

# SYMMETRIC $\alpha$ -STABLE SUBORDINATORS AND CAUCHY PROBLEMS

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ABSTRACT. We survey the results in Nane (E. Nane, *Higher order PDE's and iterated processes*, Trans. American Math. Soc. (to appear)) and Baeumer, Meerschaert, and Nane (B. Baeumer, M.M. Meerschaert and E. Nane, *Brownian subordinators and fractional Cauchy problems*: Submitted (2007)) which deal with PDE connection of some iterated processes, and obtain a new probabilistic proof of the equivalence of the higher order PDE's and fractional in time PDE's.

## 1. Introduction

In recent years, starting with the articles of Burdzy [9, 10], researchers have shown interest in iterated processes in which one changes the time parameter with one-dimensional Brownian motion.

To define **iterated Brownian motion**  $Z_t$ , due to Burdzy [9], started at  $z \in \mathbb{R}$ , let  $X_t^+$ ,  $X_t^-$  and  $Y_t$  be three independent one-dimensional Brownian motions, all started at 0. **Two-sided Brownian motion** is defined to be

$$X_t = \begin{cases} X_t^+, & t \geq 0 \\ X_{(-t)}^-, & t < 0. \end{cases}$$

Then iterated Brownian motion started at  $z \in \mathbb{R}$  is  $Z_t = z + X(Y_t)$ ,  $t \geq 0$ .

**1.1. BM versus IBM.** This process is not Markovian or Gaussian but, it has many properties analogous to those of Brownian motion. We list a few;

IBM  $Z_t$  has stationary (but not independent) increments, and is a **self-similar process** of index  $1/4$ . **Laws of the iterated logarithm (LIL)** holds: usual LIL by Burdzy [9] shows

$$\limsup_{t \rightarrow \infty} \frac{Z(t)}{t^{1/4}(\log \log(1/t))^{3/4}} = \frac{2^{5/4}}{3^{3/4}} \quad a.s.$$

Chung-type LIL is obtained by Khoshnevisan and Lewis [19] and Hu et al. [17]. Khoshnevisan and Lewis [18] extended results of Burdzy [10], to develop a **stochastic calculus** for iterated Brownian motion. Burdzy and Khosnevisan [12] showed that IBM can be used to model diffusion in a crack. Local times of this process was studied

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by Burdzy and Khosnevisan [11], Csáki, Csörgö, Földes, and Révész [13], Shi and Yor [32], Xiao [34], and Hu [16]. Bañuelos and DeBlassie [6] studied the **distribution of exit place** for iterated Brownian motion in cones. DeBlassie [14] studied the lifetime asymptotics of iterated Brownian motion in cones and Bounded domains. Nane [24, 25, 28, 29] extended some of the results of DeBlassie.

**1.2. PDE connection.** In addition to the above properties of IBM, there is an interesting connection between iterated Brownian motion and the **biharmonic operator**  $\Delta^2$ . Allouba and Zheng [2] show that if we replace the outer process  $X(t)$  in the definition of iterated Brownian motion with a continuous Markov process with  $L_x$  as the semigroup generator, then  $u(t, x) = E_x[f(Z_t)] := E[f(Z_t)|Z_0 = x]$  solves the Cauchy initial value problem

$$(1.1) \quad \frac{\partial}{\partial t} u(t, x) = \frac{L_x f(x)}{\sqrt{\pi t}} + L_x^2 u(t, x); \quad u(0, x) = f(x), \quad t > 0, x \in \mathbb{R}^d.$$

When  $Z_t$  is an iterated Brownian motion, this was also obtained by DeBlassie [14] by a different method. Let  $Z_t^1 = X(|Y_t|)$ , then Allouba and Zheng [1] shows  $E_x[f(Z_t)] = E_x[f(Z_t^1)]$ .

**1.3. Fractional Cauchy problems.** Nigmatullin [30] gave a physical derivation of the fractional kinetic equation for some special  $\beta$

$$(1.2) \quad \frac{\partial^\beta}{\partial t^\beta} u(t, x) = L_x u(t, x); \quad u(0, x) = f(x)$$

where  $0 < \beta < 1$  and  $L_x$  is the generator of some continuous Markov process  $X_0(t)$  started at  $x = 0$ . Here  $\partial^\beta g(t)/\partial t^\beta$  is the Caputo fractional derivative in time, which can be defined as the inverse Laplace transform of  $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$ , with  $\tilde{g}(s) = \int_0^\infty e^{-st} g(t) dt$  the usual Laplace transform.

Mathematical study of equation (1.2) was initiated by [33, 20, 21]. The existence and uniqueness of solutions to equation (1.2) was proved in [20, 21]. This equation was also used by Zaslavsky [35] for Hamiltonian chaos.

Baeumer and Meerschaert [4] and Meerschaert and Scheffler [23] show that the fractional Cauchy problem (1.2) is related to a certain class of subordinated stochastic processes; take  $D_t$  to be the stable subordinator, a Lévy process with strictly increasing sample paths such that  $E[e^{-sD_t}] = e^{-ts^\beta}$ , see for example Bertoin [8]. Define the inverse or hitting time or first passage time process

$$(1.3) \quad E_t = \inf\{x > 0 : D(x) > t\}.$$

The subordinated process  $Z_t = X_0(E_t)$  occurs as the scaling limit of a continuous time random walk (also called a renewal reward process), in which iid random jumps are separated by iid positive waiting times (Meerschaert and Scheffler (2004)[23]). Theorem 3.1 in Baeumer and Meerschaert [4] shows that, in the case  $p(t, x) = T(t)f(x)$

is a bounded continuous semigroup on a Banach space, the formula

$$u(t, x) = \int_0^\infty p((t/s)^\beta, x) g_\beta(s) ds = \frac{t}{\beta} \int_0^\infty p(x, s) g_\beta\left(\frac{t}{s^{1/\beta}}\right) s^{-1/\beta-1} ds$$

yields a solution to the fractional Cauchy problem (1.2). Here,  $g_\beta(t)$  is the smooth density of the stable subordinator, with  $\tilde{g}_\beta(s) = \int_0^\infty e^{-st} g_\beta(t) dt = e^{-s^\beta}$ .

## 2. Brownian subordinator and fractional Cauchy problems

We give a probabilistic proof of the following theorem. A variation of this result was realized by Orsingher and Benghin [31] for a version of iterated Brownian motion.

**Theorem 2.1** (Baeumer, Meerschaert and Nane (2007)[5]). *Let  $L_x$  be the generator of a Markov semigroup  $T(t)f(x) = E_x[f(X_t)]$ , and take  $f \in D(L_x)$  the domain of the generator. Then, both the higher order Cauchy problem (1.1) and the fractional Cauchy problem (1.2) with  $\beta = 1/2$ , have the same solution*

$$(2.1) \quad u(t, x) = E_x[f(Z_t)] = \frac{2}{\sqrt{4\pi t}} \int_0^\infty T(s)f(x) \exp\left(-\frac{s^2}{4t}\right) ds.$$

*Proof.*  $E_t$  is the inverse of a  $1 - 1/\alpha$  stable subordinator.  $E_t$  then is the local time of symmetric stable process of index  $\alpha$ . In the case  $\alpha = 2$ , local time of Brownian motion is the same as  $\sup_{0 < s < t} B_s$ . On the other hand,  $\sup_{0 < s < t} B_s$  and  $|B_t|$  are same in distribution by the reflection principle. Hence,  $E_t$  and  $|B_t|$  have the same one-dimensional distributions, implying the result of the theorem.  $\square$

We obtain the following corollary of our theorem

**Corollary 2.2** (Baeumer, Meerschaert and Nane (2007)[5]). *For any continuous Markov process  $X(t)$ , both the Brownian-time subordinated process  $X(|Y_t|)$  and the process  $X(E_t)$  subordinated to the inverse  $1/2$ -stable subordinator have the same one-dimensional distributions. Hence they are both stochastic solutions to the fractional Cauchy problem (1.2), or equivalently, to the higher order Cauchy problem (1.1).*

In contrast to the previous corollary, we have

**Theorem 2.3.** *Let  $Y$  be a symmetric stable process of index  $1 < \alpha < 2$ , and  $E_t$  is the inverse of a stable subordinator of index  $1 - 1/\alpha$ . The processes  $X(E_t)$  and  $X(|Y_t|)$  do not have same one-dimensional distribution.*

*Proof.* Let  $L_1^0$  be the local time at  $x = 0$ .  $L_1^0$  has the same one-dimensional distributions as  $E_t$ . Lemma 1 in Hawkes [15] implies that

$$(2.2) \quad P[L_1^0 > \lambda] \sim C_1 \lambda^{-\alpha/2} \exp(-C_{\alpha h} \lambda^\alpha).$$

Proposition 4 in Bertoin [8] shows

$$(2.3) \quad P[Y_1 > u] \sim P\left[\sup_{0 \leq s \leq 1} Y_s > u\right] \sim cu^{-\alpha}.$$

The results in equations (2.2) and (2.3) establish that in the case  $Y_t$  is a symmetric stable process of index  $\alpha < 2$ ,  $|Y_t|$  and  $E_t$  do not have the same one-dimensional distributions.  $\square$

When the outer process is Lévy process we have uniqueness of the solutions in Theorem 2.1. The proof relies on a Laplace-Fourier transform argument.

**Theorem 2.4** (Baeumer, Meerschaert and Nane (2007)[5]). *Suppose that  $X(t) = x + X_0(t)$  where  $X_0(t)$  is a Lévy process starting at zero. If  $L_x$  is the generator of the semigroup  $T(t)f(x) = E_x[(f(X_t))]$  on  $L^1(\mathbb{R}^d)$ , then for any  $f \in D(L_x)$ , both the initial value problem (1.1), and the fractional Cauchy problem (1.2) with  $\beta = 1/2$ , have the same unique solution given by (2.1).*

An easy extension of the argument for Theorem 2.4 shows that, under the same conditions, for any  $n = 2, 3, 4, \dots$  both the Cauchy problem

$$(2.4) \quad \frac{\partial u(t, x)}{\partial t} = \sum_{j=1}^{n-1} \frac{t^{1-j/n}}{\Gamma(j/n)} L_x^j f(x) + L_x^n u(t, x); u(0, x) = f(x)$$

and the fractional Cauchy problem (1.2) with  $\beta = 1/n$  have the same unique solution given by  $u(t, x) = \int_0^\infty p((t/s)^\beta, x) g_\beta(s) ds$  with  $\beta = 1/n$ . Hence the process  $Z_t = X(E_t)$  is also the stochastic solution to this higher order Cauchy problem.

### 3. Other subordinators

An  $\alpha$ -time process is a Markov process subordinated to the absolute value of an independent one-dimensional symmetric  $\alpha$ -stable process:  $Z_t = B(|S_t|)$ , where  $B_t$  is a Markov process and  $S_t$  is an independent symmetric  $\alpha$ -stable process both started at 0. Let  $Z_t^x = x + Z_t$  the process started at  $x$ .

This process is self similar with index  $1/2\alpha$  when the outer process  $X$  is a Brownian motion. In this case, Nane [27] defined the local time of this process and obtained laws of the iterated logarithm for the local time for large time.

#### 3.1. PDE-connection:

**Theorem 3.1** (Nane (2005) [26]). *Let  $T(s)f(x) = E[f(X^x(s))]$  be the semigroup of the continuous Markov process  $X^x(t)$  and let  $L_x$  be its generator. Let  $\alpha = 1$ . Let  $f$  be a bounded measurable function in the domain of  $L_x$ , with  $D_{ij}f$  bounded and Hölder continuous for all  $1 \leq i, j \leq n$ . Then  $u(t, x) = E[f(Z_t^x)]$  solves*

$$\frac{\partial^2}{\partial t^2} u(t, x) = -\frac{2L_x f(x)}{\pi t} - L_x^2 u(t, x); u(0, x) = f(x).$$

For  $\alpha = l/m \neq 1$  rational: the PDE is more complicated since kernels of symmetric  $\alpha$ -stable processes satisfy a higher order PDE:

$$\left(\frac{\partial^2}{\partial s^2}\right)^l + (-1)^{l+1} \frac{\partial^{2m}}{\partial t^{2m}} p_t^\alpha(0, s) = 0.$$

We also have to assume that we can take the operator out of the integral. This is valid for  $\alpha = 1/m$ ,  $m = 2, 3, \dots$ , by a Lemma in Nane [26].

**Theorem 3.2** (Nane (2005)[26]). *Let  $\alpha \in (0, 2)$  be a rational with  $\alpha = l/m$ , where  $l$  and  $m$  are relatively prime. Let  $T(s)f(x) = E[f(X^x(s))]$  be the semigroup of the continuous Markov process  $X^x(t)$  and let  $L_x$  be its generator. Let  $f$  be a bounded measurable function in the domain of  $L_x$ , with  $D^\gamma f$  bounded and Hölder continuous for all multi index  $\gamma$  such that  $|\gamma| = 2l$ . Then  $u(t, x) = E[f(Z_t^x)]$  solves*

$$\begin{aligned} (-1)^{l+1} \frac{\partial^{2m}}{\partial t^{2m}} u(t, x) &= -2 \sum_{i=1}^l \left( \frac{\partial^{2l-2i}}{\partial s^{2l-2i}} p_t^\alpha(0, s) \Big|_{s=0} \right) L_x^{2i-1} f(x) - L_x^{2l} u(t, x); \\ u(0, x) &= f(x). \end{aligned}$$

For some other connections of PDE's and iterated processes, see papers by Nane [26] and Allouba and Zheng [2], Allouba [1], Baeumer et al. [5] and references therein.

#### 4. Open Problems

**Question 1.** Looking at the governing PDE for subordinators other than Brownian motion, are there any fractional in time PDE which has the same solution as the higher order PDE?

**Question 2.** Are there PDE connections of the iterated processes in bounded domain as the PDE connection of Brownian motion in bounded domains?

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