.1. *For $l \geq 2$, the $l-1$ elements $[x_1^i, x_2^{l-i}]$ for $i = 1, \ldots, l-1$ constitute a spanning set for $B_{2,2}[l]$.*

Proof. First note that every element of $B_{2,2}[l]$ can be expressed as a linear combination of the brackets $[a, x_1]$ and $[b, x_2]$, where a and b are monomials with degree no less than 1. To see this, consider an arbitrary bracket of monomials in $B_{2,2}[l]$. This bracket may be written as $[P, q_1 q_2 \cdots q_n]$, where $n \geq 2$ and q_i represents either x_1 or x_2.

Then we have

$$[P, q_1 q_2 \cdots q_n] = [P q_1 \cdots q_{n-1}, q_n] + [q_n P q_1 \cdots q_{n-2}, q_{n-1}] + \cdots + [q_2 \cdots q_n P, q_1].$$

As every element of $B_{2,2}[l]$ is a linear combination of such brackets of monomials, and each bracket of monomials is a sum of brackets of the desired form, every element of $B_{2,2}[l]$ is a linear combination of brackets of the desired form.

Consider $[a, x_1]$. Write $a = x_1^k a_1$, where a_1 begins with x_2 or is equal to 1. Then we have $[a_1, x_1^{k+1}] = \sum_{j=0}^{k} [x_1^{k-j} a_1 x_1^j, x_1]$. Notice that all the terms in the summation are equivalent in $B_{2,2}[l]$ because we can cyclically permute either term of the bracket. So $[x_1^k a_1, x_1] = \frac{1}{k+1}[a_1, x_1^{k+1}]$.

If a_1 is not 1, then a_1 can be written as $x_2^m x_1^n a_2$, where a_2 begins with x_2 or is equal to 1. So by the same argument as above,

$$[a_1, x_1^{k+1}] = [x_2^m x_1^n a_2, x_1^{k+1}] = [x_1^n a_2 x_2^m, x_1^{k+1}] = \frac{1}{n+1}[a_2 x_2^m, x_1^{k+1+n}]$$

Continuing this process will eventually transfer all powers of x_1 to the right side of the bracket, showing that $[a, x_1]$ is a constant multiple of $[x_2^{l-i}, x_2^i] = -[x_1^i, x_2^{l-i}]$ for some i from 1 to $l-1$.

A similar argument shows that $[b, x_2]$ is a constant multiple of $[x_1^i, x_2^{l-i}]$. Recalling that every element of $B_{2,2}[l]$ is a linear combination of brackets of the form $[a, x_1]$ and $[b, x_2]$, the proposition is proved. □

Theorem 2.1. *For $l \geq 2$, the $l-1$ elements of the form $[x_1^i, x_2^{l-i}]$ for $i = 1, \ldots, l-1$ constitute a basis for $B_{2,2}[l]$, so for any $i, j \geq 1$, $B_{2,2}[(i,j)] = \mathbb{C} \cdot [x_1^i, x_2^j]$.*

Proof. We will show that $\dim B_{2,2}[l] \geq l-1$. Since we have already found $l-1$ generators for $B_{2,2}[l]$, we conclude that $\dim B_{2,2}[l]$ must be equal to $l-1$, and thus the spanning set we found must be a basis for $B_{2,2}[l]$.

We claim that $[x_1^{l-1}, x_2]$ is non-zero, i.e. $[x_1^{l-1}, x_2]$ is not in $[[A_2, A_2], A_2]$. Note that $[[A_2, A_2], A_2]$ is spanned by elements of the form $[[m_1, m_2], m_3]$, where m_1, m_2, and m_3 are monomials in A_2, and the only brackets of this form which contain either $x_1^{l-1} x_2$ or $x_2 x_1^{l-1}$ are either of the form $[[x_1^i, x_2], x_1^{l-1-i}]$ or $[[x_2, x_1^i], x_1^{l-1-i}] = -[[x_1^i, x_2], x_1^{l-1-i}]$. In these brackets, the coefficients of $x_1^{l-1} x_2$ and $x_2 x_1^{l-1}$ are always equal. Therefore, no linear combination of these brackets can give opposite signs on $x_1^{l-1}x_2$ and $x_2x_1^{l-1}$, as in $[x_1^{l-1}, x_2]$. Hence, $[x_1^{l-1}, x_2]$ is not in $[[A_2, A_2], A_2]$.

Consider the Lie algebra $\mathfrak{gl}(2, \mathbb{C})$. Then it has an action on $B_{2,2}[l]$ since it has a natural action on the generators $\{x_1, x_2\}$. By direct computation, we can see that $[x_1^{l-1}, x_2]$ is a highest weight vector for $\mathfrak{gl}(2, \mathbb{C})$ with weight $(l-1, 1)$. From the representation theory of $\mathfrak{gl}(2, \mathbb{C})$, it follows that this vector generates an $(l-1)$-dimensional irreducible representation of $\mathfrak{gl}(2, \mathbb{C})$ contained in $B_{2,2}[l]$.

Hence, $\dim B_{2,2}[l] \geq l - 1$, and the conclusion follows from the argument given in the beginning of the proof. □

2.3. **The $n = 3$ Case.**

Proposition 2.2. *For $l \geq 2$, the $l^2 - 1$ non-zero elements of the form $[x_1^{i_1}x_2^{i_2}, x_3^{i_3}]$ and $[x_1^{i_1}x_3^{i_3}, x_2^{i_2}]$ for $i_1 + i_2 + i_3 = l$ constitute a spanning set for $B_{3,2}[l]$.*

Proof. By a similar argument as before, every element of $B_{3,2}$ can be expressed as a linear combination of the brackets $[a, x_1]$, $[b, x_2]$, and $[c, x_3]$, where a, b, and c are monomials with degree no less than 1.

Consider $[a, x_1]$. This may be written as a constant multiple of a bracket of the form $[x_1^{i_1}, a_1]$, where a_1 is a product of only x_2's and x_3's. We then write $[x_1^{i_1}, a_1]$ as the sum of $[a_1x_1^{i_1-1}, x_1]$ with brackets of the form $[x_1^{i_1}d_1, x_2]$ and $[x_1^{i_1}d_2, x_3]$, where d_1 and d_2 are products of only x_2's and x_3's.

We can write $[x_1^{i_1}d_1, x_2] = k_1[x_1^{i_1}x_3^{i_3}, x_2^{i_2}]$ and $[x_1^{i_1}d_2, x_3] = k_2[x_1^{i_1}x_2^{i_2}, x_3^{i_3}]$. Noting that $[a_1x_1^{i-1}, x_1]$ is a constant multiple of $[x_1^{i_1}, a_1]$, we may now solve for $[x_1^{i_1}, a_1]$, realizing it as a linear combination of $[x_1^{i_1}x_3^{i_3}, x_2^{i_2}]$ and $[x_1^{i_1}x_2^{i_2}, x_3^{i_3}]$. Therefore $[a, x_1]$ is a linear combination of $[x_1^{i_1}x_3^{i_3}, x_2^{i_2}]$ and $[x_1^{i_1}x_2^{i_2}, x_3^{i_3}]$.

By performing a similar analysis on $[b, x_2]$ and $[c, x_3]$, we find that every element of $B_{3,2}$ can be expressed as a linear combination of $[x_1^{i_1}x_3^{i_3}, x_2^{i_2}]$, $[x_1^{i_1}x_2^{i_2}, x_3^{i_3}]$, and $[x_2^{i_2}x_3^{i_3}, x_1^{i_1}]$. Noting that $[x_1^{i_1}x_3^{i_3}, x_2^{i_2}] + [x_1^{i_1}x_2^{i_2}, x_3^{i_3}] + [x_2^{i_2}x_3^{i_3}, x_1^{i_1}] = 0$, we have that every element of $B_{3,2}$ can be expressed as a linear combination of $[x_1^{i_1}x_3^{i_3}, x_2^{i_2}]$ and $[x_1^{i_1}x_2^{i_2}, x_3^{i_3}]$. □

By using a similar method to the two variable case, we can consider the $\mathfrak{gl}(3, \mathbb{C})$ action and find that the element $[x_1^{l-1}, x_2]$ (which we showed to be nonzero when considering the two variable case) is a highest weight vector of weight $(n-1, 1, 0)$. It then follows from the representation theory of $\mathfrak{gl}(3, \mathbb{C})$ that this vector generates a representation of dimension $l^2 - 1$, and hence the dimension of $B_{3,2}[l]$ is at least $l^2 - 1$. Combining with Proposition 2.2 we have

Theorem 2.2. *For any $\mathbf{i} = (i_1, i_2, i_3) \in (\mathbb{Z}_{>0})^3$, $[x_1^{i_1}x_2^{i_2}, x_3^{i_3}]$ and $[x_1^{i_1}x_3^{i_3}, x_2^{i_2}]$ constitute a basis for $B_{3,2}[\mathbf{i}]$.*

Remark 2.1. The proof of Proposition 2.1 mimics the manipulation of 1-forms done by Feigin and Shoikhet [FS], except in the language of brackets.

Theorems 2.1 and 2.2 can be obtained from Theorems 1.3 and 1.4 in [FS], respectively. But here we have given new direct proofs of these theorems without using the results of [FS]. We have not been able to generalize these proofs to the case $n \geq 4$.

3. The structure of $B_{n,2}$ for general n

3.1. **The Main Theorem about $B_{n,2}$.** Let P_n be the set of all permutations of $1, 2, \dots, n$ which have the form $(2,3)^{\delta_2}(3,4)^{\delta_3}\cdots(n-1,n)^{\delta_{n-1}}$, where (i,j) is the permutation of i and j, and $\delta_i = 0$ or 1. From now on, let $\mathbf{i} = (i_1, \dots, i_n) \in (\mathbb{Z}_{>0})^n$.

Theorem 3.1. *The basis elements for $B_{n,2}[\mathbf{i}]$ are the 2^{n-2} brackets given by $[x_{p(1)}^{i_{p(1)}}\cdots x_{p(n-1)}^{i_{p(n-1)}}, x_{p(n)}^{i_{p(n)}}]$ for $p \in P_n$.*

In particular, we have

$$\dim B_{n,2}[\mathbf{i}] = 2^{n-2}.$$

Remark 3.1. Note that if some i_s is zero then a basis of $B_{n,2}[\mathbf{i}]$ is given by Theorem 3.1 for a smaller number of variables. Thus, Theorem 3.1 provides a basis of $B_{n,2}[l]$ for any l, and thus a homogeneous basis of $B_{n,2}$. An interesting property of this basis is that it consists of elements whose monomials are non-redundant (i.e. every letter occurs only once in some power).

The proof of Theorem 3.1 is given in the next three subsections.

3.2. **The Feigin-Shoikhet Isomorphism.** We will use the isomorphism in [FS] between $B_{n,2}$ and $\Omega^{\text{even}+}_{\text{closed}}(\mathbb{C}^n)$, the closed even differential forms with positive degree, to prove the main theorem. Recall ϕ_n is a homomorphism of algebras:

$$\phi_n : A_n \to \Omega^{\text{even}}(\mathbb{C}^n)_*$$

which takes $x_i \in A_n$ to $x_i \in \Omega^0(\mathbb{C}^n)$ and

$$\phi_n(x_i x_j) = x_i * x_j = x_i x_j + dx_i \wedge dx_j.$$

Feigin and Shoikhet proved that ϕ_n induces an isomorphism

$$\phi_n : B_{n,2} \to \Omega^{\text{even}+}_{\text{closed}}(\mathbb{C}^n).$$

Now let $w \in \Omega^p(\mathbb{C}^n)$ be a p-form. We say that w has multidegree $\mathbf{i}$ if every x_s occurs i_s times in every monomial of w. Define $\Omega^p(\mathbb{C}^n)[\mathbf{i}]$ to be the space of all forms of multidegree $\mathbf{i}$.

The main theorem is then a consequence of the following two lemmas:

Lemma 3.1. $\dim \Omega^{\text{even}+}_{\text{closed}}(\mathbb{C}^n)[\mathbf{i}] = 2^{n-2}$.

Lemma 3.2. *The 2^{n-2} brackets described in the main theorem are linearly independent.*

3.3. **Proof of Lemma 3.1.** We first prove a more basic lemma, from which Lemma 3.1 will follow.

Lemma 3.3. $\dim \Omega^p_{\text{closed}}(\mathbb{C}^n)[\mathbf{i}] = \binom{n-1}{p-1}$.

Proof. By the Poincaré Lemma, the De Rham differential defines an isomorphism $d : \Omega^{p-1}(\mathbb{C}^n)[\mathbf{i}]/\Omega^{p-1}_{\text{closed}}(\mathbb{C}^n)[\mathbf{i}] \to \Omega^p_{\text{closed}}(\mathbb{C}^n)[\mathbf{i}]$.

Hence, if $D(p) := \dim \Omega^p_{\text{closed}}(\mathbb{C}^n)[\mathbf{i}]$, we have the recurrence relation:

$$D(p) = \binom{n}{p-1} - D(p-1), \text{ and } D(0) = 0.$$

A simple inductive argument shows $D(p) = \binom{n-1}{p-1}$, as desired. □

Lemma 3.1 now follows from a simple combinatorial identity:

$$\dim \Omega^{\text{even}+}_{\text{closed}}(\mathbb{C}^n)[\mathbf{i}] = \sum_{k=1}^{\infty} \dim \Omega^{2k}_{\text{closed}}(\mathbb{C}^n)[\mathbf{i}] = \sum_{k=1}^{\infty} \binom{n-1}{2k-1} = 2^{n-2}.$$

3.4. **Proof of Lemma 3.2.** We begin by computing the image under the map ϕ_n of the brackets with the form given in the statement of the main theorem.

Lemma 3.4. *We have*

$$\phi_n(x_1^{i_1} \dots x_n^{i_n}) = x_1^{i_1} \dots x_n^{i_n} \sum_{\substack{S \subset \{1,\dots,n\} \\ |S| \ even}} \bigwedge_{k \in S} i_k \frac{dx_k}{x_k},$$

where the indices in the wedge product are in increasing order.

Proof. We prove this by induction. For $n = 1$, we have $\phi_n(x_1^{i_1}) = x_1^{i_1}$, as desired. Assume the lemma is true for n. Then

$$\phi_n(x_1^{i_1} \cdots x_{n+1}^{i_{n+1}}) = \phi_n(x_1^{i_1} \cdots x_n^{i_n}) * x_{n+1}^{i_{n+1}} = \left(x_1^{i_1} \cdots x_n^{i_n} \sum_{\substack{S \subset \{1,\dots,n\} \\ |S| \text{ even}}} \bigwedge_{k \in S} i_k \frac{dx_k}{x_k} \right) * x_{n+1}^{i_{n+1}}.$$

Note that in the expansion of the last expression, the sum of the $2l$-forms comes from $d[(2l-2)\text{-forms}] \wedge dx_{n+1}^{i_{n+1}} + 2l\text{-forms} \wedge x_{n+1}^{i_{n+1}}$. The first term gives all $2l$-forms which contain dx_{n+1}, whereas the second term gives all $2l$-forms which do not contain dx_{n+1}. Together, all possible $2l$-forms appear in the expansion. These forms correspond to the subsets S of $\{1, \dots, n+1\}$ with exactly $2l$ elements. It is not hard to see that the coefficients of these forms are precisely the ones in the lemma. □

By direct computation, we have:

Corollary 3.1. *Let* $\omega_S(x_1, \dots, x_n) = 2x_1^{i_1} \cdots x_n^{i_n} \bigwedge_{k \in S} i_k \frac{dx_k}{x_k}$ *for* $S \subset \{1, \dots, n\}$. *Then for* $i_1, \dots, i_n > 0$*, we have*

$$\phi_n([x_1^{i_1} \cdots x_{n-1}^{i_{n-1}}, x_n^{i_n}]) = \sum_{\substack{n \in S \subset \{1,\dots,n\} \\ |S| \ even}} \omega_S(x_1, \dots, x_n),$$

where the indices in the wedge product are in increasing order.

Notice that for $p \in P_n$, we have

$$\phi_n([x_{p(1)}^{i_{p(1)}} \cdots x_{p(n-1)}^{i_{p(n-1)}}, x_{p(n)}^{i_{p(n)}}]) = \sum_{\substack{p(n) \in S \subset \{1,\dots,n\} \\ |S| \text{ even}}} \epsilon(S,p)\ \omega_S(x_{p(1)}, \dots, x_{p(n)}),$$

where $\epsilon(S, p) = \pm 1$, depending on the choice of S and p.

Denote $\omega_S(x_{p(1)}, \dots, x_{p(n)})$ by ω_S^p. We are now ready to prove Lemma 3.2.

Proof of lemma 3.2. We proceed by induction on the number of variables. It is easy to see that the lemma is true for $n = 2, 3$ from the results in Section 2. Assume the lemma is true up to $n \geq 3$. We now prove the lemma for $n+1$ variables.

Let P^1_{n+1} be the set of permutations which contain $(n, n+1)$ (i.e., with $\delta_n = 1$), and P^2_{n+1} be its complement in P_{n+1}. Then we have $P_{n+1} = P^1_{n+1} \cup P^2_{n+1}$.

By applying the isomorphism ϕ_{n+1}, it is enough to show that the 2^{n-1} forms:

$$\omega^p = \sum_{\substack{p(n+1)\in S\subset\{1,\dots,n+1\}\\ |S| \text{ even}}} \epsilon(S,p)\ \omega_S^p$$

are linearly independent.

For any $p \in P_{n+1}^1$, the components of ω^p which do not contain dx_{n+1} are precisely the forms in the n variable case which appear in $\omega^{p'}$, where $p' \circ (n, n+1) = p$. Hence, the ω^p for $p \in P_{n+1}^1$ are linearly independent.

Furthermore, since every form which appears in ω_S^p for $p \in P_{n+1}^2$ contains dx_{n+1}, we only need to show that the forms ω^p with $p \in P_{n+1}^2$ are linearly independent.

Let $\mathcal{S} = \{S \subset \{1,\dots,n+1\} | 1, n+1 \in S \text{ and } |S| \text{ is even}\}$. For any $p \in P_{n+1}^2$, $\sum_{S\in\mathcal{S}} \epsilon(S,p)\omega_S^p$ is a linear combination of even forms containing $dx_1 \wedge dx_{n+1}$. It is enough to show that these 2^{n-2} sums are linearly independent.

It suffices to prove the invertibility of the $2^{n-2} \times 2^{n-2}$ matrix where each row represents a bracket $p \in P_{n+1}^2$, each column represents a form $S \in \mathcal{S}$, and whose entries are the $\epsilon(S,p)$'s. For the rows, we choose the order recursively, beginning with the identity permutation. Given the first 2^k elements, the next 2^k elements are given by composition with $(k+2, k+3)$.

For the columns, we will represent the form $dx_{j_1} \wedge \cdots \wedge dx_{j_m}$ by the ordered m-tuple $(j_1, \dots, j_m)$. We again choose the order recursively, beginning with $(1, n+1)$. Given the first 2^k columns, the next 2^k columns are given by appending $k+2, k+3$ to the first 2^{k-1} columns and by replacing $k+2$ with $k+3$ in the next 2^{k-1} columns.

We prove the invertibility of this matrix by induction on n. When $n = 3$, the matrix is given by $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, which is clearly invertible. Assume it is true for $n \geq 3$.

Divide the matrix into equal fourths. Call the submatrices α_n, β_n, γ_n, δ_n. Note that α_n is the matrix for the n variable case. Now further divide each of these submatrices into four more equal quadrants. Call them $\alpha_n^{1,1}$, $\alpha_n^{1,2}$, $\alpha_n^{2,1}$, $\alpha_n^{2,2}$, etc. In the case of α_n, we have $\alpha_n^{1,1} = \alpha_{n-1}$, $\alpha_n^{1,2} = \beta_{n-1}$, $\alpha_n^{2,1} = \gamma_{n-1}$, $\alpha_n^{2,2} = \delta_{n-1}$.

Because changing the position of n in the permutation has no effect on the sign of the forms which do not contain n, we have $\alpha_n = \gamma_n$ (and $\alpha_{n-1} = \gamma_{n-1}$). We also have $\alpha_n^{1,1} = \beta_n^{1,1}$ and $\alpha_n^{1,2} = \beta_n^{1,2}$ because the permutations in those rows leave $n-1$ and n fixed. By similar analysis of the permutations, we can show the matrix has the form:

$$\begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} = \begin{pmatrix} \alpha_{n-1} & \beta_{n-1} & \alpha_{n-1} & \beta_{n-1} \\ \alpha_{n-1} & \delta_{n-1} & * & \beta_{n-1} \\ \alpha_{n-1} & \beta_{n-1} & -\alpha_{n-1} & \beta_{n-1} \\ \alpha_{n-1} & \delta_{n-1} & * & \delta_{n-1} \end{pmatrix}.$$

Subtracting the last 2^{n-3} rows from the first 2^{n-3} rows gives

$$\begin{pmatrix} \alpha_{n-1} & \beta_{n-1} & \alpha_{n-1} & \beta_{n-1} \\ \alpha_{n-1} & \delta_{n-1} & * & \beta_{n-1} \\ 0 & 0 & 2\alpha_{n-1} & 0 \\ 0 & 0 & * & \beta_{n-1} - \delta_{n-1} \end{pmatrix}.$$

It remains to show that α_n, α_{n-1}, and $\beta_{n-1}-\delta_{n-1}$ are invertible. α_n and α_{n-1} are invertible by the induction hypothesis. We see that $\beta_{n-1}-\delta_{n-1}$ is invertible by subtracting the last half of the rows from the first half in the invertible matrix

$$\alpha_n = \begin{pmatrix} \alpha_{n-1} & \beta_{n-1} \\ \alpha_{n-1} & \delta_{n-1} \end{pmatrix}.$$

□

3.5. **Finite order case.**

Theorem 3.2. *For any* $\mathbf{i}=(i_1,\dots,i_n)\in(\mathbb{Z}_{>0})^n$,

$$B^R_{n,2}[\mathbf{i}] = \begin{cases} 0 & \text{if } i_s \geq m_s \text{ for some } s; \\ B_{n,2}[\mathbf{i}] & \text{if } i_s < m_s \text{ for all } s. \end{cases}$$

Proof. It is clear that if $i_s < m_s$ for all s then the relations have no effect, so the statement of the theorem holds. Now assume that for some s, $i_s \geq m_s$. Then the images in $B^R_{n,2}$ of all the basis elements from Theorem 3.1 are zero. But these elements must span $B^R_{n,2}[\mathbf{i}]$, which implies that this space is zero, as desired. □

Remark 3.2. In this proof it is important that the basis elements involve only non-redundant monomials, see remark 3.1.

4. The structures of $B_{2,m}[r,1]$ and $B_{2,m}[r,2]$

Let W_n be the Lie algebra of polynomial vector fields on $\mathbb{C}^n$. In [FS], Feigin and Shoikhet described an action of W_n on $B_{n,k}$.

From now on, let $A=A_2$ be the free algebra generated by x, y and $L_i = L_i(A)$. We denote by $L_i[r,s]$ (and $B_{2,m}[r,s]$) the space of elements of L_i (and $B_{2,m}$) with multi-degree (r,s), that is, consisting of monomials having r copies of x and s copies of y. The purpose of this section is to compute the bases of $B_{2,m}[r,1]$ and $B_{2,m}[r,2]$ which will help us to find the W_2-module structures of $B_{2,3}$ and $B_{2,4}$, and obtain some information about the structure of $B_{2,m}$ for general m in the subsequent sections.

Define $\mathrm{ad}_a b = [a,b]$. Then we introduce the following elements:

$$\begin{aligned} b^{(l)}_{i,j,k} &= \mathrm{ad}^i_x \circ \mathrm{ad}_y \circ \mathrm{ad}^j_x \circ \mathrm{ad}_{x^k}(y^l); \\ b^{(l)}_{i,j} &= \mathrm{ad}^i_x \circ \mathrm{ad}_{x^j}(y^l). \end{aligned}$$

Notice that $b^{(l)}_{i,j,k}$ is an element in $L_{i+j+3}[i+j+k, l+1]$, and $b^{(l)}_{i,j}$ is an element in $L_{i+2}[i+j,l]$. For simplicity, when $l=1$, we denote $b^{(1)}_{i,j,k}$ by $b_{i,j,k}$ and $b^{(1)}_{i,j}$ by $b_{i,j}$.

4.1. **Structure of $B_{2,m}[r,1]$.**

Theorem 4.1. *For $m \geq 2$ we have:*

$$B_{2,m}[r,1] = \begin{cases} 0, & r \leq m-2; \\ \mathbb{C}\cdot b_{m-2,r-m+2}, & r \geq m-1. \end{cases}$$

First we prove two lemmas.

Lemma 4.1. *For $r \geq 1$ and $s \geq 0$, the linear map $\frac{\partial}{\partial x}:\ A_2[r,s]\to A_2[r-1,s]$ is surjective.*

Proof. We do induction on s. For $s = 0$ the statement is obviously true. Now suppose $s > 0$. Every monomial in $A_2[r-1, s]$ has the form myx^a, where $0 \leq a \leq r-1$ and m is a monomial in $A_2[r-a-1, s-1]$. By the induction hypothesis there exists a polynomial p such that $\frac{\partial}{\partial x} p = m$.

Now we show by induction on a that there exists a polynomial q such that $\frac{\partial}{\partial x} q = myx^a$. If $a = 0$, put $q = py$. Suppose $a > 0$, then by the induction hypothesis there exists a polynomial f in $A_2[r, s]$ such that $\frac{\partial}{\partial x} f = pyx^{a-1}$. Then $q = pyx^a - af$ will be a solution. □

Lemma 4.2. *The kernel of the map* $\frac{\partial}{\partial x} : A[r, 1] \to A[r-1, 1]$ *is* $\mathbb{C} \cdot b_{r-1,1}$.

Proof. By Lemma 4.1, $\dim \ker \frac{\partial}{\partial x} = \dim A[r, 1] - \dim A[r-1, 1] = (r+1) - r = 1$. The element $b_{r-1,1}$ is in the kernel of $\frac{\partial}{\partial x}$, and it is non-zero. Thus the lemma is proved. □

Now we can prove the theorem.

Proof of Theorem 4.1. We first prove that for $m \leq r+1$ the element $b_{m-2,r-m+2}$ spans $B_{2,m}[r, 1]$. We do this by induction on r. For $r = 1$, $[x, y]$ spans $B_{2,2}[1, 1]$. Now suppose $r > 1$. The statement is true for $m = r+1$ obviously. Suppose $m \leq r$. For any $w \in B_{2,m}[r, 1]$, we have $\frac{\partial}{\partial x} w \in B_{2,m}[r-1, 1]$. By induction hypothesis $\frac{\partial}{\partial x} w = cb_{m-2,r-m+1}$ for some constant c. Let $p = \frac{c}{r-m+2} b_{m-2,r-m+2}$. By Lemma 4.2 we have $w - p \in \ker \frac{\partial}{\partial x} \subseteq L_{r+1}$. Since $m \leq r$, $w = p \in B_{2,m}$. So the statement is proved by induction.

Therefore we have $\dim B_{2,m}[r, 1] \leq 1$ for $m \leq r+1$, and $\dim B_{2,m}[r, 1] = 0$ for $m > r+1$. Since $\sum_{1 \leq m \leq r+1} \dim B_{2,m}[r, 1] = \dim A_2[r, 1] = r+1$ and $\dim B_{2,1}[r, 1] = 1$, we have $\dim B_{2,m}[r, 1] = 1$ for $m \leq r+1$. □

4.2. **Structure of $B_{2,m}[r, 2]$.**

Theorem 4.2. *For $m \geq 2$ we have:*

$$\dim B_{2,m}[r, 2] = \begin{cases} m-1, & m \leq r+1; \\ \lfloor \frac{r+1}{2} \rfloor, & m = r+2. \end{cases}$$

A basis of $B_{2,m}[r, 2]$ for $m \leq r+1$ is given by the $m-1$ elements

$$b_{i,j,r-m+3} \text{ for } i+j = m-3, \text{ and } b^{(2)}_{m-2,r-m+2}.$$

Before starting the proof we will prove several lemmas.

Lemma 4.3. *The set $S_r = \{b_{i,j,1} | i+j = r-1, j \text{ is even }\}$ is a basis of $L_{r+2}[r, 2]$.*

Proof. At first, we prove elements in S_r are independent by induction on r. For $r = 1$ the claim is obvious. Assume it is true for $r-1$. If r is even, then these elements have the form $[x, b_{i,j,1}]$ where $i+j = r-2$ and j is even. These elements are independent by the induction hypothesis because $\sum \alpha_{i,j} [x, b_{i,j,1}] = [x, \sum \alpha_{i,j} b_{i,j,1}]$ has the leading monomial xm where m is the leading monomial of $\sum \alpha_{i,j} b_{i,j,1}$.

If r is odd, by a similar argument as the even case, we only need to show that the element $b_{0,r-1,1}$ is independent from the others. Since it is the only element which has the monomial $2yx^r y$, the conjectured basis elements are independent by induction.

Now we show that elements in S_r span $L_{r+2}[r,2]$. It is enough to show that $b_{0,r-1,1}$ with even r is in $[x, L_{r+1}]$. Applying Jacobi identity repeatedly, we obtain

$$\begin{aligned} b_{0,r-1,1} &= [[[y,x],x],[x,\dots[x,y]\dots]] + [x, L_{r+1}] \\ &= \dots = [[[y,x],\dots],x],[x,\dots[x,y]\dots]] + [x, L_{r+1}], \end{aligned}$$

where the last element has equal number i of copies of x in the first and the second major brackets. But this element is zero, so the original element $b_{0,r-1,1}$ is in $[x, L_{r+1}]$. □

Lemma 4.4. *The set $S'_r = S_r \cup \{b_{i,j,2} | i+j = r-2\} \cup \{b^{(2)}_{r-1,1}\}$ is a basis of $L_{r+1}[r,2]$. In particular, the set $S'_r - S_r$ is a basis of $B_{2,r+1}[r,2]$.*

Proof. At first, we prove by induction on r that the elements in S'_r are independent. When $r = 1$, it is easy to see. Now suppose $r > 1$.

If r is even, all the elements in S'_r except the element $b_{0,r-2,2}$ will have the form $[x,b]$ where b is in S'_{r-1} which is the basis of $L_r[r-1,2]$. As in the proof of Lemma 4.3, we only need to show $b_{0,r-2,2}$ is independent from the others. Since its leading monomial is $2yx^ry$, which is not found in the others, the elements in S'_r are independent.

If r is odd, the elements in S'_r are $[x,b]$ where $b \in S'_{r-1}$, and two other elements $b_{0,r-2,2}$, $b_{0,r-1,1}$. We observe that $b_{0,r-1,1}$ has a leading monomial $2yx^ry$ which no other elements in S'_r have, therefore $b_{0,r-1,1}$ is independent from them.

Now let $r = 2i+1$. By direct computation, we have

$$\begin{aligned} b_{0,r-2,2} &= (2i-1)xyx^{2i} + (-2i^2+3i)x^2yx^{2i-1}y + 0xyx^{2i-1}yx + \cdots, \\ b_{1,r-3,2} &= 2xyx^{2i} + (-2i+2)x^2yx^{2i-1}y + 0xyx^{2i-1}yx + \cdots, \\ b_{2,r-3,1} &= 0xyx^{2i} + 2x^2yx^{2i-1}y - 4xyx^{2i-1}yx + \cdots. \end{aligned}$$

Since these monomials are not present in the other elements of S'_r, we have that the element $b_{0,r-2,2}$ is independent from the other elements. Therefore by induction all the elements of S'_r are independent.

Now we show that S'_r is a spanning set by induction on r. For $r = 1$ the statement is true. Assuming the statement for $r-1$, we obtain that $[x, L_r[r-1,2]]$ is spanned by elements $[x,b]$ for $b \in S'_{r-1}$. Since the space $[y, L_r[r,1]]$ is spanned by $b_{0,r-1,1}$ we only need to show that $[x^2, b_{0,r-3,1}]$ is in the spanning space of S'_r. By repeatedly applying Jacobi identity, we have:

$$\begin{aligned} [x^2, b_{0,r-3,1}] &= [[x^2,y],[x,\dots[x,y]\dots]] + [y, L_r[r,1]] \\ &= [[[x^2,y],x],[x,\dots[x,y]\dots]] + [x, L_r[r-1,2]] + [y, L_r[r,1]] \\ &= \dots = [\dots[x^2,y],x],\dots],x],y] + [x, L_r[r-1,2]] + [y, L_r[r,1]]. \end{aligned}$$

So the set S'_r spans $L_{r+1}[r,2]$ and we proved the lemma. □

Lemma 4.5. *The linear map $\frac{\partial}{\partial x} : A_2[r,2] \to A_2[r-1,2]$ has the property*

$$\ker \frac{\partial}{\partial x} \cap [A,A] \subseteq L_{r+1}[r,2].$$

Proof. From Lemma 4.3 and 4.4 it follows that

$$\dim B_{2,r+1}[r,2] = r, \dim B_{2,r+2}[r,2] = \lfloor \frac{r+1}{2} \rfloor.$$

We have the induced linear map $\frac{\partial}{\partial x}|_{B_{2,r+1}[r,2]} : B_{2,r+1}[r,2] \to B_{2,r+1}[r-1,2]$ which is surjective because $\frac{\partial}{\partial x} b_{i,j,2} = 2b_{i,j,1}$. So

$$\dim \ker \frac{\partial}{\partial x}|_{B_{2,r+1}[r,2]} = \dim B_{2,r+1}[r,2] - \dim B_{2,r+1}[r-1,2] = r - \lfloor \frac{r}{2} \rfloor = \lfloor \frac{r+1}{2} \rfloor.$$

Also $\frac{\partial}{\partial x}$ maps $B_{2,r+2}[r,2]$ to zero, so $\dim(\ker \frac{\partial}{\partial x} \cap L_{r+1}[r,2]) = 2\lfloor \frac{r+1}{2} \rfloor$.

If r is odd, $2\lfloor \frac{r+1}{2} \rfloor = r+1$, so $\ker \frac{\partial}{\partial x} \subseteq L_{r+1}[r,2]$. If r is even, $2\lfloor \frac{r+1}{2} \rfloor = r$. In this case we consider the induced map $\frac{\partial}{\partial x}|_{B_{2,1}[r,2]} : B_{2,1}[r,2] \to B_{2,1}[r-1,2]$. These spaces are the spaces of cyclic words, so $\dim \ker \frac{\partial}{\partial x}|_{B_{2,1}[r,2]} \geq \dim B_{2,1}[r,2] - \dim B_{2,1}[r-1,2] = \lceil \frac{r+1}{2} \rceil - \lceil \frac{r}{2} \rceil = 1$ if r is even. So for even r, a one-dimensional subspace of $\ker \frac{\partial}{\partial x}$ lies in $B_{2,1}$. Therefore $\ker \frac{\partial}{\partial x} \cap [A,A] \subseteq L_{r+1}[r,2]$. □

Now we prove Theorem 4.2.

Proof of Theorem 4.2. Let us prove that any element w of $B_{2,m}[r,2]$ is a linear combination of the conjectured basis elements. We do induction on r.

If $r = 1$, both $[x, y^2]$ and $[y,[x,y]]$ are basis elements. If $r > 1$, we have $\frac{\partial}{\partial x} w$ which has degree $r-1$ in x. So by induction hypothesis $\frac{\partial}{\partial x} w = \sum_{i+j=m-3} \alpha_{i,j} b_{i,j,r-m+2} + \alpha b^{(2)}_{m-2,r-m+1}$. Put $p = \sum_{i+j=m-3} \frac{\alpha_{i,j}}{r-m+3} b_{i,j,r-m+3} + \frac{\alpha}{r-m+2} b^{(2)}_{m-2,r-m+2}$, then $\frac{\partial}{\partial x}(w-p) = 0$. So $w - p \in \ker \frac{\partial}{\partial x} \cap [A,A] \subseteq L_{r+1}$ by Lemma 4.5. So $w = p$ in L_m/L_{m+1} $(m \leq r)$, and p is a combination of basis elements. Therefore the required elements span $B_{2,m}[r,2]$ and $\dim B_{2,m} \leq m-1$ $(2 \leq m \leq r)$.

We know that $\dim B_{2,1} = \lceil \frac{r+1}{2} \rceil$, $\dim B_{2,r+2} = \lfloor \frac{r+1}{2} \rfloor$ and $\dim B_{2,m} \leq m-1$ $(2 \leq m \leq r)$. But these numbers have to sum to $\dim A_2[r,2] = \frac{(r+1)(r+2)}{2}$, so $\dim B_{2,m} = m-1$ $(2 \leq m \leq r)$ and the found spanning elements actually form a basis for $B_{2,m}[r,2]$. □

5. The multiplicities of $\mathcal{F}_{(p,1)}$ and $\mathcal{F}_{(p,2)}$ in $B_{2,m}$

We consider the W_n-modules on which the Euler vector field $e = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ is semisimple with finite-dimensional eigenspaces and has its eigenvalues bounded from below. Let W_n^0 be the subalgebra of W_n of vector fields vanishing at the origin.

Let $\mathcal{F}_D = \mathrm{Hom}_{U(W_n^0)}(U(W_n), F_D)$ be the irreducible W_n-module coinduced from a $\mathfrak{gl}(n,\mathbb{C})$-module F_D where D is a Young diagram having more than one column. (For reference about modules $\mathcal{F}_D$ see [FF] or [F]; for reference about Schur modules F_D see [Ful]). Let (p,k), where $p \geq k$ are positive integers, denote a two-row Young diagram with p boxes in the first row and k boxes in the second row.

In this section we prove

Theorem 5.1. *For $m \geq 3$, the W_2-module $B_{2,m}$ has in its Jordan-Hölder series one copy of the module $\mathcal{F}_{(m-1,1)}$, $\lfloor \frac{m-2}{2} \rfloor$ copies of $\mathcal{F}_{(m-1,2)}$, and $\lfloor \frac{m-3}{2} \rfloor$ copies of $\mathcal{F}_{(m-2,2)}$. The rest of the irreducible W_2-modules in the Jordan-Hölder series of $B_{2,m}$ are of the form $\mathcal{F}_{(p,k)}$ where $k \geq 3$.*

Proof. If $B_{2,m}$ contains a module $\mathcal{F}_{(p)}$, then $\dim \mathcal{F}_{(p)}[r,0] = 1$ which contradicts $\dim B_{2,m}[r,0] = 0$. Similarly, $B_{2,m}$ cannot contain the module of exact one-forms. Therefore all the irreducible W_2-modules contained in $B_{2,m}$ are of the form $\mathcal{F}_{(p,k)}$ where $k \geq 1$.

At first, we find the multiplicities of the modules $\mathcal{F}_{(p,1)}$ in $B_{2,m}$. Notice that for modules $\mathcal{F}_{(p,k)}$ where $k \geq 2$, we have $\mathcal{F}_{(p,k)}[r,1] = 0$. We also have

$$\dim \mathcal{F}_{(p,1)}[r,1] = \begin{cases} 0, & r \leq p-1; \\ 1, & r \geq p. \end{cases}$$

Comparing this to Theorem 4.1, we obtain that $B_{2,m}$ has one copy of $\mathcal{F}_{(m-1,1)}$ and none of the other modules $\mathcal{F}_{(p,1)}$ where $p \neq m-1$.

Now let us find the multiplicities of the modules $\mathcal{F}_{(p,2)}$ in $B_{2,m}$. For modules $\mathcal{F}_{(p,k)}$ where $k \geq 3$ we have $\mathcal{F}_{(p,k)}[r,2] = 0$. We notice that

$$\dim \mathcal{F}_{(p,2)}[r,2] = \begin{cases} 0, & r \leq p-1; \\ 1, & r \geq p. \end{cases} \tag{1}$$

We also have

$$\dim \mathcal{F}_{(m-1,1)}[r,2] = \begin{cases} 0, & r \leq m-3; \\ 1, & r = m-2; \\ 2, & r \geq m-1. \end{cases}$$

By Theorem 4.2 we have

$$\dim B_{2,m}[r,2] - \dim \mathcal{F}_{(m-1,1)}[r,2] = \begin{cases} 0, & r \leq m-3; \\ \lfloor \frac{m-3}{2} \rfloor, & r = m-2; \\ m-3, & r \geq m-1. \end{cases}$$

From formula (1), we have that $B_{2,m}$ has $\lfloor \frac{m-3}{2} \rfloor$ copies of $\mathcal{F}_{(m-2,2)}$ and $\lfloor \frac{m-2}{2} \rfloor$ copies of $\mathcal{F}_{(m-1,2)}$, which together with the module $\mathcal{F}_{(m-1,1)}$ account for the dimensions of $B_{2,m}[r,2]$. Finally, we remark that there may be some copies of the modules $\mathcal{F}_{(p,k)}$ with $k \geq 3$ in $B_{2,m}$ which we cannot detect with the help of the structures of $B_{2,m}[r,1]$ and $B_{2,m}[r,2]$. □

We make the statement of this theorem more precise with the following

Proposition 5.1. *The module $\mathcal{F}_{(m-1,1)}$ is the last term of the Jordan-Hölder series of $B_{2,m}$, i.e. there is a projection map $B_{2,m} \twoheadrightarrow \mathcal{F}_{(m-1,1)}$.*

Proof. For $m \geq 4$, consider the subspaces $M_i := [A, [A, \dots [L_2, L_{m-i-2}] \dots]/L_{m+1}$ $(0 \leq i \leq m-4)$ of $B_{2,m}$. They are W_2-submodules of $B_{2,m}$ because W_2 acts on $B_{2,m}$ by derivations. So the quotient space $D_{2,m} := L_m/(L_{m+1} + M_0 + \cdots + M_{m-4})$ is a W_2-module.

We claim that $D_{2,m}$ is isomorphic to $\mathcal{F}_{(m-1,1)}$ as a W_2-module. Take an element $[p_1, [p_2, \dots [p_{m-1}, p_m] \dots]$ of $D_{2,m}$. By the relations in $B_{2,3}$, we can assume that p_m is either x or y. We notice that modulo M_i we can interchange the polynomials p_{i+1} and p_{i+2} in the expression $[p_1, [p_2, \dots [p_{m-1}, p_m] \dots]$. By such permutations, we can make p_1 either x or y. Similarly, using the relations in $B_{2,3}$ and permutations, we can make each of the elements $p_2, p_3, \dots, p_{m-2}$ either x or y. Moreover, using permutations, we can order $p_1, \dots, p_{m-2}$ so that $p_1, \dots, p_k = x$ and $p_{k+1}, \dots, p_{m-2} = y$ for some $0 \leq k \leq m-2$.

For the elements of $D_{2,m}$, we introduce the notation $c_{a,b,i,j} := \mathrm{ad}_x^a \circ \mathrm{ad}_y^b \circ \mathrm{ad}_{x^i}(y^j)$. From the previous considerations, we obtain that $D_{2,m}[l]$ is spanned by the elements $c_{a,m-a-2,i,l-m-i+2}$, where $0 \leq a \leq m-2$ and $1 \leq i \leq l-m-1$. The number of these spanning elements of $D_{2,m}$ is $(m-1)(l-m-1)$.

In particular, $D_{2,m}[m]$ is spanned by the $m-1$ elements $e_i = c_{i-1,m-i-1,1,1}$, where $1 \leq i \leq m-1$. We notice that in $D_{2,m}$ we have

$$y\frac{\partial}{\partial x}e_i = \sum_{j=1}^{i-1} \mathrm{ad}_x^{j-1} \circ \mathrm{ad}_y(c_{i-j-1,m-i-1,1,1}) = (i-1)c_{i-2,m-i,1,1} = (i-1)e_{i-1}.$$

We notice that e_1 is not zero in A since it has a leading monomial xy^{m-1} with coefficient $(-1)^{m-2} \neq 0$. We also notice that e_1 has multi-degree $(1, m-1)$ in L_m and $L_{m+1}[m] = M_i[m] = 0$ for $0 \leq i \leq m-4$. It follows that e_1 is not zero in the quotient space $D_{2,m}$.

From this we derive that $e_1, e_2, \ldots, e_{m-1}$ are independent in $D_{2,m}$ because for $1 \leq k \leq m-1$ if $a_k \neq 0$ we have

$$(y\frac{\partial}{\partial x})^{k-1}(\sum_{i<k} a_i e_i + a_k e_k) = a_k e_1 \neq 0 \tag{2}$$

Therefore $e_1, \ldots, e_{m-1}$ form a basis of $D_{2,m}$.

Now we show that the W_2-module $D_{2,m}$ is irreducible. Suppose it is not . Then it has a W_2-submodule S. Because $D_{2,m}$ starts in degree (eigenvalue of the Euler operator) m, S has to start in degree at least m. We notice that the irreducible modules in the Jordan-Hölder series of $D_{2,m}$ which start in degree m have the sum of their dimensions in degree m equal to $m-1 = \dim D_{2,m}[m]$. Therefore the sum of their dimensions in a degree $l > m$ will be $(m-1)(l-m+1)$. But we already showed that $\dim D_{2,m}[l] \leq (m-1)(l-m+1)$. Therefore all the irreducible modules in the Jordan-Hölder series of $D_{2,m}$ start in degree m. But the equality (2) shows that $D_{2,m}[m]$ belongs to a single W_2-submodule of $D_{2,m}$ generated by e_1. Therefore $D_{2,m}$ is isomorphic to an irreducible W_2-module, which starts in degree m and has dimension $m-1$ in this degree. So this module is $\mathcal{F}_D$ where $D = (p, k)$ with $p + k = m$ and $p - k = m - 2$. This is $\mathcal{F}_{(m-1,1)}$. □

6. The structures of $B_{2,3}$ and $B_{2,4}$

In this section we find the W_2-module structures of $B_{2,3}$ and $B_{2,4}$. We will use characters of W_2-modules which are formal power series in letters s, t. The character of a W_2-module M will be given by char $M = \sum \dim M[a,b] s^a t^b$, where $M[a,b]$ denotes the subspace of elements of M with weights a, b of the operators $x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}$.

First we compute the characters of the irreducible modules $\mathcal{F}_{(n,m)}$ for Young diagrams (n,m).

Proposition 6.1. *The character of $\mathcal{F}_{(n,m)}$ is given by*

$$\text{char } \mathcal{F}_{(n,m)} = s^m t^m \frac{t^{n-m} + t^{n-m-1}s + \cdots + s^{n-m}}{(1-s)(1-t)}.$$

Proof. This is true since to form an element of $\mathcal{F}_{(n,m)}[a,b]$ we have firstly to use m copies of x and m copies of y to produce the part $(dx \wedge dy)^{\otimes m}$; this accounts for the multiple $s^m t^m$ in the character formula. Next we have to choose $0 \leq i \leq n-m$ copies of x and $n-m-i$ copies of y to produce the symmetric part $(dx)^i \cdot (dy)^{n-m-i}$ of the tensor part of an element of $\mathcal{F}_{(n,m)}[a,b]$; this accounts for the sum $t^{n-m} + t^{n-m-1}s + \cdots + s^{n-m}$ in the numerator of the character formula. Lastly, we have to add a polynomial part to our element by multiplying it by s^{a-m-i} and t^{b-n+i}; this is accounted for by the multiples $\frac{1}{1-s} = \sum_{l\geq 0} s^l$ and $\frac{1}{1-t} = \sum_{l\geq 0} t^l$ in the character formula. □

By multiplying char $\mathcal{F}_{(n,m)}$ by $(1-s)(1-t)$, we obtain a polynomial with a leading monomial $s^n t^m$. Since all these polynomials for different diagrams (n,m) have different leading monomials, they are independent. Therefore the characters of different $\mathcal{F}_{(n,m)}$ are linearly independent.

Theorem 6.1. *The W_2-module $B_{2,3}$ is isomorphic to $\mathcal{F}_{(2,1)}$.*

Proof. From the results about $B_{2,2}$ we know that $[A[A,A]A,A] \subseteq L_3$. Since $[\mathbb{C}, L_2] = 0$ we have that $B_{2,3}$ is a quotient of $(S(\mathbb{C}^2)/\mathbb{C}) \otimes B_{2,2}$. By definition we have that $S(\mathbb{C}^2)$ is isomorphic to $\mathcal{F}_{(0,0)}$. By the results of [FS] we also have that $B_{2,2}$ is isomorphic to $\mathcal{F}_{(1,1)}$. So $B_{2,3}$ is a quotient of $(\mathcal{F}_{(0,0)}/\mathbb{C}) \otimes \mathcal{F}_{(1,1)}$. Therefore the irreducible modules in the Jordan-Hölder series of $B_{2,3}$ will be found among the irreducible modules contained in the module $(\mathcal{F}_{(0,0)}/\mathbb{C}) \otimes \mathcal{F}_{(1,1)}$. To find them, we compute the character of the last module:

$$\begin{aligned} & \text{char } (\mathcal{F}_{(0,0)}/\mathbb{C}) \otimes \mathcal{F}_{(1,1)} \\ = \; & \left(\frac{1}{(1-s)(1-t)} - 1\right)\frac{st}{(1-s)(1-t)} \\ = \; & \sum_{k\geq 0} st \frac{(s^k + s^{k-1}t + \cdots + t^k)}{(1-s)(1-t)} - \text{char } \mathcal{F}_{(1,1)} \\ = \; & \sum_{p\geq 1} \text{char } \mathcal{F}_{(p,1)} - \text{char } \mathcal{F}_{(1,1)} = \sum_{p\geq 2} \text{char } \mathcal{F}_{(p,1)}. \end{aligned}$$

But we know from Theorem 5.1 that the only copy of $\mathcal{F}_{(p,1)}$ in $B_{2,3}$ is $\mathcal{F}_{(2,1)}$. Therefore $B_{2,3}$ is isomorphic to $\mathcal{F}_{(2,1)}$.

□

Theorem 6.2. *The W_2-module $B_{2,4}$ has in its Jordan-Hölder series only two irreducible W_2-modules, $\mathcal{F}_{(3,1)}$ and $\mathcal{F}_{(3,2)}$ and each with multiplicity* 1*.*

Proof. From the results about $B_{2,2}$, we know that $[A[A,A]A,A] \subseteq L_3$. Since $[\mathbb{C}, L_3] = 0$ we have that $B_{2,4}$ is a quotient of $(S(\mathbb{C}^2)/\mathbb{C}) \otimes B_{2,3}$.

Since $S(\mathbb{C}^2)$ is isomorphic to $\mathcal{F}_{(0,0)}$ and from Theorem 6.1, we know that $B_{2,4}$ is a quotient of $(\mathcal{F}_{(0,0)}/\mathbb{C}) \otimes \mathcal{F}_{(2,1)}$. Therefore the irreducible modules in the Jordan-Hölder series of $B_{2,4}$ will be found among the irreducible modules contained in $(\mathcal{F}_{(0,0)}/\mathbb{C}) \otimes \mathcal{F}_{(2,1)}$. By a similar computation to the one in the proof of Theorem 6.1, we have

$$\text{char } (\mathcal{F}_{(0,0)}/\mathbb{C}) \otimes \mathcal{F}_{(2,1)} = \sum_{p\geq 3} \text{char } \mathcal{F}_{(p,1)} + \sum_{p\geq 2} \text{char } \mathcal{F}_{(p,2)}.$$

But we know from Theorem 5.1 that the only copy of $\mathcal{F}_{(p,1)}$ in $B_{2,4}$ is $\mathcal{F}_{(3,1)}$ and the only copy of $\mathcal{F}_{(p,2)}$ is $\mathcal{F}_{(3,2)}$. Therefore, the Jordan-Hölder series of the module $B_{2,4}$ contains exactly two irreducible W_2-modules $\mathcal{F}_{(3,1)}$ and $\mathcal{F}_{(3,2)}$.

□

Now we show that $B_{2,4}$ is not a direct sum of the modules $\mathcal{F}_{(3,1)}$ and $\mathcal{F}_{(3,2)}$ in its Jordan-Hölder series.

Proposition 6.2. *The W_2-module $B_{2,4}$ is isomorphic to a nontrivial extension of $\mathcal{F}_{(3,2)}$ by $\mathcal{F}_{(3,1)}$.*

Proof. For a W_2-module M, we denote by $M[k]$ the weight space of M for weight k of the Euler vector field in two variables. Notice that $B_{2,4}$ has a W_2-submodule $C_{2,4} = [[A_2, A_2], [A_2, A_2]]/L_5$. The lowest weight of $C_{2,4}$ is 5 and the lowest weight vectors are $a[[x^2, y], [x, y]] + b[[x, y^2], [x, y]]$. Therefore $C_{2,4}$ is isomorphic to $\mathcal{F}_{(3,2)}$. Since we have $\dim \mathcal{F}_{(3,2)}[4] = 2$, it follows that $[[x^2, y], [x, y]]$ and $[[x, y^2], [x, y]]$ form a basis of $C_{2,4}[4]$. From Theorem 6.2 it follows that the W_2-module $B_{2,4}/C_{2,4}$ is isomorphic to $\mathcal{F}_{(3,1)}$. So we have an exact sequence of W_2-modules

$$0 \to \mathcal{F}_{(3,2)} \to B_{2,4} \to \mathcal{F}_{(3,1)} \to 0.$$

We will now show that this sequence does not split. Since the diagram $(3, 2)$ has 5 cells, $\dim \mathcal{F}_{(3,2)}[4] = 0$. Then if we had $B_{2,4} \cong \mathcal{F}_{(3,1)} \oplus \mathcal{F}_{(3,2)}$, the entire space $B_{2,4}[4]$ would belong to the copy of $\mathcal{F}_{(3,1)}$ in $B_{2,4}$ which we denote by F. Notice that $[x, [x, [x, y]]]$, $[x, [y, [x, y]]]$, $[y, [y, [x, y]]]$ are in $B_{2,4}[4]$, so

$$s = -3y^2 \frac{\partial}{\partial y}[x, [x, [x, y]]] - x^2 \frac{\partial}{\partial y}[y, [y, [x, y]]] + 2xy \frac{\partial}{\partial x}[x, [x, [x, y]]]$$

is in F.

By using Jacobi identity and relations in $B_{2,3}$, we have:

$$\begin{aligned}
-3y^2 \frac{\partial}{\partial y}[x, [x, [x, y]]] &= -3[x, [x, [x, y^2]]], \\
-x^2 \frac{\partial}{\partial y}[y, [y, [x, y]]] &= [[x, y], [x^2, y]] - 2[y, [x^2, [x, y]]], \\
2xy \frac{\partial}{\partial x}[x, [x, [x, y]]] &= [[x, y], [x^2, y]] + 2[x, [y, [x^2, y]]] + 3[x, [x, [x, y^2]]].
\end{aligned}$$

Adding them up, we obtain $s = 4[[x, y], [x^2, y]]$ which is a nonzero element in $B_{2,4}$. Since s belongs to $F \cap C_{2,4}$, we have that $F \cap C_{2,4} \neq 0$ which contradicts our assumption that $B_{(2,4)} = F \oplus C_{2,4}$. So as a W_2-module, $B_{2,4}$ is isomorphic to a nontrivial extension of $\mathcal{F}_{(3,2)}$ by $\mathcal{F}_{(3,1)}$. □

To completely characterize $B_{2,4}$ as a W_2-module, we prove

Proposition 6.3. *All the nontrivial extensions of $\mathcal{F}_{(3,2)}$ by $\mathcal{F}_{(3,1)}$ are isomorphic.*

Proof. Firstly we construct such a nontrivial extension abstractly. We have the Lie algebra W_n of polynomial vector fields on V^*, where $V = \mathbb{C}^n$. We denote by W_n^0 the subalgebra of W_n of vector fields vanishing at the origin. For every Young diagram D, we have a corresponding representation F_D of $\mathfrak{gl}(n, \mathbb{C})$, and a corresponding representation of W_n^0 in which linear vector fields $\sum a_{ij} x_i \frac{\partial}{\partial x_j}$ act as matrices (a_{ij}) and higher-order vector fields act by zero. Suppose that D, E are two Young diagrams such that if we align their left upper corners the set-theoretic difference $E - D$ is equal to one box (an example of such a pair of diagrams is $E = (3, 2)$, $D = (3, 1)$). It is known that in this case there exists a nonzero homomorphism $F_D \otimes V \to F_E$, which is unique up to scaling.

We construct a representation Y of W_n^0 as follows. As a vector space $Y := F_D \oplus F_E$. Linear vector fields which correspond to $\mathfrak{gl}(n, \mathbb{C})$ act on Y as in the direct sum of the representations F_D, F_E of $\mathfrak{gl}(n, \mathbb{C})$. Cubic and higher vector fields act by zero. It remains to describe how quadratic vector fields act. They form a space $S^2V\otimes V^*$, which has a unique invariant projection to V. So we can define an action of $S^2V\otimes V^*$ on $F_D\oplus F_E$ by using this projection and the map $F_D\otimes V\to F_E$ (this action will map the subspace F_D to F_E and the subspace F_E to 0).

Now we define the representation $\mathcal{F}_Y := \mathrm{Hom}_{U(W_n^0)}(U(W_n),Y)$. Then we have an exact sequence

$$0\to\mathcal{F}_E\to\mathcal{F}_Y\to\mathcal{F}_D\to 0.$$

From now on, let us fix the Young diagrams $D=(3,1)$, $E=(3,2)$ and the corresponding representations $Y,\mathcal{F}_Y$ of W_2^0 and W_2.

Now we prove that any W_2-module M for which there is a short exact sequence

$$0\to\mathcal{F}_{(3,2)}\to M\to\mathcal{F}_{(3,1)}\to 0$$

which does not split is isomorphic to $\mathcal{F}_Y$. Suppose we have such a module M. We have $M[4]\cong F_{(3,1)}$ and $M[5]\cong F_{(3,2)}\oplus F_{(3,1)}\otimes V$, which is isomorphic to $F_{(3,2)}(1)\oplus(F_{(3,2)}(2)\oplus F_{(4,1)})$ (the 1 and 2 in parentheses denote the first and the second copy). So as $\mathfrak{sl}(2,\mathbb{C})$-modules, $M[4]\cong V_2$ and $M[5]\cong V_1(1)\oplus(V_1(2)\oplus V_3)$ where the subscripts denote the highest weights. Now we have the degree 1 part $W[1]$ (quadratic vector fields) of $W:=W_2$ acting from $M[5]^*$ to $M[4]^*$. As an $\mathfrak{sl}(2,\mathbb{C})$-module, we have a decomposition $W[1]=V_1\oplus V_3$. Let us pick a nonzero element f in $(V_1(1)\oplus V_1(2))^*\subset M[5]^*$ of weight 1 which is killed by the lowest vector (of weight -3) of $V_3\subset W[1]$. This is a scalar linear equation, so f exists (and is unique up to a scalar since the above equation is nontrivial). It generates a copy of V_1 inside $M[5]^*$, which we call N. Moreover, since the extension is nontrivial, $W[1]$ acts nontrivially on N. Thus, $N^\perp\oplus M[\geq 6]\subset M$ is a W_2^0-submodule, and the quotient module $M/(N^\perp\oplus M[\geq 6])=N^*\oplus M[4]$ is isomorphic to Y.

Therefore we have a natural W_2-homomorphism $M\to\mathcal{F}_Y$, which is an isomorphism in degrees 4 and 5. Hence it is an isomorphism (as there are only 2 terms in the Jordan-Hölder series of M). The proposition is proved. □

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7. Appendix: $B_2(A)$ for a general associative algebra A

by Pavel Etingof

The goal of this appendix is to generalize some of the results of Feigin and Shoikhet [FS] to the case of any associative algebra.

7.1. **The algebra $R(A)$.** Let A be an associative algebra over $\mathbb{C}$. Let $D(A)=A\oplus A$, regarded as a supervector space, where the first copy of A is even and the second one is odd. For $a\in A$, let us denote the elements $(a,0),(0,a)$ of $D(A)$ by x_a,ξ_a, respectively.

Define the supercommutative algebra $R(A)$ to be the quotient of the symmetric algebra $SD(A)$ by the relations

$$x_ax_b-x_{ab}+\xi_a\xi_b=0$$

and

$$x_a \xi_b + \xi_a x_b - \xi_{ab} = 0.$$

This is a DG algebra, with $dx_a = \xi_a, d\xi_a = 0$.

It is clear that the quotient of $R(A)$ by the ideal I generated by the odd elements is $A_{\rm ab}$, the abelianization of A. Thus $R(A)$ is a certain super-extension of the abelianization of A. More precisely, let $\Omega(A_{\rm ab})$ be the DG algebra of Kähler differential forms for the abelianization $A_{\rm ab}$ of A. It is defined by the same generators as $R(A)$ but with defining relations

$$x_a x_b - x_{ab} = 0$$

and

$$x_a \xi_b + \xi_a x_b - \xi_{ab} = 0$$

with $dx_a = \xi_a$, $d\xi_a = 0$. Thus, denoting by ${\rm gr}R(A)$ the associated graded algebra of $R(A)$ under the filtration by powers of I, we obtain that there is a natural surjective homomorphism $\eta : \Omega(A_{\rm ab}) \to {\rm gr}R(A)$. It is not always an isomorphism.

Definition 7.1. We will say that A is pseudosmooth if $A_{\rm ab}$ is a regular finitely generated algebra (i.e. ${\rm Spec}(A_{\rm ab})$ is a smooth affine algebraic variety X), and η is an isomorphism.

Proposition 7.1. *A is pseudosmooth if and only if $R(A)$ is isomorphic, as a DG algebra, to the algebra $\Omega(X)$ of regular differential forms on a smooth affine algebraic variety X.*

Proof. Suppose that $R(A) = \Omega(X)$. Then $A_{\rm ab} = \Omega(X)/(d\Omega(X)) = \mathcal{O}_X$, and η is clearly an isomorphism. Conversely, if $A_{\rm ab} = \mathcal{O}_X$ for smooth X and η is an isomorphism then the projection $R(A) \to A_{\rm ab}$ splits, and this splitting uniquely extends to an isomorphism of DG algebras $\Omega(X) \to R(A)$. □

7.2. **The Fedosov products.** For any DG algebra S introduce the Fedosov product on S by

$$f * g = f \cdot g + (-1)^{|f|} df \cdot dg,$$

and the inverse Fedosov product by

$$f \circ g = f \cdot g - (-1)^{|f|} df \cdot dg,$$

and let $S_*, S_\circ$ be the algebra S equipped with the Fedosov product, respectively the inverse Fedosov product.

Obviously, the operations of passing to the Fedosov and inverse Fedosov product in a differential algebra are inverse to each other, hence the terminology.

7.3. **The universal property.** It turns out that the algebra $R(A)$ has the following universal property.

Proposition 7.2. *For any supercommutative DG algebra S, one has a natural isomorphism ${\rm Hom}_{DG}(R(A), S) \to {\rm Hom}(A, S_{*0})$, where S_{*0} is the even part of S_*.*

Proof. It is clear that any homomorphism $f : R(A) \to S$ is determined by the elements $y_a = f(x_a)$, and the elements y_a define a homomorphism if and only if they satisfy the equations $y_a * y_b - y_{ab} = 0$. This implies the statement. □

7.4. **Relation with noncommutative differential forms.** In fact, the algebra $R(A)$ can be obtained from noncommutative differential forms on A ([CQ]). Namely, let $\Omega_{\rm nc}(A) = A \otimes T(\bar{A})$ denote the DG algebra of noncommutative differential forms on A (here $\bar{A} = A/\mathbb{C}$); it is the span of formal expressions $a_0 da_1 \cdots da_n$.

Proposition 7.3. *The algebra $R(A)$ is naturally isomorphic to the abelianization (in the supersense) of the DG algebra $\Omega_{\rm nc}(A)_\circ$.*

Proof. It suffices to show that if S is a supercommutative DG algebra, then $\mathrm{Hom}_{DG}(R(A), S) = \mathrm{Hom}_{DG}(\Omega_{\rm nc}(A)_\circ, S)$.

But $\mathrm{Hom}_{DG}(\Omega_{\rm nc}(A)_\circ, S) = \mathrm{Hom}_{DG}(\Omega_{\rm nc}(A), S_*) = \mathrm{Hom}_{DG}(A, S_{*0})$, and the result follows from the universal property of $R(A)$. □

7.5. **Description of $R(A)$ using a presentation of A.** Let V be a vector space. Then $SV \otimes \wedge V$ is naturally a differential algebra (the De Rham complex of V^*). Suppose that $A = TV/(L)$, where $L \subset TV$ is a set of relations.

Let $g : TV \to (SV \otimes \wedge V)_{*0}$ be the homomorphism defined by the condition that $g(v) = v \in SV$ for $v \in V$.

Proposition 7.4. *We have $R(A) = (SV \otimes \wedge V)/(g(L) \cup dg(L))$.*

In particular, we see that $R(TV) = SV \otimes \wedge V$.

Proof. We have

$$\mathrm{Hom}(A, S_{*0}) = \{f \in \mathrm{Hom}_{DG}(SV \otimes \wedge V, S) : f(g(L)) = 0\},$$

which implies the desired statement by Proposition 7.2. □

7.6. **The quotient of A by triple commutators.**

Proposition 7.5. *We have a natural isomorphism of algebras*

$$\phi : A/A[[A, A], A]A \to R(A)_{*0}.$$

Proof. We have a natural homomorphism ϕ given by $\phi(a) = x_a$. Let us show that it is an isomorphism. As shown in [FS], ϕ is an isomorphism for $A = TV$. On the other hand, $A/A[[A, A], A]A$ is the quotient of $TV/TV[[TV, TV], TV]TV$ by the additional relations L. Thus, it suffices to show that $R(A)_{*0}$ is obtained from $(SV \otimes \wedge V)_{*0}$ by imposing additional relations $g(L)$. These relations clearly hold, so we need to show that there is no others.

Thus, by Proposition 7.4, we need to show that in the algebra $(SV \otimes \wedge V)_{*0}/(g(L))$, we have $a \cdot g(b) = 0$ and $c \cdot dg(b) = 0$ for all $b \in L$, $a \in (SV \otimes \wedge V)_0$ and $c \in (SV \otimes \wedge V)_1$.

The first equality follows since $a \cdot g(b) = \frac{1}{2}(a * g(b) + g(b) * a)$. To prove the second equality, note that since c is odd, we have $c = \sum c_j \cdot dv_j$, hence $c \cdot dg(b) = \sum c_j \cdot dv_j \cdot dg(b)$, and $dv \cdot dg(b) = \frac{1}{2}(v * g(b) - g(b) * v) = 0$. □

Proposition 7.6. *The map ϕ of Proposition 7.5 maps $[A, A]$ onto the image of d in $R(A)_{*0}$.*

Proof. It is shown in [FS] that if $A = F$ is a free algebra then the statement holds. This implies that it holds for any associative algebra. □

Let $\mathrm{gr}(A)$ be the associated graded Lie algebra of A with respect to its lower central series filtration. Let $Z(A) = A[[A, A], A]A/([A, A] \cap A[[A, A], A]A)$. Thus, $Z(A) \subset B_1(A)$.

Proposition 7.7. *(i) $Z(A)$ is central in the Lie algebra* $\mathrm{gr}(A)$.

(ii) The space $B_1(A)/Z(A)$ is isomorphic, via ϕ, to $R(A)_0/R(A)_0^{\mathrm{exact}}$.

Proof. Part (i) follows from Lemma 2.2.1 of [FS] (this lemma is proved in [FS] for the free algebra but applies without changes to any associative algebra). Part (ii) follows from Proposition 7.6. □

7.7. **The first cyclic homology.** Let A be an associative algebra, and $W(A)$ be the subspace of $\wedge^2 A$ spanned by the elements

$$ab \wedge c + bc \wedge a + ca \wedge b.$$

We have a natural map $[,] : \wedge^2 A/W(A) \to [A, A]$ given by $a \wedge b \to [a, b]$. Recall [Lo] that the first cyclic homology $HC_1(A) \subset \wedge^2 A/W(A)$ is the kernel of this map.

Define the map $\zeta : \wedge^2 A/W(A) \to R(A)_1/R(A)_1^{\mathrm{exact}}$ by the formula

$$\zeta(a \wedge b) = d\phi(a) \cdot \phi(b).$$

It is easy to see that this map is well defined. Moreover, if $u \in HC_1(A)$ then $\zeta(u)$ closed. Thus, we obtain a map $\zeta : HC_1(A) \to H^{\mathrm{odd}}(R(A))$. Denote by $Y(A)$ the image of this map.

7.8. **Pseudoregular DG algebras.** Let S be a commutative DG algebra. Let $S' = S/S^{\mathrm{exact}}$. Define the linear map $\theta : \wedge^2 S_0 \to S_1'$ by the formula $\theta(a, b) = da \cdot b$. This is skew-symmetric because $da \cdot b + db \cdot a = d(ab)$. It is clear that the kernel $\ker\theta$ contains the elements

$$\kappa(a, b, c) := ab \wedge c + bc \wedge a + ca \wedge b,$$

where $a, b, c \in S_0$, and the elements $a \wedge b$ where a is exact. Denote the span of these two types of elements by E.

Let us say that S_0 is *pseudoregular* if $S_1 = S_0 dS_0$ (implying that θ is surjective), and $\ker\theta = E$.

7.9. **Pseudoregularity of the De Rham DG algebra of a smooth variety.** Let X be a smooth affine algebraic variety over $\mathbb{C}$. Denote by $\mathcal{O}_X$ the algebra of regular functions on X, and by $\Omega(X)$ the DG algebra of regular differential forms on X.

Theorem 7.1. *The algebra $S := \Omega(X)$ is pseudoregular.*

Remark 7.1. This was proved in [FS] in the special case when X is the affine space $\mathbb{C}^n$.

Proof. It is obvious that $S_1 = S_0 dS_0$. We need to show that θ identifies $\wedge^2 S_0/E$ with S_1'. To do so, write $\kappa(a, b, c)$ in the form

$$\kappa(a, b, c) = ab \wedge c - a \wedge bc - b \wedge ca.$$

From this we see that modulo the span of E, any element of $\wedge^2 S_0$ can be reduced to an element of $\mathcal{O}_X \otimes S_0$ (where $\mathcal{O}_X$ is viewed as the subspace of 0-forms in the space S_0 of even forms).

Furthermore, by modding out by $\kappa(a, b, c)$ we factor out a subspace of $\mathcal{O}_X \otimes S_0$ which is spanned by $ab \otimes g - a \otimes bg - b \otimes ga$, $a, b \in \mathcal{O}_X$, $g \in S_0$. The corresponding quotient space is the Hochschild homology $HH_1(\mathcal{O}_X, S_0)$. Since S_0 is a projective module over $\mathcal{O}_X$ (as X is smooth), we have $HH_1(\mathcal{O}_X, S_0) = HH_1(\mathcal{O}_X, \mathcal{O}_X) \otimes_{\mathcal{O}_X} S_0$, which by the Hochschild-Kostant-Rosenberg theorem ([Lo]) equals $\Omega^1(X)\otimes_{\mathcal{O}_X} S_0$. In fact, the relevant projection $\mathcal{O}_X\otimes_{\mathbb{C}} S_0\to\Omega^1(X)\otimes_{\mathcal{O}_X} S_0$ is simply given by the formula $a\otimes g\to da\otimes g$.

Further, for any $a,b,c\in\mathcal{O}_X$, $f\in S_0$ we have, modulo E:

$$a\wedge db\cdot dc\cdot f=a\cdot db\cdot dc\wedge f=-b\cdot da\cdot dc\wedge f=-b\wedge da\cdot dc\cdot f,$$

which proves that in fact, modulo E, the space $\Omega^1(X)\otimes_{\mathcal{O}_X} S_0$ gets projected onto its quotient space S_1, and the resulting projection map $\mathcal{O}_X\otimes_{\mathbb{C}} S_0\to S_1$ is given by $a\otimes g\to da\cdot g$. Moreover, it is clear that we project further down to $S_1'=S_1/S_1^{\mathrm{exact}}$, because the space of exact elements of S_1 is spanned by elements of the form $da\wedge f$, where f is an exact element of S_0 and $a\in\mathcal{O}_X$, and such an element is the image of $a\wedge f$, which belongs to E. The theorem is proved. □

7.10. **The structure of $B_2(A)$ for pseudosmooth algebras.** The main result of the appendix is the following theorem.

Theorem 7.2. *Let A be a pseudosmooth algebra. Then*

(i) $B_2(A)$ is naturally isomorphic to $R(A)_1'/Y(A)$; in particular, if $R(A)$ has no odd cohomology, then $B_2(A)=R(A)_0^{\mathrm{exact}}$.

(ii) $([A,A]\cap A[[A,A],A]A)/[[A,A],A]$ is naturally isomorphic to $H^{\mathrm{odd}}(R(A))/Y(A)$.

(iii) In terms of the identification of (i) and Proposition 7.6(ii), the bracket map $\wedge^2(B_1(A)/Z(A))\to B_2(A)$ is given by the formula $a\wedge b\to da\cdot b$.

Proof. According to [FS], proof of Lemma 1.2, we have an exact sequence

$$\cdots\to HC_1(A)\to\wedge^2(A/[A,A])/(ab\wedge c+bc\wedge a+ca\wedge b)\to[A,A]/[[A,A],A]\to 0.$$

By Proposition 7.7, this implies that we have an exact sequence

$$\cdots\to HC_1(A)\to\wedge^2(R(A)_0')/(ab\wedge c+bc\wedge a+ca\wedge b)\to[A,A]/[[A,A],A]\to 0.$$

Since A is pseudosmooth, by Proposition 7.1 the middle term has the form

$$\wedge^2\Omega'_{\mathrm{even}}(X)/(a\wedge bc+b\wedge ca+c\wedge ab),$$

where X is the spectrum of A_{ab}. By Theorem 7.1, this equals $\Omega_{\mathrm{odd}}(X)/\Omega_{\mathrm{odd}}^{\mathrm{exact}}(X)$. Clearly, the space $HC_1(A)$ maps onto $Y(A)\subset\Omega_{\mathrm{odd}}(X)/\Omega_{\mathrm{odd}}^{\mathrm{exact}}(X)$. This implies the first and third statements. The second statement follows from the first one and Proposition 7.6. □

Remark 7.2. In the special case when A is a free algebra, Theorem 7.2 is proved in [FS]. In this case, one has $[A,A]\cap A[[A,A],A]A=[[A,A],A]$. However, in general this equality does not have to hold. For example, let A be the algebra generated by two elements x,y with the only relation $xy=1$. Then it is easy to show that $HC_1(A)=0$ (see e.g. [EG], Section 5.4), and $R(A)=\Omega(X)$, where X is the curve defined by the equation $xy=1$ in the plane (i.e $X=\mathbb{C}^*$). This algebra is commutative (even with the $*$-product), since X is 1-dimensional. Thus, $A/A[[A,A],A]A$ is commutative, and hence $[A,A]\subset A[[A,A],A]A$. However, it follows from Theorem 7.2 that the space $[A,A]/[[A,A],A]$ is 1-dimensional. In fact, one may check that it is spanned by the element $[x,y]$.

7.11. **A sufficient condition of pseudosmoothness.**

Proposition 7.8. *Let $f_1,\dots,f_m\in A_n$ be a set of elements, such that their images $\bar f_1,\dots,\bar f_m$ in $\mathbb{C}[x_1,\dots,x_n]$ form a regular sequence defining a smooth complete intersection X in $\mathbb{C}^n$ (of codimension m). Then the algebra $A:=A_n/(f_1,\dots,f_m)$ is pseudosmooth, and $R(A)$ is isomorphic to $\Omega(X)$.*

Proof. We have $A_{\text{ab}} = \mathcal{O}_X$, and because of the complete intersection condition, η is an isomorphism. Thus A is pseudosmooth, and by Proposition 7.1, $R(A)$ isomorphic to $\Omega(X)$. □

7.12. **Examples.** The above results allow one to compute $B_2(A)$ for specific algebras A.

Proposition 7.9. *Suppose that $L \subset SV \subset TV$. Then $g(L) = L \subset SV$, and hence $R(A)$ is naturally isomorphic to the algebra of Kähler differential forms $\Omega(A_{\text{ab}})$. In particular, if A_{ab} is regular then A is pseudosmooth.*

Proof. Obvious. □

Example 7.1. Let A be the free algebra in three generators x, y, z modulo the relation $x^2 + y^2 + z^2 = 1$ (noncommutative 2-sphere). Let us compute the space $B_2(A)$ as a representation of $SO(3)$ acting on this algebra. From Proposition 7.9 we find that A is pseudosmooth, and $R(A)$ is the algebra of polynomial differential forms on the usual commutative quadric Q. In this case, we have no odd cohomology, so by Theorem 7.2, $B_2(A)$ is the space of exact 2-forms. The space of exact 2-forms is a subspace of codimension 1 in the space of all 2-forms, since $H^2(Q)$ is 1-dimensional. The space of all 2-forms is isomorphic to the space of functions as an $SO(3)$-module, since there is an invariant symplectic form on the quadric (the area form). Now, we have

$$\text{Fun}(Q) = V_0 \oplus V_2 \oplus V_4 \oplus \cdots ,$$

where V_{2i} is the $(2i+1)$-dimensional representation of $SO(3)$. Thus,

$$B_2(A) = V_2 \oplus V_4 \oplus \cdots$$

Let us now consider more general examples. As before, assume that $L \subset SV$, and suppose that $A = TV/(L)$ is a pseudosmooth algebra (i.e., A_{ab} is regular), such that $R(A)$ has no odd cohomology. Suppose further that L is fixed by a reductive subgroup $G \subset GL(V)$, such that $R(A)$ is a direct sum of irreducible representations of G with finite multiplicities. In this case, one can define the character-valued Hilbert series $F(z), E(z), H(z)$ of the graded representations $R(A)$, $R(A)^{\text{exact}}$, and the cohomology $H(A)$ of $R(A)$. Then we have the equations

$$z(F - E - H) = E,$$

which implies that

$$E = \frac{z(F - H)}{1 + z}. \tag{3}$$

This formula is useful because often F and H are known explicitly.

Example 7.2. Let $\mathfrak{g}$ be a simple Lie algebra with root system R and Weyl group W, and let G the corresponding simply connected group. Let r be the rank of G, $p_1, \dots, p_r$ be homogeneous generators of the ring $(S\mathfrak{g})^G$, and $d_i = \deg(p_i)$. Let b_i be generic complex numbers, and let $A(\mathfrak{g}, b)$ be the quotient of the tensor algebra $T\mathfrak{g}$ by the relations $p_i = b_i$. Note that the algebra from Example 7.1 is the special case of $A(\mathfrak{g}, b)$ for $\mathfrak{g} = \mathfrak{sl}(2)$.

Let us calculate the decomposition of the space $B_2(A)$ into irreducible representations of G. We have $B_2(A) = \oplus_{V \in \text{Irr}(G)} N_V \otimes V$, where $N_V = \text{Hom}_G(V, B_2(A))$.

By formula (3), we have

$$\dim N_V = \frac{1}{2}(E_V(1) + E_V(-1)),$$

with

$$E_V(z) = \frac{z}{1+z}(F_V(z) - H_V(z)),$$

where F_V and H_V are contributions of V into F and H, respectively. It remains to find $F_V(z)$ and $H_V(z)$.

By Proposition 7.9, we find that $R(A)$ is the algebra of polynomial differential forms on G/H, where H is a maximal torus in G. Thus we have $H_V(z) = 0$ unless $V = \mathbb{C}$,

$$H_{\mathbb{C}}(z) = \prod_{i=1}^{r} \frac{z^{2d_i} - 1}{z^2 - 1}$$

is the Poincaré polynomial of G/H, and

$$F_V(z) = \sum_{j \geq 0} z^j \dim \mathrm{Hom}_H(V, \wedge^j(\mathfrak{g}/\mathfrak{h})),$$

where $\mathfrak{h} = \mathrm{Lie}H$. More explicitly,

$$F_V(z) = C.T.(\chi_{V^*} \cdot \prod_{\alpha \in R}(1 + ze^{\alpha})),$$

where χ_{V^*} is the character of V^*, and C.T. means the constant term.

In the case $\mathfrak{g} = \mathfrak{sl}(2)$, this recovers the answer from Example 7.1.

Corollary 7.1. *Let $\nu(R)$ be the number of subsets of R with zero sum. Then*

$$\dim B_2(A)^G = \frac{1}{4}(\nu(R) - |W|).$$

Proof. It is easy to show that $F_{\mathbb{C}}(-1) = H_{\mathbb{C}}(-1) = |W|$, and $F'_{\mathbb{C}}(-1) = H'_{\mathbb{C}}(-1) = -|R||W|/2$, thus $E_{\mathbb{C}}(-1) = 0$. So $\dim B_2(A)^G = \frac{1}{2}E_{\mathbb{C}}(1) = \frac{1}{4}(F_{\mathbb{C}}(1) - H_{\mathbb{C}}(1))$. But we have $H_{\mathbb{C}}(1) = |W|$, and $F_{\mathbb{C}}(1) = \nu(R)$. The corollary follows. ☐

For example, $\dim B_2(A)^G$ is 0 for $\mathfrak{g} = \mathfrak{sl}(2)$, 1 for $\mathfrak{g} = \mathfrak{sl}(3)$, and 32 for $\mathfrak{g} = \mathfrak{sl}(4)$.

Example 7.3. Let $P \in \mathbb{C}\langle x, y\rangle$ be a noncommutative polynomial in two variables x, y, and $\bar{P}$ be the abelianization of P (i.e., the image of P in the polynomial algebra $\mathbb{C}[x, y]$). Denote by A_P the algebra $\mathbb{C}\langle x, y\rangle/(P)$. Assume that the curve $X_{\bar{P}}$ given by the equation $\bar{P}(x, y) = 0$ is smooth. Then $A = A_P$ is pseudosmooth, and thus Theorem 7.2 applies to A. Moreover, since $X_{\bar{P}}$ is a curve, the algebra $A/A[[A, A], A]A$ is commutative, and hence $[A, A] \subset A[[A, A], A]A$. Thus $B_2(A) = ([A, A] \cap A[[A, A], A]A)/[[A, A], A] = H^1(X_{\bar{P}})/Y(A)$.

The space $Y(A)$ actually depends on P, not only on $\bar{P}$. For example, assume that the leading term of P is generic. In this case, by the results [EG], $HC_1(A) = 0$, and hence $B_2(A) = H^1(X_{\bar{P}})$. The same holds if the leading term of P is, say $x^p y^q$. Thus, for example, if $P = x^2y - 1$ then $B_2(A) = H^1(\mathbb{C}^*) = \mathbb{C}$. On the other hand, if $P = xyx - 1$ then in A we have $xy = xyxyx = yx$, so $A = \mathbb{C}[x, y]/(yx^2 = 1)$, and $B_2(A) = 0$ (thus, $Y(A)$ is 1-dimensional in this case).

Let us do two concrete examples.

1. P is a generic polynomial of degree d. In this case the curve $X = X_{\bar{P}}$ has genus $(d-1)(d-2)/2$ and d points at infinity. So its Euler characteristic is $\chi = 2-(d-1)(d-2)-d = -d(d-2)$, and hence $\dim B_2(A) = \dim H^1(X) = (d-1)^2$.

2. Let $P(x,y)=Q(x)y^m-1$, where Q is a monic polynomial of degree n with roots of multiplicities $p_1,\dots,p_r$. In this case the curve $X=X_{\bar{P}}$ is the Riemann surface of the function $y=Q(x)^{1/m}$. The number of components of this curve is the greatest common divisor d of p_i and m. Also, the curve is a regular covering of the line without r points of degree m. Therefore, the Euler characteristic of X is $m(1-r)$, and thus $\dim B_2(A)=\dim H^1(X)=m(r-1)+d$.

Let P be a generic nonhomogeneous noncommutative polynomial of degree d in $n\geq 1$ variables. Let $A=A_n/(P)$.

Proposition 7.10. $\dim([A,A]\cap A[[A,A],A]A)/[[A,A],A]$ *is* $(d-1)^n$ *if* n *is even, and* 0 *if* n *is odd.*

Proof. Let $\bar{P}$ be the abelianization of P, and X be the hypersurface defined by the equation $\bar{P}=0$ in $\mathbb{C}^n$. Then by Theorem 7.2 and the results of [EG], the space $([A,A]\cap A[[A,A],A]A)/[[A,A],A]$ is isomorphic to the odd cohomology $H^{\rm odd}(X)$.

Since X is generic, it is obtained by removing of a smooth projective hypersurface of degree d and dimension $n-2$ from one of degree d and dimension $n-1$. Therefore, by the Lefschetz hyperplane section theorem, X has cohomology only in degrees 0 and $n-1$. This implies the result in the case of odd n. If n is even, the dimension of the odd cohomology is $1-\chi$, where χ is the Euler characteristic of X. So it remains to find χ.

The computation of χ is well known, but we give it for the reader's convenience. We may assume that X is the hypersurface $X(d,n)$ defined by the equation

$$x_1^d+\cdots+x_n^d=1.$$

Then by forgetting x_n we get a degree d surjective map $X(d,n)\to\mathbb{C}^{n-1}$ which branches along $X(d,n-1)$ (where there is 1 instead of d preimages). Thus if $\chi(d,n)$ denotes the Euler characteristic of $X(d,n)$, then we have

$$\chi(d,n)=d-(d-1)\chi(d,n-1).$$

Since $\chi(d,1)=d$, we get by induction $\chi(d,n)=1-(1-d)^n$. Hence the dimension in question is $(d-1)^n$, as desired. □

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