

Complementary self-similarity

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Let U be a nonempty proper open subset of \mathbf{R}^n , and put $E = \partial U$. Thus E is a nonempty closed set in \mathbf{R}^n with empty interior, and we are especially interested in the case where E is connected and U has infinitely many connected components. For instance, it may be that $U = \mathbf{R}^n \setminus E$, or that U is the union of the bounded components of the complement of E .

A strong version of “complementary self-similarity” of E would ask that the components of U be similar to each other, which is to say that they can be transformed into one another by combinations of translations, orthogonal transformations, and dilations. Many standard examples such as Sierpinski gaskets and carpets have this property, since the bounded complementary components are equilateral triangles or squares, etc. There are obviously numerous extensions of this, using bilipschitz, quasiconformal, or quasisymmetric mappings, and so on. However, this does not necessarily include self-similarity of the arrangement of the components of U . As an extreme case, it may be that E has positive Lebesgue measure, and that the sum of the $(n-1)$ -dimensional Hausdorff measures of the boundaries of the components of U is finite.

One could also consider scale-invariant geometric conditions on the components of U . A very basic condition of this type would ask that each component V of U contain an open ball whose radius is at least a positive constant times the diameter of V . A related scale-invariant condition concerning the arrangement of the components of U would say that for each $x \in E$ and $0 < r \leq \text{diam } U$ there is a component of U contained in $B(x, r)$ whose diameter is at least a positive constant times r . The combination of these two conditions implies that E is *porous*, in the sense that for each $x \in E$ and $0 < r \leq \text{diam } U$ there be a point in $B(x, r) \setminus E$ whose distance to E is at least a positive constant times r . A more qualitative form of the second condition would simply ask that there be a component of U contained in $B(x, r)$ for every $x \in E$ and $r > 0$.

As another possibility, suppose that there is a $k \geq 1$ such that for each component V of U and every $x, y \in \partial V$ there is a continuous path in ∂V that connects x to y and has length $\leq k|x - y|$. This implies that every continuous path in \bar{U} with finite length $\leq L$ connecting a pair of elements of E can be modified to get a path in E with the same endpoints and length $\leq kL$. In many cases, \bar{U} is very nice, perhaps even convex, and so it may be very easy to connect elements of \bar{U} by paths of controlled length. Of course, there are analogous assertions for higher-dimensional fillings.

Although distinct components of U are disjoint, their closures may or may not be. If the closures are disjoint, then one can look for lower bounds on the distances between them. A basic scale-invariant condition would ask that

$$(1) \quad \min(\text{diam } V, \text{diam } W) \leq C \text{ dist}(V, W)$$

for some $C > 0$ and any two distinct components V, W of U .

Note that E has topological dimension $n-1$ when $E = \partial U$ for a disconnected open set $U \subseteq \mathbf{R}^n$. More precisely, the topological dimension of E is $\leq n-1$ because E has empty interior, and it is then equal to $n-1$ because E separates elements of \mathbf{R}^n .

Suppose that E is compact. For each $w \in \mathbf{R}^n \setminus E$, let π_w be the mapping from E into the unit sphere \mathbf{S}^{n-1} defined by

$$(2) \quad \pi_w(x) = \frac{x-w}{|x-w|}.$$

A famous theorem states that π_w and π_z are homotopically equivalent if and only if w and z are in the same component of the complement of E . In particular, π_w is homotopically trivial exactly when w is in the unbounded component of $\mathbf{R}^n \setminus E$. The Lipschitz constant of π_w can be estimated in terms of $1/\text{dist}(w, E)$, which one might like to be as small as possible.

Even when the complement of E is connected, there can be a lot of topological activity related to linking. There is also some discreteness involved with this, as indicated by homology or cohomology groups with integer coefficients, or by homotopy classes into nice spaces for which close mappings are homotopic. There can still be matters of linking when E is connected, U has infinitely many connected components, and $n \geq 3$, because the components of U may not be contractable. Common features of the topology of the components of U are another aspect of self-similarity of the complement of E . At any rate, one is likely to be interested in more than just homotopy classes of continuous mappings from E into the nonzero complex numbers when the topological dimension of E is strictly greater than 1.

A nice hypersurface in \mathbf{R}^n has two sides, corresponding to two complementary components. Using Cauchy integrals from ordinary complex analysis when $n = 2$ and extensions to higher dimensions based on quaternions or Clifford algebras, functions on the hypersurface can be split into two pieces, which are boundary values of holomorphic functions on the two complementary components. However, if $E \subseteq \mathbf{R}^n$ is sufficiently “large”, then the Cauchy integral of suitable measures on E determine continuous functions on \mathbf{R}^n that are holomorphic on the complement of E . One can still look at operators involving projections onto holomorphic functions on the individual complementary components, even when there are infinitely many components.

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