1

THE ASYMPTOTIC TIAN-YAU-ZELDITCH EXPANSION ON RIEMANN SURFACES WITH CONSTANT CURVATURE

CHIUNG-JU LIU

ABSTRACT. Let M be a regular Riemann surface with a metric which has constant scalar curvature ρ. We give the asymptotic expansion of the sum of the square norm of the sections of the pluricanonical bundles K_M^m. That is,

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|^2_{h_m} \sim m(1+\frac{\rho}{2m}) + O\left(e^{-\frac{(\log m)^2}{8}}\right),$$

where $\{S_0, \cdots, S_{d_m-1}\}$ is an orthonormal basis for $H^0(M, K_M^m)$ for sufficiently large m.

1. Introduction

Let M be an n-dimensional compact complex Kähler manifold with an ample line bundle L over M. Let g be the Kähler metric on M corresponding to the Kähler form $\omega_g = Ric(h)$ for some positive Hermitian h metric on L. Such a Kähler metric g is called a polarized Kähler metric. The metric h induces a Hermitian metric h_m on L^m for all positive integers m. Let $\{S_0, \cdots, S_{d_m-1}\}$ be an orthonormal basis of the space $H^0(M, L^m)$ with respect to the inner product

$$(S,T) = \int_M \langle S(x), T(x)\rangle_{h_m} dV_g, \tag{1.1}$$

where $d_m = \dim H^0(M, L^m)$ and $dV_g = \frac{\omega_g^n}{n!}$ is the volume form of g. The quantity

$$\sum_{i=0}^{d_m-1} \|S_i(x)\|^2_{h_m} \tag{1.2}$$

is related to the existence of Kähler-Einstein metrics and stability of complex manifolds. A lot of work has been done for (1.2) on compact complex Kähler manifolds. Tian [6] applied Hömander's L^2-estimate to produce peak sections and proved the C^2 convergence of the Bergman metrics. Later, Ruan [5] proved the C^∞ convergence. About the same time, Zelditch [7] and Catlin [4] separately generalized the theorem of Tian by showing there is an asymptotic expansion

$$\sum_{i=0}^{d_m-1} \|S_i(x)\|^2_{h_m} \sim a_0(x)m^n + a_1(x)m^{n-1} + a_2(x)m^{n-2} + \cdots \tag{1.3}$$

for certain smooth coefficients $a_j(x)$ with $a_0 = 1$. In [10], Lu proved that each coefficient $a_j(x)$ is a polynomial of the curvature and its covariant derivatives. In particular, $a_1 = \frac{\rho}{2}$, where ρ is the scalar curvature of M. These polynomials can be

[1] *Subject Classification.* Primary 53C55; Secondary 32Q05

Date: June 30, 2008.
Key words and phrases. Asymptotic expansion; Riemann surfaces.
Research supported by the grant of NSC A95-0918.

found by finitely many steps of algebraic operations. Recently, Song [3] generalized Zelditch's theorem on orbifolds of finite isolated singularities. The information on the singularities can be found in the expansion.

On the Riemann surfaces with bounded curvature, Lu [9] proved that there is a lower bound for (1.2). Later, the result of Lu and Tian [8] implies that on the Riemann surfaces with constant scalar curvature ρ, the asymptotic expansion (1.3) is given by

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|^2_{h_m} \sim m(1+\frac{\rho}{2m}) + O\left(\frac{1}{m^p}\right)$$

for any $p > 0$. In the current paper, we obtain a more precise result for (1.3).

Theorem 1.1. *Let M be a regular compact Riemann surface and K_M be the canonical line bundle endowed with a Hermitian metric h such that the curvature $Ric(h)$ of h defines a Kähler metric g on M. Suppose that this metric g has constant scalar curvature ρ. Then there is a complete asymptotic expansion:*

$$\sum_{i=0}^{d_m-1} \|S_i(x_0)\|^2_{h_m} \sim m(1+\frac{\rho}{2m}) + O\left(e^{-\frac{(\log m)^2}{8}}\right),$$

where $\{S_0, \cdots, S_{d_m-1}\}$ is an orthonormal basis for $H^0(M, K_M^m)$ for some $m > \max\{e^{20\sqrt{5}} + 2|\rho|, |\rho|^{4/3}, \frac{1}{\delta}, \sqrt{\frac{2}{|\rho|}}\}$, where δ is the injective radius at x_0.

Our result indicates that the asymptotic expansion (1.3) is in exponential decay. Engliš [2] has an asymptotically expansion for the Berezin transformation on any planar domain of hyperbolic type. He also showed that Berezin kernel [1] has

$$\tilde{B}(\eta, \eta) = m\left(1 + O(1)\rho_0(0)^{\frac{\pi m - 3}{2}}\right),$$

where $\rho_0(0)$ is a positive constant.

Acknowledgements. The author thanks her advisor Z. Lu for his suggestions and help during the preparation of this paper. She also thanks R. Chiang for her encouragement.

2. General set up

Let M be an n-dimensional compact complex Kähler manifold with a polarized line bundle $(L, h) \to M$. Choose the K-coordinates $(z_1, \cdots, z_n)$ on an open neighborhood U of a fixed point $x_0 \in M$. Then the Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} dz_\alpha \wedge d\bar{z}_\beta$$

satisfies

$$g_{\alpha\bar{\beta}}(x_0) = \delta_{\alpha\bar{\beta}}, \qquad \frac{\partial^{p_1+\cdots+p_n} g_{\alpha\bar{\beta}}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n}}(x_0) = 0, \tag{2.1}$$

for $\alpha, \beta = 1, \cdots, n$ and any nonnegative integers $p_1, \cdots, p_n$ with $p_1 + \cdots + p_n \neq 0$.

We also choose a local holomorphic frame e_L of the line bundle L at x_0 such that a is the local representation function of the Hermitian metric h. That is,

$$Ric(h) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log a.$$

Under the K-coordinate, the function a has the properties

$$a(x_0) = 1, \qquad \frac{\partial^{p_1+\cdots+p_n}}{\partial z_1^{p_1}\cdots\partial z_n^{p_n}}(a)(x_0) = 0 \tag{2.2}$$

for any nonnegative integers $p_1, \cdots, p_n$ with $p_1 + \cdots + p_n \neq 0$.

Let $\{S_0, \cdots, S_{d_m-1}\}$ be a basis of $H^0(M, L^m)$. Assume that at the point $x_0 \in M$,

$$S_0(x_0) \neq 0, \qquad S_i(x_0) = 0, \qquad i = 1, \cdots, d_m - 1.$$

If the set $\{S_0, \cdots, S_{d_m-1}\}$ is not an orthonormal basis, we may do the following: Let the metric matrix

$$F_{ij} = (S_i, S_j), \qquad i, j = 0, \cdots, d_m - 1$$

with respect to the inner product (1.1). By definition, (F_{ij}) is a positive definite Hermitian matrix. We can find a $d_m \times d_m$ matrix G_{ij} such that

$$F_{ij} = \sum_{k=0}^{d_m-1} G_{ik}\overline{G_{jk}}.$$

Let (H_{ij}) be the inverse of (G_{ij}). Then $\{\sum_{j=0}^{d_m-1} H_{ij}S_j\}$ forms an orthonormal basis of $H^0(M, L^m)$. The left hand side of (1.2) is equal to

$$\sum_{i=0}^{d_m-1} \| \sum_{j=0}^{d_m-1} H_{ij}S_j(x_0)\|_{h_m}^2 = \sum_{i=0}^{d_m-1} |H_{i0}|^2 \|S_0(x_0)\|_{h_m}^2. \tag{2.3}$$

Let (I_{ij}) be the inverse matrix of (F_{ij}). Denote that

$$\sum_{i=0}^{d_m-1} |H_{i0}|^2 = I_{00}. \tag{2.4}$$

In order to compute (2.4), we need a suitable choice of the basis $\{S_0, \cdots, S_{d_m-1}\}$. We select some of Tian's peak sections in our basis. The following lemma an improved version of Tian's result [6, Lemma 1.2], which is done by Lu and Tian.

Let $\mathbb{Z}_+^n$ be the set of n-tuple integers $P = (p_1, \cdots, p_n)$ such that each p_i is a nonnegative integer for $i = 1, \cdots, n$. For $P \in \mathbb{Z}_+^n$, we denote that $z^P = z_1^{p_1}\cdots z_n^{p_n}$ and $|P| = p_1 + \cdots p_n$.

Lemma 2.1 (Tian). *Suppose $Ric(g) \geq -K\omega_g$, where $K > 0$ is a constant. For $P \in \mathbb{Z}_+^n$ and an integer $p' > |P|$, let m be an integer such that*

$$m > \max\{e^{20\sqrt{n+2p'}} + 2K, e^{8(p'-1+n)}\}.$$

Then there is a holomorphic section $S_{P,m} \in H^0(M, L^m)$, satisfying

$$|\int_M \|S_{P,m}\|_{h_m}^2 dV_g - 1| \leq Ce^{-\frac{1}{8}(\log m)^2}. \tag{2.5}$$

Moreover, $S_{P,m}$ can be decomposed as

$$S_{P,m} = \tilde{S}_{P,m} - u_{P,m}$$

such that

$$\tilde{S}_{P,m}(x) = \lambda_P \eta\left(\frac{m|z|^2}{(\log m)^2}\right) z^P e_L^m = \begin{cases} \lambda_P z^P e_L^m & x \in \{|z| \leq \frac{\log m}{\sqrt{2m}}\}, \\ 0 & x \in M \setminus \{|z| \leq \frac{\log m}{\sqrt{m}}\}, \end{cases} \tag{2.6}$$

and

$$\int_M \|u_{P,m}\|^2_{h_m} dV_g \le Ce^{-\frac{1}{4}(\log m)^2}, \tag{2.7}$$

where η is a smoothly cut-off function

$$\eta(t) = \begin{cases} 1 & \text{for } 0 \le t \le \frac{1}{2}, \\ 0 & \text{for } t \ge 1. \end{cases}$$

satisfying $0 \le -\eta'(t) \le 4$ *and* $|\eta''(t)| \le 8$ *and*

$$\lambda_P^{-2} = \int_{|z| \le \frac{\log m}{\sqrt{m}}} |z^P|^2 a^m dV_g. \tag{2.8}$$

Proof. Define the weight function

$$\Psi(z) = (n + 2p')\eta\left(\frac{m|z|^2}{(\log m)^2}\right)\log\left(\frac{m|z|^2}{(\log m)^2}\right).$$

A straightforward computation gives

$$\sqrt{-1}\partial\bar{\partial}\Psi \ge -\frac{100m(n+2p')}{(\log m)^2}\omega_g. \tag{2.9}$$

By using (2.9), we can verify that

$$\langle \partial\bar{\partial}\Psi + \frac{2\pi}{\sqrt{-1}}(Ric(h^m) + Ric(g)), v \wedge \bar{v}\rangle_g \ge \frac{1}{4}m\|v\|_g^2.$$

For $P \in \mathbb{Z}^n_+$, consider the 1-form

$$w_P = \bar{\partial}\eta(\frac{m|z|^2}{(\log m)^2})z^P e_L^m.$$

Since $w_P \equiv 0$ in a neighborhood of x_0, we have

$$\int_M \|w_P\|^2_{h_m} e^{-\Psi} dV_g < +\infty.$$

By [6, Prop. 2.1], there exists a smooth L^m-valued section u_P such that $\bar{\partial}u_P = w_P$ and

$$\int_M \|u_P\|^2_{h_m} e^{-\Psi} dV_g \le \frac{4}{m}\int_M \|w_P\|^2_{h_m} e^{-\Psi} dV_g < \infty. \tag{2.10}$$

By direct computation, we get

$$\int_M \|u_P\|^2_{h_m} e^{-\Psi} dV_g \le \frac{C(\log m)^{2(p-1)}}{m^p}\int_{\frac{\log m}{\sqrt{2m}} \le |z| \le \frac{\log m}{\sqrt{m}}} a^m dV_0.$$

Under the K-coordinate, we have

$$a^m = e^{m\log a} = e^{m(-|z|^2 + O(|z|^4))}.$$

Hence we get

$$\int_M \|u_P\|^2_{h_m} e^{-\Psi} dV_g \le \frac{C_1(\log m)^{2(p-1+n)}}{m^{p+n}} e^{-\frac{1}{2}(\log m)^2}$$

for some constant C_1. Let $\tilde{S}_{P,m} = \lambda_P \eta(\frac{m|z|^2}{(\log m)^2})z^P e_L^m$ and $u_{P,m} = \lambda_P u_P$. Use a result in [10]

$$\lambda_P^2 \le C_2 m^{n+|P|}$$

for some constant C_2. Then we have

$$\int_M \|u_{P,m}\|^2_{h_m} dV_g \le C(\log m)^{2(|P|-1+n)} e^{-\frac{1}{2}(\log m)^2}.$$

Choosing $m > e^{8(p'-1+n)}$, we obtain

$$\int_M \|u_{P,m}\|^2_{h_m} dV_g \le Ce^{-\frac{1}{4}(\log m)^2}.$$

□

3. Proof of Theorem 1.1

Proof. Let M be a smooth compact Riemann surface with a metric g that has constant scalar curvature. Let x_0 be a fixed point. Let

$$U = \{x : \mathrm{dist}(x, x_0) < \delta\},$$

where δ is the injective radius at x_0. It is well known that on a Riemann surface there is an isothermal coordinate at each point on U. We may assume that there is a holomorphic function z on U and it defines the conformal structure on U. That is,

$$ds^2 = gdzd\bar{z}$$

and $g > 0$. The metric g satisfies

$$\triangle \log g = -\rho, \quad g(x_0) = 1, \quad \text{and} \quad \frac{\partial g}{\partial z}(x_0) = 0, \tag{3.1}$$

where

$$\triangle = g^{-1}\frac{\partial^2}{\partial z \partial \bar{z}}$$

is the complex Laplace of M. Since the metric g has conformal structure, it is rotationally symmetric. We can write (3.1) in polar coordinates (r, θ):

$$\frac{\partial^2 g}{\partial r^2} + \frac{1}{r}\frac{\partial g}{\partial r} - \frac{1}{g}(\frac{\partial g}{\partial r})^2 = -4\rho g^2, \qquad g(0,\theta) = 1, \quad \frac{\partial g}{\partial r}(0,\theta) = 0, \tag{3.2}$$

where $z = re^{i\theta}$, and $|z|^2 = r^2$. There exists a solution

$$g = \frac{1}{(1 + \frac{\rho}{2}|z|^2)^2} \tag{3.3}$$

to (3.2) for $|z| < \sqrt{-\frac{2}{\rho}}$ if $\rho < 0$. Suppose that there exists another solution g_1 to (3.2). We have

$$\triangle \log{(g_1/g)} = 0 \quad \text{and} \quad g_1(x_0) = 1.$$

For $\rho < 0$, let $r_0 < \sqrt{-\frac{2}{\rho}}$. Since g and g_1 are rotationally symmetric, they remain constant on $|z| = r_0$. The harmonic function $\log(g/g_1)$ is a constant on $|z| \le r_0$ by Maximum Principle. By definition, we have $g(x_0) = g_1(x_0) = 1$. Therefore, the solution in (3.3) is unique around x_0. By the same reason, $g = g_1$ on $\{\mathrm{dist}(x, x_0) \le \delta_1\}$ for some $\delta_1 < \delta$ for $\rho \le 0$.

Let a be the local representation of the metric h on K_M such that

$$-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log a = \omega_g.$$

If we normalize a and a satisfies

$$\triangle \log a = -1, \quad a(x_0) = 1, \quad \frac{\partial a}{\partial z}(x_0) = 0. \tag{3.4}$$

Since

$$-\frac{\partial^2}{\partial z \partial \bar{z}} \log a = g,$$

$\log a$ is also rotationally symmetric. Since

$$a = \begin{cases} \left(1 + \frac{\rho}{2}|z|^2\right)^{-\frac{2}{\rho}}, & \text{if } \rho \neq 0; \\ e^{-|z|^2}, & \text{if } \rho = 0. \end{cases} \tag{3.5}$$

satisfies (3.4), the local uniqueness is due to the same reason.

We need to choose sufficient large m such that $\frac{\log m}{\sqrt{m}} < \min\{\delta, \sqrt{\frac{2}{|\rho|}}\}$. With these particular solutions of g and a, we further compute

$$\begin{aligned} \lambda_0^{-2} &= \int_{|z| \leq \frac{\log m}{\sqrt{m}}} a^m g \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} \\ &= 2\int_o^{\frac{\log m}{\sqrt{m}}} (1 + \frac{\rho}{2} r^2)^{-\frac{2m}{\rho} - 2} r dr \\ &= \frac{1}{m + \frac{\rho}{2}} \left(1 - \left(1 + \frac{\rho}{2} \frac{(\log m)^2}{m}\right)^{-1 - \frac{2m}{\rho}}\right) \qquad \text{for } \rho \neq 0. \end{aligned} \tag{3.6}$$

For $m > \max\{|\rho|^{4/3}, 10\}$, we have $\left|\frac{\rho}{2} \frac{(\log m)^2}{m}\right| < 1/2$. For $\rho \neq 0$, this gives

$$\left(1 + \frac{\rho}{2} \frac{(\log m)^2}{m}\right)^{-1 - \frac{2m}{\rho}} \leq 2e^{-\frac{2m}{\rho}\left(\frac{\rho}{2}\frac{(\log m)^2}{m} - \frac{1}{2}(\frac{\rho}{2}\frac{(\log m)^2}{m})^2 + \cdots\right)} \leq Ce^{-(\log m)^2}.$$

For $\rho = 0$, we have

$$\lambda_0^{-2} = \int_{|z| \leq \frac{\log m}{\sqrt{m}}} e^{-m|z|^2} \frac{\sqrt{-1}}{2\pi} dz \wedge d\bar{z} = \frac{1}{m}(1 + O(e^{-(\log m)^2})).$$

Hence we obtain

$$\lambda_0^{-2} = \frac{1}{m + \frac{\rho}{2}} \left(1 + O\left(e^{-(\log m)^2}\right)\right). \tag{3.7}$$

From the properties of g and a, the isothermal coordinate (U, z) is a K-coordinate. According to Lemma 2.1, we may choose two peak sections

$$\begin{aligned} S_{0,m} &= \lambda_0(\eta(\frac{m|z|^2}{(\log m)^2})(dz)^m - u_0) \\ S_{1,m} &= \lambda_1(\eta(\frac{m|z|^2}{(\log m)^2})z(dz)^m - u_1) \end{aligned}$$

in $H^0(M, K_M^m)$ for some $m > e^{20\sqrt{1+4}} + 2|\rho|$. Obviously, we have $S_{0,m}(x_0) \neq 0$ and $S_{1,m}(x_0) = 0$. Let the subspace

$$V = \{S \in H^0(M, K_M^m) | S(x_0) = 0, DS(x_0) = 0\},$$

where D is a covariant derivative on K_M^m. Let $T_1, \cdots, T_{d_m - 2}$ be an orthonormal basis of V. Let

$$S_i = \begin{cases} S_{i,m} & \text{if } i = 0, 1 \\ T_{i-1} & \text{if } 2 \leq i \leq d_m - 1 \end{cases}. \tag{3.8}$$

Then $\{S_i\}_{i=0}^{d_m-1}$ forms a basis for $H^0(M, K_M^m)$. Locally, each T_i has the form $f_i(dz)^m$ for some holomorphic function f_i defined in U. The holomorphic function f_i has Taylor expansion as $f_i = \sum_{\alpha=2}^{\infty} b_{i\alpha} z^\alpha$ in U, since $T_i(x_0) = 0$ and $DT_i(x_0) = 0$ for $1 \le i \le d_m - 2$

Lemma 3.1. *Let $\{S_i\}_{i=0}^{d_m-1}$ be the basis of $H^0(M, K_M^m)$, defined in (3.8). For $m > e^{20\sqrt{5}} + 2|\rho|$, the Hermitian matrix*

$$(S_i, S_j) = \int_M \langle S_i(x), S_j(x)\rangle_{h_m} dV_g$$

is given by

$$\begin{aligned}
(S_0, S_0) &= 1 + O\left(e^{-\frac{(\log m)^2}{8}}\right), \\
(S_0, S_1) &= O\left(e^{-\frac{(\log m)^2}{8}}\right), \\
(S_1, S_1) &= 1 + O\left(e^{-\frac{(\log m)^2}{8}}\right), \\
(S_0, S_i) &= O\left(e^{-\frac{(\log m)^2}{8}}\right), \\
(S_1, S_i) &= O\left(e^{-\frac{(\log m)^2}{8}}\right), \\
(S_i, S_j) &= \delta_{ij}
\end{aligned}$$

for $i, j = 2, \cdots, d_m - 1$.

Proof. By definition, we have $(S_i, S_j) = \delta_{ij}$ for $2 \le i, j \le d_m - 1$. The inner product of (S_i, S_i) for $0 \le i \le 1$ is directly from Lemma 2.1. Since $a^m g$ is rotationally symmetric, we have

$$\int_{|z| \le \frac{\log m}{\sqrt{m}}} \bar{z}^\alpha a^m g dV_0 = 0 \qquad \text{for arbitrary positive integer } \alpha.$$

Then we get

$$\begin{aligned}
(S_0, S_1) &= (\tilde{S}_0, \tilde{S}_1) + (\lambda_0 u_0, \tilde{S}_1) + (\tilde{S}_0, \lambda_1 u_1) + (u_0, u_1) \\
&= O\left(e^{-\frac{(\log m)^2}{8}}\right).
\end{aligned}$$

Consider

$$\begin{aligned}
(S_0, S_i) &= \int_M \langle \lambda_0 (\eta(\frac{m|z|^2}{(\log m)^2})(dz)^m - u_0), f_{i-1}(dz)^m \rangle_{h_m} dV_g \\
&\le \lambda_0 \int_{|z| \le \frac{\log m}{\sqrt{m}}} \sum_{\alpha=2}^{\infty} b_{(i-1)\alpha} \bar{z}^\alpha a^m g dV_0 + \lambda_0 \|u_0\| \cdot \|S_i\|.
\end{aligned}$$

Thus we have

$$(S_0, S_i) = O\left(e^{-\frac{(\log m)^2}{8}}\right) \qquad \text{for } 2 \le i \le d_m - 1.$$

Similarly, consider

$$(S_1, S_j) \le \lambda_0 \int_{|z| \le \frac{\log m}{\sqrt{m}}} \sum_{\alpha=2}^{\infty} b_{(i-1)\alpha} z \bar{z}^\alpha a^m g dV_0 + \lambda_1 \|u_1\| \cdot \|S_i\| \qquad \text{for } 2 \le i \le d_m - 1.$$

Since $a^m g$ is rotationally symmetric, $\int_{|z|\leq \frac{\log m}{\sqrt{m}}} z\bar{z}^\alpha a^m g dV_0 = 0$ for $\alpha \geq 2$. Hence we obtain

$$(S_1, S_i) = O\left(e^{-\frac{(\log m)^2}{8}}\right).$$

□

According to [10, Definition 3.1], the metric matrix (F_{ij}) can be represented by the block matrices

$$(F_{ij}) = \begin{pmatrix} (S_0,S_0) & (S_0,S_1) & M_{13} \\ (S_1,S_0) & (S_1,S_1) & M_{23} \\ M_{31} & M_{32} & E \end{pmatrix}, \tag{3.9}$$

where $M_{13} = ((S_0,S_2),\cdots,(S_0,S_{d_m-1}))$, $M_{23} = ((S_1,S_2),\cdots,(S_1,S_{d_m-1}))$, $M_{31} = \overline{M_{13}^T}$, $M_{32} = \overline{M_{23}^T}$, and E is a $(d_m-2)\times(d_m-2)$ identity matrix. By using [10, Lemma 3.1], we obtain

$$I_{00} = \frac{1}{(S_0,S_0)} + (\frac{1}{(S_0,S_0)})^2 \begin{pmatrix} (S_0,S_1) & M_{13} \end{pmatrix} \tilde{M}^{-1} \begin{pmatrix} (S_1,S_0) \\ M_{31} \end{pmatrix}, \tag{3.10}$$

where

$$\tilde{M} = \begin{pmatrix} (S_1,S_1) & M_{23} \\ M_{32} & E \end{pmatrix} - \frac{1}{(S_0,S_0)} \begin{pmatrix} (S_1,S_0) \\ M_{31} \end{pmatrix} \begin{pmatrix} (S_0,S_1) & M_{13} \end{pmatrix}.$$

Applying Lemma 3.1 in (3.10), we get

$$I_{00} = 1 + O\left(e^{-\frac{(\log m)^2}{8}}\right). \tag{3.11}$$

In order to evaluate the expansion of (2.3), we are left to find $\|S_0(x_0)\|^2_{h_m} = \lambda_0^2$. From (3.7), we have

$$\lambda_0^2 = m(1+\frac{\rho}{2m})\left(1 + O\big(e^{-(\log m)^2}\big)\right).$$

Therefore, the Tian-Yau-Zelditch expansion according to (2.3) on a Riemann surface with constant scalar curvature ρ is

$$\begin{aligned} I_{00}\lambda_0^2 &\\ &= (1 + O\left(e^{-\frac{(\log m)^2}{8}}\right))m(1+\frac{\rho}{2m})\left(1+O\big(e^{-(\log m)^2}\big)\right) \\ &= m(1+\frac{\rho}{2m}) + O\left(e^{-(\frac{(\log m)^2}{8})}\right) \end{aligned}$$

for $m > \max\{e^{20\sqrt{5}} + 2|\rho|, |\rho|^{4/3}, \frac{1}{\delta}, \sqrt{\frac{2}{|\rho|}}\}$. □

References

[1] F. A. Berezin, *Quantization*, Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 1116–1175.

[2] M. Engliš, *Asymptotics of the Berezin transform and quantization on planar domains*, Duke Math. J. **79** (1995), no. 1, 57–76.

[3] J. Song, *The Szegö Kernel on an Orbifold Circle Bundle.*

[4] D. Catlin, *The Bergman kernel and a theorem of Tian*, Analysis and geometry in several complex variables (Katata, 1997), 1999, pp. 1–23.

[5] W.-D. Ruan, *Canonical coordinates and Bergmann metrics*, Comm. Anal. Geom. **6** (1998), no. 3, 589–631.

[6] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Differential Geom. **32** (1990), no. 1, 99–130.

[7] S. Zelditch, *Szegő kernels and a theorem of Tian*, Internat. Math. Res. Notices (1998), no. 6, 317–331.

[8] Z. Lu and Gang Tian, *The log term of the Szegő kernel*, Duke Math. J. **125** (2004), no. 2, 351–387.

[9] Z. Lu, *On the lower bound estimates of sections of the canonical bundles over a Riemann surface*, Internat. J. Math. **12** (2001), no. 8, 891–926.

[10] ———, *On the lower order terms of the asymptotic expansion of Tian-Yau-Zelditch*, Amer. J. Math. **122** (2000), no. 2, 235–273.

E-mail address: cjliu4@ntu.edu.tw

Taida Institute of Mathematical Science, National Taiwan University, Taiwan 106