

ANOSOV ADS REPRESENTATIONS ARE QUASI-FUCHSIAN

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ABSTRACT. Let Γ be a cocompact lattice in $\mathrm{SO}(1, n)$. A representation $\rho : \Gamma \rightarrow \mathrm{SO}(2, n)$ is *quasi-Fuchsian* if it is faithful, discrete, and preserves an acausal subset in the boundary of anti-de Sitter space - a particular case is the case of *Fuchsian representations*, ie. composition of the inclusions $\Gamma \subset \mathrm{SO}(1, n)$ and $\mathrm{SO}(1, n) \subset \mathrm{SO}(2, n)$. We prove that if a representation is Anosov in the sense of Labourie (cf. [Lab06]) then it is also quasi-Fuchsian. We also show that Fuchsian representations are Anosov : the fact that all quasi-Fuchsian representations are Anosov will be proved in a second part by T. Barbot. The study involves the geometry of locally anti-de Sitter spaces: quasi-Fuchsian representations are holonomy representations of globally hyperbolic spacetimes diffeomorphic to $\mathbb{R} \times \Gamma \backslash \mathbb{H}^n$ locally modeled on AdS_{n+1} .

1. INTRODUCTION

Let $\mathrm{SO}_0(1, n)$, $\mathrm{SO}_0(2, n)$ denote the identity components of respectively $\mathrm{SO}(1, n)$, $\mathrm{SO}(2, n)$. Let Γ be a cocompact torsion free lattice in $\mathrm{SO}_0(1, n)$. For any Lie group G let $\mathrm{Rep}(\Gamma, G)$ denote the space of representations of Γ into G equipped with the compact-open topology.

In the case $G = \mathrm{SO}_0(1, n+1)$ we distinguish the *Fuchsian representations*: they are the representations obtained by composition of an embedding $\Gamma \subset \mathrm{SO}_0(1, n)$ (by Mostow rigidity, there is only one up to conjugacy if $n \geq 3$) and any faithful representation of $\mathrm{SO}_0(1, n)$ into $\mathrm{SO}_0(1, n+1)$. Their characteristic property is to be faithful, discrete, and to preserve a totally geodesic copy of \mathbb{H}^n into \mathbb{H}^{n+1} .

If we relax the last condition by only requiring the existence of a $\rho(\Gamma)$ -invariant topological $(n-1)$ -sphere in $\partial\mathbb{H}^{n+1}$ (in the Fuchsian case, the boundary of the $\rho_0(\Gamma)$ -invariant totally geodesic hypersurface $\mathbb{H}^n \subset \mathbb{H}^{n+1}$ provides such a topological sphere), we obtain the notion of *quasi-Fuchsian representation*. We denote by $\mathcal{QF}(\Gamma, \mathrm{SO}_0(1, n+1))$ the set of quasi-Fuchsian representations. It is well-known that $\mathcal{QF}(\Gamma, \mathrm{SO}_0(1, n+1))$ is a neighborhood of Fuchsian representations in the space of representations of Γ into $\mathrm{SO}_0(1, n+1)$. One way to prove this assertion, based on the Anosov character of the geodesic flow ϕ^t of the hyperbolic manifold $N = \Gamma \backslash \mathbb{T}^1 \mathbb{H}^n$ (for definitions, see § 5.1.1) goes as follows: ϕ^t is the projection of the geodesic flow $\tilde{\phi}^t$ on $\mathbb{T}^1 \mathbb{H}^n$. For every (x, v) in $\mathbb{T}^1 \mathbb{H}^n$, let $\ell^+(x, v)$, $\ell^-(x, v)$ be the extremities in $\partial\mathbb{H}^n \subset \partial\mathbb{H}^{n+1}$ of the unique geodesic tangent to (x, v) . These

maps define an ρ_0 -equivariant map $(\ell^+, \ell^-) : \mathbb{T}^1 \mathbb{H}^n \rightarrow \partial\mathbb{H}^{n+1} \times \partial\mathbb{H}^{n+1} \setminus \mathfrak{D}$ where \mathfrak{D} is the diagonal. To any x in \mathbb{H}^n attach a metric g^x on $\partial\mathbb{H}^{n+1}$ varying with x continuously and in a Γ -equivariant way - for example, take the angular metric at x , *ie.* the pull-back of the natural metric on $\mathbb{T}_x^1 \mathbb{H}^{n+1}$ by the map associating to a point p in $\partial\mathbb{H}^{n+1}$ the unit tangent vector at x of the geodesic ray starting from x and ending at p . This family of metrics satisfies the following property: *given p in $\partial\mathbb{H}^n$ and a tangent vector w to $\partial\mathbb{H}^{n+1}$ at p , the norm $g^{x^t}(w)$ increases exponentially with t when x^t describes a geodesic ray with final extremity p .* This property has the following consequence: consider the flat bundle $E_{\rho_0} = \Gamma \backslash (\mathbb{T}^1 \mathbb{H}^n \times (\partial\mathbb{H}^{n+1} \times \partial\mathbb{H}^{n+1} \setminus \mathfrak{D}))$ associated to the representation ρ_0 . Denote by $\pi_{\rho_0} : E_{\rho_0} \rightarrow \Gamma \backslash \mathbb{T}^1 \mathbb{H}^n$ the natural fibration. The map (ℓ^+, ℓ^-) defined above induces a section s_{ρ_0} of π_{ρ_0} . The flow $\Phi^t(x, v, \ell^+, \ell^-) = (\tilde{\phi}^t(x, v), \ell^+, \ell^-)$ induces a flow $\phi_{\rho_0}^t$ on E_{ρ_0} that preserves the image of s_{ρ_0} . Last but not least, the existence of the metrics g^x ensures that *as a $\phi_{\rho_0}^t$ -invariant closed subset of E_{ρ_0} , the image of s_{ρ_0} is a $\phi_{\rho_0}^t$ -hyperbolic set* (cf. § 5.1.1). When we deform ρ_0 , the flat bundle and the flow $\phi_{\rho_0}^t$ can be continuously deformed. The structural stability of hyperbolic invariant closed subsets ensures that for small deformations we still have a section s_ρ of the flat bundle E_ρ , the image of which is ϕ_ρ^t -hyperbolic. This section lifts to an equivariant map $\ell_\rho = (\ell_\rho^+, \ell_\rho^-) : \mathbb{T}^1 \mathbb{H}^n \rightarrow \partial\mathbb{H}^{n+1} \times \partial\mathbb{H}^{n+1} \setminus \mathfrak{D}$. It is quite straightforward to observe that ℓ_ρ^+ must be constant along the stable leaves of the geodesic flow, *ie.* the fibers of ℓ^+ . Therefore, it induces a continuous map $\ell^+ : \partial\mathbb{H}^n \rightarrow \partial\mathbb{H}^{n+1}$, the image of which is the a $\rho(\Gamma)$ -invariant topological $(n-1)$ -sphere in $\partial\mathbb{H}^{n+1}$.

This kind of argument has been extended in a more general framework by F. Labourie in [Lab06]: he defined, for any pair (G, Y) where G is a Lie group acting on a manifold Y , the notion of (G, Y) -Anosov representation (or simply Anosov representation when there is no ambiguity about the pair (G, Y)). For a definition, see 5.1.1. We denote by $\text{Anos}_Y(\Gamma, G)$ the space of (G, Y) -Anosov representations. By structural stability, $\text{Anos}_Y(\Gamma, G)$ is an open domain, and simple, general arguments ensure that Anosov representations are faithful, with discrete image formed by loxodromic elements. As a matter of fact, $\mathcal{QF}(\Gamma, \text{SO}_0(1, n+1))$ and $\text{Anos}_Y(\Gamma, \text{SO}_0(1, n+1))$ where $Y = \partial\mathbb{H}^{n+1} \times \partial\mathbb{H}^{n+1} \setminus \mathfrak{D}$ coincide: we sketched above a proof of one implication, but observe that the reverse implication, namely that quasi-Fuchsian representations are Anosov, is less obvious (it can be obtained by adapting the arguments given in the case $G = \text{SO}_0(2, n)$ in T. Barbot's sequel to this article [Bar07]).

Anosov representations have been studied in different situations, mostly in the case $n = 2$, *ie.* the case where Γ is a surface group:

- in [Lab06], F. Labourie proved that when G is the group $\text{SL}(n, \mathbb{R})$ and Y the frame variety, one connected component of $\text{Anos}_Y(\Gamma, G)$, the *quasi-Fuchsian component*, coincides with a connected component of $\text{Rep}(\Gamma, G)$: the Hitchin component. Moreover, he proved that these quasi-Fuchsian

representations are *hyperconvex*, *ie.* that they preserve some curve in the projective space $\mathbb{P}(\mathbb{R}^n)$ with some very strong convexity properties. In [Gui], O. Guichard then proved that conversely hyperconvex representations are quasi-Fuchsian. Beware: (G, Y) -Anosov representations are not necessarily quasi-Fuchsian; in other words, $\text{Anos}_Y(\Gamma, G)$ is not connected. See [Bar05c].

– In [BILW05], the authors also used the notion of Anosov representations for the study of representations of surface groups into the symplectic group of a real symplectic vector space with maximal Toledo invariant.

The present paper is devoted to the case where Γ is a cocompact lattice of $\text{SO}_0(1, n)$ that we deform in $G = \text{SO}_0(2, n)$. Whereas in the case of quasi-Fuchsian representations into $\text{SO}_0(1, n + 1)$ presented above the geometry of hyperbolic space \mathbb{H}^{n+1} played an important role, the study of $\text{Rep}(\Gamma, \text{SO}_0(2, n))$ deeply involves the geometry of the Lorentzian analog of \mathbb{H}^{n+1} , namely the anti-de Sitter space AdS_{n+1} . In Lorentzian geometry, appear some phenomena, latent in the Riemannian context, related to the causality notions. Whereas in hyperbolic space pair of points are only distinguished by their mutual distance, in the anti-de Sitter space we have to distinguish three types of pair of points, according to the nature of the geometry joining the two points: this geodesic may be spacelike, lightlike or timelike - in the last two cases, the points are said *causally related*.

The conformal boundary $\partial\mathbb{H}^{n+1}$ of the hyperbolic space plays an important role. Similarly, anti-de Sitter space admits a conformal boundary: the *Einstein universe* Ein_n . It is a conformal Lorentzian spacetime, also subject to a causality notion. In the following theorem, \mathcal{Y} is the subset of $\text{Ein}_n \times \text{Ein}_n$ made of non-causally related pairs, *ie.* pairs of points that can be joined by a spacelike geodesic in AdS_{n+1} :

Theorem 1.1. *Any $(\text{SO}_0(2, n), \mathcal{Y})$ -Anosov representation $\rho : \Gamma \rightarrow \text{SO}_0(2, n)$ is quasi-Fuchsian.*

The geometric ingredient of this Theorem is the fact that quasi-Fuchsian representations are precisely holonomy representations of Lorentzian manifolds locally modelled on AdS_{n+1} which are *spatially compact, globally hyperbolic* (in short, GHC) (§ 2.1). In this introduction, let's simply mention that, among many others, a characterization of these spacetimes is the fact to admit a proper time function (a time function being a function with everywhere timelike gradient). It is only recently that the relevance of this notion in constant curvature spacetimes started to be perceived, a great impetus being given by the paper [Mes07] when it was circulating in the physical and the mathematical community as well in the 90's (see also [ABB⁺07]). The classification of GHC spacetimes of constant curvature -1 is one of the main motivation of the present paper (the case of constant curvature $+1$ and 0 being already treated in respectively [Sca99], [Bar05b]), and of its sequel by T. Barbot ([Bar07]) where the converse of theorem 1.1 is proved.

2. PRELIMINARIES

2.1. Basic causality notions. We assume the reader acquainted to basic causality notions in Lorentzian manifolds like *causal* or *timelike* curves, *inextendible* causal curves, *time orientation*, *future* and *past* of subsets, *time function*, *achronal* subsets, etc... We refer to [BEE96] or [O’N83, § 14] for further details.

By *spacetime* we mean here an oriented and time oriented manifold. A spacetime is *strongly causal* if its topology admits a basis of *causally convex* neighborhoods, *ie.* neighborhoods U such that any causal curve with extremities in U is contained in U .

Recall that a spacetime (M, g) is *globally hyperbolic* (abbreviation GH) if it admits a *Cauchy hypersurface*, *ie.* an achronal subspace S which intersects every inextendible timelike curve at exactly one point - such a subspace is automatically a topological locally Lipschitz hypersurface (see [O’N83, § 14, Lemma 29]).

A globally hyperbolic spacetime is called *spatially compact* (in short GHC) if its Cauchy hypersurfaces are compact. An alternative and equivalent definition of GHC spacetimes is to require the existence of a *proper* time function.

2.2. Anti-de Sitter space. Let $\mathbb{R}^{2,n}$ be the vector space of dimension $n+2$, with coordinates (u, v, x_1, \dots, x_n) , endowed with the quadratic form:

$$q_{2,n} := -u^2 - v^2 + x_1^2 + \dots + x_n^2$$

We denote by $\langle x|y \rangle$ the associated scalar product. For any subset A of $\mathbb{R}^{2,n}$ we denote A^\perp the orthogonal of A , *ie.* the set of elements y in $\mathbb{R}^{2,n}$ such that $\langle y|x \rangle = 0$ for every x in A . We also denote by \mathcal{C}_n the isotropic cone $\{q_{2,n} = 0\}$.

Definition 2.1. The anti-de Sitter space AdS_{n+1} is $\{q_{2,n} = -1\}$ endowed with the Lorentzian metric obtained by restriction of $q_{2,n}$.

We will also consider the coordinates $(r, \theta, x_1, \dots, x_n)$ with:

$$u = r \cos(\theta), v = r \sin(\theta)$$

Observe the analogy with the definition of hyperbolic space \mathbb{H}^n - moreover, every subset $\{\theta = \theta_0\}$ is a totally geodesic copy of the hyperbolic space embedded in AdS_{n+1} . More generally, the totally geodesic subspaces of dimension k in AdS_{n+1} are connected components of the intersections of AdS_{n+1} with the linear subspaces of dimension $(k+1)$ in $\mathbb{R}^{2,n}$. In particular, geodesics are intersections with 2-planes.

Remark 2.2. We will also often need an auxiliary Euclidean metric on $\mathbb{R}^{2,n}$. Let’s fix once for all the euclidean norm $\|\cdot\|_0$ defined by:

$$\|(u, v, x_1, \dots, x_n)\|_0^2 := u^2 + v^2 + x_1^2 + \dots + x_n^2$$

2.3. Conformal model.

Proposition 2.3. *The anti-de Sitter space AdS_{n+1} is conformally equivalent to $(\mathbb{S}^1 \times \mathbb{D}^n, -d\theta^2 + ds^2)$, where $d\theta^2$ is the standard riemannian metric on $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, where ds^2 is the standard metric (of curvature +1) on the sphere \mathbb{S}^n and \mathbb{D}^n is the open upper hemisphere of \mathbb{S}^n .*

Proof. In the $(r, \theta, x_1, \dots, x_n)$ -coordinates the AdS metric is:

$$-r^2 d\theta^2 + ds_{hyp}^2$$

where ds_{hyp}^2 is the hyperbolic norm, *ie.* the induced metric on $\{\theta = \theta_0\} \approx \mathbb{H}^n$. More precisely, $\{\theta = \theta_0\}$ is a sheet of the hyperboloid $\{-r^2 + x_1^2 + \dots + x_n^2 = -1\}$. The map $(r, x_1, \dots, x_n) \rightarrow (1/r, x_1/r, \dots, x_n/r)$ sends this hyperboloid on \mathbb{D}^n , and an easy computation shows that the pull-back by this map of the standard metric on the hemisphere is $r^{-2} ds_{hyp}^2$. The proposition follows. \square

Proposition 2.3 shows in particular that $\widetilde{\text{AdS}}_{n+1}$ contains many closed causal curves. But the universal covering $\widetilde{\text{AdS}}_{n+1}$, conformally equivalent to $(\mathbb{R} \times \mathbb{D}^n, -d\theta^2 + ds^2)$, contains no periodic causal curve. It is strongly causal, but not globally hyperbolic.

2.4. Einstein universe. Einstein universe Ein_{n+1} is the product $\mathbb{S}^1 \times \mathbb{S}^n$ endowed with the metric $-d\theta^2 + ds^2$ where ds^2 is as above the standard spherical metric. The universal Einstein universe $\widetilde{\text{Ein}}_{n+1}$ is the cyclic covering $\mathbb{R} \times \mathbb{S}^n$ equipped with the lifted metric still denoted $-d\theta^2 + ds^2$, but where θ now takes value in \mathbb{R} . According to this definition, Ein_{n+1} and $\widetilde{\text{Ein}}_{n+1}$ are Lorentzian manifolds, but it is more adequate to consider them as conformal Lorentzian manifolds. We fix a time orientation: the one for which the coordinate θ is a time function on $\widetilde{\text{Ein}}_{n+1}$.

In the sequel, we denote by $p : \widetilde{\text{Ein}}_{n+1} \rightarrow \text{Ein}_{n+1}$ the cyclic covering map. Let $\delta : \widetilde{\text{Ein}}_{n+1} \rightarrow \widetilde{\text{Ein}}_{n+1}$ be a generator of the Galois group of this cyclic covering. More precisely, we select δ so that for any \tilde{x} in $\widetilde{\text{Ein}}_{n+1}$ the image $\delta(\tilde{x})$ is in the future of \tilde{x} .

Even if Einstein universe is merely a conformal Lorentzian spacetime, one can define the notion of *photons*, *ie.* (non parameterized) lightlike geodesics. We can also consider the causality relation in Ein_{n+1} and $\widetilde{\text{Ein}}_{n+1}$. In particular, we define for every x in Ein_{n+1} the *lightcone* $C(x)$: it is the union of photons containing x . If we write x as a pair (θ, \mathbf{x}) in $\mathbb{S}^1 \times \mathbb{S}^n$, the lightcone $C(x)$ is the set of pairs (θ', \mathbf{y}) such that $|\theta' - \theta| = d(\mathbf{x}, \mathbf{y})$ where d is distance function for the spherical metric ds^2 .

There is only one point in \mathbb{S}^n at distance π of \mathbf{x} : the antipodal point \mathbf{x}^* . Above this point, there is only one point in Ein_{n+1} contained in $C(x)$: the antipodal point $x^* = (\theta + \pi, \mathbf{x}^*)$. The lightcone $C(x)$ with the points x, x^* removed is the union of two components:

- the *future cone*: it is the set $C^+(x) := \{(\theta', y)/\theta < \theta' < \theta + \pi, d(x, y) = \theta' - \theta\}$,
- the *past cone*: it is the set $C^-(x) := \{(\theta', y)/\theta - \pi < \theta' < \theta, d(x, y) = \theta - \theta'\}$.

Observe that the future cone of x is the past cone of x^* , and that the past cone of x is the future cone of x^* .

According to Proposition 2.3 AdS_{n+1} (respectively $\widetilde{\text{AdS}}_{n+1}$) conformally embeds in Ein_{n+1} (respectively $\widetilde{\text{Ein}}_{n+1}$). Hence the time orientation on Ein_{n+1} selected above induces a time orientation on AdS_{n+1} and $\widetilde{\text{AdS}}_{n+1}$. Since the boundary $\partial\mathbb{D}^n$ is an equatorial sphere, the boundary $\partial\widetilde{\text{AdS}}_{n+1}$ is a copy of the Einstein universe $\widetilde{\text{Ein}}_n$. In other words, one can attach a “Penrose boundary” $\partial\widetilde{\text{AdS}}_{n+1}$ to $\widetilde{\text{AdS}}_{n+1}$ such that $\widetilde{\text{AdS}}_{n+1} \cup \partial\widetilde{\text{AdS}}_{n+1}$ is conformally equivalent to $(\mathbb{S}^1 \times \overline{\mathbb{D}}^n, -d\theta^2 + ds^2)$, where $\overline{\mathbb{D}}^n$ is the closed upper hemisphere of \mathbb{S}^n .

The restrictions of p and δ to $\widetilde{\text{AdS}}_{n+1} \subset \widetilde{\text{Ein}}_{n+1}$ are respectively a covering map over AdS_{n+1} and a generator of the Galois group of the covering; we will still denote them by p and δ .

2.5. Isometry groups. Every element of $\text{SO}(2, n)$ induces an isometry of AdS_{n+1} , and, for $n \geq 2$, every isometry of AdS_{n+1} comes from an element of $\text{SO}(2, n)$. Similarly, conformal isometries of Ein_{n+1} are projections of elements of $\text{SO}(2, n+1)$ acting on \mathcal{C}_{n+1} (still for $n \geq 2$).

In the sequel, we will only consider isometries preserving the orientation and the time orientation, *ie.* elements of the neutral component $\text{SO}_0(2, n)$ (or $\text{SO}_0(2, n+1)$).

2.6. Achronal subsets. Recall that a subset of a conformal Lorentzian manifold is achronal (respectively acausal) if there is no timelike (respectively causal) curve joining two distinct points of the subset. In $\text{Ein}_n \approx (\mathbb{R} \times \mathbb{S}^{n-1}, -d\theta^2 + ds^2)$, every achronal subset is precisely the graph of a 1-Lipschitz function $f : \Lambda_0 \rightarrow \mathbb{R}$ where Λ_0 is a subset of \mathbb{S}^{n-1} endowed with its canonical metric d . In particular, the achronal closed topological hypersurfaces in $\partial\widetilde{\text{AdS}}_{n+1}$ are exactly the graphs of the 1-Lipschitz functions $f : \mathbb{S}^{n-1} \rightarrow \mathbb{R}$: they are topological $(n-1)$ -spheres.

Similarly, achronal subsets of $\widetilde{\text{AdS}}_{n+1}$ are graphs of 1-Lipschitz functions $f : \Lambda_0 \rightarrow \mathbb{R}$ where Λ_0 is a subset of \mathbb{D}^n , and achronal topological hypersurfaces are graphs of 1-Lipschitz maps $f : \mathbb{D}^n \rightarrow \mathbb{R}$.

Stricto-sensu, there is no achronal subset in Ein_{n+1} since closed timelike curves through a given point cover the entire Ein_{n+1} . Nevertheless, we can keep track of this notion in Ein_{n+1} by defining “achronal” subsets of Ein_{n+1} as projections of genuine achronal subsets of $\widetilde{\text{Ein}}_{n+1}$. This definition is justified by the following results:

Lemma 2.4. *The restriction of p to any achronal subset of $\widetilde{\text{Ein}}_{n+1}$ is injective.*

Proof. Since the diameter of \mathbb{S}^n is π , the difference between the t -coordinates of two elements of an achronal subset of $\widetilde{\text{Ein}}_{n+1}$ is at most π . The lemma follows immediately. \square

Corollary 2.5. *Let $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2$ be two achronal subsets of $\widetilde{\text{Ein}}_{n+1}$ admitting the same projection in Ein_{n+1} . Then there is an integer k such that:*

$$\widetilde{\Lambda}_1 = \delta^k \widetilde{\Lambda}_2$$

\square

2.7. The Klein model ADS_{n+1} of the anti-de Sitter space. We now consider the quotient $\mathbb{S}(\mathbb{R}^{2,n})$ of $\mathbb{R}^{2,n} \setminus \{0\}$ by positive homotheties. In other words, $\mathbb{S}(\mathbb{R}^{2,n})$ is the double covering of the projective space $\mathbb{P}(\mathbb{R}^{2,n})$. We denote by \mathbb{S} the projection of $\mathbb{R}^{2,n} \setminus \{0\}$ on $\mathbb{S}(\mathbb{R}^{2,n})$. The projection \mathbb{S} is one-to-one in restriction to $\text{AdS}_{n+1} = \{q_{2,n} = -1\}$. The *Klein model* ADS_{n+1} of the anti-de Sitter space is the projection of AdS_{n+1} in $\mathbb{S}(\mathbb{R}^{2,n})$, endowed with the induced Lorentzian metric.

ADS_{n+1} is also the projection of the open domain of $\mathbb{R}^{2,n}$ defined by the inequality $\{q_{2,n} < 0\}$. The topological boundary of ADS_{n+1} in $\mathbb{S}(\mathbb{R}^{2,n})$ is the projection of the isotropic cone $\mathcal{C}_n = \{q_{2,n} = 0\}$; we will denote this boundary by ∂ADS_{n+1} . By construction, the projection \mathbb{S} defines an isometry between AdS_{n+1} and ADS_{n+1} . The continuous extension of this isometry is a canonical homeomorphism between $\text{AdS}_{n+1} \cup \partial\text{AdS}_{n+1}$ and $\text{ADS}_{n+1} \cup \partial\text{ADS}_{n+1}$.

For every linear subspace F of dimension $k + 1$ in $\mathbb{R}^{2,n}$, we denote by $\mathbb{S}(F) := \mathbb{S}(F \setminus \{0\})$ the corresponding projective subspace of dimension k in $\mathbb{S}(\mathbb{R}^{2,n})$. The geodesics of ADS_{n+1} are the connected components of the intersections of ADS_{n+1} with the projective lines $\mathbb{S}(F)$ of $\mathbb{S}(\mathbb{R}^{2,n})$. More generally, the totally geodesic subspaces of dimension k in ADS_{n+1} are the connected components of the intersections of ADS_{n+1} with the projective subspaces $\mathbb{S}(F)$ of dimension k of $\mathbb{S}(\mathbb{R}^{2,n})$.

Remark 2.6. In the conformal model, the spacelike geodesics of AdS_{n+1} ending at some point x of ∂AdS_{n+1} are all orthogonal to ∂AdS_{n+1} at x whereas in the Klein model spacelike geodesics ending at a given point in ∂ADS_{n+1} are not tangent one to the other. Hence the homeomorphism between $\text{AdS}_{n+1} \cup \partial\text{AdS}_{n+1}$ and $\text{ADS}_{n+1} \cup \partial\text{ADS}_{n+1}$ is not a diffeomorphism.

Definition 2.7. For every x in AdS_{n+1} , the *affine domain* $U(x)$ of ADS_{n+1} is the connected component of $\text{ADS}_{n+1} \setminus \mathbb{S}(x^\perp)$ containing x . Let $V(x)$ be the connected component of $\mathbb{S}(\mathbb{R}^{2,n}) \setminus \mathbb{S}(x^\perp)$ containing $U(x)$. The boundary $\partial U(x) \subset \partial\text{ADS}_{n+1}$ of $U(x)$ in $V(x)$ is called the *affine boundary* of $U(x)$.

Remark 2.8. Up to composition by an element of the isometry group $SO_0(2, n)$ of $q_{2,n}$, we can assume that $\mathbb{S}(x^\perp)$ is the projection of the hyperplane $\{u = 0\}$ in $\mathbb{R}^{2,n}$ and $V(x)$ is the projection of the region $\{u > 0\}$

in $\mathbb{R}^{2,n}$. The map

$$(u, v, x_1, x_2, \dots, x_{n+1}) \mapsto (t, \bar{x}_1, \dots, \bar{x}_n) := \left(\frac{v}{u}, \frac{x_1}{u}, \frac{x_2}{u}, \dots, \frac{x_n}{u} \right)$$

induces a diffeomorphism between $V(x)$ and \mathbb{R}^{n+1} mapping the affine domain $U(x)$ to the region $\{-t^2 + \bar{x}_1^2 + \dots + \bar{x}_n^2 < 1\}$. The affine boundary $\partial U(x)$ corresponds to the hyperboloid $\{-t^2 + \bar{x}_1^2 + \dots + \bar{x}_n^2 = 1\}$. The intersections between $U(x)$ with the totally geodesic subspaces of ADS_{n+1} correspond to the intersections of the region $\{-t^2 + \bar{x}_1^2 + \dots + \bar{x}_n^2 < 1\}$ with the affine subspaces of \mathbb{R}^{n+1} .

Although the real number $\langle x | y \rangle$ is well-defined only for $x, y \in \mathbb{R}^{2,n}$, its sign is well-defined for $x, y \in \mathbb{S}(\mathbb{R}^{2,n})$.

Lemma 2.9. *Let U be an affine domain in ADS_{n+1} and $\partial U \subset \partial \text{ADS}_{n+1}$ be its affine boundary. Let x be a point in ∂U , and y be a point in $U \cup \partial U$. There exists a causal (resp. timelike) curve joining x to y in $U \cup \partial U$ if and only if $\langle x | y \rangle \geq 0$ (resp. $\langle x | y \rangle > 0$).*

Proof. See e.g. [Bar05a, Proposition 5.10] or [BBZ07, Proposition 4.19]. \square

2.8. The Klein model of the Einstein universe. Similarly, Einstein universe has a Klein model: it is the projection $\mathbb{S}(\mathcal{C}_n)$ in $\mathbb{S}(\mathbb{R}^{2,n})$ of the isotropic cone \mathcal{C}_n in $\mathbb{R}^{2,n}$. The conformal Lorentzian structure can be defined in terms of the quadratic form $q_{2,n}$. In particular, an immediate corollary of Lemma 2.9 is:

Corollary 2.10. *For $\Lambda \subseteq \text{Ein}_n$, the following assertions are equivalent.*

- (1) Λ is achronal (respectively acausal)
- (2) when we see Λ as a subset of $\mathbb{S}(\mathcal{C}_n) \approx \text{Ein}_n$ the scalar product $\langle x | y \rangle$ is non-positive (respectively negative) for every distinct $x, y \in \Lambda$.

\square

In the sequel, we will frequently identify Ein_n with $\mathbb{S}(\mathcal{C}_n)$, since it is common to skip from one model to the other. For more details about the Einstein universe, see [Fra05, BCD⁺07].

Remark 2.11. The affine boundary $\partial U(x)$ defined in remark 2.8, as a domain of Ein_n , is conformally isometric to the *de Sitter space*. Hence we also call it de Sitter domain.

2.9. Unit tangent bundle. Denote by $\mathcal{E}^1 \text{AdS}_{n+1}$ (resp. $\mathcal{L}^1 \text{AdS}_{n+1}$) the tangent bundle of unit spacelike (respectively lightlike) tangent vectors. For such a vector v tangent to AdS_{n+1} at x , the geodesic issued from (x, v) has a future and past limit in the Einstein universe. We denote by $\ell^\pm : \mathcal{E}^1 \text{AdS}_{n+1} \cup \mathcal{L}^1 \text{AdS}_{n+1} \rightarrow \text{Ein}_n$ the applications which maps such a vector to its limits.

3. REGULAR AdS MANIFOLDS

3.1. AdS regular domains. Let $\tilde{\Lambda}$ be a closed achronal subset of $\partial\widetilde{\text{AdS}}_{n+1}$, and Λ be the projection of $\tilde{\Lambda}$ in ∂AdS_{n+1} . We denote by $\tilde{E}(\tilde{\Lambda})$ the *invisible domain* of $\tilde{\Lambda}$ in $\widetilde{\text{AdS}}_{n+1} \cup \partial\widetilde{\text{AdS}}_{n+1}$, that is,

$$\tilde{E}(\tilde{\Lambda}) =: \left(\widetilde{\text{AdS}}_{n+1} \cup \partial\widetilde{\text{AdS}}_{n+1} \right) \setminus \left(J^-(\tilde{\Lambda}) \cup J^+(\tilde{\Lambda}) \right)$$

where $J^-(\tilde{\Lambda})$ and $J^+(\tilde{\Lambda})$ are the causal past and the causal future of $\tilde{\Lambda}$ in $\widetilde{\text{AdS}}_{n+1} \cup \partial\widetilde{\text{AdS}}_{n+1} = (\mathbb{R} \times \overline{\mathbb{D}}^{n-1}, -d\theta^2 + ds^2)$. We denote by $\text{Cl}(\tilde{E}(\tilde{\Lambda}))$ the closure of $\tilde{E}(\tilde{\Lambda})$ in $\widetilde{\text{AdS}}_{n+1} \cup \partial\widetilde{\text{AdS}}_{n+1}$ and by $E(\Lambda)$ the projection of $\tilde{E}(\tilde{\Lambda})$ in $\text{AdS}_{n+1} \cup \partial\text{AdS}_{n+1}$ (according to Corollary 2.5, $E(\Lambda)$ only depends on Λ , not on $\tilde{\Lambda}$).

Definition 3.1. A *n-dimensional AdS regular domain* is a domain of the form $E(\Lambda)$ where Λ is the projection in ∂AdS_{n+1} of an achronal subset $\tilde{\Lambda} \subset \partial\widetilde{\text{AdS}}_{n+1}$ containing at least two points. If $\tilde{\Lambda}$ is a topological $(n-1)$ -sphere, then $E(\Lambda)$ is *GH-regular* (this definition is motivated by theorem 4.7).

Remark 3.2. For every closed achronal set $\tilde{\Lambda}$ in $\partial\widetilde{\text{AdS}}_{n+1}$, the invisible domain $\tilde{E}(\tilde{\Lambda})$ is *causally convex* in $\widetilde{\text{AdS}}_{n+1} \cup \partial\widetilde{\text{AdS}}_{n+1}$: this is an immediate consequence of the definitions. It follows that AdS regular domains are strongly causal.

Remark 3.3. Let $\tilde{\Lambda}$ be a closed achronal subset of $\partial\widetilde{\text{AdS}}_{n+1}$. Recall that $\tilde{\Lambda}$ is the graph of a 1-Lipschitz function $f : \Lambda_0 \rightarrow \mathbb{R}$ where Λ_0 is a closed subset of \mathbb{S}^{n-1} (§ 2.6). Define two functions $f^-, f^+ : \overline{\mathbb{D}}^n \rightarrow \mathbb{R}$ as follows:

$$\begin{aligned} f^-(x) &:= \text{Sup}_{y \in \Lambda_0} \{f(y) - d(x, y)\}, \\ f^+(x) &:= \text{Inf}_{y \in \Lambda_0} \{f(y) + d(x, y)\}, \end{aligned}$$

where d is the distance induced by ds^2 on $\overline{\mathbb{D}}^n$. It is easy to check that

$$\tilde{E}(\tilde{\Lambda}) = \{(\theta, x) \in \mathbb{R} \times \overline{\mathbb{D}}^n \mid f^-(x) < \theta < f^+(x)\}.$$

The following lemma is a refinement of lemma 2.4:

Lemma 3.4. *For every (non-empty) closed achronal set $\tilde{\Lambda} \subset \partial\widetilde{\text{AdS}}_{n+1}$, the projection of $\tilde{E}(\tilde{\Lambda})$ on $E(\Lambda)$ is one-to-one.*

Proof. We use the notations introduced in remark 3.3. For every $x \in \overline{\mathbb{D}}^n$, there exists a point $y \in \mathbb{S}^{n-1} = \partial\overline{\mathbb{D}}^n$ such that $d(x, y) \leq \pi/2$. Hence, for every $x \in \overline{\mathbb{D}}^n$, we have $f^+(x) - f^-(x) \leq \pi$. Hence $\tilde{E}(\tilde{\Lambda})$ lies in $E = \{(\theta, x) \in \mathbb{R} \times \overline{\mathbb{D}}^n \text{ such that } f^-(x) < \theta < f^+(x) + \pi\}$. The restriction to E of the projection of $\widetilde{\text{AdS}}_{n+1} \cup \partial\widetilde{\text{AdS}}_{n+1} = \mathbb{R} \times \overline{\mathbb{D}}^n$ on $\text{AdS}_{n+1} \cup \partial\text{AdS}_{n+1} = (\mathbb{R}/2\pi\mathbb{Z}) \times \overline{\mathbb{D}}^{n-1}$ is obviously one-to-one. \square

Definition 3.5. An achronal subset $\tilde{\Lambda}$ of $\widetilde{\text{Ein}}_{n+1}$ is *pure lightlike* if the associated subset Λ_0 of \mathbb{S}^n contains two antipodal points x_0 and x_0^* such that, for the associated 1-Lipschitz map $f : \Lambda_0 \rightarrow \mathbb{R}$ the equality $f(x_0) = f(x_0^*) + \pi$ holds.

If $\tilde{\Lambda}$ is pure lightlike, for every element x of $\overline{\mathbb{D}^n}$ we have $f^-(x) = f^+(x) = f(x_0^*) + d(x_0^*, x) = f(x_0) - d(x_0, x)$, implying that $\tilde{E}(\tilde{\Lambda})$ is empty. Conversely:

Lemma 3.6. $\tilde{E}(\tilde{\Lambda})$ is empty if and only if $\tilde{\Lambda}$ is pure lightlike. More precisely, if for some point x in \mathbb{D}^n the equality $f^+(x) = f^-(x)$ holds then $\tilde{\Lambda}$ is pure lightlike.

Proof. Assume $f^+(x) = f^-(x)$ for some x in \mathbb{D}^n . Then, since Λ_0 is compact, the upper and lower bounds are attained: there are y^\pm in Λ_0 such that:

$$\begin{aligned} f^-(x) &= f(y^-) - d(x, y^-) \\ f^+(x) &= f(y^+) + d(x, y^+) \end{aligned}$$

Hence:

$$d(y^-, y^+) \geq f(y^-) - f(y^+) = d(y^-, x) + d(x, y^+)$$

We are in the equality case of the triangular inequality. It follows that x belongs to a minimizing geodesic in \mathbb{S}^n joining y^- to y^+ . It is possible only if y^+ , y^- are antipodal one to the other, since if not the minimizing geodesic joining them is unique and contained in $\partial\mathbb{D}^n$. Moreover, $f(y^-) = f(y^+) + \pi$. The lemma follows. \square

Corollary 3.7. For every achronal topological $(n-1)$ -sphere $\tilde{\Lambda} \subset \partial\widetilde{\text{AdS}}_{n+1}$,

- (1) $\tilde{E}(\tilde{\Lambda})$ is disjoint from $\partial\widetilde{\text{AdS}}_{n+1}$ (ie. it is contained in $\widetilde{\text{AdS}}_{n+1}$);
- (2) $\text{Cl}\left(\tilde{E}(\tilde{\Lambda})\right) \cap \partial\widetilde{\text{AdS}}_{n+1} = \tilde{\Lambda}$, where $\text{Cl}\left(\tilde{E}(\tilde{\Lambda})\right)$ denotes the closure of $\tilde{E}(\tilde{\Lambda})$ in $\widetilde{\text{Ein}}_{n+1}$.

Proof. We use the notations introduced in remark 3.3. Since $\tilde{\Lambda}$ is a topological $(n-1)$ -sphere, the set Λ_0 is the whole sphere \mathbb{S}^{n-1} . For every $x \in \mathbb{S}^{n-1} = \Lambda_0$, one has $f^-(x) = f^+(x) = f(x)$. Finally, recall that $(\theta, x) \in \tilde{E}(\tilde{\Lambda})$ (resp. $(\theta, x) \in \text{Cl}(\tilde{E}(\tilde{\Lambda}))$) if and only if $f^-(x) < \theta < f^+(x)$ (resp. $f^-(x) \leq \theta \leq f^+(x)$). The corollary follows. \square

Remark 3.8. It follows from item (2) of Corollary 3.7 that the GH-regular domain $E(\Lambda)$ characterizes Λ , ie. invisible domains of different achronal $(n-1)$ -spheres are different. We call Λ the *limit set of $E(\Lambda)$* .

3.2. AdS regular domains as subsets of AdS_{n+1} . The canonical homeomorphism between $\text{AdS}_{n+1} \cup \partial\text{AdS}_{n+1}$ and $\widetilde{\text{AdS}}_{n+1} \cup \partial\widetilde{\text{AdS}}_{n+1}$ allows us to see AdS regular domains as subsets of $\widetilde{\text{AdS}}_{n+1}$.

Lemma 3.9. *Let $\Lambda \subset \partial \text{AdS}_{n+1}$ be the projection of a closed achronal subset of $\partial \widetilde{\text{AdS}}_{n+1}$ which is not pure lightlike. We see Λ and $E(\Lambda)$ in $\text{ADS}_{n+1} \cup \partial \text{ADS}_{n+1}$. Then Λ and $E(\Lambda)$ are contained in the union $U \cup \partial U$ of an affine domain and its affine boundary.*

Proof. See [Bar05a, Lemma 8.27]. □

Lemma 3.9 implies, in particular, that every AdS regular domain is contained in an affine domain of ADS_{n+1} . This allows to visualize AdS regular domains as subsets of \mathbb{R}^{n+1} (see remark 2.8).

Putting together the definition of the invisible domain $E(\Lambda)$ of a set $\Lambda \subset \partial \text{AdS}_{n+1}$ and Lemma 2.9, one gets:

Proposition 3.10. *Let $\Lambda \subset \partial \text{AdS}_{n+1}$ be the projection of a closed achronal subset of $\partial \widetilde{\text{AdS}}_{n+1}$ which is not pure lightlike. If we see Λ and $E(\Lambda)$ in the Klein model $\text{ADS}_{n+1} \cup \partial \text{ADS}_{n+1}$, then*

$$E(\Lambda) = \{y \in \text{ADS}_{n+1} \cup \partial \text{ADS}_{n+1} \text{ such that } \langle y | x \rangle < 0 \text{ for every } x \in \Lambda\}.$$

Remark 3.11. A nice (and important) corollary of this Proposition is that the invisible domain $E(\Lambda)$ associated with a set Λ is always geodesically convex: any geodesic joining two points in $E(\Lambda)$ is contained in $E(\Lambda)$.

4. GLOBALLY HYPERBOLIC ADS SPACETIMES

4.1. Cosmological time functions. In any spacetime (M, g) , one can define the *cosmological time function* as follows (see [AGH98]):

Definition 4.1. The cosmological time function of a spacetime (M, g) is the function $\tau : M \rightarrow [0, +\infty]$ defined by

$$\tau(x) := \text{Sup}\{L(c) \mid c \in \mathcal{R}^-(x)\},$$

where $\mathcal{R}^-(x)$ is the set of past-oriented causal curves starting at x , and $L(c)$ is the Lorentzian length of the causal curve c .

This function is in general very badly behaved. For example, in the case of Minkowski space, the cosmological time function is everywhere infinite.

Definition 4.2. A spacetime (M, g) is *CT-regular* with cosmological time function τ if

- (1) M has *finite existence time*, $\tau(x) < \infty$ for every x in M ,
- (2) for every past-oriented inextendible causal curve $c : [0, +\infty) \rightarrow M$,
 $\lim_{t \rightarrow \infty} \tau(c(t)) = 0$.

Theorem 4.3 ([AGH98]). *CT-regular spacetimes are globally hyperbolic.*

A very nice feature of CT-regularity is that it is preserved by isometries (and thus, by Galois automorphisms):

Proposition 4.4. *Let $(\widetilde{M}, \widetilde{g})$ be a CT-regular spacetime. Let Γ be a discrete group of isometries of $(\widetilde{M}, \widetilde{g})$ preserving the time orientation and without fixed points. Then, the action of Γ on $(\widetilde{M}, \widetilde{g})$ is properly discontinuous. Furthermore, the quotient spacetime (M, g) is CT-regular. More precisely, if $p : \widetilde{M} \rightarrow M$ denote the quotient map, the cosmological times $\tilde{\tau} : \widetilde{M} \rightarrow [0, +\infty)$ and $\tau : M \rightarrow [0, +\infty)$ satisfy:*

$$\tilde{\tau} = \tau \circ p$$

Sketch of proof. Γ clearly preserves the cosmological time and its level sets. These level sets are metric spaces on which Γ acts isometrically, and hence, properly discontinuously. It follows quite easily that Γ acts properly discontinuously on the entire \widetilde{M} .

The proof of the identity $\tilde{\tau} = \tau \circ p$ is straightforward: it follows from the Γ -invariance of $\tilde{\tau}$ and the fact that inextendible causal curves in M are precisely the projections by p of inextendible causal curves in \widetilde{M} . \square

4.2. GH-regular AdS spacetimes are CT-regular. Let Λ be a non-pure lightlike topological achronal $(n-1)$ -sphere in $\partial \text{AdS}_{n+1}$.

Proposition 4.5. *The AdS regular domain $E(\Lambda)$ is CT-regular.*

Proof. Recall that Λ is, by definition, the projection of an achronal topological sphere $\tilde{\Lambda} \subset \partial \widetilde{\text{AdS}}_{n+1}$, and that $E(\Lambda)$ is the projection of the invisible domain $\tilde{E}(\tilde{\Lambda})$ of $\tilde{\Lambda}$ in $\widetilde{\text{AdS}}_{n+1} \cup \partial \widetilde{\text{AdS}}_{n+1}$. We will prove that $\tilde{E}(\tilde{\Lambda})$ has regular cosmological time. Since the projection of $\tilde{E}(\tilde{\Lambda})$ on $E(\Lambda)$ is one-to-one (lemma 3.4), this will imply that $E(\Lambda)$ also has regular cosmological time. We denote by $\tilde{\tau}$ the cosmological time of $\tilde{E}(\tilde{\Lambda})$.

Let x be a point in $\tilde{E}(\tilde{\Lambda})$. On the one hand, according to corollary 3.7, $\text{Cl}(\tilde{E}(\tilde{\Lambda}))$ is a compact subset of $\widetilde{\text{AdS}}_{n+1} \cup \partial \widetilde{\text{AdS}}_{n+1}$, and the intersection $\text{Cl}(\tilde{E}(\tilde{\Lambda})) \cap \partial \widetilde{\text{AdS}}_{n+1}$ equals $\tilde{\Lambda}$. On the other hand, since x is in the invisible domain of $\tilde{\Lambda}$, the set $J^-(x)$ is disjoint from $\tilde{\Lambda}$. Therefore $J^-(x) \cap \text{Cl}(\tilde{E}(\tilde{\Lambda}))$ is a compact subset of $\widetilde{\text{AdS}}_{n+1}$. Therefore $J^-(x) \cap \text{Cl}(\tilde{E}(\tilde{\Lambda}))$ is conformally equivalent to a compact causally convex domain in $(\mathbb{R} \times \mathbb{D}^n, -d\theta^2 + ds^2)$, with a bounded conformal factor since everything is compact. It follows that the lengths of the past-directed causal curves starting at x contained in $\tilde{E}(\tilde{\Lambda})$ is bounded (in other words, $\tilde{\tau}(x)$ is finite), and that, for every past-oriented inextendible causal curve $c : [0, +\infty) \rightarrow \tilde{E}(\tilde{\Lambda})$ with $c(0) = x$, one has $\tilde{\tau}(c(t)) \rightarrow 0$ when $t \rightarrow \infty$. This proves that $\tilde{E}(\tilde{\Lambda})$ has regular cosmological time. \square

Hence, GH-regular domains and their quotients are globally hyperbolic (see Theorem 4.3, Proposition 4.4).

4.3. GHC AdS spacetimes are GH-regular.

Definition 4.6. A GH spacetime with constant curvature -1 is *maximal* (abbreviation MGH) if it admits no non-surjective embedding in another GH spacetime N with constant curvature such that each Cauchy hypersurface in M embeds in N as a Cauchy hypersurface.

Any GH spacetime with constant curvature -1 embeds in a MGH spacetime, and this maximal extension is unique up to isometry (see [CBG69]). Hence, the classification of GH spacetimes with constant curvature -1 essentially reduces to the classification of MGH ones.

Theorem 4.7. *Every $(n + 1)$ -dimensional MGHC spacetime with constant curvature -1 is isometric to the quotient of a GH-regular domain in AdS_{n+1} by a torsion-free discrete subgroup of $\text{SO}_0(2, n)$.*

This theorem was proved by Mess in his celebrated preprint [Mes07, ABB⁺07] (Mess only deals with the case where $n = 2$, but his arguments also apply in higher dimension). For the reader's convenience, we shall recall the main steps of the proof (see [Bar05a, Corollary 11.2] for more details).

Sketch of proof of Theorem 4.7. Let (M, g) be $(n + 1)$ -dimensional MGHC spacetime with constant curvature -1 . In other words, (M, g) is a locally modeled on AdS_{n+1} . The theory of (G, X) -structures provides us with a locally isometric developing map $\mathcal{D} : \widetilde{M} \rightarrow \widetilde{\text{AdS}}_{n+1}$ and a holonomy representation $\rho : \pi_1(M) \rightarrow \text{SO}_0(2, n)$. Pick a Cauchy hypersurface Σ in M , and a lift $\widetilde{\Sigma}$ of Σ in \widetilde{M} . Then $\widetilde{S} := \mathcal{D}(\widetilde{\Sigma})$ is an immersed complete spacelike hypersurface in $\widetilde{\text{AdS}}_{n+1}$. One can prove that such a hypersurface is automatically properly embedded and corresponds to the graph of a 1-Lipschitz function $f : \mathbb{D}^n \rightarrow \mathbb{R}$ in the conformal model $(\mathbb{R} \times \mathbb{D}^2, -d\theta^2 + ds^2)$. Such a function extends to a 1-Lipschitz function \bar{f} defined on the closed disc $\overline{\mathbb{D}^n}$. This shows that the boundary $\partial\widetilde{S}$ of \widetilde{S} in $\widetilde{\text{AdS}}_{n+1} \cup \partial\widetilde{\text{AdS}}_{n+1}$ is an achronal topological sphere $\widetilde{\Lambda}$ contained in $\partial\widetilde{\text{AdS}}_{n+1}$.

On the one hand, it is easy to see that the Cauchy development $D(\widetilde{S})$ coincides with the invisible domain $\widetilde{E}(\widetilde{\Lambda})$ (this essentially relies on the fact that $\widetilde{S} \cup \partial\widetilde{S}$ is the graph of the 1-Lipschitz function \bar{f} , hence an achronal set in $\widetilde{\text{AdS}}_{n+1} \cup \partial\widetilde{\text{AdS}}_{n+1}$).

On the other hand, since $\widetilde{\Sigma}$ is a Cauchy hypersurface in \widetilde{M} , the image $\mathcal{D}(\widetilde{M})$ is necessarily contained in $D(\widetilde{S}) = \widetilde{E}(\widetilde{\Lambda})$. Hence, the developing map \mathcal{D} induces an isometric embedding from M into $\Gamma \backslash D(\widetilde{S})$, where $\Gamma := \rho(\pi_1(M))$. Since M is maximal, this embedding must be onto, and thus, M is isometric to the quotient $\Gamma \backslash \widetilde{E}(\widetilde{\Lambda})$. \square

Definition 4.8. A representation $\rho : \Gamma \rightarrow \text{SO}_0(2, n)$ is GH-regular if it is the holonomy of a GH-regular spacetime, *ie.* if it is faithful, discrete and preserves a GH-regular domain. If the quotient spacetime $\rho(\Gamma) \backslash E(\Lambda)$ is spatially compact, we say that ρ is GHC-regular.

5. ANOSOV ANTI-DE SITTER MANIFOLDS ARE MGHC

5.1. Anosov representations.

5.1.1. *General definition.* Let N be a manifold equipped with a non singular flow Φ^t and an auxiliary Riemannian metric $\|\cdot\|$.

Definition 5.1. A closed subset $F \subseteq N$ is Φ^t -hyperbolic if it is Φ^t -invariant and if the tangent bundle of N admits a decomposition $TN = \Delta \oplus E^{ss} \oplus E^{uu}$ over F such that, for some positive constants a, b :

- The line bundle Δ is tangent to the flow,
- for any vector v in E^{ss} over a point p of N , and for any positive t :

$$\|d_p \Phi^t(v)\| \leq b e^{-at} \|v\|$$

- for any vector v in E^{uu} over a point p of N , and for any negative t :

$$\|d_p \Phi^t(v)\| \leq b e^{at} \|v\|$$

If F is the entire manifold N , the flow Φ^t is Anosov.

Typical examples are geodesic flows of negatively curved Riemannian manifolds. Let Γ be the fundamental group of N . Let Y be a manifold, and G be a Lie group acting smoothly on Y . Given any representation $\rho : \Gamma \rightarrow G$ one constructs the associated flat bundle E_ρ over N : it is the quotient of the product $\tilde{N} \times Y$ by the natural action of Γ , with the projection $\pi_\rho : E_\rho \rightarrow N$. The bundle E_ρ inherits a flow Φ_ρ^t from the lifting $\tilde{\Phi}^t$ of Φ^t on \tilde{N} .

Definition 5.2. A representation $\rho : \Gamma \rightarrow G$ is (G, Y) -Anosov if the flat bundle $\pi_\rho : E_\rho \rightarrow N$ admits a continuous section $s : N \rightarrow E_\rho$ such that the image of s is an invariant hyperbolic subset for Φ_ρ^t .

A very nice feature of the Anosov representations is the following proposition, which is consequence of the stability property of closed hyperbolic set (see [Lab06], proposition 2.1 for a proof):

Theorem 5.3. *Let N a compact manifold endowed with an Anosov flow Φ^t . The set of (G, Y) -Anosov representations from $\Gamma = \pi_1(N)$ to G is open in the space of representations of Γ in G , usually denoted $\text{Rep}(\Gamma, G)$ (endowed with the compact-open topology)*

5.1.2. *Anosov AdS representations.* Here, we are concerned with the case $(G, Y) = (\text{SO}_0(2, n), \mathcal{Y})$ where \mathcal{Y} is the open subset of $\text{Ein}_n \times \text{Ein}_n$ made of the pairs of points that can be joined by a spacelike geodesic. Given a $(\text{SO}_0(2, n), \mathcal{Y})$ -Anosov representation, the section $s : N \rightarrow E_\rho$ defining the Anosov property lifts to a map $\ell_\rho : \tilde{N} \rightarrow \mathcal{Y}$ which is Γ -equivariant, *ie.* $\rho_g \circ \ell_\rho = \ell_\rho \circ g$, and Φ^t -invariant. This application can be decomposed in $\ell_\rho = (\ell_\rho^+, \ell_\rho^-)$ where ℓ_ρ^+ (resp. ℓ_ρ^-) are two applications from \tilde{N} to Ein_n .

Remark 5.4. An equivalent way to formulate the (G, \mathcal{Y}) -Anosov property is to require the existence of continuous maps $\ell_\rho^\pm : \tilde{N} \rightarrow \text{Ein}_n$ and of a family of Riemannian metrics g^p depending continuously on $p \in \tilde{N}$ and defined in a neighborhood of $\ell_\rho^\pm(p)$ in Ein_n such that:

- (1) This family is Γ -equivariant, *ie.* for every γ in Γ :

$$g^{\gamma p}(d\rho(\gamma)w, d\rho(\gamma)w) = g^p(w, w)$$

where w belongs to $T_{\ell^\pm(p)} \text{Ein}_n$ and $d\rho(\gamma)$ is the differential of $\rho(\gamma)$ at $\ell^\pm(p)$.

- (2) The family increases (resp. decreases) exponentially along positive (resp. negative) orbits of $\tilde{\Phi}^t$, *ie.* for some $a, b > 0$, if w is a vector tangent at $\ell_\rho^+(p)$ (resp. $\ell_\rho^-(p)$) to Ein_n , then:

$$\begin{aligned} g^{\tilde{\Phi}^t(p)}(w, w) &\geq b^{-1} \exp(at) g^p(w, w) \\ g^{\tilde{\Phi}^t(p)}(w, w) &\leq b \exp(-at) g^p(w, w) \end{aligned}$$

5.1.3. *Basic properties of the geodesic flow.* From now, we only consider the case where the Anosov flow is the geodesic flow on the unit tangent bundle over a hyperbolic manifold: Γ is a cocompact torsion free lattice of $\text{SO}_0(1, n)$, N is the quotient $\text{T}^1 \mathbb{H}^n$ by Γ and Φ^t is the geodesic flow ϕ^t on $\Gamma \backslash \text{T}^1 \mathbb{H}^n$, projection of the geodesic flow $\tilde{\phi}^t$ of $\text{T}^1 \mathbb{H}^n$. Let's remind few well-known properties of geodesic flows on hyperbolic manifolds and fix some notations:

- (1) The orbit by $\tilde{\phi}^t$ of (x, v) in $\text{T}^1 \mathbb{H}^n$ is the set of points (x^t, v^t) where x^t describes at unit speed the geodesic tangent to (x, v) and v^t is tangent to this geodesic at x^t ,
- (2) We denote by $\ell^+(x, v)$ the future extremity in $\partial \mathbb{H}^n$ of the geodesic tangent to (x, v) , and by $\ell^-(x, v)$ the past extremity of this geodesic. The fibers of ℓ^+ (respectively of ℓ^-) are called the *stable leaves* (respectively the *unstable leaves*).
- (3) If $(x, v), (x', v')$ belong to the same stable leaf, then there is some T such that the hyperbolic distance between $\tilde{\phi}^{t+T}(x, v)$ and $\tilde{\phi}^t(x', v')$ exponentially tend to 0 when $t \rightarrow +\infty$.
- (4) Geodesics in \mathbb{H}^n are characterized by their extremities, which are distinct. Hence the map $(\ell^+, \ell^-) : \text{T}^1 \mathbb{H}^n \rightarrow \partial \mathbb{H}^n \times \partial \mathbb{H}^n$ induces an identification between the orbit space of $\tilde{\phi}$ and $\partial \mathbb{H}^n \times \partial \mathbb{H}^n \setminus \mathfrak{D}$ where \mathfrak{D} is the diagonal.
- (5) Every element γ of Γ is loxodromic: it admits one attractive fixed point x_γ^+ in $\partial \mathbb{H}^n$ and one repelling fixed point x_γ^- . The geodesic with extremities x_γ^+ and x_γ^- is the unique geodesic of \mathbb{H}^n preserved by γ . There is a real number $T > 0$ such that for every (x, v) tangent to the γ -invariant geodesic and such that $\ell^+(x, v) = x_\gamma^+, \ell^-(x, v) = x_\gamma^-$ we have: $\tilde{\phi}^T(x, v) = \gamma(x, v)$.
- (6) Attractive fixed points of elements of Γ are dense in $\partial \mathbb{H}^n$.

- (7) Periodic geodesics are dense in N , *ie.* pairs (x_γ^+, x_γ^-) are dense in $\partial\mathbb{H}^n \times \partial\mathbb{H}^n \setminus \mathfrak{D}$.

5.2. Fuchsian representations are Anosov. The representation $\rho : \Gamma \rightarrow \mathrm{SO}_0(2, n)$ is *Fuchsian* if it is faithful, discrete, and that $\rho(\Gamma)$ admits a global fixed point in AdS_{n+1} . Up to conjugacy in $\mathrm{SO}_0(2, n)$ every Fuchsian representation is the inclusion $\rho_0 : \Gamma \subset \mathrm{SO}_0(1, n) \subset \mathrm{SO}_0(2, n)$. In this § we prove that Fuchsian representations are $(\mathrm{SO}_0(2, n), \mathcal{Y})$ -Anosov.

5.2.1. De Sitter domains in Ein_n . For any x in AdS_{n+1} the associated de Sitter domain $\partial U(x)$ (cf. remark 2.11) is the open subset of Ein_n comprising limits of spacelike geodesics starting at x . If (x, v) is a unit spacelike tangent vector to AdS_{n+1} — *ie.* $q_{2,n}(x) = -1$, $\langle x|v \rangle = 0$ and $q_{2,n}(v) = -1$ — then $x + v \subseteq \mathcal{C}_n$ is a representant of $\ell^+(x, v)$ (see § 2.9). Hence $\partial U(x)$ is simply the projection on the sphere $\mathbb{S}(\mathbb{R}^{2,n})$ of

$$\mathcal{U}_x := \{x + v; v \in \{x\}^\perp, q_{2,n}(v) = 1\} \subseteq \mathcal{C}$$

An inverse map for this projection can be constructed from the application $s_x : \mathbb{R}^{2,n} \setminus \{x\}^\perp \rightarrow \mathbb{R}^{2,n}$ which maps a point $y \in \mathbb{R}^{2,n} \setminus \{x\}^\perp$ to the unique colinear point $s_x(y)$ in $H_x = \{z; \langle z|x \rangle = -1\}$, *ie.* $s_x(y) = -y/\langle y|x \rangle$. This map induces a diffeomorphism $\tilde{s}_x : \partial U(x) \subseteq \mathrm{Ein}_n \rightarrow \mathcal{U}_x$.

5.2.2. Construction of the metric. For each choice of a point $V \in \mathbb{R}^{2,n}$ of norm -1 such that $\langle x|V \rangle = 0$ we construct a metric $g^{x,V}$ on \mathcal{U}_x as follows. For any choice of $\zeta \in \mathcal{U}_x$ such that $\zeta = x + v$ ($v \in T_x \mathrm{AdS}_{n+1}$), we define a unit timelike tangent vector $\tau_\zeta^{x,V}$ to \mathcal{U}_x at ζ by

$$\tau_\zeta^{x,V} := \frac{V - \langle V|v \rangle v}{\langle V|v \rangle^2 + 1}$$

Let $\bar{g}_\zeta^{x,V}$ be the metric on \mathcal{U}_x obtained by changing the sign of $\tau_\zeta^{x,V}$ in the metric induced by $q_{2,n}$ on $T_\zeta \mathcal{U}_x$. More precisely:

$$\bar{g}_\zeta^{x,V}(w, w) := q_{2,n}(w, w) + 2\langle w|\tau_\zeta^{x,V} \rangle^2$$

The pull-back of this metric by the section s_x is a Riemannian metric $g^{x,V} = s_x^* \bar{g}^{x,V}$ on $\partial U(x)$ for each choice of V in AdS_{n+1} .

Remarks 5.5.

- (1) In the terminology of [BB05] the metric $g^{x,V}$ is the Wick rotation performed on the de Sitter metric of $\partial U(x)$ along the gradient of the time function $\zeta \rightarrow \langle V|\zeta \rangle$.
- (2) The previous construction is $\mathrm{SO}_0(2, n)$ -equivariant in the sense that if γ is any isometry of $\mathbb{R}^{2,n}$,

$$\gamma \partial U(x) = \partial U(\gamma(x))$$

$$\text{and } g_{\gamma(\zeta)}^{\gamma x, \gamma V}(d\gamma w, d\gamma w) = g_\zeta^{x,V}(w, w)$$

5.2.3. *The inclusion is Anosov.* The group $\rho_0(\Gamma)$ preserves an element V of AdS_{n+1} and the stable spacelike hypersurface $\mathbb{S}(V^\perp) \cap \text{AdS}_{n+1}$ isometric to \mathbb{H}^n . It gives a natural inclusion $\mathbb{T}^1 \mathbb{H}^n \subseteq \mathcal{E}^1 \text{AdS}_{n+1}$. We define the maps $\ell_{\rho_0}^\pm : \mathbb{T}^1 \mathbb{H}^n \rightarrow \text{Ein}_n$ as the restrictions of ℓ^\pm to $\mathbb{T}^1 \mathbb{H}^n$. They are both $\tilde{\phi}^t$ -invariant and Γ -equivariant. Since $\mathbb{H}^n \subseteq \text{AdS}_{n+1}$ is spacelike, $\ell_{\rho_0}^+(x, v)$ and $\ell_{\rho_0}^-(x, v)$ are joined by a spacelike geodesic, implying that the map $\ell_{\rho_0} := (\ell_{\rho_0}^-, \ell_{\rho_0}^+)$ takes its value in $\mathcal{Y} \subseteq \text{Ein}_n \times \text{Ein}_n$. In order to prove that the representation ρ_0 is Anosov, we only need to check the hyperbolicity property, as formulated in Remark 5.4, for the family of metrics $g^{(x,v)} := g^{x,V}$. It is the matter of the following proposition which only establishes the expanding property at $\ell_{\rho_0}^+(x, v)$, the contracting property at $\ell_{\rho_0}^-(x, v)$ is similar.

Proposition 5.6. *Let (x, v) be an element of $\mathbb{T}^1 \mathbb{H}^n$ and ν a vector tangent to Ein_n at $\ell_{\rho_0}^+(x, v)$. Then $g^{\tilde{\phi}_{\rho_0}^t(x,v)}(\nu, \nu) = \exp(2t)g^{(x,v)}(\nu, \nu)$.*

Proof. Let (x, v) be an element of $\mathbb{T}^1 \mathbb{H}^n$, and x^t be the base-point of $\tilde{\phi}^t(x, v)$, i.e. $x^t = (\cosh t)x + (\sinh t)v$. While the limit vector $\zeta = \ell_{\rho_0}^+(\tilde{\phi}^t(x, v))$ doesn't change in the Einstein universe, its representant in $\mathcal{U}_{x^t} \subset \mathcal{C}_n \subset \mathbb{R}^{2,n}$ vary with t ; the exponential expanding behaviour comes from the changes in the derivative of the maps s_{x^t} .

The representant of $\zeta = \ell_{\rho_0}^+(x, v)$ in \mathcal{U}_x is $x + v$. The tangent vector ν is the image under the derivative of s_x of a vector w tangent to $\mathcal{U}_x \cap \mathcal{C}_n$. In particular, we have $\langle w|x \rangle = \langle w|v \rangle = 0$. Its representant in \mathcal{U}_{x^t} is the image w^t of w under the derivative of s_{x^t} at $x + v$, i.e. :

$$\begin{aligned} w^t &= d_{x+v}s_{x^t}(w) = \frac{(x+v)\langle w|x^t \rangle - w\langle x+v|x^t \rangle}{\langle x+v|x^t \rangle^2} \\ &= \frac{0 + we^{-t}}{e^{-2t}} \\ &= e^t w \end{aligned}$$

Since $\langle V|v \rangle = 0$, the vector $\tau_\zeta^{x^t, V}$ is V for every t . The proposition follows. \square

5.3. Anosov representations are GH-regular. We proceed to the proof that Anosov representations are GH-regular. Our goal in this § is to prove that the applications $\ell_\rho^\pm : \tilde{N} \rightarrow \text{Ein}_n$ have the same image and that this common image is an acausal hypersphere of Ein_n .

Lemma 5.7. *The application ℓ_ρ^+ (resp. ℓ_ρ^-) is constant along the leaves of the stable (resp. unstable) foliation of \tilde{N} .*

Proof. Corollary of item (3) in § 5.1.3 (and from the compactness of $\Gamma \backslash \mathbb{H}^n$). \square

Therefore, ℓ_ρ^\pm induce Γ -equivariant maps $\tilde{\ell}_\rho^\pm : \partial \mathbb{H}^n \rightarrow \text{Ein}_n$.

Proposition 5.8. *Let α be the map from $\mathbb{T}^1 \mathbb{H}^n$ to itself which sends the vector (x, v) to $(x, -v)$. Then $\ell_\rho^+ = \ell_\rho^- \circ \alpha$.*

Before proving this proposition we need a few lemmas :

Lemma 5.9. *Let γ be an element of Γ . Then $\bar{\ell}_\rho^+(x_\gamma^+)$ (resp. $\bar{\ell}_\rho^-(x_\gamma^-)$) is an attractive (resp. repelling) fixed point of $\rho(\gamma)$ (cf. item (5) in § 5.1.3).*

Proof. Let $(x, v) \in \mathbb{T}^1 \mathbb{H}^n$ such that $\ell^+(x, v) = x_\gamma^+$ and $\ell^-(x, v) = x_\gamma^-$. The images $\zeta^\pm = \bar{\ell}_\rho^\pm(x_\gamma^\pm) = \ell_\rho^\pm(x, v)$ are obviously fixed points of $\rho(\gamma)$. For some $T > 0$ we have $\tilde{\phi}^{nT}(x, v) = \gamma^{nT}(x, v)$. Consider the family of metrics $g^{(x,v)}$ appearing in the alternative definition of Anosov representations (Remark 5.4). Then, for every vector w tangent to $\ell_\rho^+(x_\gamma^+)$ in Ein_n :

$$g_{\zeta^+}^{\tilde{\phi}^{nT}(x,v)}(w, w) \geq a \exp(nT) g_{\zeta^+}^{(x,v)}(w, w)$$

On the other hand, since this family of metrics is Γ -equivariant:

$$g_{\zeta^+}^{\tilde{\phi}^{nT}(x,v)}(w, w) = g_{\zeta^+}^{\gamma^n(x,v)}(w, w) = g^{(x,v)}(d_{\zeta^+} \rho(\gamma)^{-n}(w), d_{\zeta^+} \rho(\gamma)^{-n}(w))$$

Hence ζ^+ is an attractive fixed point. Similarly, ζ^- is repelling. \square

Lemma 5.10. *The image $\rho(\gamma)$ of an element of Γ admits exactly one attractive fixed point in Ein_n .*

Proof. Let x^+ be an attractive fixed point of $\rho(\gamma)$ in Ein_n : there exists a neighborhood U of x^+ in Ein_n such that for all $y \in U$, $\rho(\gamma)^n y \rightarrow x^+$. The convex hull of U in $\mathbb{P}(\mathbb{R}^{2,n})$ satisfies the same property, but it is also a neighborhood of x^+ in $\mathbb{P}(\mathbb{R}^{2,n})$ (it follows from the fact that in any affine chart of $\mathbb{P}(\mathbb{R}^{2,n})$ around x the Einstein space is a one sheet hyperboloid). Hence x^+ is also an attractive fixed point in the projective space $\mathbb{P}(\mathbb{R}^{2,n})$. The lemma follows since attractive fixed points of projective automorphisms of $\mathbb{P}(\mathbb{R}^{2,n})$ are unique. \square

Proof of proposition 5.8. It follows from lemmas 5.9, 5.10 that this proposition is true when the geodesic tangent to (x, v) is preserved by a non trivial element of Γ . The general case follows from the density of periodic orbits (item (7) in § 5.1.3). \square

Corollary 5.11. *The applications ℓ_ρ^\pm have the same image. They are homeomorphisms between $\partial \mathbb{H}^n$ and a topological acausal $(n-1)$ -sphere Λ_ρ .*

Proof. The equality of the images is an immediate consequence of proposition 5.8. We only have to show that the application ℓ_ρ^+ (for example) is injective. Let (x, v) and (y, w) be two points of $\mathbb{T}^1 \mathbb{H}^n$, belonging to two different stable leaves. Hence, there exists a point (z, ν) which is in (x, v) 's stable leaf and (y, w) 's unstable one. We thus have $(\ell_\rho^+(x, v), \ell_\rho^-(\alpha(y, w))) = (\ell_\rho^+(z, \nu), \ell_\rho^-(z, \nu)) \subseteq \mathcal{Y}$. In particular, $\ell_\rho^+(x, v)$ and $\ell_\rho^-(\alpha(y, w)) = \ell_\rho^+(y, w)$ are joined by a spacelike geodesic and must be different. \square

Proof of Theorem 1.1. The fact that Anosov representations are GH-regular with acausal limit set follows from the last corollary and propositions 4.5 and 4.4. \square

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