Equivalent metrics and compactifications

Young Deuk Kim
School of Mathematical Sciences
Seoul National University
Seoul, 151-747, Korea
(ydkimus@yahoo.com)

November 20, 2018

Abstract

Let (X, d) be a metric space and $m \in X$. Suppose that $\phi : X \times X \to \mathbf{R}$ is a nonnegative symmetric function. We define a metric $d^{\phi,m}$ on X which is equivalent to d. If $d^{\phi,m}$ is totally bounded, its completion is a compactification of (X, d). As examples, we construct two compactifications of $(\mathbf{R}^s, d_E)$, where d_E is the Euclidean metric and $s \geq 2$.

key words. equivalent metric; completion; compactification
Mathematics Subject Classifications (2000). 54E35, 54D35

1 The metric $d^{\phi,m}$

Let (X, d) be a metric space and $m \in X$. Suppose that $\phi : X \times X \to \mathbf{R}$ is a nonnegative symmetric function. As usual, two metrics d_1 and d_2 on a set X are called equivalent if (X, d_1) and (X, d_2) are homeomorphic. In this section, we will define a metric $d^{\phi,m}$ on X which is equivalent to d.

For each $x, y \in X$, let

$$\delta^{\phi,m}(x, y) = \min \left\{ d(x, y),\ \frac{1}{1 + d(m, x)} + \phi(x, y) + \frac{1}{1 + d(m, y)} \right\}.$$

And for each $x, y \in X$ and $n \in \mathbf{N}$, let

$$\Gamma^n_{x,y} = \{ (x_0, \cdots, x_n) \mid x_0 = x, x_n = y \text{ and } x_i \in X \text{ for all } i \}$$

and

$$\Gamma_{x,y} = \bigcup_{n \in \mathbf{N}} \Gamma^n_{x,y}.$$

Notice that $\Gamma_{x,y} \neq \emptyset$ for all $x, y \in X$. In the following definition, the infimum runs over all elements of $\Gamma_{x,y}$.

Definition 1.1 *Suppose that $x, y \in X$. Let*

$$d^{\phi,m}(x,y) = \inf_{\Gamma_{x,y}} \sum_{i=1}^{n} \delta^{\phi,m}(x_{i-1}, x_i). \tag{1}$$

For the sake of simplicity, we will simply write d^ϕ, δ^ϕ to denote $d^{\phi,m}$, $\delta^{\phi,m}$ respectively. In particular, we write eq. (1) as

$$d^{\phi}(x,y) = \inf_{\Gamma_{x,y}} \sum_{i=1}^{n} \delta^{\phi}(x_{i-1}, x_i).$$

Notice that $(x,y) \in \Gamma_{x,y}$, and therefore

$$d^{\phi}(x,y) = \inf_{\Gamma_{x,y}} \sum_{i=1}^{n} \delta^{\phi}(x_{i-1}, x_i) \leq \delta^\phi(x,y) \leq d(x,y). \tag{2}$$

Notice also that d^ϕ is nonnegative. Therefore from eq. (2), we have

$$d^\phi(x,x) = 0 \quad \text{for all } x \in X. \tag{3}$$

The following subset $\Delta_{x,y}$ of $\Gamma_{x,y}$ is useful in the proof of Lemma 1.1.

$$\Delta_{x,y} = \{\, (x_0, \cdots, x_n) \in \Gamma_{x,y} \mid \delta^\phi(x_{i-1}, x_i) \neq d(x_{i-1}, x_i) \text{ for some } 1 \leq i \leq n \,\}.$$

Lemma 1.1 *Suppose that $d^\phi(x,y) \neq d(x,y)$. Then*

$$d^\phi(x,y) \geq \frac{1}{2(1 + d(m,x))}.$$

Proof. Suppose that $d^\phi(x,y) \neq d(x,y)$. By eq. (3) we have $x \neq y$, and by eq. (2) we have

$$d^\phi(x,y) < d(x,y). \tag{4}$$

If $(x_0, \cdots, x_n) \in \Gamma_{x,y} - \Delta_{x,y}$, then

$$\sum_{i=1}^{n} \delta^{\phi}(x_{i-1}, x_i) = \sum_{i=1}^{n} d(x_{i-1}, x_i) \geq d(x,y).$$

Therefore from eq. (4), we have $\Delta_{x,y} \neq \emptyset$ and

$$d^{\phi}(x,y) = \inf_{\Delta_{x,y}} \sum_{i=1}^{n} \delta^{\phi}(x_{i-1}, x_i). \tag{5}$$

Suppose that $(x_0, \cdots, x_n) \in \Delta_{x,y}$. Let k be the smallest integer such that $\delta^\phi(x_k, x_{k+1}) \neq d(x_k, x_{k+1})$. Notice that if $k \geq 1$ then

$$\delta^\phi(x_{i-1}, x_i) = d(x_{i-1}, x_i) \quad \text{for all } 1 \leq i \leq k.$$

If $d(x_0, x_k) \geq 1 + d(m, x_0)$ then we have $k \geq 1$, and therefore

$$\begin{aligned}
\sum_{i=1}^{n} \delta^{\phi}(x_{i-1}, x_i) &\geq \sum_{i=1}^{k} \delta^{\phi}(x_{i-1}, x_i) \\
&= \sum_{i=1}^{k} d(x_{i-1}, x_i) \\
&\geq d(x_0, x_k) \\
&\geq 1 + d(m, x_0) \\
&= 1 + d(m, x). \qquad (6)
\end{aligned}$$

If $d(x_0, x_k) < 1 + d(m, x_0)$ then

$$1 + d(m, x_k) \leq 1 + d(m, x_0) + d(x_0, x_k) < 2 + 2d(m, x_0).$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{n} \delta^{\phi}(x_{i-1}, x_i) &\geq \delta^{\phi}(x_k, x_{k+1}) \\
&= \frac{1}{1 + d(m, x_k)} + \phi(x_k, x_{k+1}) + \frac{1}{1 + d(m, x_{k+1})} \\
&> \frac{1}{1 + d(m, x_k)} \\
&> \frac{1}{2(1 + d(m, x_0))} \\
&= \frac{1}{2(1 + d(m, x))}. \qquad (7)
\end{aligned}$$

Hence from eq. (5), (6) and (7), we have

$$d^{\phi}(x, y) \geq \min\left\{1 + d(m, x),\ \frac{1}{2(1 + d(m, x))}\right\} = \frac{1}{2(1 + d(m, x))}.$$

∎

Now we show that d^{ϕ} is a metric on X.

Theorem 1.1 *d^{ϕ} is a metric on X.*

Proof. From eq. (1) and (3), recall that d^{ϕ} is nonnegative and $d^{\phi}(x, x) = 0$ for all $x \in X$. Suppose that $d^{\phi}(x, y) = 0$. By Lemma 1.1, we have $d(x, y) = d^{\phi}(x, y) = 0$. Thus $x = y$.

Suppose that $x, y \in X$. Notice that $(x_0, x_1, \cdots, x_n) \in \Gamma_{x,y}$ if and only if $(x_n, x_{n-1}, \cdots, x_0) \in \Gamma_{y,x}$. Since ϕ is symmetric, so is δ^{ϕ}. Therefore

$$\sum_{i=1}^{n} \delta^{\phi}(x_{i-1}, x_i) = \sum_{i=1}^{n} \delta^{\phi}(x_{n+1-i}, x_{n-i}) \quad \text{for all } (x_0, x_1, \cdots, x_n) \in \Gamma_{x,y}.$$

Hence $d^\phi(x,y) = d^\phi(y,x)$.

Suppose that $x, y, z \in X$ and $\epsilon > 0$. There exist $(x_0, x_1, \cdots, x_n) \in \Gamma_{x,y}$ and $(y_0, y_1, \cdots, y_m) \in \Gamma_{y,z}$ such that

$$\sum_{i=1}^{n} \delta^\phi(x_{i-1}, x_i) < d^\phi(x,y) + \frac{\epsilon}{2} \quad \text{and} \quad \sum_{j=1}^{m} \delta^\phi(y_{j-1}, y_j) < d^\phi(y,z) + \frac{\epsilon}{2}.$$

Notice that $(x_0, \cdots, x_n = y = y_0, \cdots, y_m) \in \Gamma_{x,z}$. Therefore

$$\begin{aligned} d^\phi(x,z) &\leq \sum_{i=1}^{n} \delta^\phi(x_{i-1}, x_i) + \sum_{j=1}^{m} \delta^\phi(y_{j-1}, y_j) \\ &< d^\phi(x,y) + \frac{\epsilon}{2} + d^\phi(y,z) + \frac{\epsilon}{2} \\ &= d^\phi(x,y) + d^\phi(y,z) + \epsilon. \end{aligned}$$

Since ϵ is arbitrary, we have $d^\phi(x,z) \leq d^\phi(x,y) + d^\phi(y,z)$. ∎

By the following lemma, the identity map from (X, d^ϕ) to (X, d) is continuous.

Lemma 1.2 *For all $x \in X$, there exists an open ball B_x in (X, d^ϕ), with center x, such that $d^\phi(y,z) = d(y,z)$ for all $y, z \in B_x$.*

Proof. For each $x \in X$, let

$$B_x = \left\{ y \in X \mid d^\phi(y,x) < \frac{1}{8(1 + d(m,x))} \right\}.$$

Suppose that $y \in B_x$. By Lemma 1.1, we have $d^\phi(x,y) = d(x,y)$, and therefore

$$\begin{aligned} d(m,y) &\leq d(m,x) + d(x,y) \\ &= d(m,x) + d^\phi(x,y) \\ &< d(m,x) + \frac{1}{8(1 + d(m,x))} \\ &< d(m,x) + 1 + d(m,x) \\ &= 1 + 2d(m,x). \end{aligned} \tag{8}$$

Suppose that $y, z \in B_x$. From eq. (8), we have $1 + d(m,y) < 2 + 2d(m,x)$. Therefore

$$\begin{aligned} d^\phi(y,z) &\leq d^\phi(y,x) + d^\phi(x,z) \\ &< \frac{1}{8(1 + d(m,x))} + \frac{1}{8(1 + d(m,x))} \\ &= \frac{1}{4(1 + d(m,x))} \\ &< \frac{1}{2(1 + d(m,y))}. \end{aligned}$$

Hence by Lemma 1.1, we have $d^{\phi}(y,z) = d(y,z)$. ∎

By the following corollary, d^{ϕ} is equivalent to d for all ϕ and m.

Corollary 1.1 *The identity map from (X, d^{ϕ}) to (X, d) is a homeomorphism.*

Proof. By eq. (2) and Lemma 1.2, it is trivial. ∎

2 The compactification

A compactification of a topological space X is a compact Hausdorff space Y containing X as a subspace such that $\overline{X} = Y$. It is known that every metric space has a compactification (see [6], §38). With the equivalent metric in the previous section, we are able to construct various compactifications of a metric space.

Let (X, d) be a metric space. Suppose that $m \in X$ and $\phi : X \times X \to \mathbf{R}$ is a nonnegative symmetric function. To get a compactification, we assume that

$$(X, d^{\phi}) = (X, d^{\phi,m}) \quad \text{is totally bounded,}$$

ie. there is a finite covering by ϵ balls for every $\epsilon > 0$. Then our compactification of (X, d) is the completion $(\overline{X}, \rho)$ of the totally bounded metric space (X, d^{ϕ}).

Notice that X is a dense subset of $\overline{X}$ and $(\overline{X}, \rho)$ is a compact metric space (see [6], §45 and [3], §XIV.3 for details). $\overline{X}$ can be considered as the set of equivalence classes of all Cauchy sequences in (X, d^{ϕ}) with the equivalence relation (see [4], §V.7)

$$x_i \sim y_i \quad \text{if and only if} \quad \lim_{i\to\infty} d^{\phi}(x_i, y_i) = 0,$$

where a point x in X is identified to the equivalence class of constant Cauchy sequence $\{x\}$.

Suppose that $\{x_i\}, \{y_i\} \in \overline{X}$. The metric ρ is given by

$$\rho(\{x_i\}, \{y_i\}) = \lim_{i\to\infty} d^{\phi}(x_i, y_i).$$

In particular, we have

$$\rho(\{x\}, \{y\}) = d^{\phi}(x, y) \quad \text{for all } x, y \in X.$$

In 2002, the author had tried to apply this compactification to the research on the tameness conjecture of Marden([5]) which was proved by Agol([1]) and Calegari-Gabai([2]) in 2004, independently. The author think that the compactification could be useful in the study of Teichmüller space. In the next two sections, we apply the compactification to the Euclidean metric space $\mathbf{R}^s$ with $s \geq 2$.

3 The standard compactification of $(\mathbf{R}^s, d_E)$

Let $O = (0, \cdots, 0) \in \mathbf{R}^s$. We write d_E to denote the Euclidean metric on $\mathbf{R}^s$. In this section, as an example of the compactification in Section 2, we construct a compactification of $(\mathbf{R}^s, d_E)$, which will be called *the standard compactification*, which is homeomorphic to the Euclidean closed unit ball

$$B^s = \{x \in \mathbf{R}^s \mid d_E(O, x) \leq 1\}.$$

Notice that we need to define a nonnegative symmetric function $\phi : \mathbf{R}^s \times \mathbf{R}^s \to \mathbf{R}$ such that

$$(\mathbf{R}^s, d^\phi) = (\mathbf{R}^s, d_E^{\phi,O})$$

is totally bounded, where we wrote d^ϕ to denote $d_E^{\phi,O}$ for the sake of simplicity.

For all $m \in \mathbf{N}$, let

$$a_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$$

and

$$S_m = \{x \in \mathbf{R}^s \mid d_E(O, x) = a_m\}.$$

Note that a_m is an increasing sequence and $\lim_{m\to\infty} a_m = \infty$.

For all $p, q \in \mathbf{N}$, let $h_{p,q} : S_p \to S_q$ be the homeomorphism defined by

$$h_{p,q}(x) = \frac{a_q}{a_p}\, x \quad \text{for all } x \in S_p.$$

Notice that if $h_{p,q}(x) = y$ then $h_{q,p}(y) = x$. We define the nonnegative symmetric function ϕ as follows.

Definition 3.1

$$\phi(x,y) = \begin{cases} 0 & \textit{if } \ h_{p,q}(x) = y \textit{ for some } p, q \in \mathbf{N} \\ \frac{1}{a_m}\, d_E(x,y) = d_E\left(\frac{x}{a_m}, \frac{y}{a_m}\right) & \textit{if } \ x, y \in S_m \textit{ for some } m \in \mathbf{N} \\ d_E(x,y) & \textit{otherwise} \end{cases}$$

Suppose that $x \in \mathbf{R}^s$ and $r > 0$. We write $B_r(x)$ to denote the Euclidean open ball with center x and radius r, and $B_r^\phi(x)$ to denote the open ball in $(\mathbf{R}^s, d^\phi)$. Now we show that $(\mathbf{R}^s, d^\phi)$ is totally bounded.

Lemma 3.1 *$(\mathbf{R}^s, d^\phi)$ is totally bounded.*

Proof. Let $\epsilon > 0$. We may assume that $\epsilon < 1$. Choose $k \in \mathbf{N}$ such that

$$\frac{1}{1+k} < \frac{\epsilon}{4} \quad \text{and} \quad \frac{1}{1+a_k} < \frac{\epsilon}{4}, \tag{9}$$

and let

$$B_{k+1} = \{x \in \mathbf{R}^s \mid d_E(O, x) \leq a_{k+1}\}.$$

Since B_{k+1} is compact in $(\mathbf{R}^s, d_E)$, so is in $(\mathbf{R}^s, d^\phi)$ by Corollary 1.1. Therefore we can cover B_{k+1} with finite number of ϵ-balls in $(\mathbf{R}^s, d^\phi)$. Notice that $S_k \subset B_{k+1}$. Since S_k is also compact in $(\mathbf{R}^s, d_E)$, we can cover S_k with finite number of Euclidean $\frac{\epsilon}{4}$-balls with centers $x_1, x_2, \cdots, x_N \in S_k$. From eq. (2), we have

$$S_k \subset \bigcup_{i=1}^{N} B_{\frac{\epsilon}{4}}(x_i) \subset \bigcup_{i=1}^{N} B_\epsilon(x_i) \subset \bigcup_{i=1}^{N} B_\epsilon^\phi(x_i).$$

Note that if $z \in S_k$ then there exists $x_i \in \{x_1, x_2, \cdots, x_N\} \subset S_k$ such that

$$d_E(z, x_i) < \frac{\epsilon}{4}.$$

To show that $(\mathbf{R}^s, d^\phi)$ is totally bounded, it is enough to show that if $x \notin B_{k+1}$ then there exists $x_i \in \{x_1, x_2, \cdots, x_N\}$ such that $d^\phi(x, x_i) < \epsilon$. Suppose that $x \notin B_{k+1}$. There exists $m \in \mathbf{N}$ such that

$$a_m \leq d_E(O, x) < a_{m+1}.$$

Since $x \notin B_{k+1}$, we have $k < m$. Let

$$y = \frac{a_m}{d_E(O, x)} x \in S_m.$$

From eq. (9), we have

$$d_E(x, y) < \frac{1}{1+m} < \frac{1}{1+k} < \frac{\epsilon}{4}. \tag{10}$$

Let z be the point in S_k such that $h_{k,m}(z) = y$. Choose $x_i \in \{x_1, x_2, \cdots, x_N\}$ such that

$$d_E(z, x_i) < \frac{\epsilon}{4}. \tag{11}$$

From eq. (2), (9), (10) and (11), we have

$$\begin{aligned} d^\phi(x, x_i) &\leq d^\phi(x, y) + d^\phi(y, z) + d^\phi(z, x_i) \\ &\leq d_E(x, y) + \delta^\phi(y, z) + d_E(z, x_i) \\ &< \frac{\epsilon}{4} + \frac{1}{1+a_m} + \frac{1}{1+a_k} + \frac{\epsilon}{4} \\ &< \epsilon. \end{aligned}$$

∎

Since $(\mathbf{R}^s, d^\phi)$ is totally bounded, its completion $(\overline{\mathbf{R}^s}, \rho) = (\overline{\mathbf{R}^s}, \rho_\phi)$ is a compactification of $(\mathbf{R}^s, d_E)$, where we wrote simply ρ to denote ρ_ϕ for the sake of simplicity. Recall that an element of $(\overline{\mathbf{R}^s}, \rho)$ is an equivalence class of Cauchy sequence in $(\mathbf{R}^s, d^\phi)$, where two Cauchy sequences $\{x_i\}$ and $\{y_i\}$ are equivalent if and only if

$$\lim_{i\to\infty} d^\phi(x_i, y_i) = 0.$$

Notice that if $\{x_i\}$ is a Cauchy sequence in $(\mathbf{R}^s, d^\phi)$ which converges to x, then $\{x_i\}$ and the constant Cauchy sequence $\{x\}$ are equivalent. Notice also that if $\{y_i\}$ is a subsequence of a Cauchy sequence $\{x_i\}$, then they are equivalent.

Since for all $x \in S_1$, we have

$$d^\phi(a_i x, a_j x) \leq \delta^\phi(a_i x, a_j x) \leq \frac{1}{1+a_i} + \frac{1}{1+a_j},$$

it is clear that $\{a_i x\}$ is a Cauchy sequence in $(\mathbf{R}^s, d^\phi)$. By Lemma 1.1, we can show that $\{a_i x\}$ is not equivalent to any constant Cauchy sequence (see the proof of Lemma 3.4). Furthermore, we have

Lemma 3.2 *If $\{x_i\}$ is a Cauchy sequence in $(\mathbf{R}^s, d^\phi)$ which is not equivalent to a constant Cauchy sequence, then it is equivalent to $\{a_i x\}$ for some $x \in S_1$.*

Proof. Suppose that $\{x_i\}$ is a Cauchy sequence in $(\mathbf{R}^s, d^\phi)$ which is not equivalent to a constant Cauchy sequence. If $\{x_i\}$ is bounded in $(\mathbf{R}^s, d_E)$, then it has a convergent subsequence $\{y_i\}$, which converges to a point y in $(\mathbf{R}^s, d_E)$. Notice that $\{y_i\}$ converges to y in $(\mathbf{R}^s, d^\phi)$, too. Therefore $\{x_i\}$ is equivalent to $\{y_i\}$, and hence to the constant Cauchy sequence $\{y\}$. This is a contradiction.

Since $\{x_i\}$ is unbounded in $(\mathbf{R}^s, d_E)$, we can choose a subsequence of x_i, which we will call x_i again, such that

$$0 < d_E(O, x_i) < d_E(O, x_{i+1}) \quad \text{for all } i \in \mathbf{N}$$

and there exists at most one x_i such that

$$a_m \leq d_E(O, x_i) < a_{m+1}$$

for each $m \in \mathbf{N}$. Notice that $m \to \infty$ as $i \to \infty$. Since

$$\frac{1}{d_E(O, x_i)}\, x_i \in S_1$$

for all $i \in \mathbf{N}$ and (S_1, d_E) is compact, x_i has a subsequence, which we will call x_i again, such that

$$\frac{x_i}{d_E(O, x_i)} \text{ converges to } x \text{ for some } x \in S_1.$$

Suppose that $a_m \leq d_E(O, x_i) < a_{m+1}$. Let $y_i = a_m x$. Notice that $\{y_i\}$ is a subsequence of $\{a_i x\}$. Let

$$z_i = \frac{a_m}{d_E(O, x_i)} x_i.$$

Since $d_E(x_i, z_i) \leq \frac{1}{m+1}$, we have

$$\begin{aligned} & \lim_{i\to\infty} d^\phi(x_i, y_i) \\ \leq\ & \lim_{i\to\infty} \left(d^\phi(x_i, z_i) + d^\phi(z_i, y_i)\right) \end{aligned}$$

$$
\begin{aligned}
&\le \lim_{i\to\infty}\left(d_E(x_i,z_i)+\delta^{\phi}(z_i,y_i)\right)\\
&\le \lim_{i\to\infty}\left(\frac{1}{1+m}+\frac{1}{1+a_m}+d_E\left(\frac{x_i}{d_E(O,x_i)},x\right)+\frac{1}{1+a_m}\right)\\
&= 0.
\end{aligned}
$$

Therefore $\{x_i\}$ and $\{y_i\}$ are equivalent, and thus $\{x_i\}$ is equivalent to $\{a_i x\}$. ∎

To show that $(\overline{\mathbf{R}^s},\rho)$ is homeomorphic to (B^s,d_E), we define a function

$$h:(B^s,d_E)\to(\overline{\mathbf{R}^s},\rho)$$

as follows.

$$
h(x)=\begin{cases}\frac{1}{1-d_E(O,x)}\,x \text{ (the constant Cauchy sequence)} & \text{if } d_E(O,x)<1\\ \{a_i x\} & \text{if } d_E(O,x)=1\end{cases}
$$

Notice that

$$h\left(\frac{1}{1+d_E(O,y)}\,y\right)=y$$

for all $y\in\mathbf{R}^s$. Therefore from Lemma 3.2, it is clear that h is surjective. We will need the following lemma to show that h is injective.

Lemma 3.3 *Suppose that $d_E(O,x)\ge 1$ and $d_E(O,y)\ge 1$. Let $(x_0,x_1,\cdots,x_m)\in\Gamma_{x,y}$ with $d_E(O,x_i)<1$ for all $1\le i\le m-1$. Then*

$$\sum_{i=1}^{m}\delta^{\phi}(x_{i-1},x_i)\ge d_E\left(\frac{x}{d_E(O,x)},\frac{y}{d_E(O,y)}\right).$$

Proof. Notice that we may assume

$$\frac{x}{d_E(O,x)}\ne\frac{y}{d_E(O,y)}.$$

If $m=1$ then

$$
\begin{aligned}
&\sum_{i=1}^{m}\delta^{\phi}(x_{i-1},x_i)\\
=\;&\delta^{\phi}(x,y)\\
=\;&\min\left\{d_E(x,y),\frac{1}{1+d_E(O,x)}+\phi(x,y)+\frac{1}{1+d_E(O,y)}\right\}\\
\ge\;&\min\left\{d_E(x,y),\frac{1}{1+d_E(O,x)}+d_E\left(\frac{x}{d_E(O,x)},\frac{y}{d_E(O,y)}\right)+\frac{1}{1+d_E(O,y)}\right\}\\
\ge\;&d_E\left(\frac{x}{d_E(O,x)},\frac{y}{d_E(O,y)}\right).
\end{aligned}
$$

Suppose that $m \neq 1$. Notice that

$$\delta^{\phi}(x_{i-1}, x_i) = d_E(x_{i-1}, x_i) \quad \text{for all } 1 \leq i \leq m$$

and therefore

$$\sum_{i=1}^{m} \delta^{\phi}(x_{i-1}, x_i) \geq \sum_{i=1}^{m} d_E(x_{i-1}, x_i) \geq d_E(x, y) \geq d_E\left(\frac{x}{d_E(O,x)}, \frac{y}{d_E(O,y)}\right).$$

∎

Now we show that h is injective.

Lemma 3.4 *h is injective.*

Proof. Suppose that $h(x) = h(y)$. We will show that $x = y$. If $d_E(O, x) < 1$ and $d_E(O, y) < 1$, then

$$\frac{1}{1 - d_E(O,x)}\, x = \frac{1}{1 - d_E(O,y)}\, y \tag{12}$$

and therefore

$$\frac{1}{1 - d_E(O,x)}\, d_E(O,x) = \frac{1}{1 - d_E(O,y)}\, d_E(O,y).$$

Hence $d_E(O, x) = d_E(O, y)$. Thus from eq. (12), we have $x = y$.

If $d_E(O, x) = 1$ and $d_E(O, y) = 1$, then the Cauchy sequences $\{a_i x\}$ and $\{a_i y\}$ are equivalent. Suppose that $x \neq y$. We will get a contradiction. Let

$$(x_0, x_1, \cdots, x_m) \in \Gamma_{a_i x, a_i y}.$$

Using Lemma 3.3, we can show that

$$\sum_{i=1}^{m} \delta^{\phi}(x_{i-1}, x_i) \geq d_E(x, y).$$

and therefore

$$d^{\phi}(a_i x, a_i y) \geq d_E(x, y) > 0 \quad \text{for all } i. \tag{13}$$

Hence $\lim_{i\to\infty} d^{\phi}(a_i x, a_i y) \neq 0$. This is a contradiction.

Suppose that $d_E(O, x) < 1$, $d_E(O, y) = 1$ and

$$\lim_{i\to\infty} d^{\phi}\left(\frac{1}{1 - d_E(O,x)}\, x,\ a_i y\right) = 0.$$

We will get a contradiction. Notice that if i is large enough, then

$$d^{\phi}\left(\frac{1}{1 - d_E(O,x)}\, x,\ a_i y\right) \neq d_E\left(\frac{1}{1 - d_E(O,x)}\, x,\ a_i y\right).$$

Therefore by Lemma 1.1, for large enough i, we have

$$d^{\phi}\left(\frac{1}{1-d_E(O,x)}\,x,\,a_i y\right) \geq \frac{1}{2\left(1+d_E\left(O,\frac{1}{1-d_E(O,x)}x\right)\right)} > 0.$$

Hence

$$\lim_{i\to\infty} d^{\phi}\left(\frac{1}{1-d_E(O,x)}\,x,\,a_i y\right) \neq 0.$$

This is a contradiction. ∎

Since h is bijective, we can consider its inverse function. Recall Lemma 3.2 and let

$$k : (\overline{\mathbf{R}^s}, \rho) \to (B^s, d_E)$$

be the function defined by

$$k(\{x_i\}) = \begin{cases} \frac{1}{1+d_E(O,x)}\,x & \text{if } \{x_i\} = \{x\} \text{ is a constant Cauchy sequence} \\ x & \text{if } x_i = a_i x \text{ for some } x \in S_1. \end{cases}$$

It is easy to show that k is the inverse function of h. In the following two lemmas, we will show that h and k are continuous. Therefore $(\overline{\mathbf{R}^s}, \rho)$ is homeomorphic to (B^s, d_E).

Lemma 3.5 *h is continuous.*

Proof. Suppose that $x_n \to x$ in (B^s, d_E). We will show that $h(x_n) \to h(x)$ in $(\overline{\mathbf{R}^s}, \rho)$. If $d_E(O,x) < 1$, then it is trivial to show that $h(x_n) \to h(x)$ in $(\mathbf{R}^s, d_E)$. Therefore from eq. (2), we have $h(x_n) \to h(x)$ in $(\mathbf{R}^s, d^{\phi})$, and hence in $(\overline{\mathbf{R}^s}, \rho)$.

Suppose that $d_E(O,x) = 1$. Notice that it is enough to consider only the following two cases,

(a) $d_E(O,x_n) = 1$ for all n

(b) $d_E(O,x_n) < 1$ for all n.

For the case (a), we have

$$\begin{aligned} \rho(h(x_n), h(x)) &= \lim_{i\to\infty} d^{\phi}(a_i x_n, a_i x) \\ &\leq \lim_{i\to\infty}\left(\frac{1}{1+a_i} + d_E(x_n, x) + \frac{1}{1+a_i}\right) \\ &= d_E(x_n, x). \end{aligned}$$

Therefore if $x_n \to x$ in (B^s, d_E), then $h(x_n) \to h(x)$ in $(\overline{\mathbf{R}^s}, \rho)$.

For the case (b), if

$$a_m \leq d_E(O, h(x_n)) = d_E\left(O, \frac{1}{1-d_E(O,x_n)}\,x_n\right) < a_{m+1},$$

let

$$z_n = \frac{a_m}{d_E(O, h(x_n))}\, h(x_n) = \frac{a_m}{d_E(O, x_n)}\, x_n.$$

Notice that $z_n \in S_m$, and $m \to \infty$ as $n \to \infty$. Therefore from eq. (2), we have

$$\begin{aligned}
& \lim_{n\to\infty} \rho(h(x_n), h(x)) \\
= & \lim_{n\to\infty} \lim_{i\to\infty} d^{\phi}(h(x_n), a_i x) \\
\leq & \lim_{n\to\infty} \lim_{i\to\infty} \left(d^{\phi}(h(x_n), z_n) + d^{\phi}(z_n, a_m x) + d^{\phi}(a_m x, a_i x) \right) \\
\leq & \lim_{n\to\infty} \lim_{i\to\infty} \left(d_E(h(x_n), z_n) + \delta^{\phi}(z_n, a_m x) + \delta^{\phi}(a_m x, a_i x) \right) \\
\leq & \lim_{n\to\infty} \lim_{i\to\infty} \left(\frac{1}{1+m} + \frac{1}{1+a_m} + d_E\left(\frac{h(x_n)}{d_E(O, h(x_n))}, x \right) + \frac{1}{1+a_m} \right. \\
& \left. \qquad + \frac{1}{1+a_m} + \frac{1}{1+a_i} \right) \\
\leq & \lim_{n\to\infty} d_E\left(\frac{x_n}{d_E(O, x_n)}, x \right) \\
= & \ 0.
\end{aligned}$$

Therefore $h(x_n) \to h(x)$ as $n \to \infty$. ∎

Lemma 3.6 *k is continuous.*

Proof. Suppose that $\mathbf{x}_n = \{x_{n,i}\}$ converges to $\mathbf{x} = \{x_i\}$ in $(\overline{\mathbf{R}^s}, \rho)$. We will show that $k(\mathbf{x}_n)$ converges to $k(\mathbf{x})$ in $(\mathbf{B}^s, d_E)$.

Suppose that $\mathbf{x}$ is equivalent to a constant Cauchy sequence $\{x\}$ in $(\mathbf{R}^s, d^{\phi})$. If $\mathbf{x}_n$ is equivalent to $\{a_i x_n\}$ with $x_n \in S_1$ for infinitely many n, then choose a subsequence of $\mathbf{x}_n$, which we will call $\mathbf{x}_n$ again, such that $\mathbf{x}_n = \{a_i x_n\}$ with $x_n \in S_1$. Notice that there exists $I > 0$, which does not depend on n, such that

$$d_E(a_i x_n, x) \geq \frac{1}{2(1 + d_E(O, x))} \quad \text{for all } i > I.$$

Therefore by Lemma 1.1, we have

$$d^{\phi}(a_i x_n, x) \geq \frac{1}{2(1 + d_E(O, x))} \quad \text{for all } i > I.$$

Hence $\mathbf{x}_n$ does not converges to $\mathbf{x}$ in $(\overline{\mathbf{R}^s}, \rho)$. This is a contradiction. Therefore $\mathbf{x}_n = \{x_n\}$ is a constant Cauchy sequence in $(\mathbf{R}^s, d^{\phi})$ for large enough n. Since x_n converges to x in $(\mathbf{R}^s, d^{\phi})$, by Corollary 1.1, x_n converges to x in $(\mathbf{R}^s, d_E)$. Therefore

$$k(\mathbf{x}_n) = \frac{1}{1 + d_E(O, x_n)}\, x_n \quad \text{converges to} \quad k(\mathbf{x}) = \frac{1}{1 + d_E(O, x)}\, x.$$

If $\mathbf{x} = \{x_i\}$ is not equivalent to a constant Cauchy sequence in $(\mathbf{R}^s, d^\phi)$, then by Lemma 3.2, we may assume $x_i = a_i x$ for some $x \in S_1$. Notice that we may consider only the following two cases.

(a) For all n, $x_{n,i} = a_i x_n$ for some $x_n \in S_1$.

(b) For all n, $\mathbf{x}_n = \{x_n\}$ is a constant Cauchy sequence.

For the case (a), from eq. (13) we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \rho(\mathbf{x}_n, \mathbf{x}) \\ &= \lim_{n\to\infty} \lim_{i\to\infty} d^\phi(a_i x_n, a_i x) \\ &\geq \lim_{n\to\infty} d_E(x_n, x) \\ &= \lim_{n\to\infty} d_E(k(\mathbf{x}_n), k(\mathbf{x})). \end{aligned}$$

For the case (b), suppose that

$$\lim_{n\to\infty} d_E\left(\frac{1}{1+d_E(O,x_n)}\, x_n, x\right) \neq 0.$$

We will get a contradiction. Choose a subsequence $\{y_n\}$ of $\{x_n\}$ such that

$$\frac{1}{1+d_E(O,y_n)}\, y_n \to y \neq x \quad \text{in} \quad (B^s, d_E).$$

Since h is continuous and injective, we have

$$y_n = h\left(\frac{1}{1+d_E(O,y_n)}\, y_n\right) \to h(y) \neq h(x) = \mathbf{x} \quad \text{in} \quad (\overline{\mathbf{R}^s}, \rho).$$

Therefore $\lim_{n\to\infty} \rho(\mathbf{y}_n, \mathbf{x}) \neq 0$. This is a contradiction. ∎

4 A compactification of $(\mathbf{R}^s, d_E)$ which is not equivalent to the standard compactification

Two compactifications Y_1 and Y_2 of a topological space X are called equivalent if there exists a homeomorphism $h : Y_1 \to Y_2$ such that $h(x) = x$ for all $x \in X$. Recall that $s \geq 2$. In this section, we construct a compactification of $(\mathbf{R}^s, d_E)$ which is homeomorphic to the closed unit ball (B^s, d_E), but not equivalent to the standard compactification $(\overline{\mathbf{R}^s}, \rho_\phi)$ in Section 3. We define a nonnegative symmetric function $\psi : \mathbf{R}^s \times \mathbf{R}^s \to \mathbf{R}$ as follows. Choose $0 < \delta < \frac{\pi}{4}$ and let

$$A^+ = \{x \in S_1 \mid \angle xO\mathbf{a}_1 \leq \delta\}, \quad A^- = \{x \in S_1 \mid \angle xO(-\mathbf{a}_1) \leq \delta\},$$

where $\mathbf{a}_1 = (1, 0, \cdots, 0)$ and $-\mathbf{a}_1 = (-1, 0, \cdots, 0) \in \mathbf{R}^s$. For each $x \in S_1$, let

$$P_x = \{t\mathbf{a}_1 + t'x \in \mathbf{R}^s \mid t, t' \in \mathbf{R}\}.$$

We define an infinite ray $L_x \subset P_x$ starting from x as follows. See Figure 1, where

$$\theta = \frac{\pi}{\pi - 2\delta}(\angle xO\mathbf{a}_1 - \delta).$$

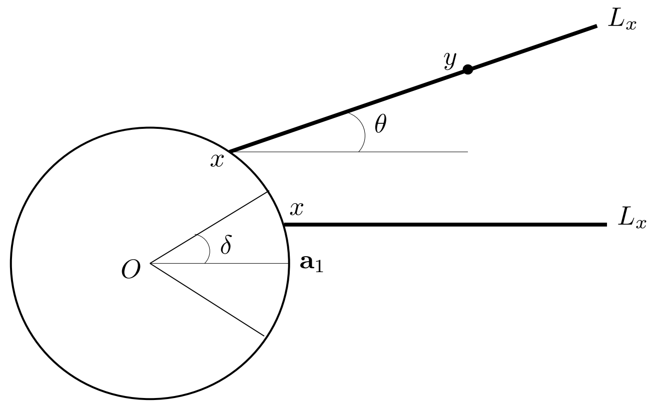

Figure 1: L_x

$$L_x = \begin{cases} \{x + t\mathbf{a}_1 \mid t \geq 0\} & \text{if } x \in A^+ \\ \{x + t\mathbf{a}_1 \mid t \leq 0\} & \text{if } x \in A^- \\ \{x, y \in P_x \mid \angle(y-x)O\mathbf{a}_1 = \frac{\pi}{\pi - 2\delta}(\angle xO\mathbf{a}_1 - \delta)\} & \text{if } x \in S_1 \setminus (A^+ \cup A^-) \end{cases}$$

and let $L = \{L_x \mid x \in S_1\}$. Notice that

(i) If $\angle xO\mathbf{a}_1 = \frac{\pi}{2}$, then $L_x = \{tx \mid t \geq 1\}$.

(ii) For all $x \in S_1$, the angle between two rays L_x and $\{tx \mid t \geq 1\}$ is not greater than δ.

(iii) For all $y \in \mathbf{R}^s$ with $d_E(O, y) \geq 1$, there exists unique ray in L which is through y.

For all $p, q \in \mathbf{N}$, let $h_{p,q} : S_p \to S_q$ be the homeomorphism defined by

$$h_{p,q}(x) = \text{ the intersection of } S_q \text{ and the ray in } L \text{ which is through } x.$$

In particular, we have $h_{p,p}(x) = x$, and if $h_{p,q}(x) = y$ then $h_{q,p}(y) = x$. The nonnegative symmetric function ψ is defined as follows.

Definition 4.1

$$\psi(x, y) = \begin{cases} 0 & \textit{if } \ h_{p,q}(x) = y \textit{ for some } p, q \in \mathbf{N} \\ d_E\left(h_{m,1}(x), h_{m,1}(y)\right) & \textit{if } \ x, y \in S_m \textit{ for some } m \in \mathbf{N} \\ d_E(x, y) & \textit{otherwise.} \end{cases}$$

Similarly as in Section 3, we can show that $(\mathbf{R}^s, d^\psi) = (\mathbf{R}^s, d_E^{\psi,O})$ is totally bounded, and its completion $(\overline{\mathbf{R}^s}, \rho_\psi)$ is homeomorphic to (B^s, d_E) by the following homeomorphism $h : (B^s, d_E) \to (\overline{\mathbf{R}^s}, \rho_\psi)$,

$$h(x) = \begin{cases} \frac{1}{1-d_E(O,x)}\, x & \text{if } d_E(O,x) < \frac{1}{2} \\ y \in L_{\frac{x}{d_E(O,x)}} \text{ such that} & \\ \qquad d_E\left(\frac{x}{d_E(O,x)}, y\right) = \frac{d_E(O,x)-\frac{1}{2}}{1-d_E(O,x)} & \text{if } \frac{1}{2} \le d_E(O,x) < 1 \\ \{h_{1,i}(x)\} & \text{if } d_E(O,x) = 1. \end{cases}$$

Suppose that $A, B \subset \mathbf{R}^s$. Let

$$d_E(A,B) = \inf\{d_E(x,y) \mid x \in A,\ y \in B\}.$$

In spherical coordinate system the distance between $(\rho_1, \phi_1, \theta_1)$ and $(\rho_2, \phi_2, \theta_2)$ is

$$\sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2\{\sin\phi_1 \sin\phi_2 \cos(\theta_1 - \theta_2) + \cos\phi_1 \cos\phi_2\}}. \tag{14}$$

The following two Lemmas are useful to show that h is a homeomorphism.

Lemma 4.1 *Suppose that $x, y \in S_1$. Then*

$$d_E(L_x, L_y) \ge \frac{1}{2\sqrt{2}}\, d_E(x,y).$$

Proof.

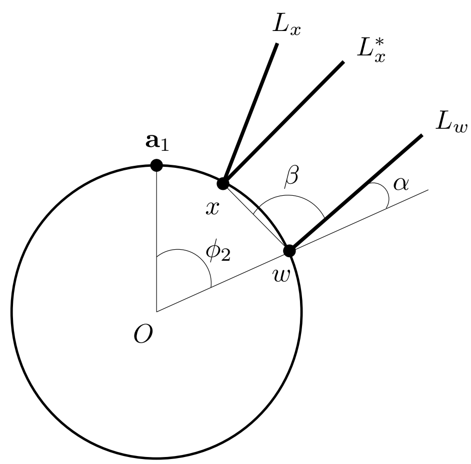

Figure 2: Since $0 \le \alpha \le \delta < \frac{\pi}{4}$, we have $\frac{\pi}{4} < \beta \le \frac{3\pi}{4}$.

We may assume that $x \ne y$. Since there exists a 3-dimensional subspace which contains O, $\mathbf{a}_1$, $-\mathbf{a}_1$, x and y, we may assume that $\mathbf{R}^s = \mathbf{R}^3$. In spherical coordinates (ρ, ϕ, θ), let $O = (0,0,0)$, $\mathbf{a}_1 = (1,0,0)$, $-\mathbf{a}_1 = (1,\pi,0)$, $x = (1, \phi_1, \theta_1)$

and $y = (1, \phi_2, \theta_2)$. By exchanging x and y if necessary, we may assume that

$$0 \le \phi_1 \le \frac{\pi}{2} \quad \text{and} \quad \phi_1 \le \phi_2.$$

Suppose that $\phi_2 \le \frac{\pi}{2}$. Let $z = (1, \phi_1, \theta_2)$ and $w = (1, \phi_2, \theta_1)$. Since $d_E(x, y) \le d_E(x, z) + d_E(z, y)$ and $d_E(x, w) = d_E(z, y)$, we have

$$d_E(x, z) \ge \frac{1}{2} d_E(x, y) \quad \text{or} \quad d_E(x, w) \ge \frac{1}{2} d_E(x, y).$$

If $d_E(x, z) \ge \frac{1}{2} d_E(x, y)$, let P be the plane containing x and z which is perpendicular to $\mathbf{a}_1$. Let L'_x be the projection of L_x to the plane P and so is L'_y. Notice that we have

$$d_E(L_x, L_y) \ge d_E(L'_x, L'_y) \ge d_E(x, z) \ge \frac{1}{2}\, d_E(x, y).$$

Suppose that $d_E(x, w) \ge \frac{1}{2} d_E(x, y)$. Let L^*_x be the ray starting from x with the same direction as L_w. Since $L_w = \{(\rho, \phi, \theta_1) \mid (\rho, \phi, \theta_2) \in L_y\}$, from eq. (14) and Figure 2, we have

$$d_E(L_x, L_y) \ge d_E(L_x, L_w) \ge d_E(L^*_x, L_w) \ge \frac{1}{\sqrt{2}}\, d_E(x, w) \ge \frac{1}{2\sqrt{2}}\, d_E(x, y).$$

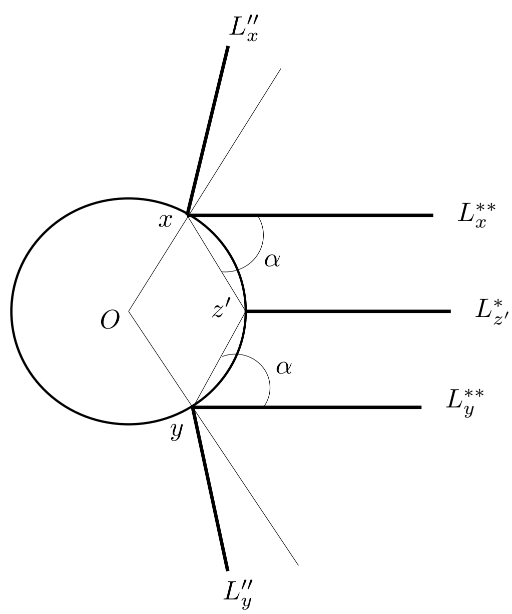

Figure 3: $\frac{\pi}{4} \le \alpha \le \frac{\pi}{2}$

Suppose that $\phi_2 > \frac{\pi}{2}$. Let P' be the plane which contains the greatest circle in S_1 through the points x and y. Let z' be the point on the greatest circle such that

$$\angle xOz' = \angle yOz' \le \frac{\pi}{2}.$$

Let L''_x be the projection of L_x to the plane P' and so is L''_y. Let

$$L^*_{z'} = \{tz' \mid t \ge 1\}.$$

Let L^{**}_x be the ray from x to the direction of $L^*_{z'}$ and so is L^{**}_y. From Figure 3, we have

$$\begin{aligned} d_E(L_x, L_y) &\ge d_E(L''_x, L''_y) \\ &\ge d_E(L^{**}, L^{**}_y) \\ &= d_E(L^{**}_x, L^*_{z'}) + d_E(L^*_{z'}, L^{**}_y) \\ &\ge \frac{1}{\sqrt{2}}\, d_E(x, z') + \frac{1}{\sqrt{2}}\, d_E(z', y) \\ &\ge \frac{1}{\sqrt{2}}\, d_E(x, y). \end{aligned}$$

∎

Lemma 4.2 *Suppose that* $a_m \le d_E(O, x) < a_{m+1}$*. Let* y *be the intersection of* S_m *and the ray in* L *which is through* x*. Then we have*

$$d_E(y, x) \le \frac{1}{\cos\delta}\,\frac{1}{m+1} \le \frac{\sqrt{2}}{m+1}.$$

Proof. Recall that $0 < \delta < \frac{\pi}{4}$. From Figure 4, the proof is trivial. ∎

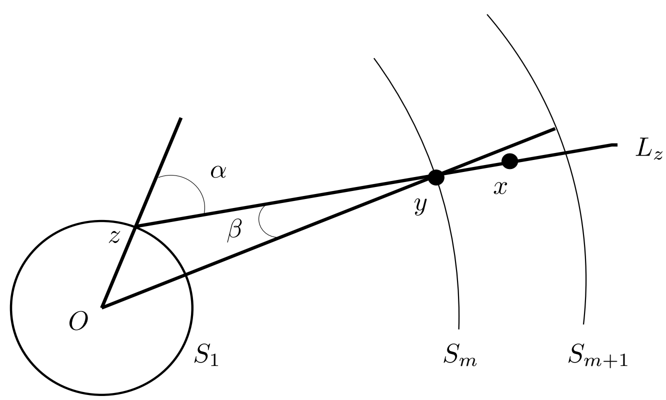

Figure 4: $\beta < \alpha \le \delta < \frac{\pi}{4}$

The following Lemma is also useful to show that h is a homeomorphism. Similarly as Lemma 3.3, we can prove this lemma.

Lemma 4.3 *Suppose that* $d_E(O,x) \geq 1$ *and* $d_E(O,y) \geq 1$. *Suppose also that* $x \in L_{x'}$ *and* $y \in L_{y'}$ *with* $x', y' \in S_1$. *Let* $(x_0, x_1, \cdots, x_m) \in \Gamma_{x,y}$ *with* $d_E(O, x_i) < 1$ *for all* $1 \leq i \leq m-1$. *Then*

$$\sum_{i=1}^{m} \delta^{\psi}(x_{i-1}, x_i) \geq d_E\left(L_{x'}, L_{y'}\right).$$

Now we show that the compactification $(\overline{\mathbf{R}^s}, \rho_\psi)$ of $(\mathbf{R}^s, d_E)$ is not equivalent to the standard compactification $(\overline{\mathbf{R}^s}, \rho_\phi)$ in Section 3.

Proposition 4.1 $(\overline{\mathbf{R}^s}, \rho_\psi)$ *and* $(\overline{\mathbf{R}^s}, \rho_\phi)$ *are not equivalent compactifications.*

Proof. Suppose that they are equivalent. There exists a homeomorphism

$$h : (\overline{\mathbf{R}^s}, \rho_\psi) \to (\overline{\mathbf{R}^s}, \rho_\phi)$$

such that $h(x) = x$ for all $x \in \mathbf{R}^s$. Choose a point $\mathbf{b}_1 \in S_1$ such that

$$\angle \mathbf{b}_1 O \mathbf{a}_1 = \frac{\delta}{2}.$$

Let $\mathbf{a} = \{\mathbf{a}_i\}$ and $\mathbf{b} = \{\mathbf{b}_i\}$, where

$$\mathbf{a}_i = h_{1,i}(\mathbf{a}_1) = (a_i, 0, \cdots, 0) \quad \text{and} \quad \mathbf{b}_i = h_{1,i}(\mathbf{b}_1)$$

for all $i \in \mathbf{N}$. Notice that

$$\sin\frac{\delta}{2} \leq d_E(\mathbf{a}_i, \mathbf{b}_i) \leq \frac{\delta}{2} \quad \text{for all } i.$$

Suppose that $(x_0, x_1, \cdots, x_m) \in \Gamma_{\mathbf{a}_i, \mathbf{b}_i}$. Using Lemma 4.1 and 4.3, we can show that

$$\sum_{i=1}^{m} \delta^{\psi}(x_{i-1}, x_i) \geq \frac{1}{2\sqrt{2}} d_E(\mathbf{a}_1, \mathbf{b}_1) \geq \frac{1}{2\sqrt{2}} \sin\frac{\delta}{2} > 0.$$

Therefore

$$d^{\psi}(\mathbf{a}_i, \mathbf{b}_i) \geq \frac{1}{2\sqrt{2}} \sin\frac{\delta}{2} \quad \text{for all } i,$$

and hence $\rho_\psi(\mathbf{a}, \mathbf{b}) \neq 0$. Thus $\mathbf{a} \neq \mathbf{b}$ in $(\overline{\mathbf{R}^s}, \rho_\psi)$.

But we have

$$\begin{aligned}
\rho_\phi(h(\mathbf{a}), h(\mathbf{b})) &= \lim_{i\to\infty} \rho_\phi(h(\mathbf{a}_i), h(\mathbf{b}_i)) \\
&= \lim_{i\to\infty} \rho_\phi(\mathbf{a}_i, \mathbf{b}_i) \\
&= \lim_{i\to\infty} d^{\phi}(\mathbf{a}_i, \mathbf{b}_i) \\
&\leq \lim_{i\to\infty} \delta^{\phi}(\mathbf{a}_i, \mathbf{b}_i) \\
&\leq \lim_{i\to\infty} \left(\frac{1}{1+a_i} + \frac{1}{a_i} d_E(\mathbf{a}_i, \mathbf{b}_i) + \frac{1}{1+a_i} \right) \\
&\leq \lim_{i\to\infty} \left(\frac{2}{1+a_i} + \frac{\delta}{2a_i} \right) \\
&= 0.
\end{aligned}$$

Therefore $h(\mathbf{a}) = h(\mathbf{b})$ in $(\overline{\mathbf{R}^s}, \rho_\phi)$. This is a contradiction. ∎

References

[1] I. Agol, *Tameness of hyperbolic 3-manifolds*, preprint, arXiv:math. GT/0405568.

[2] D. Calegari and D. Gabai, *Shrinkwrapping and taming of hyperbolic 3-manifolds*, J. Amer. Math. Soc. **19**(2006), no.2, 385-446.

[3] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[4] S. A. Gaal, *Point set toplogy*, Academic Press, New York and London, 1964.

[5] A. Marden, *The geometry of finitely generated Kleinian groups*, Ann. of Math. (2) **99**(1974), 383-462.

[6] J. R. Munkres, *Topology at first course*, Prentice-Hall, New Jersey, 1975.