On the second Paneitz-Branson invariant

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ABSTRACT. We define the second Paneitz-Branson operator on a compact Einsteinian manifold of dimension $n \ge 5$ and we give sufficient conditions that make it attained.

1. Introduction

In 1983, Paneitz [11] discovered a conformally invariant fourth order operator on 4-dimensional Riemannian manifolds. Branson[2] extended the notion to Riemannian manifolds of dimension $n \geq 5$. This operator has geometrical roots, it is associated to the notion of the *Q*-curvature which can be seen as the analogue of the scalar curvature for the conformal Laplacian. Let (M, g) be a Riemannian manifold; the Paneitz-Branson operator reads as

$$P_g(u) = \Delta^2 u - div \left(\frac{(n-2)^2 + 4}{2(n-1)(n-2)}S_g - \frac{4}{n-2}Ric_g\right) du + \frac{n-4}{2}Q_g u$$

where Ric_g and S_g denote respectively the Ricci curvature and the scalar curvature of g and where

$$Q_g = \frac{1}{2(n-1)} \Delta S_g + \frac{n^3 - 4n^2 + 16(n-1)}{8(n-1)^2(n-2)^2} S_g^2 - \frac{2}{(n-2)^2} |Ric_g|^2.$$

The conformal property of the Paneitz-Branson expresses as: let $\tilde{g} = \varphi^{\frac{4}{n-4}}g$ be a conformal metric to g, where $\varphi > 0$ is smooth function on M. Then

$$P_g(u\varphi) = \varphi^{N-1} P_g(u)$$

where $N = \frac{2n}{n-4}$.

Observe that when (M,g) is Einstein, the Paneitz-Branson operator is reduced to

$$P_g(u) = \Delta^2 u + \alpha \Delta u + \overline{\alpha} u$$

where

$$\Delta = -div\nabla$$

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and

$$\alpha = \frac{n^2 - 2n - 4}{2n(n-1)} S_g \ , \ \overline{\alpha} = \frac{(n-4)(n^2 - 4)}{16n(n-1)^2} S_g^2.$$

Notice that

(1.1)
$$\frac{\alpha^2}{4} - \overline{\alpha} = \frac{S_g^2}{n^2(n-1)^2}.$$

Let $H_2^2(M)$ be the standard Sobolev space, which is the completion of the space

$$C_2^2(M) = \left\{ \varphi \in C^\infty(M), \ \|\varphi\|_{2,2} < \infty \right\}$$

with respect to the norm

$$\|\varphi\|_{2,2} = \left(\|\Delta\varphi\|_2^2 + \|\nabla\varphi\|_2^2 + \|u\|_2^2\right)^{\frac{1}{3}}$$

Let $Gr_k(H_2^2)$ be the k-dimensional Grassmannian manifold in $H_2^2(M)$ i.e. the set of all subspaces of $H_2^2(M)$ of dimension $k \ge 1$.

Denote by [g] the conformal class of the metric g i.e. $\tilde{g} \in [g], \tilde{g} = ug$ with u > 0 a smooth function on M. The minimax characterization of the eigenvalue of order $k \geq 1$ of the Paneitz-Branson operator P_g is given by

$$\lambda_k(g) = \inf_{V \in G^r(H_2^2)} \sup_{v \in V - \{0\}} \frac{\int_M v P_g(v) dv_g}{\int_M v^2 dv_g}.$$

Similarly to the Yamabe invariant of higher order introduced by Amman and Humbert([1]), we define the Paneitz-Branson invariant.

Definition 1. Let $k \in N^*$. The k^{th} Paneitz-Branson invariant is defined by

$$\mu_k(M,g) = \inf_{\widetilde{g} \in [g]} \lambda_k(\widetilde{g}) Vol(M,\widetilde{g})^{\frac{4}{n}}.$$

In a recent paper [1] Amman and Humbert introduced the Yamabe invariant of high order $\mu_k(M,g)$, $k \ge 1$ and studied $\mu_2(M,g)$, mainly they showed that contrary to the standard Yamabe invariant $\mu_1(M,g)$ the second invariant $\mu_2(M,g)$ cannot be attained by a metric if the manifold (M,g) is connected. To find a minimizer to $\mu_2(M,g)$, they enlarge the class [g] of conformal metric to what they called the generalized conformal metric to gi.e. $\tilde{g} \in [g]$ if $\tilde{g} = u^{2^*-2}g$ where $u \in L^{2^*}(M)$ and $u \ge 0$ not indentically null and where $2^* = \frac{2n}{n-2}$.

The goal of this paper is to study the second Paneitz-Branson invariant on Einsteinian manifolds we seek for situations where this latter is attained. Observe that to have positive solutions in case of the Yamabe invariant it suffices to remark that for any $u \in H_1^2(M)$, $|u| \in H_1^2(M)$ and $|\nabla |u|| = |\nabla u|$ which is no longer true in the case of the Branson-Paneitz operator because of the term $\int_M (\Delta u)^2 dv_g$ and also if $u \in H_2^2(M)$, |u| is not necessary in

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 $H_2^2(M)$. The condition(1.1) implies by ([10] Theorem1.1) that Paneitz-Branson operator is coercive i.e.

$$\int_{M} u P_g(u) dv_g \ge \Lambda \, \|u\|_{2,2}$$

where the left hand side of this inequality has to be understood in the distribution sense and where $\Lambda > 0$ is a constant.

Hereafter, the space $H_2^2(M)$ will be endowed with the norm

$$\|u\| = \left(\int_M u P_g(u) dv_g\right)^{\frac{1}{2}}$$

which is equivalent to the norm $\|.\|_{2,2}$.

 $\|,\|_p$ will denote the L^p -norm with respect to the Riemannian measure dv_q .

The main results we obtain are

Theorem 1. If the compact manifold (M, g) is Einstein and of dimension $n \ge 12$ then $\mu_2(M, g)$ is attained by a generalized metric.

Theorem 2. Let (M,g) be a compact Einstein manifold of positive scalar curvature and of dimension $n \ge 5$. Assume that $\mu_2(M,g)$ is attained by a generalized metric $u^{N-2}g$ with $u \in L^N(M)$ and $u \ge 0$ not identically null.

Then there exist a nodal solution $w \in C^{4,\alpha}(M)$ ($\alpha < N-2$) to the equation $P_g(w) = \mu_2(M,g)u^{N-2}w$ such that |w| = u.

Our paper is organized as follows

In the first section we give some properties of the first and second eigenvalues of the Branson-Paneitz operator. In the second one we establish a Sobolev inequality related to the second Branson-Paneitz invariant $\mu_2(M,g)$. The third section is devoted to the existence of a minimizer to $\mu_2(M,g)$. In the fourth section an estimation of $\mu_2(M,g)$ is given in terms of $\mu_1(M,g)$ and of the best constant K_2 in the Sobolev embedding of $H_2^2(\mathbb{R}^n)$ in $L^N(\mathbb{R}^n)$. In the fifth section we give a sufficient condition which assures the strong convergence of a sequence of solutions. In the last section, we analyze situations where nodal solutions exist and by the way we deduce that $\mu_2(M,g)$ is not attained by a classical conformal metric.

Now, we quote some facts which will be of use in the sequel of this paper.

Lemma 1. ([4]) Let (M, g) be a compact Riemannian manifold of dimension $n \geq 5$, for any $\epsilon > 0$ there exists a constant $A(\epsilon)$ such that every $u \in H_2^2(M)$ fulfills

$$\|u\|_N^2 \le (K_2^2 + \epsilon) \|\Delta u\|_2^2 + A(\epsilon) \|u\|_2^2$$

with $N = \frac{2n}{n-4}$ and $K_2^{-2} = \pi^2 n(n-1)(n^2-4)\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}$ where Γ denotes the Euler function. **Lemma 2.** ([7])Let (S^n, h) be the standard unit sphere of \mathbb{R}^{n+1} , $n \geq 5$, and let P be the Paneitz-Branson operator on (S^n, h) , then

$$K_2^{-2} = \inf_{u \in C^{\infty}(S^n) - \{0\}} \frac{\int_{S^n} u P(u) dv_h}{\left(\int_{S^n} |u|^N dv_h\right)^{\frac{2}{N}}}$$

Lemma 3. ([7])Let(M,g) be a smooth compact n-dimensional $(n \ge 5)$ Riemannian manifold, α a positive real number, let b be a real valued functions defined on M and $u \in H_2^2(M)$ be a weak solution of

$$\Delta^2 u + \alpha \Delta u + \frac{\alpha^2}{4}u = bu.$$

If $b \in L^{\frac{n}{4}}(M)$, then $u \in L^{s}(M)$ for all $s \ge 1$.

2. First and second eigenvalues for a generalized metric

Let $L_{+}^{N}(M)$ be the space of L^{N} -integrable non negative functions which are not identically 0. Denote by $Gr_{k}^{u}(H_{2}^{2})$ the set of all k-dimensional subspaces $(k \geq 1)$ of $H_{2}^{2}(M)$ which are the span of the functions $u_{1}, ..., u_{k}$ if and only if $u_{1|_{M-u_{1}^{-1}(0)}}, ..., u_{k|_{M-u_{k}^{-1}(0)}}$ are linearly independent.

Definition 2. A generalized metric conform to a metric g is of the form $\tilde{g} = ug$ with $u \in L^N_+(M)$.

Definition 3. For any generalized metric $\tilde{g} = u^{\frac{N-2}{2}}g$ of a Riemannian metric g we define the eigenvalue of order $k \geq 1$ to the Branson-Paneitz operator P_g by

$$\lambda_k(\widetilde{g}) = \inf_{V \in Gr_k^u(H_2^2(M))} \sup_{v \in V - \{0\}} \frac{\int_M v P_g(v) dv_g}{\int_M u^{N-2} v^2 dv_g}.$$

We need the following lemma which is first given in ([1]) for sequences in $H_1^2(M)$ but its proof remains inchanged and we reproduce it here for reason of completness.

Lemma 4. If $u \in L^N_+(M)$ and (v_n) is a sequence in $H^2_2(M)$ which converges weakly to v, then

(2.1)
$$\int_{M} u^{N-2} \left| v_m^2 - v^2 \right| dv_g \to 0.$$

PROOF. Letting A be any real positive number, we put $u_A = \inf(u, A)$. Then $(u_A)_A$ is a monotone sequences which converges pointwisely almost everywhere to v, so by Lebesgue monotone convergence theorem, we get

$$\int_{M} (u^{N-2} - u_A^{N-2})^{\frac{N}{N-2}} dv_g \to 0.$$

On the other hand, we have

$$\int_{M} u^{N-2} \left| v_m^2 - v^2 \right| dv_g \le \int_{M} u_A^{N-2} \left| v_m^2 - v^2 \right| dv_g$$

$$+ \int_{M} \left(u^{N-2} - u^{N-2}_{A} \right) (|v_{m}| + |v|)^{2} dv_{g}$$

Using the Hölder inequality, we obtain

$$\int_{M} u^{N-2} \left| v_{m}^{2} - v^{2} \right| dv_{g} \leq A^{N-2} \int_{M} \left| v_{m}^{2} - v^{2} \right| dv_{g}$$
$$+ \left(\int_{M} \left| u^{N-2} - u_{A}^{N-2} \right|^{\frac{N}{N-2}} dv_{g} \right)^{\frac{N-2}{N}} \left(\int_{M} (|v_{m}| + |v|)^{N} dv_{g} \right)^{\frac{2}{N}}.$$

Taking account of the boundedness and the strong convergence of (v_m) to v in $L^2(M)$ we get the result. \Box

Proposition 1. Let $\tilde{g} = u^{\frac{N-2}{2}}g$ be any generalized conformal metric to a metric g. The equation

$$(2.2) P_g v = \lambda_1 u^{N-2} v$$

has a solution of class $C^{4,\alpha}(M)$ ($0 < \alpha < N-2$) with the constraint

$$\int_M u^{N-2} v^2 dv_g = 1.$$

PROOF. Let (v_m) be a minimizer sequence of $\lambda_1(\tilde{g})$ with the constraint $\int_M u^{N-2} v_m^2 dv_g = 1$. The sequence (v_m) is bounded in $H_2^2(M)$ and by passing to a subsequences also labelled (v_m) , there exists $v \in H_2^2(M)$ such that

(i) $v_m \to v$ weakly in $H_2^2(M)$ (ii) $v_m \to v$ strongly in $L^2(M)$

From (i), we obtain

$$\|v\| \le \liminf \|v_m\|$$

and by Lemma4 we get

$$\int_M u^{N-2} v^2 dv_g = \lim_{m \to \infty} \int_M u^{N-2} v_m^2 dv_g = 1$$

and we derive that $||v||^2 = \lambda_1(\tilde{g}).$

Consequently v is a non trivial weak solution of the equation (2.2).

By Lemma3, $v \in L^{s}(M)$ for any $s \geq 1$ and it follows that $v \in C^{4,\alpha}(M)$, with $\alpha < N-2$.

2.1. Positivity of solutions. Now we are going to show that the equation (2.2) admits a positive solution.

Proposition 2. If the scalar curvature S_g of the Einsteinian manifold (M,g) is positive, the equation

$$(2.3) P_g f = \lambda_1 u^{N-2} v$$

has a positive solution with the constraint

(2.4)
$$\int_{M} u^{N-2} v^2 dv_g = 1.$$

PROOF. Let v be a solution to the equation(2.3) and let f be the solution of the equation

$$\Delta f + \frac{\alpha}{2}f = \left|\Delta v + \frac{\alpha}{2}v\right|$$

with $\alpha > 0$. Clearly $f \in C^{2,\alpha}(M)$ $(\alpha < N-2)$.

If $\Delta v + \frac{\alpha}{2}v \ge 0$ (resp. $\Delta v + \frac{\alpha}{2}v \le 0$) we have f = v (resp. f = -v). If it is not the case, putting $w = f \pm v$, we get

(2.5)
$$\Delta w \pm \frac{\alpha}{2}w = \left|\Delta f + \frac{\alpha}{2}f\right| \pm \left(\Delta v + \frac{\alpha}{2}v\right) \ge 0$$

so $\Delta(-w) \pm \frac{\alpha}{2} (-w) \leq 0$. The maximum principle asserts that $-w = v \pm f$ attains a maximum $M \geq 0$ then w is a constant function but this is excluded since $-\alpha \ M \leq 0$ implies that M = 0. Consequently $f > |v| \geq 0$. Let $k \geq 0$ be a real number such that $\int_M u^{N-2} (kf)^2 dv_g = 1$, then

Let $k \ge 0$ be a real number such that $\int_M u^{N-2} (kf)^2 dv_g = 1$, then 0 < k < 1. Now letting $\hat{f} = kf$ and taking account of the equation(2.3) we get

$$\begin{split} \int_{M} \left(\left(\Delta \widehat{f} \right)^{2} + \alpha \left| \nabla \widehat{f} \right|^{2} + \overline{\alpha} \widehat{f}^{2} \right) dv_{g} - \lambda_{1}(\widetilde{g}) &= \\ k^{2} \int_{M} \left(\left(\Delta f \right)^{2} + \alpha \left| \nabla f \right|^{2} + \overline{\alpha} \widehat{f}^{2} \right) dv_{g} - \lambda_{1}(\widetilde{g}) &= \\ k^{2} \int_{M} \left(\left(\Delta f + \frac{\alpha}{2} f \right)^{2} - \frac{\alpha^{2}}{4} f^{2} + \overline{\alpha} \widehat{f}^{2} \right) dv_{g} - \lambda_{1}(\widetilde{g}) &= \\ k^{2} \int_{M} \left(\left(\Delta v + \frac{\alpha}{2} v \right)^{2} - \frac{\alpha^{2}}{4} f^{2} + \overline{\alpha} \widehat{f}^{2} \right) dv_{g} - \lambda_{1}(\widetilde{g}) &= \\ (k^{2} - 1)\lambda_{1}(\widetilde{g}) + \left(\overline{\alpha} - \frac{\alpha^{2}}{4} \right) \int_{M} \left(\widehat{f}^{2} - v^{2} \right) dv_{g} \leq 0. \end{split}$$

Consequently

(2.6)
$$\int_{M} \left(\left(\Delta \widehat{f} \right)^{2} + \alpha \left| \nabla \widehat{f} \right|^{2} + \overline{\alpha} \widehat{f}^{2} \right) dv_{g} = \lambda_{1}(\widetilde{g}).$$

Proposition 3. Let $u \in L^N_+(M)$, if $v \in H^2_2(M)$ is a weak solution of the equation

(2.7)
$$P_g v = \lambda_1(\widetilde{g}) u^{N-2} v$$

with

(2.8)
$$\int_M u^{N-2} v^2 dv_g = 1$$

then there is a weak solution $w \in H^2_2(M)$ of the equation

(2.9)
$$P_g w = \lambda'_2(\tilde{g}) u^{N-2} w$$

with the constraints

$$\int_M u^{N-2} w^2 dv_g = 1, \int_M u^{N-2} v w dv_g = 0$$

where

$$\lambda_2'(\widetilde{g}) = \inf_E \frac{\int_M w P_g w dv_g}{\int_M u^{N-2} w^2 dv_g}$$

and

$$E = \left\{ u^{\frac{N-2}{2}}w \, : \, w \in H_2^2(M) / \, u^{\frac{N-2}{2}}w \not\cong 0, \ \int_M u^{N-2}vwdv_g = 0 \ and \ \int_M u^{N-2}w^2dv_g = 1 \ \right\}.$$

PROOF. First, we show that the set E is non empty. Let v, $s \in H_2^2(M)$ noncolinear such that $\int_M u^{N-2}v^2 dv_g = 1$, $\int_M u^{N-2}s^2 dv_g = 1$. Necessarily $u^{\frac{N-2}{2}}v \not\cong 0$ and $u^{\frac{N-2}{2}}s \not\cong 0$. Observe that $\int_M u^{N-2}vsdv_g \neq 1$, since if it is not the case the equality is attained in the the Hölder inequality and this possible if and only if there a real constant c such that v = cs.

Putting $w = \alpha v + \beta s$ with $\alpha, \beta \in R$, we obtain

$$u^{N-2}w = \alpha u^{N-2}v + \beta u^{N-2}s$$

so to get

$$\int_{M} u^{N-2} v w dv_g = \alpha + \beta \int_{M} u^{N-2} v s dv_g = 0$$

and

$$\int_M u^{N-2} w^2 dv_g = 1$$

we let

$$\beta = -\frac{\alpha}{\int_M u^{N-2} v s dv_g}$$

and

$$1 = \int_{M} u^{N-2} (\alpha v + \beta s)^{2} dv_{g}$$
$$= \alpha^{2} + \beta^{2} + 2\alpha\beta \int_{M} u^{N-2} v s dv_{g}$$

We obtain

$$\alpha = \pm \left(\frac{\int_M u^{N-2} v s dv_g}{1 - \int_M u^{N-2} v s dv_g} \right)^{\frac{1}{2}}$$

and

and

$$\beta = \pm \frac{1}{\left(\left(1 - \int_M u^{N-2} v s dv_g\right) \int_M u^{N-2} v s dv_g\right)^{\frac{1}{2}}}$$

Now we will show that w is a weak non trivial solution of the equation(2.9). Let (w_n) be a minimizer sequence of $\lambda'_2(\tilde{g})$ such that

$$\int_{M} u^{N-2} w_m^2 dv_g = 1$$
$$\int_{M} u^{N-2} w_m v dv_g = 0.$$

Then the sequences (w_m) is bounded in $H_2^2(M)$ and there is $w \in H_2^2(M)$ a weak solution of the equation(2.9). It remains to verify that $\int_M u^{N-2}w^2 = 1$ and also $\int_M u^{N-2}wv dv_g = 0$. The first equality follows from Lemma4 the second one is true since the function $\varphi = u^{N-2}v \in L^{\frac{N}{N-1}}(M)$.

Proposition 4. Suppose that the solutions v and w of the equations (2.7) and (2.9) are as in proposition6, then $\lambda_2(\tilde{g}) = \lambda'_2(\tilde{g})$.

Proof. The weak solution $w \in H_2^2(M)$ of the equation

$$P_g w = \lambda_2'(\tilde{g}) u^{N-2} w$$

is a minimizer of

$$\lambda_2'(\widetilde{g}) = \inf_{w \in E} \frac{\int_M w P_g w dv_g}{\int_M u^{N-2} w^2 dv_g}$$

where

$$E = \left\{ u^{\frac{N-2}{2}} w : w \in H_2^2(M) \text{ s. t. } u^{\frac{N-2}{2}} w \not\cong 0 , \int_M u^{N-2} v w dv_g = 0 \text{ and } \int_M u^{N-2} w^2 dv_g = 1 \right\}.$$

Since $u^{\frac{N-2}{2}}v$ and $u^{\frac{N-2}{2}}w$ are linearly independent it follows that $V_o = \operatorname{span}(v, w) \in Gr_2^u(H_2^2(M)).$

Putting

$$f = \lambda v + \mu w$$
 with $(\lambda, \mu) \in \mathbb{R}^2 - \{(0, 0)\}$

we evaluate

$$s = \frac{\int_M f P_g f dv_g}{\int_M u^{N-2} f^2 dv_g}$$

on the plane V_o .

We obtain

$$s = \frac{\lambda^2 \int_M v P_g(v) \, dv_g + \mu^2 \int_M w P_g(w) \, dv_g}{\lambda^2 + \mu^2}$$
$$= \frac{\lambda^2}{\lambda^2 + \mu^2} \lambda_1(\widetilde{g}) + \frac{\mu^2}{\lambda^2 + \mu^2} \lambda'_2(\widetilde{g})$$
$$= \lambda_1(\widetilde{g}) \cos^2 \theta + \lambda'_2(\widetilde{g}) \sin^2 \theta$$

with $\theta \in \mathbb{R}$.

On the other hand, we have

$$\frac{ds}{d\theta} = (\lambda'_2(\widetilde{g}) - \lambda_1(\widetilde{g}))\sin 2\theta$$

and noting that

$$\lambda_1(\widetilde{g}) \le \lambda_2'(\widetilde{g})$$

we get easily

$$\min s(\theta) = \lambda_1(\widetilde{g}) \text{ and } \max s(\theta) = \lambda'_2(\widetilde{g}).$$

Consequently

$$\lambda_2'(\widetilde{g}) = \sup_{w \in V_o} \frac{\int_M w P_g(w) dv_g}{\int_M u^{N-2} w^2 dv_g}.$$

On the other hand the infimum of $\sup_{w \in V - \{0\}} \frac{\int_M w P_g(w) dv_g}{\int_M u^{N-2} w^2 dv_g}$ on all the subspaces of $Gr_2^u(H_2^2(M))$ is attained by $V_o = span(v, w)$.

Hence

$$\lambda_2'(\widetilde{g}) = \lambda_2(\widetilde{g}).$$

Proposition 5. If $u \in C^{\infty}(M)$ with $u \ge 0$ not identically 0. Then any weak solution of the equation

$$(2.10) P_q v = \mu u^{N-2} v$$

is of class $C^{\infty}(M)$, $\mu \in \mathbb{R}$.

PROOF. Let $u \in C^{\infty}(M)$, $u \ge 0$ and not identically 0 and v a weak solution of the equation(2.10). We have

$$(\Delta + a)(\Delta + b)v = \mu u^{N-2}v$$

with $a = \frac{\alpha - \sqrt{\alpha^2 - 4\overline{\alpha}}}{2}$ and $b = \frac{\alpha + \sqrt{\alpha^2 - 4\overline{\alpha}}}{2}$. Putting

$$z = (\Delta + b)v$$

we get

$$(\Delta + a)z = \mu u^{N-2}v$$

and since $v \in H_2^2(M)$, $u^{N-2}v \in H_2^2(M)$ so $z \in H_4^2(M)$. Recurrently, for any $k \ge 2$ we obtain $v \in H_k^2$. Now, classical regularity theorem allows us to conclude that $v \in C^{\infty}(M)$.

3. A Sobolev inequality related to $\mu_2(M,g)$

The Sobolev inequality given by Lemma1 which allows to avoid concentration phenomena for the minimizing sequence of the first Paneitz-Branson invaiant $\mu_1(M,g)$ is not sufficient in the case of the second Paneitz-Branson invariant $\mu_2(M,g)$, we propose the following Sobolev type inequality.

Proposition 6. Let (M, g) be a Riemannian manifold of dimension $n \ge 5$. For any $\epsilon > 0$ there is a constant $A(\epsilon)$ such that, for any $u \in L^N_+(M)$ and any $v \in H^2_2(M)$, we have

$$\int_{M} u^{N-2} v^2 dv_g \le \left(2^{-\frac{4}{n}} (K_2^2 + \epsilon) \int_{M} (\Delta v)^2 dv_g + A(\epsilon) \int_{M} v^2 dv_g\right) \left(\int_{M} u^N dv_g\right)^{\frac{2}{N}}$$

PROOF. For any $\epsilon > 0$, put

$$B(\epsilon) = A(\epsilon)K_2^{-2}(1+\epsilon)^{-1}$$

and let

$$G(u,v) = \frac{\int_M (\Delta v)^2 dv_g + B(\epsilon) \int_M v^2 dv_g}{\int_M u^{N-2} v^2 dv_g} \left(\int_M u^N dv_g \right)$$

where $u \in L^N_+(M)$ and $v \in H^2_2(M) - \{0\}$ such that $\int_M u^{N-2} v^2 dv_g \neq 0$.

Obviously G(u, v) is continuous on $L^N_+(M) \times H^2_2(M) - \{0\}$. So $I(u, V) = \sup_{v \in V - \{0\}} G(u, v)$ depends continuously on $u \in L^N_+(M)$ and $V \in Gr^u_2(H^2_2(M))$. We must show that

$$I(u,V) \ge 2^{\frac{4}{n}} K_2^{-2} (1+\epsilon)^{-1}$$

for all $u \in C^{\infty}(M)$, u > 0 and $V \in Gr_2^u(C^{\infty}(M))$. Without lost of generality, we suppose that $\int_M u^{N-2}v^2 dv_g = 1$. On the

other hand the operator

(3.1)
$$v \to Q(v) = u^{\frac{2-N}{2}} \Delta^2(u^{\frac{2-N}{2}}v) + B(\epsilon)u^{2-N}v$$

is a fourth order elliptic and self adjoint with respect to the inner product in $L^2(M)$. Q has a discrete spectrum $\lambda_1 \leq \lambda_2 \leq \dots$

The corresponding eigenfunctions $\varphi_1, \varphi_2, \dots$ are smooth functions on M. Letting $v_i = u^{\frac{2-N}{2}} \varphi_i$, we get

$$\int_{M} (\Delta v_i)^2 dv_g + B(\epsilon) \int_{M} v_i^2 dv_g = \lambda_i \int_{M} u^{N-2} v_i^2 dv_g$$

with

$$\int_M u^{N-2} v_i v_j dv_g = 0.$$

Let \widetilde{P}_g be the operator defined on $C^\infty(M)$ by $\widetilde{P}_g u = \Delta_g^2 u + B(\epsilon) u$ and let Ω_1 and Ω_2 be two non empty open disjoint sets in M and let v_1 and v_2 be two non trivial solutions to the equation

(3.2)
$$\widetilde{P}_g v_i = \lambda_2 u^{N-2} v_i$$

i = 1, 2 with supports included respectively in $\overline{\Omega}_1$ and $\overline{\Omega}_2$, the closer sets of Ω_1 and Ω_2 r and where λ_2 is the second eigenvalue of the operator Q defined above. By multiplying if necessary v_1 and v_2 by constants, we assume that $\int_M u^{N-2} v_1^2 dv_g = \int_M u^{N-2} v_2^2 dv_g = 1.$

Using the Hölder inequality and the Sobolev one given in Lemma1, we get

$$2 = \int_{M} u^{N-2} v_{1}^{2} dv_{g} + \int_{M} u^{N-2} v_{2}^{2} dv_{g}$$

$$\leq \left(\int_{\Omega_{1}} u^{N} dv_{g}\right)^{1-\frac{2}{N}} \left(\int_{M} |v_{1}|^{N} dv_{g}\right)^{\frac{2}{N}} + \left(\int_{\Omega_{2}} u^{N} dv_{g}\right)^{1-\frac{2}{N}} \left(\int_{M} |v_{2}|^{N} dv_{g}\right)^{\frac{2}{N}}$$

$$\leq \left(\int_{\Omega_{1}} u^{N} dv_{g}\right)^{1-\frac{2}{N}} K_{2}^{2}(1+\epsilon) \left(\int_{M} (\Delta v_{1})^{2} dv_{g} + B(\epsilon) \int_{M} v_{1}^{2} dv_{g}\right)$$

$$+ \left(\int_{\Omega_{2}} u^{N} dv_{g}\right)^{1-\frac{2}{N}} K_{2}^{2}(1+\epsilon) \left(\int_{M} (\Delta v_{2})^{2} dv_{g} + B(\epsilon) \int_{M} v_{2}^{2} dv_{g}\right).$$

And since v_1 and v_2 are solutions to the equation (3.2), we obtain

$$2 \le K_2^2 (1+\epsilon)\lambda_2 \left(\left(\int_{\Omega_1} u^N dv_g \right)^{1-\frac{2}{N}} + \left(\int_{\Omega_2} u^N dv_g \right)^{1-\frac{2}{N}} \right)$$

Using Hölder inequality, we get

$$\left(\int_{\Omega_1} u^N dv_g\right)^{1-\frac{2}{N}} + \left(\int_{\Omega_2} u^N dv_g\right)^{1-\frac{2}{N}} \le 2^{\frac{2}{N}} \left(\int_{\Omega_1} u^N dv_g + \int_{\Omega_2} u^N dv_g\right)$$
so

$$\lambda_2 \ge 2^{\frac{4}{n}} (1+\epsilon)^{-1} K_2^{-2}.$$

Letting $V = span(v_1, v_2)$, we obtain for any $(\alpha, \beta) \in \mathbb{R}^2 - (0, 0)$,

$$\begin{split} G(u, \alpha v_1 + \beta v_2) &= \frac{\int_M \left[(\Delta (\alpha v_1 + \beta v_2))^2 + B(\epsilon) (\alpha v_1 + \beta v_2)^2 \right] dv_g}{\int_M u^{n-2} (\alpha v_1 + \beta v_2)^2 dv_g} \\ &= \frac{\alpha^2 \int_M \left((\Delta v_1)^2 + B(\epsilon) v_1^2 \right) dv_g + \alpha^2 \int_M \left((\Delta v_2)^2 + B(\epsilon) v_2^2 \right) dv_g}{\alpha^2 \int_M u^{N-2} v_1^2 dv_g + \beta^2 \int_M u^{N-2} v_2^2 dv_g} \\ &= \lambda_2. \end{split}$$

Then

$$I(u,V) = \sup_{(\alpha,\beta)\in R^2 - (0,0)} G(u,\alpha v_1 + \beta v_2) = \lambda_2$$

and the proof of the proposition is achieved.

In the particular case of the standard unit (S^n, h) sphere of \mathbb{R}^{n+1} , we obtain

Proposition 7. Let (S^n, h) be the unit sphere of \mathbb{R}^{n+1} , $n \geq 5$, and let P be the Paneitz-Branson operator on (S^n, h) . For any $u \in L^N_+(S^n)$ and any $v \in H^2_2(S^n)$, we have

$$\int_{S^n} u^{N-2} v^2 dv_h \le 2^{-\frac{4}{n}} K_2^2 \int_{S^n} v P(v) dv_h \left(\int_{S^n} u^N dv_h \right)^{\frac{2}{N}}.$$

PROOF. The proof is similar to that of the proposition 6, by using the Sobolev inequality given by Lemma 2 instead of that given by Lemma 1. \Box

As corollary of proposition7, we get the following Sobolev inequality on the Euclidean space \mathbb{R}^n .

Corollary 1. Let $C_c^{\infty}(\mathbb{R}^n)$ be the space of functions of classe C^{∞} and of compact supports on \mathbb{R}^n . For any $u \in L^N_+(\mathbb{R}^n)$ and any $v \in H^2_2(\mathbb{R}^n)$, we have

$$\int_{R^{n}} u^{N-2} v^{2} dx \leq 2^{-\frac{4}{n}} K_{2}^{2} \int_{R^{n}} (\Delta v)^{2} dx \left(\int_{M} u^{N} dx \right)^{\frac{2}{N}}$$

where dx denotes the Euclidean measure on \mathbb{R}^n .

PROOF. Since \mathbb{R}^n is conformal to $S^n - \{p\}$, where p is any point of S^n and the Paneitz-Branson is a conformal invariant the corollory1 follows from proposition7.

Proposition 8. If $\mu_1(M,g)K_2^2 < 1$, then $\mu(M,g) = \mu_1(M,g)$

Proof.

$$\begin{split} \mu_1(M,g) &= \inf_{\widetilde{g} \in [g]} \lambda_1(\widetilde{g}) \left(vol(M) \right)^{\frac{4}{n}} \\ &= \inf_{\substack{u \in C^{\infty}(M) \ v \in C^{\infty}(M) - \{0\}}} \inf_{\substack{\int_M v P_g(v) dv_g \\ J_M u^{N-2} v^2 dv_g}} \left(\int_M u^N dv_g \right)^{\frac{4}{n}} \\ &\leq \inf_{v \in C^{\infty}(M) - \{0\}} \frac{\int_M v P_g(v) dv_g}{\left(\int_M |v|^N dv_g\right)^{\frac{2}{N}}} = \mu(M,g). \end{split}$$

The inequality in the other sense requires a variational method. Let $g_m =$ $u_m^{\frac{N-2}{2}}g$ with $u_m \in L^N_+(M)$, a minimizer sequence of $\mu_1(M,g)$ i.e.

$$\mu_1(M,g) = \lim_{m \to \infty} \lambda_1(g_m) \left(vol(M,g) \right)^{\frac{4}{n}}.$$

Considering the Yamabe functional

$$Y(u,v) = \frac{\int_M v P_g(v) dv_g}{\int_M u^{N-2} v^2 dv_g} \left(\int_M u^N dv_g \right)^{\frac{4}{n}}$$

with $v \in H_2^2(M) - \{0\}$ and $u \in L^N_+(M)$, we write, for any $\lambda \in \mathbb{R}^*$

$$Y(\lambda u, v) = \frac{\int_M v P_g(v) dv_g}{\lambda^{N-2} \int_M u^{N-2} v^2 dv_g} \lambda^{\frac{2N}{n}} \left(\int_M u^N dv_g \right)^{\frac{4}{n}} = Y(u, v).$$

So, we can choose the sequence (u_m) such that $\int_M u_m^N dv_g = 1$ and there is a subsequence of (u_m) still labelled by (u_m) converging weakly to $u \ge 0$ in $L^N(M).$

On the other hand, by Proposition 3 for any $u_m \in L^N_+(M)$ there is $v_m \in$ $H_2^2(M)$ solutions of the equation

$$P_g(v_m) = \lambda_{1,m} u_m^{N-2} v_m$$

with the constraint

$$\int_M u_m^{N-2} v_m^2 dv_g = 1.$$

Obviously (v_m) is bounded in $H_2^2(M)$ so there is $v \in H_2^2(M)$ such that $v_m \to v$ weakly in $H_2^2(M)$, $v_m \to v$ a.e. in M. Since $\lim_{m\to\infty} \lambda_{1,m} = \mu_1(M,g)$, v is a weak solution of the equation

$$P_g(v) = \mu_1(M, g) u^{N-2} v$$
.

Now, we are going to show that v satisfies the condition

$$\int_M u^{N-2} v^2 dv_g = 1.$$

It is obvious that

$$\int_{M} u^{N-2} v^2 dv_g \le 1$$

So we have to show the inequality in the other sense, to do so, we consider

$$\begin{split} \int_{M} u^{N-2} v^2 dv_g &= \int_{M} u_m^{N-2} v_m^2 dv_g - \int_{M} (u_m^{N-2} v_m^2 - u^{N-2} v^2) dv_g \\ &= 1 - \int_{M} (u_m^{N-2} v_m^2 - u^{N-2} v^2) dv_g. \end{split}$$

Now, since

$$\left| u_m^{N-2} v_m^2 - u_m^{N-2} (v_m - v)^2 \right| \le C u_m^{N-2} \left| v_m + v \right| \left| v \right|$$

where C is a postive constant

we get

$$u_m^{N-2}v_m^2 - u_m^{N-2}(v_m - v_2)^2 \to u^{N-2}v^2$$
 in $L^1(M)$

and

$$\int_{M} \left(u_m^{N-2} v_m^2 - u^{N-2} v^2 \right) dv_g \to \int_{M} u_m^{N-2} (v_m - v_2)^2 dv_g.$$

 \mathbf{so}

(3.3)
$$\int_{M} u^{N-2} v^2 dv_g = 1 - \int_{M} u_m^{N-2} (v_m - v)^2 dv_g + o(1).$$

where o(1) is a sequence converging to 0 as $m \to +\infty$. Using simultaneously the Hölder inequality and the Sobolev inequality given by Lemma1, we get

$$\begin{split} \int_{M} u_{m}^{N-2} (v_{m} - v)^{2} dv_{g} &\leq \left(K_{2}^{2} + \epsilon\right) \|\Delta \left(v_{m} - v\right)\|_{2}^{2} + A(\epsilon) \|v_{m} - v\|_{2}^{2} \\ &\leq \left(K_{2}^{2} + \epsilon\right) \left(\int_{M} \left(v_{m} P_{g}(v_{m}) - v P_{g}(v)\right) dv_{g}\right) + o(1) \\ &\leq \left(K_{2}^{2} + \epsilon\right) \mu_{1}(M, g) \int_{M} \left(u_{m}^{N-2} v_{m}^{2} - u^{N-2} v^{2}\right) dv_{g} + o(1) \\ &\leq \left(K_{2}^{2} + \epsilon\right) \mu_{1}(M, g) \left(1 - \int_{M} u^{N-2} v^{2} dv_{g}\right) + o(1). \end{split}$$

Taking account of (3.3), we have

$$\int_{M} u^{N-2} v^2 dv_g \ge 1 - (K_2^2 + \epsilon) \mu_1(M, g) \left(1 - \int_{M} u^{N-2} v^2 dv_g\right) + o(1).$$

Then

$$(1 - (K_2^2 + \epsilon) \mu_1(M, g)) \int_M u^{N-2} v^2 dv_g \ge 1 - 2^{-\frac{4}{n}} (K_2^2 + \epsilon) \mu_1(M, g)$$

+o(1).

So if

$$\mu_1(M,g)K_2^2 < 1$$

we get

$$\int_M u^{N-2} v^2 dv_g \ge 1.$$

Consequently

$$\mu_1(M,g) > 0.$$

Let $\overline{u} = a |v|$ with a > 0 and

$$\int_{M} \overline{u}^{N} dv_{g} = a^{N} \int_{M} v^{N} dv_{g} = 1$$

then

$$\begin{split} \mu_1(M,g) &\leq \frac{\int_M v P_g(v) dv_g}{\int_M \overline{u}^{N-2} v^2 dv_g} \\ &\leq \frac{\mu_1(M,g) \int_M u^{N-2} v^2 dv_g}{a^{N-2} \int_M v^N dv_g} \\ &\leq a^2 \mu_1(M,g) \int_M u^{N-2} v^2 dv_g \\ &\leq \mu_1(M,g) \int_M u^{N-2} \overline{u}^2 dv_g \end{split}$$

The Hölder inequality implies that

$$\begin{split} \mu_1(M,g) &\leq \mu_1(M,g) \left(\int_M u^N dv_g\right)^{1-\frac{2}{N}} \left(\int_M \overline{u}^N dv_g\right)^{\frac{2}{N}} \\ &\leq \mu_1(M,g). \end{split}$$

So the equality is attained in the Hölder inequality and this is possible only if

$$\overline{u} = cu$$

with c > 0 is a constant which implies that

$$c = 1$$

and

$$u = \overline{u} = a |v|.$$

Also

$$a^{N-2} \int_M v^N dv_g = \int_M u^{N-2} v^2 dv_g = 1$$

and

$$\frac{a^N \int_M v^N dv_g}{a^2} = 1$$

Finally since a > 0, we get

a = 1

hence

$$u = |v|$$
.

That means that v is a weak solution in $H_2^2(M)$ to the equation

$$P_g(v) = \mu_1(M,g) |v|^{N-2} v$$

The condition $\int_M v^N dv_g = 1$ implies that v is non trivial. Consequently

$$\mu_1(M,g) = \frac{\int_M P_g(v) dv_g}{\int_M |v|^N dv_g} \ge \mu(M,g).$$

4. Existence of a minimizer to $\mu_2(M,g)$

Proposition 9. If $\mu_2(M,g)K_2^2 2^{-\frac{4}{n}} < 1$, then $\mu_2(M,g)$ is attained by a generalized metric $u^{N-2}g$, $u \in L^N_+(M)$.

PROOF. Let $g_m = u_m^{N-2}g$ with $u_m \in C^{\infty}(M)$ and $u_m > 0$ be a minimizing sequence of $\mu_2(M, g)$. Since we can assume that

$$\int_{M} u_m^N dv_g = 1$$

we have

$$\lim_{m} \lambda_{2,m} = \mu_2(M,g)$$

By Proposition3, there are $v_m, w_m \in H^2_2(M)$ such that

(4.1)
$$P(v_m) = \lambda_{1,m} u_m^{N-2} v_m$$

and

$$(4.2) P(w_m) = \lambda_{2,m} u_m^{N-2} w_m$$

with the normalized conditions

(4.3)
$$\int_{M} u_m^{N-2} v_m^2 dv_g = \int_{M} u_m^{N-2} v_m^2 dv_g = 1, \quad \int_{M} u_m^{N-2} v_m w_m dv_g = 0.$$

First, we have for any integer $m \ge 1$,

$$\lambda_{1,m} < \lambda_{2,m}.$$

Since, if $\lambda_{1,m} = \lambda_{2,m}$; w_m is a minimizer of $\lambda_{1,m}$. On other hand taking account of the coerciveness of the Paneitz operator P and applying the Lax-Milgram theorem, we get easily that the first eigenvalue $\lambda_{1,m}$ of P_g is simple, so $w_m = \alpha v_m$ with a real $\alpha \neq 0$. Thus by (4.3), we get that

$$\int_M u_m^{N-2} v_m^2 dv_g = 0$$

a contradiction. The sequences $(v_m)_m$ and $(w_m)_m$ are bounded in $H_2^2(M)$ so there are functions $v, w \in H_2^2(M)$ and subsequences still denoted by $(v_m)_m$ and $(w_m)_m$ converging weakly to v and w, respectively, in $H_2^2(M)$. This latter facts and the weak convergence of (u_m) to u in $L^N(M)$ allow us to write in the weak sense that

$$(4.4) P_g(v) = \nu u^{N-2} v$$

and w

(4.5)
$$P_g(w) = \mu_2(M, g)u^{N-2}w$$

where $\nu = \lim_{m \to \infty} \lambda_{1,m} \leq \mu_2(M,g)$.

Now, we are going to show that v, w fulfill respectively the conditions $\int_M u^{N-2}v^2 dv_g = \int_M u^{N-2}w^2 dv_g = 1$ and by the way v, w are not identically null. To do so, we borrow ideas and notations from([1]). Set

$$S_m = \{\lambda_m v_m + \mu_m w_m : \ (\lambda_m, \mu_m) \in R^2, \ \lambda_m^2 + \mu_m^2 = 1, \ \lambda_m \mu_m > \alpha > 0 \}$$
$$S = \{\lambda v + \mu w : \ (\lambda, \mu) \in R^2, \ \lambda^2 + \mu^2 = 1 \}$$

and let $\overline{w}_m = \lambda_m v_m + \mu_m w_m$, $\overline{w} = \lambda v + \mu w$ where up to a subsequence $(\lambda_m, \mu_m) \to (\lambda, \mu)$.

Obviously, we have

(4.6)
$$\int_{M} u^{N-2} \overline{w}^2 dv_g \leq \liminf_{m} \inf_{m} \int_{M} u_m^{N-2} \overline{w}_m^2 dv_g = 1.$$

For the inequality in the other sense, we have

$$\begin{split} \int_M u^{N-2}\overline{w}^2 dv_g &= \int_M u_m^{N-2}\overline{w}_m^2 dv_g - \int_M (u_m^{N-2}\overline{w}_m^2 - u^{N-2}\overline{w}^2) dv_g \\ &= 1 - \int_M (u_m^{N-2}v_m^2 - u^{N-2}v^2) dv_g. \end{split}$$

Now, since

(4.7)
$$\left|u_m^{N-2}\overline{w}_m^2 - u_m^{N-2}(\overline{w}_m - \overline{w})^2\right| \le C u_m^{N-2} \left|\overline{w}_m + \overline{w}\right| \left|\overline{w}\right|$$

where C > 0 is some constant

we get

$$\left|u_m^{N-2}\overline{w}_m^2 - u_m^{N-2}(\overline{w}_m - \overline{w})^2\right| \to u^{N-2}\overline{w}^2 \text{ in } L^1(M)$$

and

(4.8)
$$\int_M \left(u_m^{N-2} \overline{w}_m^2 - u^{N-2} \overline{w}^2 \right) dv_g \to \int_M u_m^{N-2} (\overline{w}_m - \overline{w})^2 dv_g.$$

Thus

$$\int_{M} u^{N-2} \overline{w}^2 dv_g = 1 - \int_{M} u_m^{N-2} (\overline{w}_m - \overline{w})^2 dv_g + o(1)$$

where o(1) is a sequence which goes to 0 as $m \to +\infty$.

Using the Sobolev inequality given by proposition(6), and taking account of

 $\|u_m\|_N = 1$

we get

$$\int_{M} u_m^{N-2} (\overline{w}_m - \overline{w})^2 dv_g \le 2^{-\frac{4}{n}} \left(K_2^2 + \epsilon \right) \left\| \Delta (\overline{w}_m - \overline{w}) \right\|_2^2 + A(\epsilon) \left\| \overline{w}_m - \overline{w} \right\|_2^2.$$

Now by the Brezis-Lieb lemma([3]) and the fact that $\|\overline{w}_m - \overline{w}\|_2 \to 0$ as $m \to +\infty$, we obtain

$$(4.9) \quad \int_{M} u_m^{N-2} (\overline{w}_m - \overline{w})^2 dv_g \le 2^{-\frac{4}{n}} \left(K_2^2 + \epsilon \right) \left(\left\| \Delta \overline{w}_m \right\|_2^2 - \left\| \Delta \overline{w} \right\|_2^2 \right) + o(1).$$

By the fact that

$$\overline{w}_m - \overline{w} \to 0$$
 in $H^2_q(M), q = 0, 1$ as $m \to +\infty$

we have

$$\begin{split} \|\Delta \overline{w}_{m}\|_{2}^{2} - \|\Delta \overline{w}\|_{2}^{2} &= \int_{M} \left(\overline{w}_{m} P_{g}(\overline{w}_{m}) - \overline{w} P(\overline{w})\right) dv_{g} + o(1). \\ &= \lambda_{1,m} \lambda_{m}^{2} \int_{M} u_{m}^{N-2} v_{m}^{2} + \lambda_{2,m} \mu_{m}^{2} \int_{M} u_{m}^{N-2} w_{m}^{2} - \nu(M,g) \lambda^{2} \int_{M} u^{N-2} v^{2} - \mu_{2} \mu^{2} \int_{M} u^{N-2} w^{2} + o(1). \end{split}$$

Taking into account of (4.6), we get

$$\begin{split} \|\Delta \overline{w}_{m}\|_{2}^{2} - \|\Delta \overline{w}\|_{2}^{2} &\leq \lambda_{2,m} \left(\int_{M} u_{m}^{N-2} \overline{w}_{m}^{2} - \int_{M} u^{N-2} \overline{w}^{2} dv_{g} \right) \\ &+ \left(\lambda_{1,m} \lambda_{m}^{2} - \nu(M,g) \lambda^{2} \right) \int_{M} u^{N-2} v^{2} + \left(\lambda_{2,m} \mu_{m}^{2} - \mu_{2} \mu^{2} \right) \int_{M} u^{N-2} w^{2} + o(1) \\ &\leq \mu_{2}(M,g) \int_{M} (u_{m}^{N-2} \overline{w}_{m}^{2} - u^{N-2} \overline{w}^{2}) dv_{g} + o(1). \end{split}$$

Consequently

$$\int_{M} u^{N-2} \overline{w}^2 dv_g \ge 1 - 2^{-\frac{4}{n}} \left(K_2^2 + \epsilon \right) \mu_2(M,g) \int_{M} (u_m^{N-2} \overline{w}_m^2 - u^{N-2} \overline{w}^2) dv_g + o(1)$$
 and since

$$\int_{M} u_m^{N-2} \overline{w}_m^2 dv_g = 1$$

we obtain

$$\left(1 - 2^{-\frac{4}{n}} \left(K_2^2 + \epsilon\right) \mu_2(M, g)\right) \int_M u^{N-2} \overline{w}^2 dv_g \ge 1 - 2^{-\frac{4}{n}} \left(K_2^2 + \epsilon\right) \mu_2(M, g) + o(1).$$

So if

$$K_2^2 \mu_2(M,g) 2^{-\frac{4}{n}} < 1$$

we choose $\epsilon > 0$ sufficiently small and get

$$\int_{M} u^{N-2} \overline{w}^2 dv_g \ge 1.$$

The inequality (4.9), the Lieb-Brezis lemma ([3]) and the strong convergence of the sequence $(\overline{w}_m)_m$ to \overline{w} in $H^2_q(M)$, q = 0, 1, we get

$$\int_{M} u_m^{N-2} (\overline{w}_m - \overline{w})^2 dv_g \leq 2^{-\frac{4}{n}} \left(K_2^2 + \epsilon \right) \left(\left\| \Delta \overline{w}_m \right\|_2^2 - \left\| \Delta \overline{w} \right\|_2^2 \right) + o(1)$$

$$\leq 2^{-\frac{4}{n}} \left(K_2^2 + \epsilon \right) \left(\lambda_{2,m} \int_{M} u_m^{N-2} \overline{w}_m^2 dv_g - \mu_2(M,g) \int_{M} u^{N-2} \overline{w}^2 dv_g \right)$$

$$+ o(1)$$

$$\leq 2^{-\frac{4}{n}} \left(K_2^2 + \epsilon \right) \left[\left(\lambda_{2,m} - \mu_2(M,g) \right) \int_M u_m^{N-2} \overline{w}_m^2 dv_g \right. \\ \left. + \left. \mu_2(M,g) \int_M \left(u_m^{N-2} \overline{w}_m^2 - u^{N-2} \overline{w}^2 \right) dv_g \right] + o(1).$$

Since $\lambda_{2,m} \to \mu_2(M,g)$ as $m \to +\infty$ and

(4.10)
$$\int_{M} \left(u_m^{N-2} \overline{w}_m^2 - u^{N-2} \overline{w}^2 \right) dv_g = \int_{M} u_m^{N-2} \left(\overline{w}_m - \overline{w} \right)^2 dv_g + o(1)$$

we obtain

$$\left(1 - 2^{-\frac{4}{n}} \left(K_2^2 + \epsilon\right) \mu_2(M, g)\right) \int_M u_m^{N-2} \left(\overline{w}_m - \overline{w}\right)^2 dv_g \le o(1).$$

So if

$$K_2^2 \mu_2(M,g) 2^{-\frac{4}{n}} < 1$$

we get

$$\lim_{m \to +\infty} \int_M u_m^{N-2} \left(\overline{w}_m - \overline{w}\right)^2 dv_g = 0.$$

Hence by the equality (4.10), we get

$$\int_{M} u_m^{N-2} \overline{w}_m^2 dv_g \to \int_{M} u^{N-2} \overline{w}^2 dv_g.$$

 So

$$\int_{M} u^{N-2} \overline{w}^2 dv_g = 0$$

and since

$$\int_M u^{N-2} v^2 dv_g = \int_M u^{N-2} w^2 dv_g = 1$$

and

$$\lambda^2 + \mu^2 = 1, \lambda \mu \neq 0$$

it follows that

$$\int_{M} u^{N-2} v w dv_g = 0.$$

Thus the functions $u^{\frac{N-2}{2}}v$, $u^{\frac{N-2}{2}}w$ are linearly independent.

5. An estimation to $\mu_2(M,g)$

Mimicking which is done in [1], we establish the following lemma.

Lemma 5. If the manifold (M,g) is of dimensional $n \ge 12$, then $\mu_2(M,g) < \left[\mu_1(M,g)^{\frac{n}{4}} + \left(K_2^{-2}\right)^{\frac{n}{4}}\right]^{\frac{4}{n}}$.

To prove this lemma, we need the following elementary inequality.

Lemma 6. [1] For any real numbers x > 0, y > 0 and p > 2, there is a constant C > 0 such that

$$(x+y)^p \le x^p + y^p + C(x^{p-1}y + xy^{p-1}).$$

PROOF. (of Lemma[?]) Let $x_o \in M$, $\delta > 0$ sufficiently small and $B_{x_o}(\delta)$ the ball of center x_o and of radius δ and $\eta \in C^{\infty}$ -function

$$\eta(x) = \begin{cases} 1 & \text{if } x \in B_{x_o}(\delta) \\ 0 & \text{if } x \notin B_{x_o}(2\delta) \end{cases}$$

Put

$$\varphi_{\epsilon} = \eta (r^2 + \epsilon^2)^{-\frac{n-4}{2}}$$

where η is a bumping function, obviously $\varphi_{\epsilon} \in H_2^2(M)$. For any n > 6 and $\epsilon \to 0$, a calculation done in [4] leads to

(5.1)
$$Y(\varphi_{\epsilon}) \to K^{-2} - \epsilon^2 c_o + O(\epsilon^3)$$

where

$$Y(v) = \frac{\int_{M} v P_g(v) dv_g}{\left(\int_{M} |v|^N dv_g\right)^{\frac{2}{N}}}$$

 $c_o > 0$ and

$$K_2^{-2} = \frac{n(n+2)(n-2)(n-4)}{16}\omega_{n-1}^{\frac{4}{n}}$$

 ω_{n-1} denotes the volume of the unit Euclidean sphere. Consider the function

(5.2)
$$v_{\epsilon} = c_{\epsilon}\varphi_{\epsilon}$$

with $c_\epsilon>0$ is such that $\int_M v_\epsilon^N dv_g=1$. Standard computations give (5.3) $c_\epsilon=c_o\epsilon^{\frac{n-4}{2}}$

with $c_o > 0$.

Denote also by v a smooth positive solution of the equation

$$P_g(v) = \mu_1(M, g)v^{N-1}$$

with $\|v\|_N = 1$. . Put

$$u_{\epsilon} = Y(v_{\epsilon})^{\frac{1}{N-2}}v_{\epsilon} + \mu_1(M,g)^{\frac{1}{N-2}}v_{\epsilon}$$

For any $(\lambda, \mu) \in \mathbb{R}^2 - \{(0, 0)\}$, we have

$$\int_{M} (\lambda v_{\epsilon} + \mu v) P_{g} (\lambda v_{\epsilon} + \mu v) dv_{g})^{2} dv_{g}$$
$$= \int_{M} (\lambda^{2} v_{\epsilon} P(v_{\epsilon}) dv_{g} + \mu^{2} v P(v) + 2\lambda \mu v_{\epsilon} P(v)) dv_{g}.$$

Since $\int_{M} v P_g(v) dv_g = \mu_1(M, g)$ and $\int_{M} v_{\epsilon} P_g(v_{\epsilon}) dv_g = Y(v_{\epsilon})$, we get $\int_{M} (\lambda v_{\epsilon} + \mu v) P_g(\lambda v_{\epsilon} + \mu v) dv_g = \lambda^2 Y(v_{\epsilon}) + \mu^2 \mu_1(M, g)$ $+ 2\lambda \mu \mu_1(M, g) \int_{M} v_{\epsilon} v^{N-1} dv_g.$

and

$$\begin{split} \int_{M} u_{\epsilon}^{N-2} (\lambda v_{\epsilon} + \mu v)^2 dv_g &= \lambda^2 \int_{M} u_{\epsilon}^{N-2} v_{\epsilon}^2 dv_g + \mu^2 \int_{M} u_{\epsilon}^{N-2} v^2 dv_g \\ &+ 2\lambda \mu \int_{M} u_{\epsilon}^{N-2} v_{\epsilon} v dv_g \\ \geq \lambda^2 Y(v_{\epsilon}) \int_{M} v_{\epsilon}^N dv_g + \mu^2 \mu_1(M,g) \int_{M} v^N dv_g + 2\lambda \mu \int_{M} u_{\epsilon}^{N-2} v_{\epsilon} v dv_g \\ &= \lambda^2 Y(v_{\epsilon}) + \mu^2 \mu_1(M,g) + 2\lambda \mu \int_{M} u_{\epsilon}^{N-2} v_{\epsilon} v dv_g. \end{split}$$

We have also

$$\int_{M} u_{\epsilon}^{N-2} v_{\epsilon} v dv_{g} \ge \mu_{1}(M,g) \int_{M} v_{\epsilon} v^{N-1} dv_{g}$$

so, if $\lambda \mu \geq 0$

$$\frac{\int_M (\lambda v_{\epsilon} + \mu v) P_g(\lambda v_{\epsilon} + \mu v) dv_g}{\int_M u_{\epsilon}^{N-2} (\lambda v + \mu v_{\epsilon})^2 dv_g} \le 1.$$

In the case $\lambda \mu < 0$ and $N - 2 \in (0, 1]$ i.e. $n \ge 12$, we have

$$\begin{split} u_{\epsilon}^{N-2} &= \left(Y(v_{\epsilon})^{\frac{1}{N-2}}v_{\epsilon} + \mu_1(M,g)^{\frac{1}{N-2}}v\right)^{N-2} \\ &\leq Y(v_{\epsilon})v_{\epsilon}^{N-2} + \mu_1(M,g)v^{N-2}. \end{split}$$

Consequently

$$\begin{split} \int_{M} u_{\epsilon}^{N-2} (\lambda v_{\epsilon} + \mu v)^2 dv_g &= \lambda^2 \int_{M} u_{\epsilon}^{N-2} v_{\epsilon}^2 dv_g + \mu^2 \int_{M} u_{\epsilon}^{N-2} v^2 dv_g \\ &+ 2\lambda \mu \int_{M} u_{\epsilon}^{N-2} v_{\epsilon} v dv_g \geq \lambda^2 Y(v_{\epsilon}) + \mu^2 \mu_1(M,g) \\ &+ 2\lambda \mu Y(v_{\epsilon}) \int_{M} v_{\epsilon}^{N-1} v dv_g + 2\lambda \mu \mu_1(M,g) \int_{M} v_{\epsilon} v^{N-1} dv_g \\ &\geq \lambda^2 Y(v_{\epsilon}) + \mu^2 \mu_1(M,g) - C \left(\int_{M} v_{\epsilon}^{N-1} v dv_g + \int_{M} v_{\epsilon} v^{N-1} dv_g \right) \\ \text{e } C \geq 0 \text{ is a constant independent of } \epsilon \end{split}$$

where C > 0 is a constant independent of ϵ .

Now taking account of (5.2) and (5.3) we get, for any $(\lambda, \mu) \in \mathbb{R}^2$ – $\{(0,0)\},\$

$$\frac{\int_{M} (\lambda v_{\epsilon} + \mu v) P_{g}(\lambda v_{\epsilon} + \mu v) dv_{g}}{\int_{M} u_{\epsilon}^{N-2} (\lambda v + \mu v_{\epsilon})^{2} dv_{g}} \leq 1 + O(\epsilon^{\frac{n-4}{2}}).$$

By Lemma6, we obtain

$$\begin{split} \int_{M} u_{\epsilon}^{N} dv_{g} &\leq Y(v_{\epsilon})^{\frac{n}{4}} \int_{M} v_{\epsilon}^{N} dv_{g} + \mu_{1}(M,g)^{\frac{n}{4}} \int_{M} v^{N} dv_{g} \geq \\ &+ C \left(\int_{M} v_{\epsilon}^{N-1} v dv_{g} + \int_{M} v v_{\epsilon}^{N-1} dv_{g} \right) \\ &= Y(v_{\epsilon})^{\frac{n}{4}} + \mu_{1}(M,g)^{\frac{n}{4}} + C \left(\int_{M} v_{\epsilon}^{N-1} v dv_{g} + \int_{M} v_{\epsilon} v^{N-1} dv_{g} \right). \end{split}$$

And by the relation (5.1), we deduce that

$$\left(\int_{M} u_{\epsilon}^{N} dv_{g}\right)^{\frac{4}{n}} \leq \left[\mu_{1}(M,g)^{\frac{n}{4}} + (K_{2}^{-2})^{\frac{n}{4}} - c.\epsilon^{2} + o(\epsilon^{3}) + o(\epsilon^{\frac{n-4}{2}})\right]^{\frac{4}{n}}$$
$$= \left[\mu_{1}(M,g)^{\frac{n}{4}} + (K_{2}^{-2})^{\frac{n}{4}}\right]^{\frac{4}{n}} - c.\epsilon^{2} + o(\epsilon^{3}) \qquad (\frac{n-4}{2} \geq 4).$$

where c > 0 is a constant.

Hence, for any $(\lambda, \mu) \in \mathbb{R}^2 - \{(0, 0)\},\$

$$\frac{\int_{M} (\lambda v_{\epsilon} + \mu v) P_{g}(\lambda v_{\epsilon} + \mu v) dv_{g}}{\int_{M} u_{\epsilon}^{N-2} (\lambda v + \mu v_{\epsilon})^{2} dv_{g}} \left(\int_{M} u_{\epsilon}^{N} dv_{g} \right)^{\frac{4}{n}} \leq \left[\mu_{1}(M,g)^{\frac{n}{4}} + (K_{2}^{-2})^{\frac{n}{4}} \right]^{\frac{4}{n}} - c.\epsilon^{2} + O(\epsilon^{3})$$

 \mathbf{SO}

$$\mu_2(M,g) < \left[\mu_1(M,g)^{\frac{n}{4}} + \left(K_2^{-2}\right)^{\frac{n}{4}}\right]^{\frac{4}{n}}.$$

6. Strong convergence

Lemma 7. Suppose that $\mu_2 K_2^2 2^{-\frac{4}{n}} < 1$. Then the sequence $(v_m)_m$ (resp. $(w_m)_m$) of solutions of the equations (4.1) (resp. of solutions of the equations (4.2)) has a bounded subsequence on M.

PROOF. Let as in the section $u_m \in L^N_+(M)$ and $v_m, w_m \in H^2_2(M)$ solutions respectively of the equations

$$P(v_m) = \lambda_{1,m} u_m^{N-2} v_m$$

and

(6.2)
$$P(w_m) = \lambda_{2,m} u_m^{N-2} w_m.$$

 Set

$$S_m = \{ \lambda_m v_m + \mu_m w_m : \ (\lambda_m, \mu_m) \in R^2, \ \lambda_m^2 + \mu_m^2 = 1, \ \lambda_m \mu_m > \alpha > 0 \}$$
$$S = \{ \lambda v + \mu w : \ (\lambda, \mu) \in R^2, \ \lambda^2 + \mu^2 = 1 \}$$

and let $\overline{w}_m = \lambda_m v_m + \mu_m w_m$, $\overline{w} = \lambda v + \mu w$ where up to a subsequence $(\lambda_m, \mu_m) \to (\lambda, \mu)$.

First we are going to show that the sequence $(\overline{w}_m)_m$ is uniformally bounded on the manifold M. Suppose by contradiction that $(\overline{W}_m)_m$ is unbounded. Then, for every m there exists a point $x_m \in M$ such that

$$\overline{w}_m(x_m) = \max_{x \in M} \overline{w}_m = \xi_m \to +\infty \text{ as } m \to +\infty.$$

Given $\delta > 0$ less than the injectivity radius of (M, g), we let \widetilde{w}_m and \widetilde{u}_m be the functions defined on the Euclidean ball of center 0 and radius $\delta \xi_m$, $B_o(\delta \xi_m)$, by

$$\widetilde{w}_m(x) = \frac{1}{\xi_m} \overline{w}_m(\exp_{x_m}(\frac{x}{\xi_m}))$$

where \exp_m is the exponential map at x_m . Denote by

$$g_m(x) = \left(\exp_{x_m}\right)^* g(\frac{x}{\xi_m})$$

the Riemannian metric on the ball $B_o(\delta \xi_m)$. Clearly, if E is the Euclidean metric, $g_m \to E$ in C^2 on any compact set.

Now, since the functions v_m and w_m are solutions respectively of the equations (6.1) and (6.2) then multiplying by $\overline{w}_m \in H_2^2(M)$ and integrating over the geodesic ball $\exp_{x_m}(B_o(\delta \xi_m))$, we obtain

$$\int_{B_{o}(\delta\xi_{m})} \widetilde{w}_{m} \Delta_{g_{m}}^{2} \widetilde{w}_{m} dv_{g_{m}} + \frac{\alpha}{\xi_{m}^{2}} \int_{B_{o}(\delta\xi_{m})} \widetilde{w}_{m} \Delta \widetilde{w}_{m} dv_{g_{m}}$$
$$+ \frac{a}{\xi_{m}^{4}} \int_{B_{o}(\delta\xi_{m})} \widetilde{w}_{m}^{2} dv_{g_{m}} \leq \lambda_{2,m} \int_{B_{o}(\delta\xi_{m})} \widetilde{u}_{m}^{N-2} \widetilde{w}_{m}^{2} dv_{g_{m}}.$$

Thus the fact that $|\tilde{w}_m| \leq 1$ on $B_o(\delta \xi_m)$ and standard elliptic theory lead after passing to a subsequence to

$$\widetilde{w}_m \to \widetilde{w} \quad \text{in } C^4_{loc}(\mathbb{R}^n).$$

Independently, we have, for any R > 0

$$\begin{split} \int_{B_o(R\delta)} \widetilde{u}_m^{N-2} dx &= \int_{B_o(R\delta)} u_m^{N-2} (\exp_{x_m}(\frac{x}{R})) \widetilde{w}_m^2 dv_{g_m} + o(1) \\ &= \int_{B(x_m,R\delta)} u_m^{N-2} (\exp_{x_m}(\frac{x}{R})) dv_g + o(1) \\ &\leq \int_M u_m^{N-2}(x) dv_g + o(1). \end{split}$$

So $\widetilde{u}_m \to \widetilde{u}$ weakly in $L^N_{loc}(\mathbb{R}^n)$

$$\int_{B_o(R\delta)} u_m^N(\exp_{x_m}(\frac{x}{R})) dv_{g_m} = \int_{B(x_m,R\delta)} u_m^N(x) dv_g \le \int_M u_m^N(x) dv_g$$

Now letting $m \to \infty$, we get

$$\int_{\mathbb{R}^n} (\Delta_E \widetilde{w})^2 \, dx \le \mu_2 \int_{\mathbb{R}^n} u^{N-2} \widetilde{w}^2 \, dx$$

and by the Sobolev inequality given by Corollary1, we obtain that

$$\int_{\mathbb{R}^n} \left(\Delta_E \widetilde{w}\right)^2 dx \le \mu_2 2^{-\frac{4}{n}} K_2^2 \int_{\mathbb{R}^n} \left(\Delta_E \widetilde{w}\right)^2 dx \left(\int_{\mathbb{R}^n} u^N dx\right)^{\frac{2}{N}} \\ \le \mu_2 2^{-\frac{4}{n}} K_2^2 \int_{\mathbb{R}^n} \left(\Delta_E \widetilde{w}\right)^2 dx.$$

Consequently

$$\mu_2 K_2^2 2^{-\frac{4}{n}} \ge 1$$

which contradicts the inequality of the hypothesis, so the sequence (\overline{w}_m) is bounded on M.

Corollary 2. The sequence $(v_m)_m$ (resp. $(w_m)_m$) given in (4.1) (resp. in (4.2)) converges strongly in $L^N(M)$.

PROOF. Let $\epsilon > 0$, the Hölder inequality leads to

$$\int_{M} |v_n - v|^N \, dv_g \le \left(\int_{M} |v_n - v|^{N-\epsilon} \, dv_g \right)^{\frac{1}{2}} \left(\int_{M} |v_n + v|^{N+\epsilon} \, dv_g \right)^{\frac{1}{2}}.$$

By the boundedness of the sequence $(v_n)_n$ in M and the strong convergence of the latter to v in $L^{N-\epsilon}(M)$, we get that $v_n \to v$ in $L^N(M)$. The same is also true for the sequence $(w_n)_n$.

Corollary 3. The functions $u^{\frac{N-2}{2}}v$ and $u^{\frac{N-2}{2}}w$ are linearly independent.

Indeed, since the sequence $(v_m)_m$ (resp. $(v_m)_m$) converges strongly to v (resp.to w) in $L^N(M)$, we pass to the limit in the last equality in (4.3) and get $\int_M u^{N-2}vwdv_g = 0$

As a corollary of Lemma 7 and Corollary 2, we obtain our main result

Theorem 3. If the Einsteinian manifold (M,g) is of dimension $n \ge 12$, then $\mu_2(M,g)$ is attained by a generalized metric.

7. Nodals solutions

The same arguments as in the proof of Lemma 3.3 [1] allow us to state.

Lemma 8. Let $u \in L^N_+(M)$ with $||u||_N = 1$. Suppose that $w_1, w_2 \in H^2_2(M) - \{0\}$, such that $w_1 \ge 0, w_2 \ge 0$ satisfy

(7.1)
$$\int_{M} w_1 P(w_1) dv_g \le \mu_2(M,g) \int_{M} u^{N-2} w_1^2 dv_g$$

(7.2)
$$\int_{M} w_2 P(w_2) dv_g \le \mu_2(M,g) \int_{M} u^{N-2} w_2^2 dv_g$$

If $(M - w_1^{-1}(0)) \cap (M - w_2^{-1}(0))$ has measure 0, then there exist constants a > 0 and b > 0 such that $u = aw_1 + bw_2$ and the equalities in (7.1) and (7.2) hold.

Now we establish the existence of a nodal solution.

Theorem 4. Let v and w as in the Proposition(3) and suppose that the scalar curvature of (M;g) is positive and $\mu_2(M,g) \neq 0$ and attained by a general metric $\tilde{g} = u^{N-2}g$ with $u \in L^N_+(M)$. Then u = |w| and in particular the equation

(7.3)
$$P_g(w) = \mu_2(M,g) |w|^{N-2} w$$

has a nodal solution.

PROOF. Let v and w as in the Proposition(3). Without lost of generality, we choose $u \in L^N_+(M)$ with $\int_M u^N dv_g = 1$, hence $\lambda_2(\tilde{g}) = \mu_2(M,g)$. As in the proof of Proposition9 we have $\lambda_1(\tilde{g}) < \lambda_2(\tilde{g})$. Suppose that the solution of the equation(7.3) is not nodal, by taking -w if w is non positive, we assume that $w \ge 0$. On the other hand since the scalar curvature of (M,g)is positive, by Proposition(2) the equation

$$P_q(v) = \lambda_1(\widetilde{g})u^{N-2}v$$

has a positive solution and by Proposition 3 with the constraints

$$\int_{M} u^{N-2} v^2 dv_g = \int_{M} u^{N-2} w^2 dv_g = 1$$
$$\int_{M} u^{N-2} v w dv_g = 0.$$

and

This latter equality implies that the set
$$(M - v^{-1}(0)) \cap (M - w^{-1}(0))$$
 has
measure 0. So by Lemma8, we get equalities in (7.3), a contradiction with
the fact that $\lambda_1(\tilde{g}) < \lambda_2(\tilde{g})$. Consequently w is a nodal function.

Suppose that the compact manifold M splits into two non empty disjoint domains Ω_1 and Ω_2 such that $M = \Omega_1 \cup \Omega_2 \cup F$ with measure(F) = 0. Let v_1 and v_2 be positive solutions to the equation $P_g(v_i) = \lambda_2 u^{N-2} v_i$, such that $v_i = 0$ and $\Delta v_i = 0$ on $\partial \Omega_i$, where λ_2 is the second eigenvalue of the Paneitz-Branson operator P_g . By Lemma8, there exist constants a > 0 and b > 0 such that $u = av_1 + bv_2$. It follows that u is of class $C^{o,\alpha}(M)$ with $\alpha \in (0, N-2)$. Observe that the nodal set $u^{-1}(0) \subset v_1^{-1}(0) \cap v_2^{-1}(0) \subset F$.

Now we follows the proof in [1]. Let $h \in C^{\infty}(M)$ with support in $M - u^{-1}(0)$ and put $u_t = u + th$. Since u is continuous and positive on the support of h, then $u_t > 0$ for t close to 0. The same arguments as the proof in the Proposition 3.3 in [1] we obtain that |w| = u on $M - u^{-1}(0)$. Independently since the nodal set $u^{-1}(0)$ is negligible and u, |w| are continuous, then |w| = u on M.

Corollary 4. $\mu_2(M,g)$ is not attained by a classical conformal metric.

Since if it is not the case, u > 0 and w such that |w| = u is not nodal.

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