THE KÜNNETH FORMULA FOR NUCLEAR
DF-SPACES AND HOCHSCHILD COHOMOLOGY

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ABSTRACT. We consider complexes $(\mathcal{X}, d)$ of nuclear Fréchet spaces and continuous boundary maps d_n with closed ranges and prove that, up to topological isomorphism, $(H_n(\mathcal{X}, d))^* \cong H^n(\mathcal{X}^*, d^*)$, where $(H_n(\mathcal{X}, d))^*$ is the strong dual space of the homology group of $(\mathcal{X}, d)$ and $H^n(\mathcal{X}^*, d^*)$ is the cohomology group of the strong dual complex $(\mathcal{X}^*, d^*)$. We use this result to establish the existence of topological isomorphisms in the Künneth formula for the cohomology of complete nuclear DF-complexes and in the Künneth formula for continuous Hochschild cohomology of nuclear $\hat{\otimes}$-algebras which are Fréchet spaces or DF-spaces for which all boundary maps of the standard homology complexes have closed ranges. We describe explicitly continuous Hochschild and cyclic cohomology groups of certain tensor products of $\hat{\otimes}$-algebras which are Fréchet spaces or nuclear DF-spaces.

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1. INTRODUCTION

Künneth formulae for bounded chain complexes $\mathcal{X}$ and $\mathcal{Y}$ of Fréchet and Banach spaces and continuous boundary maps with closed ranges were established, under certain topological assumptions, in [16, 7, 8]. Recall that in the category of nuclear Fréchet spaces short exact sequences are topologically pure and objects are strictly flat, and so the Künneth formula can be used for calculation of continuous Hochschild

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homology $\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y)$ if boundary maps of the standard homology complexes have closed ranges. To compute the continuous cyclic-type Hochschild cohomology of Fréchet algebras one has to deal with complexes of complete DF-spaces. In the recent paper [21] the author showed that, for a continuous morphism $\varphi : \mathcal{X} \to \mathcal{Y}$ of complexes of complete nuclear DF-spaces, a surjective map of cohomology groups $H^n(\varphi) : H^n(\mathcal{X}) \to H^n(\mathcal{Y})$ is automatically open. In this paper we establish relations between topological properties of the homology of complexes of Fréchet spaces and of the cohomology of their strong dual complexes. We use these properties to show the existence of a topological isomorphism in the Künneth formula for complexes of complete nuclear DF-spaces and continuous boundary maps with closed ranges and thereby to describe explicitly the continuous Hochschild and cyclic homology and cohomology of $\mathcal{A}\hat{\otimes}\mathcal{B}$ for certain $\hat{\otimes}$-algebras $\mathcal{A}$ and $\mathcal{B}$ which are Fréchet spaces or nuclear DF-spaces.

In Theorem 3.5 and Corollary 3.6, for a complex of nuclear Fréchet spaces or of complete nuclear DF-spaces $(\mathcal{X}, d)$ and continuous boundary maps d_n with closed ranges, we establish that there is a topological isomorphism, $(H_n(\mathcal{X}, d))^* \cong H^n(\mathcal{X}^*, d^*)$, where $(H_n(\mathcal{X}, d))^*$ is the dual space of the homology group of $(\mathcal{X}, d)$ and $H^n(\mathcal{X}^*, d^*)$ is the cohomology group of the dual complex $(\mathcal{X}^*, d^*)$.

In Theorem 4.4 and Theorem 4.3, for bounded chain complexes $\mathcal{X}$ and $\mathcal{Y}$ of complete nuclear DF-spaces or of nuclear Fréchet spaces such that all boundary maps have closed ranges, we prove that, up to topological isomorphism,

$$H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})] \text{ and}$$

$$H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong \bigoplus_{m+q=n} [H^m(\mathcal{X}^*)\hat{\otimes}H^q(\mathcal{Y}^*)].$$

In Corollary 4.2, for bounded chain complexes $(\mathcal{X}, d_{\mathcal{X}})$ of Banach spaces and $(\mathcal{Y}, d_{\mathcal{Y}})$ of Fréchet spaces such that all boundary maps have closed ranges, and $H_n(\mathcal{X})$ and Ker $(d_{\mathcal{X}})_n$ are strictly flat in $\mathcal{B}an$ for all n, we prove that, up to topological isomorphism,

$$H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})]$$

and, up to isomorphism of linear spaces,

$$H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})]^*.$$

The Künneth formulae for the continuous Hochschild homology $\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y)$ and cohomology $H^n((\mathcal{C}_\sim(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y))^*)$ are proved in Theorem 5.4 for the underlying category of complete nuclear DF-spaces and for the underlying category of nuclear Fréchet spaces. In these underlying categories, for unital $\hat{\otimes}$-algebras $\mathcal{A}$ and $\mathcal{B}$, for a unital $\mathcal{A}$-$\hat{\otimes}$-bimodule X and a unital $\mathcal{B}$-$\hat{\otimes}$-bimodule Y, under the assumption that all boundary maps of the standard homology complexes $\mathcal{C}_\sim(\mathcal{A}, X)$ and $\mathcal{C}_\sim(\mathcal{B}, Y)$ have closed ranges, we show that, up to topological isomorphism,

$$\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y) \cong \bigoplus_{m+q=n} [\mathcal{H}_m(\mathcal{A}, X)\hat{\otimes}\mathcal{H}_q(\mathcal{B}, Y)] \text{ and}$$

$$H^n((\mathcal{C}_\sim(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y))^*) \cong$$
$$\bigoplus_{m+q=n} [H^m((\mathcal{C}_\sim(\mathcal{A}, X))^*)\hat{\otimes}H^q((\mathcal{C}_\sim(\mathcal{B}, Y))^*)].$$

In Theorem 5.5 we prove the Künneth formulae for the continuous Hochschild homology groups of Banach and Fréchet algebras under some topological assumptions. In Section 6 we describe explicitly the continuous cyclic-type homology and cohomology of certain tensor products of $\hat{\otimes}$-algebras which are Banach or Fréchet or nuclear Fréchet or nuclear DF-spaces.

2. Definitions and notation

We recall some notation and terminology used in homology and in the theory of topological algebras. Homological theory can be found in any relevant textbook, for instance, MacLane [22], Loday [17] for the pure algebraic case and Helemskii [12] for the continuous case.

Throughout the paper $\hat{\otimes}$ is the projective tensor product of complete locally convex spaces. By $X^{\hat{\otimes}n}$ we mean the n-fold projective tensor power $X\hat{\otimes}\ldots\hat{\otimes}X$ of X and id denotes the identity operator.

We use the notation $\mathcal{B}an$, $\mathcal{F}r$ and $\mathcal{LCS}$ for the categories whose objects are Banach spaces, Fréchet spaces and complete Hausdorff locally convex spaces respectively, and whose morphisms in all cases are continuous linear operators. For topological homology theory it is important to find a suitable category for the underlying spaces of the

algebras and modules. In [12] Helemskii constructed homology theory
for the following categories Φ of underlying spaces, for which he used
the notation $(\Phi, \hat{\otimes})$.

DEFINITION 2.1. *([12, Section II.5]) A suitable category for underly-
ing spaces of the algebras and modules is an arbitrary complete sub-
category Φ of $\mathcal{LCS}$ having the following properties:*
    *(i) if Φ contains a space, it also contains all those spaces topologi-
cally isomorphic to it;*
    *(ii) if Φ contains a space, it also contains any of its closed subspaces
and the completion of any its Hausdorff quotient spaces;*
    *(iii) Φ contains the direct sum and the projective tensor product of
any pair of its spaces;*
    (iv) Φ contains $\mathbb{C}$.

Besides $\mathcal{B}an$, $\mathcal{F}r$ and $\mathcal{LCS}$ important examples of suitable categories
Φ are the categories of complete nuclear spaces [29, Proposition 50.1],
nuclear Fréchet spaces and complete nuclear DF-spaces [21].
    By definition a $\hat{\otimes}$-algebra is a complete Hausdorff locally convex
algebra with jointly continuous multiplication. A left $\hat{\otimes}$-module X
over a $\hat{\otimes}$-algebra $\mathcal{A}$ is a complete Hausdorff locally convex space X
together with the structure of a left $\mathcal{A}$-module such that the map
$\mathcal{A} \times X \to X$, $(a, x) \mapsto a \cdot x$ is jointly continuous. For a $\hat{\otimes}$-algebra
$\mathcal{A}$, $\hat{\otimes}_{\mathcal{A}}$ is the projective tensor product over $\mathcal{A}$ of left and right $\mathcal{A}$-
$\hat{\otimes}$-modules (see [10], [12, II.4.1]). The category of left [unital] $\mathcal{A}$-$\hat{\otimes}$-
modules is denoted by $\mathcal{A}$-mod [$\mathcal{A}$-unmod] and the category of [unital]
$\mathcal{A}$-$\hat{\otimes}$-bimodules is denoted by $\mathcal{A}$-mod-$\mathcal{A}$ [$\mathcal{A}$-unmod-$\mathcal{A}$].
    Let $\mathcal{K}$ be a category. A *chain complex* $\mathcal{X}_{\sim}$ in the category $\mathcal{K}$ is a
sequence of $X_n \in \mathcal{K}$ and morphisms d_n (called *boundary maps*)

$$\cdots \leftarrow X_n \xleftarrow{d_n} X_{n+1} \xleftarrow{d_{n+1}} X_{n+2} \leftarrow \ldots$$

such that $d_n \circ d_{n+1} = 0$ for every n. The *cycles* are the elements of

$$Z_n(\mathcal{X}) = \mathrm{Ker}\ (d_{n-1} : X_n \to X_{n-1}).$$

The *boundaries* are the elements of

$$B_n(\mathcal{X}) = \mathrm{Im}\ (d_n : X_{n+1} \to X_n).$$

The relation $d_{n-1} \circ d_n = 0$ implies $B_n(\mathcal{X}) \subset Z_n(\mathcal{X})$. The *homology
groups* are defined by

$$H_n(\mathcal{X}_{\sim}) = Z_n(\mathcal{X})/B_n(\mathcal{X}).$$

As usual, we will often drop the subscript n of d_n. If there is a need to distinguish between various boundary maps on various chain complexes, we will use subscripts, that is, we will denote the boundary maps on $\mathcal{X}$ by $d_{\mathcal{X}}$. A chain complex $\mathcal{X}$ is called *bounded* if $X_n = \{0\}$ whenever n is less than a certain fixed integer $N \in \mathbb{Z}$.

Given $E \in \mathcal{K}$ and a chain complex $(\mathcal{X}, d)$ in $\mathcal{K}$, we can form the chain complex $E \hat{\otimes} \mathcal{X}$ of the locally convex spaces $E \hat{\otimes} X_n$ and boundary maps $\mathrm{id}_{\mathrm{E}} \otimes \mathrm{d}$. Definitions of the totalization $\mathrm{Tot}(\overline{\mathcal{M}})$ of a bounded bicomplex $\overline{\mathcal{M}}$ and the tensor product $\mathcal{X} \hat{\otimes} \mathcal{Y}$ of bounded complexes $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{F}r$ can be found in [12, Definitions II.5.23-25]. Recall that $\mathcal{X} \hat{\otimes} \mathcal{Y} \overset{\mathrm{def}}{=} \mathrm{Tot}(\overline{\mathcal{X} \hat{\otimes} \mathcal{Y}})$ of a bounded bicomplex $\overline{\mathcal{X} \hat{\otimes} \mathcal{Y}}$.

We recall here the definition of a strictly flat locally convex space in a suitable category Φ which is equivalent to that given in [12, Chapter VII]. Note that it can be seen as a special case of the corresponding notion for $\hat{\otimes}$-modules, where the $\hat{\otimes}$-algebra is taken to be the complex numbers $\mathbb{C}$.

DEFINITION 2.2. *A locally convex space $G \in \Phi$ is* strictly flat *in Φ if for every short exact sequence*

$$0 \to X \to Y \to Z \to 0$$

of locally convex spaces from Φ and continuous linear operators, the short sequence

$$0 \to G \hat{\otimes} X \to G \hat{\otimes} Y \to G \hat{\otimes} Z \to 0$$

is also exact.

EXAMPLE 2.3. (i) *Nuclear Fréchet spaces are strictly flat in $\mathcal{F}r$* [6, Theorems A.1.5 and A.1.6]. (ii) *Finite-dimensional Banach spaces and $L^1(\Omega, \mu)$ are strictly flat in $\mathcal{B}an$* [30, Theorem III.B.2] *and in $\mathcal{F}r$* [26, Proposition 4.4].

If E is a topological vector space E^* denotes its dual space of continuous linear functionals. For a subset V of E, the *polar* of V is

$$V^0 = \{g \in E^* : |g(x)| \leq 1 \text{ for all } x \in V\}.$$

Throughout the paper, E^* will always be equipped with the strong topology unless otherwise stated. The *strong topology* is defined on E^* by taking as a basis of neighbourhoods of 0 the family of polars of all bounded subsets of E; see [29, II.19.2].

Let $\mathcal{A}$ be a $\hat{\otimes}$-algebra. A complex of $\mathcal{A}$-$\hat{\otimes}$-modules and their morphisms is called *admissible* if it splits as a complex in $\mathcal{LCS}$ [12, III.1.11].

A complex of $\mathcal{A}$-$\hat{\otimes}$-modules and their morphisms is called *weakly admissible* if its strong dual complex splits.

For $Y \in \mathcal{A}$-mod-$\mathcal{A}$ a complex

$$0 \longleftarrow Y \xleftarrow{\varepsilon} P_0 \xleftarrow{\phi_0} P_1 \xleftarrow{\phi_1} P_2 \longleftarrow \cdots \qquad (0 \leftarrow Y \leftarrow \mathcal{P})$$

is called a *projective resolution of Y in $\mathcal{A}$-mod-$\mathcal{A}$* if it is admissible and all the modules in $\mathcal{P}$ are projective in $\mathcal{A}$-mod-$\mathcal{A}$ [12, Definition III.2.1].

For any $\hat{\otimes}$-algebra $\mathcal{A}$, not necessarily unital, $\mathcal{A}_+$ is the $\hat{\otimes}$-algebra obtained by adjoining an identity to $\mathcal{A}$. For a $\hat{\otimes}$-algebra $\mathcal{A}$, the algebra $\mathcal{A}^e = \mathcal{A}_+ \hat{\otimes} \mathcal{A}_+^{op}$ is called the *enveloping algebra of $\mathcal{A}$*, where $\mathcal{A}_+^{op}$ is the *opposite algebra* of $\mathcal{A}_+$ with multiplication $a \cdot b = ba$.

A module $Y \in \mathcal{A}$-mod is called *flat* if for any admissible complex $\mathcal{X}$ of right $\mathcal{A}$-$\hat{\otimes}$-modules the complex $\mathcal{X} \hat{\otimes}_{\mathcal{A}} Y$ is exact. A module $Y \in \mathcal{A}$-mod-$\mathcal{A}$ is called *flat* if for any admissible complex $\mathcal{X}$ of $\mathcal{A}$-$\hat{\otimes}$-bimodules the complex $\mathcal{X} \hat{\otimes}_{\mathcal{A}^e} Y$ is exact.

For $Y, X \in \mathcal{A}$-mod-$\mathcal{A}$, we shall denote by $\mathrm{Tor}_n^{\mathcal{A}^e}(X, Y)$ the nth homology of the complex $X \hat{\otimes}_{\mathcal{A}^e} \mathcal{P}$, where $0 \leftarrow Y \leftarrow \mathcal{P}$ is a projective resolution of Y in $\mathcal{A}$-mod-$\mathcal{A}$, [12, Definition III.4.23].

DEFINITION 2.4. *A short exact sequence of locally convex spaces from Φ and continuous operators*

$$0 \to Y \xrightarrow{i} Z \xrightarrow{j} W \to 0$$

is called topologically pure *in Φ if for every $X \in \Phi$ the sequence*

$$0 \to X \hat{\otimes} Y \xrightarrow{\mathrm{id}_X \hat{\otimes} i} X \hat{\otimes} Z \xrightarrow{\mathrm{id}_X \hat{\otimes} j} X \hat{\otimes} W \to 0$$

is exact.

By [2, II.1.8f and Remark after II.1.9], an extension of Banach spaces is topologically pure in $\mathcal{B}an$ if and only if it is weakly admissible in $\mathcal{B}an$. In the category of Fréchet spaces the situation with topologically pure extensions is more interesting. Firstly, it is known that extensions of nuclear Fréchet spaces are topologically pure (see [6, Theorems A.1.6 and A.1.5]. Note that nuclear Fréchet spaces are reflexive, and therefore a short sequence of nuclear Fréchet spaces is weakly admissible if and only if it is admissible. It is shown in [20, Lemma 2.4] that in $\mathcal{F}r$ the weak admissibility of an extension implies topological purity of the extension, but is not equivalent to the topological purity of the extension [20, Section 2]. Recall that extensions of Fréchet algebras

$0 \to Y \to Z \to W \to 0$ such that Y has a left or right bounded approximate identity are topologically pure [20, Lemma 2.5].

3. Topological isomorphism between $(H_n(\mathcal{X}, d))^*$ and $H^n(\mathcal{X}^*, d^*)$ in the category of complete nuclear DF-spaces

DF-spaces were introduced by A. Grothendieck in [9]. It is well known that the strong dual of a Fréchet space is a complete DF-space and that nuclear Fréchet spaces and complete nuclear DF-spaces are reflexive [23, Theorem 4.4.12]. Moreover, the correspondence $E \leftrightarrow E^*$ establishes a one-to-one relation between the nuclear Fréchet spaces and complete nuclear DF-spaces [23, Theorem 4.4.13]. It is known that there exist closed linear subspaces of DF-spaces that are not DF-spaces. For nuclear spaces, however, we have the following.

LEMMA 3.1. [23, Proposition 5.1.7] *Each closed linear subspace F of the strong dual of a nuclear Fréchet space E is also the strong dual of a nuclear Fréchet space.*

In a locally convex space a subset is called a *barrel* if it is absolutely convex, absorbent and closed. Every locally convex space has a neighbourhood base consisting of barrels. A locally convex space is called a *barrelled* space or a *t-space* if every barrel is a neighbourhood [27]. A Hausdorff barrelled locally convex space with the further property that its closed bounded subsets are compact is called a Montel space. In particular, each nuclear Fréchet space is an FM-space, that is, a Fréchet space that is Montel.

Let E and F be locally convex topological vector spaces and let $T : E \to F$ be a continuous linear operator. If T is open, then it is also called a *homomorphism*. Let E^* and F^* have the strong dual topologies. If $T^* : F^* \to E^*$ is an open continuous linear operator then T^* is called a *strong homomorphism*.

LEMMA 3.2. *Let X and Y be nuclear Fréchet spaces and let $T : X \to Y$ be a continuous linear operator such that* Im T *is closed. Then up to topological isomorphism,* Im $T^* \cong (X/\mathrm{Ker}\ T)^*$ *is the strong dual of a nuclear Fréchet space and hence is barrelled.*

Proof. In view of the algebraic identification Im $T^* \cong (X/\mathrm{Ker}\ T)^*$, the adjoint of the quotient map $X \to X/\mathrm{Ker}\ T$ is the inclusion map Im $T^* \to X^*$. Since nuclear Fréchet spaces are in particular FM-spaces and the adjoint of a homomorphism between FM-spaces is

a strong homomorphism, the relative topology and the strong dual topology of Im T^* (as the dual space of $X/\operatorname{Ker} T$) coincide. Hence, up to topological isomorphism, Im $T^* \cong (X/\operatorname{Ker} T)^*$. By [27, Corollary IV.3.1], the strong dual of a nuclear Fréchet space is barrelled.   □

Further we will need the following version of the open mapping theorem.

COROLLARY 3.3. [21, Corollary 3.6] *Let E and F be nuclear Fréchet spaces and let E^* and F^* be the strong duals of E and F respectively. Then a continuous linear operator T of E^* onto F^* is open.*

For a continuous morphism of chain complexes $\psi_\sim : \mathcal{X}_\sim \to \mathcal{P}_\sim$ in $\mathcal{F}r$, a surjective map $H_n(\varphi) : H_n(\mathcal{X}) \to H_n(\mathcal{Y})$ is automatically open, see [12, Lemma 0.5.9]. In the category of complete nuclear DF-spaces it was proved be the author in [21, Lemma 3.5].

The following result is known for Banach and Fréchet spaces.

PROPOSITION 3.4. [7, Corollary 4.9] *Let $(\mathcal{X}, d)$ be a chain complex of Fréchet (Banach) spaces and continuous linear operators and $(\mathcal{X}^*, d^*)$ the strong dual cochain complex. Then the following are equivalent:*

(1) $H_n(\mathcal{X}, d) = \operatorname{Ker} d_{n-1} / \operatorname{Im} d_n$ *is a Fréchet (Banach) space;*
(2) $B_n(\mathcal{X}, d) = \operatorname{Im} d_n$ *is closed in X_n;*
(3) d_n *has closed range;*
(4) *the dual map $d^n = d_n^*$ has closed range;*
(5) $B^{n+1}(\mathcal{X}^*, d^*) = \operatorname{Im} d_n^*$ *is strongly closed in $(X_{n+1})^*$;*

In the category of Banach spaces (1) - (5) are equivalent to:

(6) $B^{n+1}(\mathcal{X}^*, d^*)$ *is a Banach space;*
(7) $H^{n+1}(\mathcal{X}^*, d^*) = \operatorname{Ker} d_{n+1}^* / \operatorname{Im} d_n^*$ *is a Banach space.*

Moreover, whenever $H_n(\mathcal{X}, d)$ and $H^n(\mathcal{X}^*, d^*)$ are Banach spaces, up to topological isomorphism,

$$H^n(\mathcal{X}^*, d^*) \cong H_n(\mathcal{X}, d)^*.$$

The next theorem shows that certain niceties of the theory of nuclear DF-spaces allow us to generalize this result to nuclear Fréchet spaces.

THEOREM 3.5. *Let $(\mathcal{X}, d)$ be a chain complex of Fréchet spaces and continuous linear operators and let $(\mathcal{X}^*, d^*)$ be its strong dual complex. Suppose that, for a certain n, either d_n and d_{n-1} have closed ranges or d_n^* and d_{n-1}^* have closed ranges.*
(i) *Then, up to isomorphism of linear spaces,*

$$(H_n(\mathcal{X}, d))^* \cong H^n(\mathcal{X}^*, d^*).$$

(ii) *If in addition $(\mathcal{X}, d)$ is a chain complex of* nuclear *Fréchet spaces, then $H_n(\mathcal{X}, d)$ is a nuclear Fréchet space and, up to topological isomorphism,*

$$(H_n(\mathcal{X}, d))^* \cong H^n(\mathcal{X}^*, d^*) \text{ and } H_n(\mathcal{X}, d) \cong (H^n(\mathcal{X}^*, d^*))^*.$$

Proof. We will give a proof of (ii), case (i) being simpler. By [5, Theorem 8.6.13], d_n has closed range if and only if d_n^* has closed range. Thus d_{n-1}, d_n, d_{n-1}^* and d_n^* have closed ranges. We consider the following commutative diagram as in [22, Lemma V.10.3].

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \downarrow & & & & \\
0 & \to & B_n(\mathcal{X}) & \overset{i_n}{\hookrightarrow} & Z_n(\mathcal{X}) & \overset{\sigma_n}{\twoheadrightarrow} & H_n(\mathcal{X}) & \to & 0 \\
 & & \tilde{d}_n \uparrow & & j_n \downarrow & & & & \\
 & & X_{n+1} & \overset{d_n}{\to} & X_n & & & & \\
 & & & & d_{n-1} \downarrow & & & & \\
 & & & & X_{n-1} & & & &
\end{array}
$$

where i_n and j_n are the natural inclusions and σ_n is the quotient map. The notation $\tilde{d}$ is an instance of one we shall use repeatedly, and thus we adopt the following definition. Given a continuous linear map $\theta : E \to F$, the map $\tilde{\theta}$ is the surjective map $\tilde{\theta} : E \to \operatorname{Im}\theta$ defined by $\tilde{\theta}(t) = \theta(t)$. Here again all the maps have closed ranges.

We form the dual diagram and add the kernel of d_n^*, $Z^n(\mathcal{X}^*) = \operatorname{Ker} d_n^*$, and the image of d_{n-1}^*, $B^n(\mathcal{X}^*) = \operatorname{Im} d_{n-1}^*$ which is closed by assumption.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \uparrow & & & & \\
0 & \leftarrow & (B_n(\mathcal{X}))^* & \overset{i_n^*}{\leftarrow} & (Z_n(\mathcal{X}))^* & \overset{\sigma_n^*}{\leftarrow} & (H_n(\mathcal{X}))^* & \leftarrow & 0 \\
 & & \tilde{d}_n^* \downarrow & & j_n^* \uparrow & & & & \\
 & & X_{n+1}^* & \overset{d_n^*}{\leftarrow} & X_n^* & \overset{i_Z}{\leftarrow} & Z^n(\mathcal{X}^*) & \leftarrow & 0 \\
 & & & & d_{n-1}^* \uparrow & & \uparrow i_B & & \\
 & & & & X_{n-1}^* & \overset{\widetilde{d_{n-1}^*}}{\longrightarrow} & B^n(\mathcal{X}^*) & \to & 0
\end{array}
$$

where $\widetilde{d_{n-1}^*} : X_{n-1}^* \to \operatorname{Im} d_{n-1}^* : \gamma \to [d_{n-1}^*](\gamma) = \gamma \circ d_{n-1}$. This diagram commutes and has exact rows and columns. By [20, Lemma 2.3], the exactness of a complex in $\mathcal{F}r$ is equivalent to the exactness of its dual complex. Thus the exactness of the first line follows from

[20, Lemma 2.3]; of the second line from the definition of $Z^n(\mathcal{X}^*)$; of the first column from [5, Corollary 8.6.11] since the surjectivity of $\tilde{d}_n$ implies the injectivity of $\tilde{d}_n^*$, and of the second column from [20, Lemma 2.3]. Commutativity only needs to be checked for the square involving the two added terms, namely $Z^n(\mathcal{X}^*)$ and $B^n(\mathcal{X}^*)$, and this is obvious. By Lemma 3.1 and Lemma 3.2, $Z^n(\mathcal{X}^*) = \operatorname{Ker} d_n^*$ and $B^n(\mathcal{X}^*) = \operatorname{Im} d_{n-1}^*$ are the strong duals of nuclear Fréchet spaces. Therefore this diagram is one of strong duals of nuclear Fréchet spaces and continuous linear operators with closed ranges.

By Lemma 3.2, $\operatorname{Im} \sigma_n^*$ is a strong dual of a nuclear Fréchet space. Therefore, by Corollary 3.3, the continuous linear surjective operator

$$\widetilde{\sigma_n^*} : (H_n(\mathcal{X}))^* \to \operatorname{Im} \sigma_n^* : \gamma \mapsto \sigma_n^*(\gamma)$$

is open.

Let us define a map

$$\varphi : Z^n(\mathcal{X}^*) \to (H_n(\mathcal{X}))^*$$

by the formula $\varphi = \widetilde{\sigma_n^*}^{-1} \circ j_n^* \circ i_Z$, where $\widetilde{\sigma_n^*}^{-1}$ is the inverse of the topological isomorphism $\widetilde{\sigma_n^*}$. It is now a standard diagram-chasing argument to show that φ is well defined and surjective. Let us give this argument. An element $z \in Z^n(\mathcal{X}^*)$ is sent by $d_n^* \circ i_Z$ to 0 in X_{n+1}^* and therefore, since $\tilde{d}_n^*$ is injective, $[i_n^* \circ j_n^* \circ i_Z](z) = 0$. Hence the element $[j_n^* \circ i_Z](z)$ of $(Z_n(\mathcal{X}))^*$ belongs to $\operatorname{Ker} i_n^* = \operatorname{Im} \sigma_n^*$, by exactness of the first line of the diagram. Thus φ is a well defined continuous linear operator. To show that this map is surjective, starting with $v \in (H_n(\mathcal{X}))^*$, we get $u = \sigma_n^*(v) \in (Z_n(\mathcal{X}))^*$, and, since j_n^* is surjective, there is $t \in \mathcal{X}_n^*$ such that $j_n^*(t) = u$. It is easy to see that $t \in \operatorname{Ker} d_n^*$ and therefore it lifts uniquely to $z \in Z^n(\mathcal{X}^*)$ and $\varphi(z) = v$.

One can see that $i_B(B^n(\mathcal{X}^*)) \subseteq \operatorname{Ker} \varphi$, since $\widetilde{d_{n-1}^*}$ is surjective and, for any $y \in X_{n-1}^*$, $[j_n^* \circ d_{n-1}^*](y) = 0$. Suppose $z \in \operatorname{Ker} \varphi$, hence $[j_n^* \circ i_Z](z) = 0$. It implies that $i_Z(z) \in \operatorname{Ker} j_n^* = \operatorname{Im} d_{n-1}^*$, so that there is $y \in X_{n-1}^*$ such that $d_{n-1}^*(y) = i_Z(z)$. Since i_Z is injective, $z = i_B(\widetilde{d_{n-1}^*}(y))$. Thus $\operatorname{Ker} \varphi = i_B[B^n(\mathcal{X}^*)]$.

By Corollary 3.3, the continuous surjective linear operator between strong dual of nuclear Fréchet spaces

$$\varphi : Z^n(\mathcal{X}^*) \to (H_n(\mathcal{X}))^*$$

is open and, up to topological isomorphism,

$$(H_n(\mathcal{X}, d))^* \cong H^n(\mathcal{X}^*, d^*).$$

By [23, Theorem 4.4.13], up to topological isomorphism,

$$H_n(\mathcal{X}, d) \cong (H^n(\mathcal{X}^*, d^*))^*.$$

$\square$

COROLLARY 3.6. *Let $(\mathcal{Y}, d)$ be a cochain complex of complete nuclear DF-spaces and continuous operators. Suppose that, for a certain n, d_n and d_{n-1} have closed ranges. Then $H^n(\mathcal{Y}, d)$ is a complete nuclear DF-space, $H_n(\mathcal{Y}^*, d^*)$ is a nuclear Fréchet space and, up to topological isomorphism,*

$$(H^n(\mathcal{Y}, d))^* \cong H_n(\mathcal{Y}^*, d^*) \text{ and } H^n(\mathcal{Y}, d) \cong (H_n(\mathcal{Y}^*, d^*))^*.$$

Proof. By [23, Theorem 4.4.13], the complex $(\mathcal{Y}, d)$ is the strong dual of the chain complex $(\mathcal{Y}^*, d^*)$ of nuclear Fréchet spaces and continuous linear operators. By [23, Theorem 4.4.12], complete nuclear DF-spaces are reflexive, and therefore the statement follows from Theorem 3.5 and Proposition 3.4. $\square$

4. THE KÜNNETH FORMULA FOR FRÉCHET AND COMPLETE NUCLEAR DF-COMPLEXES

In this section we prove the existence of a topological isomorphism in the Künneth formula for the cohomology groups of complete nuclear DF-complexes (Theorem 4.3). To start with we state the result by F. Gourdeau, M.C. White and the author on the Künneth formula for Fréchet and Banach chain complexes. Note that similar results are true for cochain complexes. One can see that to obtain the Künneth formula in the category of Fréchet spaces and continuous operators, we need the following notions of strict flatness (Def. 2.2) and of the topological purity of short exact sequences of Fréchet spaces (Def. 2.4). These conditions allow us to deal with the known problems in the category of Fréchet spaces that the projective tensor product of injective continuous linear operators is not necessarily injective and the range of an operator is not always closed.

THEOREM 4.1. [7, Theorem 5.2 and Corollary 4.9] *Let $\mathcal{X}$ and $\mathcal{Y}$ be bounded chain complexes in $\mathcal{F}r$ (in $\mathcal{B}an$) such that all boundary maps have closed ranges. Suppose that the following exact sequences of Fréchet (Banach) spaces are topologically pure for all n:*

(1) $$0 \to Z_n(\mathcal{X}) \xrightarrow{i_n} X_n \xrightarrow{\tilde{d}_{n-1}} B_{n-1}(\mathcal{X}) \to 0$$

and

(2) $$0 \to B_n(\mathcal{X}) \xrightarrow{j_n} Z_n(\mathcal{X}) \xrightarrow{\sigma_n} H_n(\mathcal{X}) \to 0.$$

where i_n and j_n are the natural inclusions and σ_n is the quotient map. Suppose also that $Z_n(\mathcal{X})$ and $B_n(\mathcal{X})$ are strictly flat in $\mathcal{F}r$ (in $\mathcal{B}an$) for all n. Then, up to topological isomorphism,

$$H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})],$$

and, in addition, for complexes of Banach spaces, there is also a topological isomorphism

$$H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})]^*.$$

COROLLARY 4.2. *Let $\mathcal{X}$ and $\mathcal{Y}$ be bounded chain complexes of Banach spaces and of Fréchet spaces respectively such that all boundary maps have closed ranges, $H_n(\mathcal{X})$ and $B_n(\mathcal{X})$ are strictly flat in $\mathcal{B}an$ for all n. Then, up to topological isomorphism,*

$$H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})]$$

and, up to isomorphism of linear spaces,

$$H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})]^*.$$

If, in addition, $\mathcal{Y}$ is a complex of Banach spaces, then both the above isomorphisms are topological.

Proof. In the category of Banach spaces, by [12, Proposition VII.1.17], $B_n(\mathcal{X})$ and $H_m(\mathcal{X})$ strictly flat implies that $Z_n(\mathcal{X})$ is strictly flat as well. By [26, Proposition 4.4], $B_n(\mathcal{X})$, $H_m(\mathcal{X})$ and $Z_n(\mathcal{X})$ are also strictly flat in $\mathcal{F}r$. By [7, Lemma 4.3], strict flatness of $B_n(\mathcal{X})$ and $H_m(\mathcal{X})$ in $\mathcal{B}an$ implies that the short exact sequences (1) and (2) of Banach spaces are weakly admissible. By [20, Lemma 2.4], the short exact sequences (1) and (2) are topologically pure in $\mathcal{F}r$. The statement follows from Theorem 4.1 and Theorem 3.5.

By Proposition 3.4, in the case that both $\mathcal{X}$ and $\mathcal{Y}$ are from $\mathcal{B}an$, we have a topological isomorphism $H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong (H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}))^*$. $\square$

The topological isomorphism (3) for homology groups under the assumptions of Part (i) of the following theorem is already known, see

M. Karoubi [16]. To get the isomorphism for cohomology groups of dual complexes he required $H^n(\mathcal{X}^*)$ to be finite-dimensional.

THEOREM 4.3. *Let $\mathcal{X}$ and $\mathcal{Y}$ be bounded chain complexes in $\mathcal{F}r$ such that all boundary maps have closed ranges.*

(i) *Suppose that one of complexes, say $\mathcal{X}$, is a complex of nuclear Fréchet spaces. Then, up to topological isomorphism,*

$$(3) \qquad H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})]$$

and, up to isomorphism of linear spaces,

$$H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})]^* \cong \bigoplus_{m+q=n} [H^m(\mathcal{X}^*)\hat{\otimes}[H_q(\mathcal{Y})]^*].$$

(ii) *Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are complexes of nuclear Fréchet spaces. Then, up to topological isomorphism,*

$$H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong H^n(\mathcal{X}^*\hat{\otimes}\mathcal{Y}^*) \cong \bigoplus_{m+q=n} [H^m(\mathcal{X}^*)\hat{\otimes}H^q(\mathcal{Y}^*)].$$

Proof. (i) Suppose that $\mathcal{X}$ is a complex of nuclear Fréchet spaces. Since all boundary maps have closed ranges, $Z_n(\mathcal{X})$ and $B_n(\mathcal{X})$ are nuclear Fréchet spaces. By Theorem A.1.6 and Theorem A.1.5 of [6], $Z_n(\mathcal{X})$ and $B_n(\mathcal{X})$ are strictly flat all n in $\mathcal{F}r$ and the short exact sequences (1) and (2) are topologically pure in $\mathcal{F}r$. The first part of the statement follows from Theorem 4.1. By Theorem 3.5, up to isomorphism of linear spaces, $H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) = (H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}))^*$. Thus, up to isomorphism of linear spaces,

$$H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})]^*$$

By assumption, $H_m(\mathcal{X})$ is a nuclear Fréchet space for all m. By [15, Theorem 21.5.9] and by Theorem 3.5, up to topological isomorphism,

$$[H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})]^* \cong [H_m(\mathcal{X})]^*\hat{\otimes}[H_q(\mathcal{Y})]^* \cong H^m(\mathcal{X}^*)\hat{\otimes}[H_q(\mathcal{Y})]^*$$

for all m, q.

(ii) Since $\mathcal{X}$ and $\mathcal{Y}$ are complexes of nuclear Fréchet spaces, by [15, Theorem 21.5.9], up to topological isomorphism, $(\mathcal{X}\hat{\otimes}\mathcal{Y})^* \cong \mathcal{X}^*\hat{\otimes}\mathcal{Y}^*$, and so

$$H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong H^n(\mathcal{X}^*\hat{\otimes}\mathcal{Y}^*).$$

By [29, Proposition III.50.1], the projective tensor product of nuclear Fréchet spaces is a nuclear Fréchet space. Hence $\mathcal{X}\hat{\otimes}\mathcal{Y}$ is a complex of nuclear Fréchet spaces. By (i), for all n,

$$H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})],$$

is a nuclear Fréchet space. By Proposition 3.4 and Theorem 3.5,

$$H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong (H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}))^* \cong \left(\bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})] \right)^*.$$

By [15, Theorem 21.5.9] and Theorem 3.5, since $H_m(\mathcal{X})$ and $H_q(\mathcal{Y})$ are nuclear Fréchet spaces,

$$\bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})]^* \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})]^*\hat{\otimes}[H_q(\mathcal{Y})]^*$$

$$\cong \bigoplus_{m+q=n} [H^m(\mathcal{X}^*)\hat{\otimes}H^q(\mathcal{Y}^*)].$$

$\square$

THEOREM 4.4.   (i) *Let $\mathcal{X}$ and $\mathcal{Y}$ be bounded chain complexes of complete* nuclear *DF-spaces such that all boundary maps have closed ranges. Then, up to topological isomorphism,*

$$H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}) \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})].$$

   (ii) *Let $\mathcal{X}$ be a bounded chain complex of complete* nuclear *DF-spaces such that all boundary maps have closed ranges, and let $\mathcal{Y}$ be a bounded chain complex of complete DF-spaces such that all boundary maps of its strong dual complex $\mathcal{Y}^*$ have closed ranges. Then, up to topological isomorphism,*

$$H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong H^n(\mathcal{X}^*\hat{\otimes}\mathcal{Y}^*) \cong \bigoplus_{m+q=n} H^m(\mathcal{X}^*)\hat{\otimes}H^q(\mathcal{Y}^*).$$

Proof. (i) By [23, Theorem 4.4.13], the chain complexes $\mathcal{X}$ and $\mathcal{Y}$ are the strong duals of cochain complexes $\mathcal{X}^*$ and $\mathcal{Y}^*$ of nuclear Fréchet spaces and continuous linear operators. By Proposition 3.4, all boundary maps of complexes $\mathcal{X}^*$ and $\mathcal{Y}^*$ have closed ranges. By Theorem 4.3 (ii), for the complexes $\mathcal{X}^*$ and $\mathcal{Y}^*$ of nuclear Fréchet spaces, up to topological isomorphism,

$$H_n(\mathcal{X}\hat{\otimes}\mathcal{Y}) \cong H_n((\mathcal{X}^*)^*\hat{\otimes}(\mathcal{Y}^*)^*)$$

$$\cong \bigoplus_{m+q=n} [H_m((\mathcal{X}^*)^*)\hat{\otimes}H_q((\mathcal{Y}^*)^*)] \cong \bigoplus_{m+q=n} [H_m(\mathcal{X})\hat{\otimes}H_q(\mathcal{Y})].$$

(ii) Since $\mathcal{X}$ is the complex of complete nuclear DF-spaces, then, by [15, Theorem 21.5.9], $(\mathcal{X}\hat{\otimes}\mathcal{Y})^* \cong \mathcal{X}^*\hat{\otimes}\mathcal{Y}^*$ and

$$H^n((\mathcal{X}\hat{\otimes}\mathcal{Y})^*) \cong H^n(\mathcal{X}^*\hat{\otimes}\mathcal{Y}^*).$$

By Proposition 3.4, all boundary maps of complexes $\mathcal{X}^*$ have closed ranges. By Theorem 4.3 (i), for the cochain complex of nuclear Fréchet spaces $\mathcal{X}^*$ ([23, Theorem 4.4.13]) and for the cochain complex of Fréchet spaces $\mathcal{Y}^*$,

$$H^n(\mathcal{X}^*\hat{\otimes}\mathcal{Y}^*) \cong \bigoplus_{m+q=n} H^m(\mathcal{X}^*)\hat{\otimes}H^q(\mathcal{Y}^*).$$

$\square$

5. The Künneth formula for Hochschild cohomology of $\hat{\otimes}$-algebras which are nuclear DF- or Fréchet spaces

Let $\mathcal{A}$ be a $\hat{\otimes}$-algebra and let X be an $\mathcal{A}$-$\hat{\otimes}$-bimodule. We assume here that the category of underlying spaces Φ has the properties from Definition 2.1. Let us recall the definition of the standard homological chain complex $\mathcal{C}_{\sim}(\mathcal{A}, X)$. For $n \geq 0$, let $C_n(\mathcal{A}, X)$ denote the projective tensor product $X\hat{\otimes}\mathcal{A}^{\hat{\otimes}^n}$. The elements of $C_n(\mathcal{A}, X)$ are called n-chains. Let the differential $d_n : C_{n+1} \to C_n$ be given by

$$d_n(x \otimes a_1 \otimes \ldots \otimes a_{n+1}) = x \cdot a_1 \otimes \ldots \otimes a_{n+1}$$

$$+ \sum_{k=1}^{n} (-1)^k (x \otimes a_1 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_{n+1})$$

$$+ (-1)^{n+1}(a_{n+1} \cdot x \otimes a_1 \otimes \ldots \otimes a_n)$$

with d_{-1} the null map. The space of boundaries $B_n(\mathcal{C}_{\sim}(\mathcal{A}, X)) = \mathrm{Im}\, d_n$ is denoted by $B_n(\mathcal{A}, X)$ and the space of cycles $Z_n(\mathcal{C}_{\sim}(\mathcal{A}, X)) = \mathrm{Ker}\, d_{n-1}$ is denoted by $Z_n(\mathcal{A}, X)$. The homology groups of this complex $H_n(\mathcal{C}_{\sim}(\mathcal{A}, X)) = Z_n(\mathcal{A}, X)/B_n(\mathcal{A}, X)$ are called the *continuous Hochschild homology groups of $\mathcal{A}$ with coefficients in X* and are denoted by $\mathcal{H}_n(\mathcal{A}, X)$ [12, Definition II.5.28].

We also consider the cohomology groups $H^n((\mathcal{C}_{\sim}(\mathcal{A}, X))^*)$ of the dual complex $(\mathcal{C}_{\sim}(\mathcal{A}, X))^*$ with the strong dual topology. For Banach algebras $\mathcal{A}$, $(\mathcal{C}_{\sim}(\mathcal{A}, X))^*$ is topologically isomorphic to the Hochschild cohomology $\mathcal{H}^n(\mathcal{A}, X^*)$ of $\mathcal{A}$ with coefficients in the dual $\mathcal{A}$-bimodule X^* [12, Definition I.3.2 and Proposition II.5.27].

Let $\mathcal{A}$ be in Φ and be a unital $\hat{\otimes}$-algebra. We put $\beta_n(\mathcal{A}) = \mathcal{A}^{\hat{\otimes}^{n+2}}$, $n \geq 0$ and let $d_n : \beta_{n+1}(\mathcal{A}) \to \beta_n(\mathcal{A})$ be given by

$$d_n(a_0 \otimes \ldots \otimes a_{n+2}) = \sum_{k=0}^{n+1} (-1)^k (a_0 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_{n+2}).$$

By [12, Proposition III.2.9], the complex over $\mathcal{A}$, $\pi : \beta(\mathcal{A}) \to \mathcal{A} :$ $a \otimes b \mapsto ab$, where $\beta(\mathcal{A})$ denotes

$$0 \leftarrow \beta_0(\mathcal{A}) \xleftarrow{d_0} \beta_1(\mathcal{A}) \xleftarrow{d_1} \cdots \leftarrow \beta_n(\mathcal{A}) \xleftarrow{d_n} \beta_{n+1}(\mathcal{A}) \leftarrow \ldots$$

is a projective resolution of $\mathcal{A}$-$\hat{\otimes}$-bimodule $\mathcal{A}$. $\beta(\mathcal{A})$ is called the *bar resolution* of $\mathcal{A}$. The complex has a *contracting homotopy* $s_n :$ $\beta_n(\mathcal{A}) \to \beta_{n+1}(\mathcal{A})$, $(n \geq 1)$, given by

$$s_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1},$$

which is to say that $d_n s_n + s_{n-1} d_{n-1} = 1_{\beta_n(\mathcal{A})}$.

PROPOSITION 5.1. *Let A_1 and A_2 be unital $\hat{\otimes}$-algebras, let $0 \leftarrow X \xleftarrow{\varepsilon_1} \mathcal{X}$ be a projective resolution of $X \in A_1$-unmod and $0 \leftarrow Y \xleftarrow{\varepsilon_2} \mathcal{Y}$ be a projective resolution of $Y \in A_2$-unmod. Then $0 \leftarrow X \hat{\otimes} Y \xleftarrow{\varepsilon_1 \otimes \varepsilon_2} \mathcal{X} \hat{\otimes} \mathcal{Y}$ is a projective resolution of $X \hat{\otimes} Y \in A_1 \hat{\otimes} A_2$-unmod.*

Proof. The proof requires only minor modifications of that of [22, Proposition X.7.1]. $\square$

Note that the statement of Proposition 5.1 is also true in the category of bimodules. In the next theorem we extend the result [7, Theorem 6.2] to the category of complete nuclear DF-spaces.

THEOREM 5.2. *Let the category for underlying spaces Φ be $\mathcal{F}r$ or the category of complete nuclear DF-spaces. Let $\mathcal{A}$ and $\mathcal{B}$ be unital $\hat{\otimes}$-algebras with identities $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$, let X be an $\mathcal{A}$-$\hat{\otimes}$-bimodule and let Y be a $\mathcal{B}$-$\hat{\otimes}$-bimodule. Then, up to topological isomorphism, for all $n \geq 1$,*

$$\mathcal{H}_n(\mathcal{A} \hat{\otimes} \mathcal{B}, X \hat{\otimes} Y) \cong \mathcal{H}_n(\mathcal{A} \hat{\otimes} \mathcal{B}, e_{\mathcal{A}} X e_{\mathcal{A}} \hat{\otimes} e_{\mathcal{B}} Y e_{\mathcal{B}})$$

$$\cong H_n(\mathcal{C}_{\sim}(\mathcal{A}, e_{\mathcal{A}} X e_{\mathcal{A}}) \hat{\otimes} \mathcal{C}_{\sim}(\mathcal{B}, e_{\mathcal{B}} Y e_{\mathcal{B}})).$$

If X and Y are also unital, then, up to topological isomorphism, for all $n \geq 0$,

$$\mathcal{H}_n(\mathcal{A} \hat{\otimes} \mathcal{B}, X \hat{\otimes} Y) \cong H_n(\mathcal{C}_{\sim}(\mathcal{A}, X) \hat{\otimes} \mathcal{C}_{\sim}(\mathcal{B}, Y)).$$

Proof. It is well known that, for a $\hat{\otimes}$-algebra $\mathcal{U}$ with an identity e and for a $\mathcal{U}$-$\hat{\otimes}$-bimodule Z, up to topological isomorphism, for all $n \geq 1$,

$$\mathcal{H}_n(\mathcal{U}, Z) \cong \mathcal{H}_n(\mathcal{U}, eZe),$$

where eZe is a unital $\mathcal{U}$-$\hat{\otimes}$-bimodule. Therefore, up to topological isomorphism, for all $n \geq 1$,

$$\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y) \cong \mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, e_{\mathcal{A}}Xe_{\mathcal{A}}\hat{\otimes}e_{\mathcal{B}}Ye_{\mathcal{B}}).$$

Let $\beta(\mathcal{A})$ and $\beta(\mathcal{B})$ be the bar resolutions of $\mathcal{A}$ and $\mathcal{B}$. Since the bar resolution $\beta(\mathcal{A})$ is an $\mathcal{A}$-biprojective resolution of $\mathcal{A}$ and $\beta(\mathcal{B})$ is a $\mathcal{B}$-biprojective resolution of $\mathcal{B}$, by Proposition 5.1 their projective tensor product $\beta(\mathcal{A})\hat{\otimes}\beta(\mathcal{B})$ is an $\mathcal{A}\hat{\otimes}\mathcal{B}$-biprojective resolution of $\mathcal{A}\hat{\otimes}\mathcal{B}$.

The open mapping theorem holds in the categories of Fréchet spaces and of complete nuclear DF-spaces, see Corollary 3.3 for DF-spaces, and, for a continuous morphism of chain complexes $\psi_\sim : \mathcal{X}_\sim \to \mathcal{P}_\sim$ in these categories, a surjective map $H_n(\varphi) : H_n(\mathcal{X}) \to H_n(\mathcal{Y})$ is automatically open, see [12, Lemma 0.5.9] and [21, Lemma 3.5].

For a unital $\hat{\otimes}$-algebra $\mathcal{U}$ and for a unital $\mathcal{U}$-$\hat{\otimes}$-bimodule Z, by [12, Theorem III.4.25], the Hochschild chain complex $\mathcal{C}_\sim(\mathcal{U}, Z)$ is isomorphic to $Z\hat{\otimes}_{\mathcal{U}^e}\beta(\mathcal{U})$ and, up to topological isomorphism, for all $n \geq 0$,

$$\mathcal{H}_n(\mathcal{U}, Z) \cong \mathrm{Tor}_n^{\mathcal{U}^e}(Z, \mathcal{U}) \cong H_n(Z\hat{\otimes}_{\mathcal{U}^e}\beta(\mathcal{U})).$$

By [12, Section III.3.15], the nth derived functor $\mathrm{Tor}_n^{\mathcal{U}^e}(\cdot, \mathcal{U})$ does not depend on the choice of a $\mathcal{U}$-biprojective resolution of $\mathcal{U}$. Therefore in these categories, up to topological isomorphism, for all $n \geq 0$,

$$\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, e_{\mathcal{A}}Xe_{\mathcal{A}}\hat{\otimes}e_{\mathcal{B}}Ye_{\mathcal{B}}) \cong \mathrm{Tor}_n^{(\mathcal{A}\hat{\otimes}\mathcal{B})^e}(e_{\mathcal{A}}Xe_{\mathcal{A}}\hat{\otimes}e_{\mathcal{B}}Ye_{\mathcal{B}}, \mathcal{A}\hat{\otimes}\mathcal{B})$$

$$\cong H_n((e_{\mathcal{A}}Xe_{\mathcal{A}}\hat{\otimes}e_{\mathcal{B}}Ye_{\mathcal{B}})\hat{\otimes}_{(\mathcal{A}\hat{\otimes}\mathcal{B})^e}\beta(\mathcal{A}\hat{\otimes}\mathcal{B}))$$

$$\cong H_n((e_{\mathcal{A}}Xe_{\mathcal{A}}\hat{\otimes}e_{\mathcal{B}}Ye_{\mathcal{B}})\hat{\otimes}_{(\mathcal{A}\hat{\otimes}\mathcal{B})^e}(\beta(\mathcal{A})\hat{\otimes}\beta(\mathcal{B}))).$$

By [12, Section II.5.3], one can prove that the following chain complexes are isomorphic:

$$(e_{\mathcal{A}}Xe_{\mathcal{A}}\hat{\otimes}e_{\mathcal{B}}Ye_{\mathcal{B}})\hat{\otimes}_{(\mathcal{A}\hat{\otimes}\mathcal{B})^e}(\beta(\mathcal{A})\hat{\otimes}\beta(\mathcal{B})) \cong$$

$$(e_{\mathcal{A}}Xe_{\mathcal{A}}\hat{\otimes}_{\mathcal{A}^e}\beta(\mathcal{A}))\hat{\otimes}(e_{\mathcal{B}}Ye_{\mathcal{B}}\hat{\otimes}_{\mathcal{B}^e}\beta(\mathcal{B})) \cong$$

$$\mathcal{C}_\sim(\mathcal{A}, e_{\mathcal{A}}Xe_{\mathcal{A}})\hat{\otimes}\mathcal{C}_\sim(\mathcal{B}, e_{\mathcal{B}}Ye_{\mathcal{B}})$$

Thus, up to topological isomorphism, for all $n \geq 0$,

$$\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, e_{\mathcal{A}}Xe_{\mathcal{A}}\hat{\otimes}e_{\mathcal{B}}Ye_{\mathcal{B}}) \cong$$

$$H_n((e_{\mathcal{A}}Xe_{\mathcal{A}}\hat{\otimes}e_{\mathcal{B}}Ye_{\mathcal{B}})\hat{\otimes}_{(\mathcal{A}\hat{\otimes}\mathcal{B})^e}(\beta(\mathcal{A})\hat{\otimes}\beta(\mathcal{B}))) \cong$$

$$H_n(\mathcal{C}_\sim(\mathcal{A}, e_{\mathcal{A}}Xe_{\mathcal{A}})\hat{\otimes}\mathcal{C}_\sim(\mathcal{B}, e_{\mathcal{B}}Ye_{\mathcal{B}})).$$

$\square$

REMARK 5.3. For a $\hat{\otimes}$-algebra $\mathcal{U}$ with an identity e and for a $\mathcal{U}$-$\hat{\otimes}$-bimodule Z, up to topological isomorphism, for all $n \geq 1$,

$$\mathcal{H}_n(\mathcal{U}, Z) \cong \mathcal{H}_n(\mathcal{U}, eZe),$$

where eZe is a unital $\mathcal{U}$-$\hat{\otimes}$-bimodule. Thus it is easy to see that if the boundary maps of the standard homology complex $\mathcal{C}_\sim(\mathcal{U}, Z)$ have closed ranges then the boundary maps of the standard homology complex $\mathcal{C}_\sim(\mathcal{U}, eZe)$ have closed ranges. The previous theorem and this remark show that further we may concentrate on unital bimodules.

THEOREM 5.4. *Let the category for underlying spaces Φ be $\mathcal{F}r$ or the category of complete nuclear DF-spaces. Let $\mathcal{A}$ and $\mathcal{B}$ be unital $\hat{\otimes}$-algebras, let X be a unital $\mathcal{A}$-$\hat{\otimes}$-bimodule and let Y be a unital $\mathcal{B}$-$\hat{\otimes}$-bimodule. Suppose that all boundary maps of the standard homology complexes $\mathcal{C}_\sim(\mathcal{A}, X)$ and $\mathcal{C}_\sim(\mathcal{B}, Y)$ have closed ranges. Then*

(i) *up to topological isomorphism in the category of complete nuclear DF-spaces and in the category $\mathcal{F}r$ under the assumption that either $\mathcal{A}$ and X or $\mathcal{B}$ and Y are nuclear, for all $n \geq 0$,*

$$\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y) \cong \bigoplus_{m+q=n} [\mathcal{H}_m(\mathcal{A}, X)\hat{\otimes}\mathcal{H}_q(\mathcal{B}, Y)];$$

(ii) *up to topological isomorphism in the category of complete nuclear DF-spaces and in the category $\mathcal{F}r$ under the assumption that $\mathcal{A}$, X, $\mathcal{B}$ and Y are nuclear, for all $n \geq 0$,*

$$H^n((\mathcal{C}_\sim(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y))^*) \cong \left(\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y)\right)^*$$

$$\cong \bigoplus_{m+q=n} [H^m((\mathcal{C}_\sim(\mathcal{A}, X))^*)\hat{\otimes}H^q((\mathcal{C}_\sim(\mathcal{B}, Y))^*)];$$

(iii) *up to isomorphism of linear spaces, in the category $\mathcal{F}r$ under the assumption that either $\mathcal{A}$ and X or $\mathcal{B}$ and Y are nuclear, for all $n \geq 0$,*

$$H^n((\mathcal{C}_\sim(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y))^*) \cong \left(\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y)\right)^*$$

$$\cong \bigoplus_{m+q=n} [H_m(\mathcal{C}_\sim(\mathcal{A}, X))]^*\hat{\otimes}[H_q(\mathcal{C}_\sim(\mathcal{B}, Y))]^*.$$

Proof. By Theorem 5.2, up to topological isomorphism, for all $n \geq 0$,

$$\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y) \cong H_n(\mathcal{C}_\sim(\mathcal{A}, X)\hat{\otimes}\mathcal{C}_\sim(\mathcal{B}, Y)).$$

By [29, Proposition III.50.1], the projective tensor product of nuclear Fréchet spaces is a nuclear Fréchet space. By [29, Proposition

III.50.1] and [15, Theorem 15.6.2], the projective tensor product of complete nuclear DF-spaces is a complete nuclear DF-space. Therefore $\mathcal{C}_\sim(\mathcal{A}, X)$ and $\mathcal{C}_\sim(\mathcal{B}, Y)$ are complexes of complete nuclear DF-spaces or of [nuclear] Fréchet spaces such that all boundary maps have closed ranges. The results follow from Theorem 4.3 and Theorem 4.4. $\qquad\square$

THEOREM 5.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital Banach and Fréchet algebras respectively, let X be a unital Banach $\mathcal{A}$-bimodule and let Y be a unital Fréchet $\mathcal{B}$-bimodule. Suppose that all boundary maps of the standard homology complexes $\mathcal{C}_\sim(\mathcal{A}, X)$ and $\mathcal{C}_\sim(\mathcal{B}, Y)$ have closed ranges. Suppose that $\mathcal{H}_n(\mathcal{A}, \mathcal{X})$ and $B_n(\mathcal{A}, \mathcal{X})$ are strictly flat in $\mathcal{B}an$. Then, up to topological isomorphism,*

$$\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y) \cong \bigoplus_{m+q=n} [\mathcal{H}_m(\mathcal{A}, X)\hat{\otimes}\mathcal{H}_q(\mathcal{B}, Y)],$$

and, up to isomorphism of linear spaces,

$$H^n((\mathcal{C}_\sim(\mathcal{A}\hat{\otimes}\mathcal{B}, X\hat{\otimes}Y))^*) \cong \bigoplus_{m+q=n} [\mathcal{H}_m(\mathcal{A}, X)\hat{\otimes}\mathcal{H}_q(\mathcal{B}, Y)]^*.$$

Proof. It follows from Theorem 5.2 and Corollary 4.2. $\qquad\square$

EXAMPLE 5.6. Let $\mathcal{A} = \ell^1(\mathbb{Z}_+)$ where

$$\ell^1(\mathbb{Z}_+) = \left\{ (a_n)_{n=0}^\infty : \sum_{n=0}^\infty |a_n| < \infty \right\}$$

be the unital semigroup Banach algebra of $\mathbb{Z}_+$ with convolution multiplication and norm $\|(a_n)_{n=0}^\infty\| = \sum_{n=0}^\infty |a_n|$. In [7, Theorem 7.4] we showed that all boundary maps of the standard homology complex $\mathcal{C}_\sim(\mathcal{A}, \mathcal{A})$ have closed ranges and that $\mathcal{H}_n(\mathcal{A}, \mathcal{A})$ and $B_n(\mathcal{A}, \mathcal{A})$ are strictly flat in $\mathcal{B}an$. In [7, Theorem 7.5] we describe explicitly the simplicial homology groups $\mathcal{H}_n(\ell^1(\mathbb{Z}_+^k), \ell^1(\mathbb{Z}_+^k))$ and cohomology groups $\mathcal{H}^n(\ell^1(\mathbb{Z}_+^k), (\ell^1(\mathbb{Z}_+^k))^*)$ of the semigroup algebra $\ell^1(\mathbb{Z}_+^k)$.

EXAMPLE 5.7. In [21, Theorem 5.3] we describe explicitly the cyclic-type homology and cohomology groups of amenable Fréchet algebras $\mathcal{B}$. In particular we showed that all boundary maps of the standard homology complex $\mathcal{C}_\sim(\mathcal{B}, \mathcal{B})$ have closed ranges. In [26, Corollary 9.9] Pikovskii showed that an amenable unital uniform Fréchet algebra is topologically isomorphic to the algebra $C(\Omega)$ of continuous complex-valued functions on a hemicompact k-space Ω. Recall that a Hausdorff

topological space Ω is *hemicompact* if there exists a countable exhaustion $\Omega = \bigcup K_n$ with K_n compact such that each compact subset of Ω is contained in some K_n. A Hausdorff topological space Ω is a *k-space* if a subset $F \subset \Omega$ is closed whenever $F \cap K$ is closed for every compact subset $K \subset \Omega$. For example, $C(\mathbb{R})$ is an amenable unital Fréchet algebra.

The closure in a $\hat{\otimes}$-algebra $\mathcal{A}$ of the linear span of elements of the form $\{ab - ba : a, b \in \mathcal{A}\}$ is denoted by $[\mathcal{A}, \mathcal{A}]$.

COROLLARY 5.8. *Let $\mathcal{A} = \ell^1(\mathbb{Z}_+)$ and let $\mathcal{I} = \ell^1(\mathbb{N})$ be the closed ideal of $\ell^1(\mathbb{Z}_+)$ consisting of those elements with $a_0 = 0$. Let C be an amenable unital Fréchet algebra or an amenable Banach algebra. Then*

$$\mathcal{H}_n(\ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}, \ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}) \cong \{0\} \text{ if } n > k;$$

$$\mathcal{H}^n\left(\mathcal{C}_{\sim}\left(\ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}, \ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}\right)^*\right) \cong \{0\} \text{ if } n > k;$$

up to topological isomorphism,

$$\mathcal{H}_n(\ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}, \ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}) \cong \bigoplus^{\binom{k}{n}}\left(\mathcal{I}^{\hat{\otimes}^n}\hat{\otimes}\mathcal{A}^{\hat{\otimes}^{k-n}}\right)\hat{\otimes}(\mathcal{C}/[\mathcal{C},\mathcal{C}])$$

if $n \leq k$; and, up to isomorphism of linear spaces for Fréchet algebras $\mathcal{C}$ and up to topological isomorphism for Banach algebras $\mathcal{C}$,

$$\mathcal{H}^n\left(\mathcal{C}_{\sim}\left(\ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}, \ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}\right)^*\right) \cong \bigoplus^{\binom{k}{n}}\left(\mathcal{I}^{\hat{\otimes}^n}\hat{\otimes}\mathcal{A}^{\hat{\otimes}^{k-n}}\hat{\otimes}(\mathcal{C}/[\mathcal{C},\mathcal{C}])\right)^*$$

if $n \leq k$. Moreover, for Banach algebras $\mathcal{C}$, up to topological isomorphism, for all $n \geq 0$,

$$\mathcal{H}^n(\ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}, (\ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C})^*) \cong \mathcal{H}^n(\mathcal{C}_{\sim}(\ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}, \ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C})^*).$$

Proof. By [21, Theorem 5.3], for an amenable Fréchet algebra $\mathcal{C}$, $\mathcal{H}_0(\mathcal{C}, \mathcal{C}) \cong \mathcal{C}/[\mathcal{C}, \mathcal{C}]$ and $\mathcal{H}_n(\mathcal{C}, \mathcal{C}) \cong \{0\}$ for all $n \geq 1$. Recall that an amenable Banach algebra has a bounded approximate identity.

In [7, Theorem 7.4] we showed that all boundary maps of the standard homology complex $\mathcal{C}_{\sim}(\mathcal{A}, \mathcal{A})$ have closed ranges and that $\mathcal{H}_n(\mathcal{A}, \mathcal{A})$ and $B_n(\mathcal{A}, \mathcal{A})$ are strictly flat in $\mathcal{B}an$. By [7, Proposition 7.3], up to topological isomorphism, the simplicial homology groups $\mathcal{H}_n(\mathcal{A}, \mathcal{A})$ are given by $\mathcal{H}_0(\mathcal{A}, \mathcal{A}) \cong \mathcal{A} = \ell^1(\mathbb{Z}_+)$, $\mathcal{H}_1(\mathcal{A}, \mathcal{A}) \cong \mathcal{I} = \ell^1(\mathbb{N})$, $\mathcal{H}_n(\mathcal{A}, \mathcal{A}) \cong \{0\}$ for $n \geq 2$.

Note that $\ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C} \cong \mathcal{A}\hat{\otimes}\mathcal{B}$ where $\mathcal{B} = \ell^1(\mathbb{Z}_+^{k-1})\hat{\otimes}\mathcal{C}$. We use induction on k to prove the corollary for homology groups. For $k = 1$,

the result follows from Theorem 5.5 for an amenable unital Fréchet algebra $\mathcal{C}$, and from [8, Theorem 5.5] for an amenable Banach algebra $\mathcal{C}$. The simplicial homology groups $\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{C}, \mathcal{A}\hat{\otimes}\mathcal{C})$ are given, up to topological isomorphism, by

$$
\begin{aligned}
\mathcal{H}_0(\mathcal{A}\hat{\otimes}\mathcal{C}, \mathcal{A}\hat{\otimes}\mathcal{C}) &\cong \mathcal{A}\hat{\otimes}(\mathcal{C}/[\mathcal{C},\mathcal{C}]), \\
\mathcal{H}_1(\mathcal{A}\hat{\otimes}\mathcal{C}, \mathcal{A}\hat{\otimes}\mathcal{C}) &\cong \mathcal{I}\hat{\otimes}(\mathcal{C}/[\mathcal{C},\mathcal{C}]), \\
\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{C}, \mathcal{A}\hat{\otimes}\mathcal{C}) &\cong \{0\} \quad \text{for } n \geq 2.
\end{aligned}
$$

Let $k > 1$ and suppose that the result for homology holds for $k - 1$. As $\ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C} \cong \mathcal{A}\hat{\otimes}\mathcal{B}$ where $\mathcal{B} = \ell^1(\mathbb{Z}_+^{k-1})\hat{\otimes}\mathcal{C}$, we have

$$
\mathcal{H}_n(\ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}, \ell^1(\mathbb{Z}_+^k)\hat{\otimes}\mathcal{C}) \cong \mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, \mathcal{A}\hat{\otimes}\mathcal{B}).
$$

Also, it follows from the inductive hypothesis that, for all n, the $\mathcal{H}_n(\mathcal{B}, \mathcal{B})$ are Fréchet [Banach] spaces and hence the $B_n(\mathcal{B}, \mathcal{B})$ are closed. We can therefore apply Theorem 5.5 for an amenable unital Fréchet algebra $\mathcal{C}$ and [8, Theorem 5.5] for an amenable Banach algebra $\mathcal{C}$, to get

$$
\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, \mathcal{A}\hat{\otimes}\mathcal{B}) \cong \bigoplus_{m+q=n} \left[\mathcal{H}_m(\mathcal{A}, \mathcal{A})\hat{\otimes}\mathcal{H}_q(\mathcal{B}, \mathcal{B}) \right].
$$

The terms in this direct sum vanish for $m \geq 2$, and thus we only need to consider

$$
\left(\mathcal{H}_0(\mathcal{A}, \mathcal{A})\hat{\otimes}\mathcal{H}_n(\mathcal{B}, \mathcal{B}) \right) \oplus \left(\mathcal{H}_1(\mathcal{A}, \mathcal{A})\hat{\otimes}\mathcal{H}_{n-1}(\mathcal{B}, \mathcal{B}) \right).
$$

The rest is clear. $\qquad\square$

EXAMPLE 5.9. Let $\mathcal{A}$ be the convolution Banach algebra $L^1(\mathbb{R}_+)$ of complex-valued, Lebesgue measurable functions f on $\mathbb{R}_+$ with finite L^1-norm. In [8, Theorem 4.6] we showed that all boundary maps of the standard homology complex $\mathcal{C}_\sim(\mathcal{A}, \mathcal{A})$ have closed ranges and that $\mathcal{H}_n(\mathcal{A}, \mathcal{A})$ and $B_n(\mathcal{A}, \mathcal{A})$ are strictly flat in $\mathcal{B}an$. In [8, Theorem 6.4] we describe explicitly the simplicial homology groups $\mathcal{H}_n(L^1(\mathbb{R}_+^k), L^1(\mathbb{R}_+^k))$ and cohomology groups $\mathcal{H}^n(L^1(\mathbb{R}_+^k), (L^1(\mathbb{R}_+^k))^*)$ of the semigroup algebra $L^1(\mathbb{R}_+^k)$.

COROLLARY 5.10. *Let* $\mathcal{A} = L^1(\mathbb{R}_+^k)$ *and let* C *be an amenable Banach algebra. Then*

$$
\mathcal{H}_n(L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}, L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}) \cong \{0\} \text{ if } n > k;
$$

$$
\mathcal{H}^n\left(L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}, \left(L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}\right)^* \right) \cong \{0\} \text{ if } n > k;
$$

up to topological isomorphism,

$$\mathcal{H}_n(L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}, L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}) \cong \bigoplus^{\binom{k}{n}} L^1(\mathbb{R}_+^k)\hat{\otimes}(\mathcal{C}/[\mathcal{C},\mathcal{C}]) \text{ if } n \le k;$$

and

$$\mathcal{H}^n(L^\infty(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}, (L^\infty(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C})^*) \cong \bigoplus^{\binom{k}{n}} \left[L^\infty(\mathbb{R}_+^k)\hat{\otimes}(\mathcal{C}/[\mathcal{C},\mathcal{C}])\right]^*$$

if $n \le k$.

Proof. Note that $\mathcal{A}$ and $\mathcal{C}$ have bounded approximate identities. By [21, Theorem 5.3], for an amenable Banach algebra $\mathcal{C}$, $\mathcal{H}_n(\mathcal{C},\mathcal{C}) \cong \{0\}$ for all $n \ge 1$, $\mathcal{H}_0(\mathcal{C},\mathcal{C}) \cong \mathcal{C}/[\mathcal{C},\mathcal{C}]$. Therefore all boundary maps of the standard homology complex $\mathcal{C}_\sim(\mathcal{C},\mathcal{C})$ have closed ranges.

In [8, Theorem 4.6] we showed that all boundary maps of the standard homology complex $\mathcal{C}_\sim(\mathcal{A},\mathcal{A})$ have closed ranges and that $\mathcal{H}_n(\mathcal{A},\mathcal{A})$ and $B_n(\mathcal{A},\mathcal{A})$ are strictly flat in $\mathcal{B}an$. By [8, Theorem 4.6], up to topological isomorphism, the simplicial homology groups $\mathcal{H}_n(\mathcal{A},\mathcal{A})$ are given by $\mathcal{H}_0(\mathcal{A},\mathcal{A}) \cong \mathcal{H}_1(\mathcal{A},\mathcal{A}) \cong \mathcal{A} = L^1(\mathbb{R}_+)$ and $\mathcal{H}_n(\mathcal{A},\mathcal{A}) \cong \{0\}$ for $n \ge 2$.

Note that $L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C} \cong \mathcal{A}\hat{\otimes}\mathcal{B}$ where $\mathcal{B} = L^1(\mathbb{R}_+^{k-1})\hat{\otimes}\mathcal{C}$. We use induction on k to prove the corollary for homology groups. For $k = 1$, the result follows from [8, Theorem 5.5]. The simplicial homology groups $\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{C}, \mathcal{A}\hat{\otimes}\mathcal{C})$ are given, up to topological isomorphism, by

$$\mathcal{H}_0(\mathcal{A}\hat{\otimes}\mathcal{C}, \mathcal{A}\hat{\otimes}\mathcal{C}) \cong \mathcal{H}_1(\mathcal{A}\hat{\otimes}\mathcal{C}, \mathcal{A}\hat{\otimes}\mathcal{C}) \cong \mathcal{A}\hat{\otimes}(\mathcal{C}/[\mathcal{C},\mathcal{C}])$$

and

$$\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{C}, \mathcal{A}\hat{\otimes}\mathcal{C}) \cong \{0\}$$

for $n \ge 2$.

Let $k > 1$ and suppose that the result for homology holds for $k - 1$. As $L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C} \cong \mathcal{A}\hat{\otimes}\mathcal{B}$ where $\mathcal{B} = L^1(\mathbb{R}_+^{k-1})\hat{\otimes}\mathcal{C}$, we have

$$\mathcal{H}_n(L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}, L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}) \cong \mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, \mathcal{A}\hat{\otimes}\mathcal{B}).$$

Also, it follows from the inductive hypothesis that, for all n, the $\mathcal{H}_n(\mathcal{B},\mathcal{B})$ are Banach spaces and hence the $B_n(\mathcal{B},\mathcal{B})$ are closed. We can therefore apply [8, Theorem 5.5] for an amenable Banach algebra $\mathcal{C}$, to get

$$\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}, \mathcal{A}\hat{\otimes}\mathcal{B}) \cong \bigoplus_{m+q=n} \left[\mathcal{H}_m(\mathcal{A},\mathcal{A})\hat{\otimes}\mathcal{H}_q(\mathcal{B},\mathcal{B})\right].$$

The terms in this direct sum vanish for $m \geq 2$, and thus we only need to consider

$$\left(\mathcal{H}_0(\mathcal{A}, \mathcal{A})\hat{\otimes}\mathcal{H}_n(\mathcal{B}, \mathcal{B})\right) \oplus \left(\mathcal{H}_1(\mathcal{A}, \mathcal{A})\hat{\otimes}\mathcal{H}_{n-1}(\mathcal{B}, \mathcal{B})\right).$$

For cohomology groups, by [7, Corollary 4.9],

$$\mathcal{H}^n(L^\infty(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}, (L^\infty(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C})^*) \cong \mathcal{H}^n\left(\mathcal{C}_\sim\left(L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}, L^1(\mathbb{R}_+^k)\hat{\otimes}\mathcal{C}\right)^*\right)$$

$$\cong \bigoplus^{\binom{k}{n}}\left[L^\infty(\mathbb{R}_+^k)\hat{\otimes}\left(\mathcal{C}/[\mathcal{C}, \mathcal{C}]\right)\right]^*$$

if $n \leq k$. □

6. Applications to the cyclic-type cohomology of certain Fréchet and DF algebras

In this section we give explicit formulae for the continuous cyclic-type homology and cohomology of projective tensor products of certain $\hat{\otimes}$-algebras which are Fréchet spaces or complete nuclear DF-spaces.

One can consult the books by Loday [17] or Connes [4] on cyclic-type homological theory. The *continuous bar* and *'naive' Hochschild homology of* a $\hat{\otimes}$-algebra $\mathcal{A}$ are defined respectively as

$$\mathcal{H}_*^{bar}(\mathcal{A}) = H_*(\mathcal{C}(\mathcal{A}), b') \text{ and } \mathcal{H}_*^{naive}(\mathcal{A}) = H_*(\mathcal{C}(\mathcal{A}), b),$$

where $\mathcal{C}_n(\mathcal{A}) = \mathcal{A}^{\hat{\otimes}(n+1)}$, and the differentials b, b' are given by

$$b'(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \text{ and}$$

$$b(a_0 \otimes \cdots \otimes a_n) = b'(a_0 \otimes \cdots \otimes a_n) + (-1)^n(a_n a_0 \otimes \cdots \otimes a_{n-1}).$$

Note that $\mathcal{H}_*^{naive}(\mathcal{A})$ is just another way of writing $\mathcal{H}_*(\mathcal{A}, \mathcal{A})$, the continuous homology of $\mathcal{A}$ with coefficients in $\mathcal{A}$, as described in [12, 14].

For a $\hat{\otimes}$-algebra $\mathcal{A}$, consider the mixed complex $(\bar{\Omega}\mathcal{A}_+, \tilde{b}, \tilde{B})$, where $\bar{\Omega}^n\mathcal{A}_+ = \mathcal{A}^{\hat{\otimes}(n+1)} \oplus \mathcal{A}^{\hat{\otimes}n}$ and

$$\tilde{b} = \begin{pmatrix} b & 1-\lambda \\ 0 & -b' \end{pmatrix}; \quad \tilde{B} = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}$$

where $\lambda(a_1 \otimes \cdots \otimes a_n) = (-1)^{n-1}(a_n \otimes a_1 \otimes \cdots \otimes a_{n-1})$ and $N = \text{id} + \lambda + \cdots + \lambda^{n-1}$ [17, 1.4.5]. The *continuous Hochschild homology of* $\mathcal{A}$, the *continuous cyclic homology of* $\mathcal{A}$ and the *continuous periodic cyclic homology of* $\mathcal{A}$ are defined by

$$\mathcal{HH}_*(\mathcal{A}) = H_*^b(\bar{\Omega}\mathcal{A}_+, \tilde{b}, \tilde{B}), \quad \mathcal{HC}_*(\mathcal{A}) = H_*^c(\bar{\Omega}\mathcal{A}_+, \tilde{b}, \tilde{B}) \text{ and}$$

$$\mathcal{HP}_*(\mathcal{A}) = H^p_*(\bar{\Omega}\mathcal{A}_+, \tilde{b}, \tilde{B})$$

where H^b_*, H^c_* and H^p_* are Hochschild homology, cyclic homology and periodic cyclic homology of the mixed complex $(\bar{\Omega}\mathcal{A}_+, \tilde{b}, \tilde{B})$ in the category $\mathcal{LCS}$ of locally convex spaces and continuous linear operators; see, for example, [19].

There is also a *cyclic cohomology* theory associated with a complete locally convex algebra $\mathcal{A}$, obtained when one replaces the chain complexes of $\mathcal{A}$ by their dual complexes of strong dual spaces. For example, the *continuous bar cohomology* $\mathcal{H}^n_{bar}(\mathcal{A})$ of $\mathcal{A}$ is the cohomology of the dual complex $(\mathcal{C}(\mathcal{A})^*, (b')^*)$ of $(\mathcal{C}(\mathcal{A}), b')$.

EXAMPLE 6.1. Some examples of C^*-algebras without non-zero bounded traces are:
(i) The C^*-algebra $\mathcal{K}(H)$ of compact operators on an infinite-dimensional Hilbert space H; see [1, Theorem 2]. We can also show that $C(\Omega, \mathcal{K}(H))^{tr} = 0$, where Ω is a compact space.
(ii) Properly infinite von Neumann algebras $\mathcal{U}$; see [18, Example 4.6]. This class includes the C^*-algebra $\mathcal{B}(H)$ of all bounded operators on an infinite-dimensional Hilbert space H; see also [11] for the statement $\mathcal{B}(H)^{tr} = 0$.

EXAMPLE 6.2. Let $H = \varinjlim_i H_i$ be a strict inductive limit of Hilbert spaces. Suppose that H_1 and H_{m+1}/H_m, $m = 1, 2, \ldots$, are infinite-dimensional spaces. Consider the Fréchet locally C^*-algebra $\mathcal{L}(H)$ of continuous linear operators T on H that leave each H_i invariant and satisfy $T_j P_{ij} = P_{ij} T_j$ for all $i < j$ where $T_j = T|H_j : T_j(\eta) = T(\eta)$ for $\eta \in H_j$ and P_{ij} is the projection from H_j onto H_i. By [19, Example 6.6] that, for all $n \geq 0$, $\mathcal{H}_n(\mathcal{L}(H), \mathcal{L}(H)) = \{0\}$.

COROLLARY 6.3. *Let $\mathcal{A}$ be a Fréchet algebra belonging to one of the following classes:*

(i) *$\mathcal{A} = \ell^1(\mathbb{Z}^k_+)\hat{\otimes}\mathcal{C}$, where $\mathcal{C}$ is a C^*-algebra without non-zero bounded traces;*

(ii) *$\mathcal{A} = L^1(\mathbb{R}^k_+)\hat{\otimes}\mathcal{C}$, where $\mathcal{C}$ is a C^*-algebra without non-zero bounded traces;*

(iii) *$\mathcal{A} = \ell^1(\mathbb{Z}^k_+)\hat{\otimes}\mathcal{C}$, where $\mathcal{C}$ is the Fréchet locally C^*-algebra $\mathcal{L}(H)$ of continuous linear operators on a strict inductive limit $H = \varinjlim_i H_i$ of*

Hilbert spaces such that H_1 and H_{m+1}/H_m, $m = 1, 2, \ldots$, are infinite-dimensional spaces;

(iv) *$\mathcal{A} = \mathcal{D} \hat{\otimes} \mathcal{C}$, where $\mathcal{D}$ is a unital nuclear Fréchet algebra such that all boundary maps of the standard homology complex $\mathcal{C}_\sim(\mathcal{D}, \mathcal{D})$ have closed ranges and $\mathcal{C}$ is the Fréchet locally C^*-algebra $\mathcal{L}(H)$ of continuous linear operators on a strict inductive limit $H = \varinjlim_i H_i$ of*

Hilbert spaces such that H_1 and H_{m+1}/H_m, $m = 1, 2, \ldots$, are infinite-dimensional spaces.

Then, $\mathcal{H}_n(\mathcal{A}, \mathcal{A}) \cong \{0\}$ and $\mathcal{H}^n(\mathcal{A}, \mathcal{A}) \cong \{0\}$ for all $n \geq 0$;

$$HH_n(\mathcal{A}) \cong HH^n(\mathcal{A}) \cong \{0\} \quad \text{for all} \quad n \geq 0,$$
$$HC_n(\mathcal{A}) \cong HC^n(\mathcal{A}) \cong \{0\} \quad \text{for all} \quad n \geq 0,$$

and

$$HP_m(\mathcal{A}) \cong HP^m(\mathcal{A}) \cong \{0\} \quad \text{for} \quad m = 0, 1.$$

Proof. By [3, Theorem 4.1 and Corollary 3.3], for a C^*-algebra $\mathcal{U}$ without non-zero bounded traces $\mathcal{H}^n(\mathcal{U}, \mathcal{U}^*) \cong \{0\}$ for all $n \geq 0$. By [19, Example 6.6], for the Fréchet locally C^*-algebra $\mathcal{L}(H)$, $\mathcal{H}_n(\mathcal{L}(H), \mathcal{L}(H)) \cong \{0\}$ for all $n \geq 0$.

In cases (i) and (ii) we use induction by k and apply [8, Theorem 5.5] for a C^*-algebra $\mathcal{C}$ without non-zero bounded traces, to get $\mathcal{H}_n(\mathcal{A}, \mathcal{A}) \cong \{0\}$ for all $n \geq 0$. For example, as in Corollary 5.8, note that $\ell^1(\mathbb{Z}_+^k) \hat{\otimes} \mathcal{C} \cong \ell^1(\mathbb{Z}_+) \hat{\otimes} \mathcal{B}$ where $\mathcal{B} = \ell^1(\mathbb{Z}_+^{k-1}) \hat{\otimes} \mathcal{C}$. The algebras $\ell^1(\mathbb{Z}_+)$ and $\mathcal{B}$ satisfy the conditions of Theorem 5.5. We use induction on k to prove that

$$\mathcal{H}_n(\ell^1(\mathbb{Z}_+^k) \hat{\otimes} \mathcal{C}, \ell^1(\mathbb{Z}_+^k) \hat{\otimes} \mathcal{C}) \cong \{0\}$$

for all $n \geq 0$.

In case (iii) we use induction on k and apply Theorem 5.5 for the Fréchet locally C^*-algebra $\mathcal{L}(H)$, to get $\mathcal{H}_n(\mathcal{A}, \mathcal{A}) \cong \{0\}$ for all $n \geq 0$.

In case (iv) we apply Theorem 5.4, to get $\mathcal{H}_n(\mathcal{A}, \mathcal{A}) \cong \{0\}$ for all $n \geq 0$.

The triviality of the continuous cyclic and periodic cyclic homology and cohomology groups follows from [19, Corollory 4.7]. $\square$

A $\hat{\otimes}$-algebra $\mathcal{A}$ is said to be *biprojective* if it is projective in the category of $\mathcal{A}$-$\hat{\otimes}$-bimodules [12, Def. 4.5.1]. A $\hat{\otimes}$-algebra $\mathcal{A}$ is said to be *contractible* if $\mathcal{A}_+$ is projective $\mathcal{A}$-$\hat{\otimes}$-bimodules. A $\hat{\otimes}$-algebra $\mathcal{A}$ is contractible if and only if $\mathcal{A}$ is biprojective and has an identity [12, Def. 4.5.8].

Recall that, for a $\hat{\otimes}$-algebra $\mathcal{A}$ and for an A-$\hat{\otimes}$-bimodule X, $[X, \mathcal{A}]$ is the closure in X of the linear span of elements of the form $a \cdot x - x \cdot a$; $x \in X, a \in \mathcal{A}$;

$$\mathrm{Cen}_{\mathcal{A}} X = \{x \in X : a \cdot x = x \cdot a \text{ for all } a \in \mathcal{A}\} \text{ and}$$

$$\mathrm{Cen}_{\mathcal{A}} X^* = \{f \in X^* : f(a \cdot x) = f(x \cdot a) \text{ for all } a \in \mathcal{A} \text{ and } x \in X.\}$$

THEOREM 6.4. *Let the category for underlying spaces Φ be $\mathcal{F}r$ or the category of complete nuclear DF-spaces. Let $\mathcal{A}$ and $\mathcal{B}$ be unital $\hat{\otimes}$-algebras, let Y be a unital $\mathcal{B}$-$\hat{\otimes}$-bimodule and let X be a unital $\mathcal{A}$-$\hat{\otimes}$-bimodule such that $\mathcal{H}_0(\mathcal{A}, X)$ is Hausdorff and $\mathcal{H}_n(\mathcal{A}, X) = \{0\}$ for all $n \geq 1$. Suppose that all boundary maps of the standard homology complex $\mathcal{C}_\sim(\mathcal{B}, Y)$ have closed ranges. Then*
(i) *up to topological isomorphism in the category of complete nuclear DF-spaces and in the category $\mathcal{F}r$ under the assumption that either $\mathcal{A}$ and X or $\mathcal{B}$ and Y are nuclear, for all $n \geq 0$,*

$$\mathcal{H}_n(\mathcal{A} \hat{\otimes} \mathcal{B}, \mathcal{X} \hat{\otimes} Y) \cong X/[X, \mathcal{A}] \hat{\otimes} \mathcal{H}_n(\mathcal{B}, Y);$$

(ii) *up to topological isomorphism in the category of complete nuclear DF-spaces and in the category $\mathcal{F}r$ under the assumption that $\mathcal{A}$, X, $\mathcal{B}$ and Y are nuclear, for all $n \geq 0$,*

$$H^n((\mathcal{C}_\sim(\mathcal{A} \hat{\otimes} \mathcal{B}, X \hat{\otimes} Y))^*) \cong \left(\mathcal{H}_n(\mathcal{A} \hat{\otimes} \mathcal{B}, X \hat{\otimes} Y)\right)^*$$

$$\cong \mathrm{Cen}_{\mathcal{A}} X^* \hat{\otimes} H^n((\mathcal{C}_\sim(\mathcal{B}, Y))^*);$$

(iii) *up to isomorphism of linear spaces, in the category $\mathcal{F}r$ under the assumption that either $\mathcal{A}$ and X or $\mathcal{B}$ and Y are nuclear, for all $n \geq 0$,*

$$H^n((\mathcal{C}_\sim(\mathcal{A} \hat{\otimes} \mathcal{B}, X \hat{\otimes} Y))^*) \cong \left(\mathcal{H}_n(\mathcal{A} \hat{\otimes} \mathcal{B}, X \hat{\otimes} Y)\right)^*$$

$$\cong (X/[X, \mathcal{A}])^* \hat{\otimes} (\mathcal{H}_n(\mathcal{B}, Y))^*.$$

Proof. By assumption $\mathcal{H}_0(\mathcal{A}, X)$ is Hausdorff, and so $\mathcal{H}_0(\mathcal{A}, X) \cong X/[X, \mathcal{A}]$. The result follows from Theorem 5.4. $\square$

LEMMA 6.5. *Let the category for underlying spaces Φ be $\mathcal{F}r$ or the category of complete nuclear DF-spaces. Let $\mathcal{A}$ be a contractible $\hat{\otimes}$-algebra. Then, for each $\mathcal{A}$-$\hat{\otimes}$-bimodule X, $\mathcal{H}_0(\mathcal{A}, X)$ is Hausdorff and $\mathcal{H}_n(\mathcal{A}, X) \cong \{0\}$ for all $n \geq 1$.*

Proof. In the categories of Fréchet spaces and complete nuclear DF-spaces, the open mapping theorem holds. Therefore, by [12, Theorem

III.4.25], for all $n \geq 0$ and all $\mathcal{A}$-$\hat{\otimes}$-bimodule X, up to topological isomorphism,

$$\mathcal{H}_n(\mathcal{A}, X) \cong \operatorname{Tor}_n^{\mathcal{A}^e}(X, \mathcal{A}_+).$$

Since $\mathcal{A}$ is contractible, by [12, Proposition III.3.5 and Proposition VII.1.2], $\mathcal{H}_n(\mathcal{A}, X) \cong \{0\}$ for all $n \geq 1$ and $\mathcal{H}_0(\mathcal{A}, X) \cong \operatorname{Tor}_0^{\mathcal{A}^e}(X, \mathcal{A}_+)$ is Hausdorff. □

6.1. Contractible $\hat{\otimes}$-algebras.

EXAMPLE 6.6.   A countable direct product $\prod_{i \in J} M_{n_i}(\mathbb{C})$ of full matrix algebras is contractible Fréchet algebra [28].

EXAMPLE 6.7.   Let G be a compact Lie group and let $\mathcal{E}^*(G)$ be the strong dual to the nuclear Fréchet algebra of smooth functions $\mathcal{E}(G)$ on G with the convolution product, so that $\mathcal{E}^*(G)$ is a complete nuclear DF-space. This is a $\hat{\otimes}$-algebra with respect to convolution multiplication: for $f, g \in \mathcal{E}^*(G)$ and $x \in \mathcal{E}(G)$, $< f * g, x > = < f, y >$, where $y \in \mathcal{E}(G)$ is defined by $y(s) = < g, x_s >$, $s \in G$ and $x_s(t) = x(s^{-1}t)$, $t \in G$. J.L. Taylor proved that the algebra of distributions $\mathcal{E}^*(G)$ on a compact Lie group G is contractible [28].

EXAMPLE 6.8.   Fix a real number $1 \leq R \leq \infty$ and a nondecreasing sequence $\alpha = (\alpha_i)$ of positive numbers with $\lim_{i \to \infty} \alpha_i = \infty$. The power series space

$$\Lambda_R(\alpha) = \{x = (x_n) \in \mathbb{C}^{\mathbb{N}} : \|x\|_r = \sum_n |x_n| r^{\alpha_n} < \infty \text{ for all } 0 < r < R\}$$

is a Fréchet Köthe algebra with pointwise multiplication. The topology of $\Lambda_R(\alpha)$ is determined by a countable family of seminorms $\{\|x\|_{r_k} : k \in \mathbb{N}\}$ where $\{r_k\}$ is an arbitrary increasing sequence converging to R.

By [25, Corollary 3.3], $\Lambda_R(\alpha)$ is *biprojective* if and only if $R = 1$ or $R = \infty$.

By the Grothendieck-Pietsch criterion, $\Lambda_R(\alpha)$ is *nuclear* if and only if for $\overline{\lim}_n \frac{\log n}{\alpha_n} = 0$ for $R < \infty$ and $\overline{\lim}_n \frac{\log n}{\alpha_n} < \infty$ for $R = \infty$, see [24, Example 3.4].

By [25, Proposition 3.15], for the Fréchet Köthe algebra $\Lambda_1(\alpha)$, the following conditions are equivalent: (i) $\Lambda_1(\alpha)$ is contractible, (ii) $\Lambda_1(\alpha)$ is nuclear, (iii) $\Lambda_1(\alpha)$ is unital.

By [25, Corollary 3.18], if $\Lambda_\infty(\alpha)$ is nuclear, then the strong dual $\Lambda_\infty(\alpha)^*$ is a nuclear, contractible Köthe $\hat{\otimes}$-algebra which is a DF-space.

The algebra $\Lambda_R((n))$ is topologically isomorphic to the algebra of functions holomorphic on the open disc of radius R, endowed with *Hadamard product*, that is, with "co-ordinatewise" product of the Taylor expansions of holomorphic functions.

EXAMPLE 6.9.   The algebra $\mathcal{H}(\mathbb{C}) \cong \Lambda_\infty((n))$ of entire functions, endowed with the Hadamard product, is a biprojective nuclear Fréchet algebra [25]. The strong dual $\mathcal{H}(\mathbb{C})^*$ is a nuclear contractible Köthe $\hat{\otimes}$-algebra which is a DF-space.

EXAMPLE 6.10.   The algebra $\mathcal{H}(\mathbb{D}_1) \cong \Lambda_1((n))$ of functions holomorphic on the open unit disc, endowed with the Hadamard product, is a biprojective nuclear Fréchet algebra. Moreover it is contractible, since the function $z \mapsto (1-z)^{-1}$ is an identity for $\mathcal{H}(\mathbb{D}_1)$ [25].

EXAMPLE 6.11.   The nuclear Fréchet algebra of rapidly decreasing sequences

$$s = \{x = (x_n) \in \mathbb{C}^{\mathbb{N}} : \|x\|_k = \sum_n |x_n| n^k < \infty \text{ for all } k \in \mathbb{N}\}$$

is a biprojective Köthe algebra [24]. The algebra s is topologically isomorphic to $\Lambda_\infty(\alpha)$ with $\alpha_n = \log n$ [25]. The nuclear Köthe $\hat{\otimes}$-algebra s^* of sequences of polynomial growth is contractible [28].

The space of continuous traces on a topological algebra $\mathcal{A}$ is denoted by $\mathcal{A}^{tr}$, that is,

$$\mathcal{A}^{tr} = \text{Cen}_\mathcal{A}\mathcal{A}^* = \{f \in \mathcal{A}^* : f(ab) = f(ba) \text{ for all } a, b \in \mathcal{A}\}.$$

The closure in $\mathcal{A}$ of the linear span of elements of the form $\{ab - ba : a, b \in \mathcal{A}\}$ is denoted by $[\mathcal{A}, \mathcal{A}]$. Recall that $b_0 : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ is uniquely determined by $a \otimes b \mapsto ab - ba$.

COROLLARY 6.12.   *Let the category for underlying spaces Φ be $\mathcal{F}r$ or the category of complete nuclear DF-spaces. Let $\mathcal{A}$ and $\mathcal{B}$ be unital*

$\hat{\otimes}$-*algebras such that* $\mathcal{H}_0(\mathcal{A}, \mathcal{A})$ *and* $\mathcal{H}_0(\mathcal{B}, \mathcal{B})$ *are Hausdorff, and*

$$\mathcal{H}_n(\mathcal{A}, \mathcal{A}) \cong \mathcal{H}_n(\mathcal{B}, \mathcal{B}) \cong \{0\}$$

for all $n \geq 1$. *Then*

(i) *up to topological isomorphism in the category of complete nuclear* DF-*spaces and in the category* $\mathcal{F}r$ *under the assumption that either* $\mathcal{A}$ *or* $\mathcal{B}$ *is nuclear,*

$$\mathcal{H}\mathcal{H}_0(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \mathcal{H}_0^{naive}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \mathcal{A}/[\mathcal{A}, \mathcal{A}]\hat{\otimes}\mathcal{B}/[\mathcal{B}, \mathcal{B}] \text{ and}$$
$$\mathcal{H}\mathcal{H}_n(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \mathcal{H}_n^{naive}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \{0\} \text{ for all } n \geq 1;$$

(4)

$$\mathcal{H}\mathcal{C}_{2\ell}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \mathcal{A}/[\mathcal{A}, \mathcal{A}]\hat{\otimes}\mathcal{B}/[\mathcal{B}, \mathcal{B}] \text{ and}$$
$$\mathcal{H}\mathcal{C}_{2\ell+1}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \{0\} \text{ for all } \ell \geq 0;$$

(ii) *up to topological isomorphism in the category of complete nuclear* DF-*spaces and in the category* $\mathcal{F}r$ *under the assumption that* $\mathcal{A}$ *and* $\mathcal{B}$ *are nuclear,*

$$\mathcal{H}\mathcal{H}^0(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \mathcal{H}^0_{naive}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong (\mathcal{A}\hat{\otimes}\mathcal{B})^{tr} \text{ and}$$
$$\mathcal{H}\mathcal{H}^n(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \mathcal{H}^n_{naive}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \{0\} \text{ for all } n \geq 1;$$

(5)

$$\mathcal{H}\mathcal{C}^{2\ell}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong (\mathcal{A}\hat{\otimes}\mathcal{B})^{tr} \text{ and}$$
$$\mathcal{H}\mathcal{C}^{2\ell+1}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \{0\} \text{ for all } \ell \geq 0;$$

(iii) *up to topological isomorphism in the category* $\mathcal{F}r$ *under the assumption that* $\mathcal{A}$ *and* $\mathcal{B}$ *are nuclear and up to isomorphism of linear spaces in the category of complete nuclear* DF-*spaces and in the category* $\mathcal{F}r$ *under the assumption that either* $\mathcal{A}$ *or* $\mathcal{B}$ *nuclear,*

(6)
$$\mathcal{H}\mathcal{P}_0(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \mathcal{A}/[\mathcal{A}, \mathcal{A}]\hat{\otimes}\mathcal{B}/[\mathcal{B}, \mathcal{B}] \text{ and } \mathcal{H}\mathcal{P}_1(\mathcal{A}) \cong \{0\};$$
$$\mathcal{H}\mathcal{P}^0(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong (\mathcal{A}\hat{\otimes}\mathcal{B})^{tr} \text{ and } \mathcal{H}\mathcal{P}^1(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \{0\}.$$

Proof. Since $\mathcal{A}$ and $\mathcal{B}$ are unital, $\mathcal{H}_n^{bar}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \{0\}$ for all $n \geq 0$. By Theorem 6.4, up to topological isomorphism,

$$\mathcal{H}_0^{naive}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \mathcal{A}/[\mathcal{A}, \mathcal{A}]\hat{\otimes}\mathcal{B}/[\mathcal{B}, \mathcal{B}]$$

and so is Hausdorff, and $\mathcal{H}_n^{naive}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong \{0\}$ for all $n \geq 1$. The result follows from [21, Theorem 5.3]. Note that, by definition, the Hausdorff $\mathcal{H}_0^{naive}(\mathcal{A}\hat{\otimes}\mathcal{B}) \cong (\mathcal{A}\hat{\otimes}\mathcal{B})/[\mathcal{A}\hat{\otimes}\mathcal{B}, \mathcal{A}\hat{\otimes}\mathcal{B}]$. $\qquad \square$

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