

# SOME GEOMETRIC STRUCTURES ASSOCIATED TO A $k$ -SYMPLECTIC MANIFOLD

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ABSTRACT. A canonical connection is attached to any  $k$ -symplectic manifold. We study the properties of this connection and its geometric applications to  $k$ -symplectic manifolds. In particular, we prove that, under some natural assumptions, any  $k$ -symplectic manifold admits an Ehresmann connection, discuss some corollaries of this result and find vanishing theorems for characteristic classes on a  $k$ -symplectic manifold.

## 1. INTRODUCTION

The theory of  $k$ -symplectic manifolds was initiated by A. Awane ([1]), who defined a  $k$ -symplectic structure on an  $n(k+1)$ -dimensional smooth manifold  $M$  as an  $n$ -codimensional foliation  $\mathcal{F}$  and a system of  $k$  closed 2-forms vanishing on the subbundle of  $TM$  defined by  $\mathcal{F}$  with transversal characteristic spaces (for a precise definition see § 2). The study of these structures was motivated by some mathematical and physical considerations, like the local study of Pfaffian systems and Nambu's statistical mechanics. But the interest on  $k$ -symplectic geometry has increased especially in recent years by the awareness of its relationship with polysymplectic (or multisymplectic) and  $n$ -symplectic geometry, and their applications in field theory (cf. [17], [18], [20]). In fact the  $k$ -symplectic formalism is the generalization to field theories of the standard symplectic formalism in mechanics, which is the geometric framework for describing most of autonomous mechanical systems. Especially it can be used for giving a geometric description of first order field theories in which the Lagrangian and Hamiltonian depend on the first jet (prolongation) of the field.

The definition of a  $k$ -symplectic manifold is a generalization of the notion of a symplectic manifold foliated by a Lagrangian foliation. Thus it is a natural question whether one can define an appropriate analogue of the well-known notion of bi-Lagrangian structure to the context of  $k$ -symplectic geometry. We recall that an almost bi-Lagrangian manifold is a symplectic manifold  $(M^{2n}, \omega)$  endowed with two transversal Lagrangian distributions  $L_1$  and  $L_2$ . When  $L_1$  and  $L_2$  are both integrable, we speak of bi-Lagrangian manifold. The peculiarity of these geometric structures is that a canonical symplectic connection can be attached to them. This connection was introduced by H. Hess ([13]), who was working in geometric quantization, and later on its important geometric properties were pointed out by N. B. Boyom ([9]) and I. Vaisman ([22], [23]).

In this work we consider the  $k$ -symplectic analogue of bi-Lagrangian structure and attach to a such  $k$ -symplectic manifold a canonical connection which plays the same role in  $k$ -symplectic geometry as the Hess connection. Moreover we define on a  $k$ -symplectic

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manifold a family of tensor fields which can be thought as the proper generalization in this setting of almost Kähler structures, and we prove that under some integrability assumptions, the above connection coincides with the Levi-Civita connection of a suitable compatible metric. Finally, as an application, we prove that under some certain natural assumption, any  $k$ -symplectic manifold admits an Ehresmann connection and we deduce some geometric and topological properties on the  $k$ -symplectic manifold in question.

## 2. $k$ -SYMPLECTIC STRUCTURES

A  $k$ -symplectic manifold (cf. [1], [19]) is a smooth manifold  $M$  together with  $k$  closed 2-forms  $\omega_1, \dots, \omega_k$  such that

- (1)  $C_x(\omega_1) \cap \dots \cap C_x(\omega_k) = \{0\}$ ,
- (2)  $\omega_\alpha(X, X') = 0$  for any  $X, X' \in \Gamma(T\mathcal{F})$  and for all  $\alpha \in \{1, \dots, k\}$ ,

where  $C_x(\omega) = \{v \in T_x M : \omega_x(v, w) = 0 \text{ for any } w \in T_x M\}$  and  $\mathcal{F}$  is an  $nk$ -dimensional foliation on  $M$ . It follows that  $\dim(M) = n(k+1)$ . We will usually denote by  $L$  the tangent bundle of the foliation  $\mathcal{F}$ . In terms of  $G$ -structures, a  $k$ -symplectic manifold can be defined by an integrable  $Sp(k, n; \mathbb{R})$ -structure, where  $Sp(k, n; \mathbb{R})$  denotes the  $k$ -symplectic group, defined by the set of matrices of the following type

$$\begin{pmatrix} T & & 0 & S_1 \\ & \ddots & & \vdots \\ & & T & S_k \\ 0 & & & {}^t(T^{-1}) \end{pmatrix}$$

where  $T \in Gl(n; \mathbb{R})$  and  $S_1, \dots, S_k$  are  $n \times n$  real matrices such that  $T^t S_\alpha = S_\alpha^t T$  for all  $\alpha \in \{1, \dots, k\}$ . The canonical model of these structures is the  $k$ -cotangent bundle  $(T_k^1)^* N$  of an arbitrary manifold  $N$ , which can be identified with the vector bundle  $J^1(N, \mathbb{R}^k)_0$  whose total space is the manifold of 1-jets of maps with target  $0 \in \mathbb{R}^k$ , and projection  $\tau^*(j_{x,0}^1 \sigma) = x$ . In this case, identifying  $(T_k^1)^* N$  with the Whitney sum of  $k$  copies of  $T^* N$ ,  $(T_k^1)^* N \cong T^* N \oplus \dots \oplus T^* N$ ,  $j_{x,0}^1 \sigma \mapsto (j_{x,0}^1 \sigma^1, \dots, j_{x,0}^1 \sigma^k)$ , where  $\sigma^\alpha = \pi_\alpha \circ \sigma : N \rightarrow \mathbb{R}$  is the  $\alpha$ -th component of  $\sigma$ , the  $k$ -symplectic structure on  $(T_k^1)^* N$  is given by  $\omega_\alpha = (\tau_\alpha^*)^*(\omega_0)$  and  $T\mathcal{F}_{j_{x,0}^1 \sigma} = \ker(\tau^*)_{*}(j_{x,0}^1 \sigma)$ , where  $\tau_\alpha^* : (T_k^1)^* N \rightarrow T^* N$  is the projection on the  $\alpha$ -th copy  $T^* N$  of  $(T_k^1)^* N$  and  $\omega_0$  is the standard symplectic structure on  $T^* N$ .

Returning to the general case of an arbitrary  $k$ -symplectic manifold  $(M, \omega_\alpha, \mathcal{F})$ , for each  $\alpha \in \{1, \dots, k\}$  we set

$$(2.1) \quad L_{\alpha_x} := \bigcap_{\beta \neq \alpha} C_x(\omega_\beta).$$

Then we have ([3]):

- (a) for each  $\alpha \in \{1, \dots, k\}$  the distribution  $L_\alpha = (L_{\alpha_x})_{x \in M}$  is integrable (we denote by  $\mathcal{F}_\alpha$  the foliation integral to  $L_\alpha$ );
- (b)  $L = L_1 \oplus \dots \oplus L_k$ ;
- (c) for each  $\alpha \in \{1, \dots, k\}$  the map  $i_\alpha : L_\alpha \rightarrow (N\mathcal{F})^*$ ,  $X \mapsto i_X \omega_\alpha$ , is an isomorphism, where  $N\mathcal{F}$  denotes the normal bundle of  $\mathcal{F}$ .

The standard Darboux theorem for Lagrangian foliations holds also for  $k$ -symplectic manifolds:

**Theorem 2.1** ([1]). *About any point of a  $k$ -symplectic manifold  $(M, \omega_\alpha, \mathcal{F})$ ,  $\alpha \in \{1, \dots, k\}$ , there exist local coordinates  $\{x_1, \dots, x_n, y_1, \dots, y_{kn}\}$  such that  $\omega_\alpha = \sum_{i=1}^n dx_i \wedge dy_{(\alpha-1)n+i}$*

and  $\mathcal{F}$  is described by the equations  $\{x_i = \text{const.}\}$ . In particular, for each  $\alpha \in \{1, \dots, k\}$ ,  $L_\alpha$  is generated by  $\frac{\partial}{\partial y_{(\alpha-1)n+1}}, \dots, \frac{\partial}{\partial y_{\alpha n}}$ .

Recall that a vector field  $X$  on a symplectic manifold  $(M^{2n}, \omega)$  is said to be symplectic if  $\mathcal{L}_X \omega = 0$ . For  $k$ -symplectic manifolds we prove the following lemma which will be useful in the sequel.

**Lemma 2.2.** *In any  $k$ -symplectic manifold,  $\mathcal{L}_X \omega_\alpha = 0$ , for any  $X \in \Gamma(L_\beta)$  with  $\alpha \neq \beta$ .*

*Proof.* Using the Cartan formula for the Lie derivative, we have  $\mathcal{L}_X \omega_\alpha = i_X d\omega_\alpha + di_X \omega_\alpha = di_X \omega_\alpha$ , since  $\omega_\alpha$  is closed. But, for any  $V \in \Gamma(TM)$ ,  $i_X \omega_\alpha(V) = 2\omega_\alpha(X, V) = 0$  from the definition of  $L_\alpha$ .  $\square$

### 3. A CANONICAL CONNECTION ON $k$ -SYMPLECTIC MANIFOLDS

Let  $(M, \omega_\alpha, \mathcal{F})$ ,  $\alpha \in \{1, \dots, k\}$ , be a  $k$ -symplectic manifold. In what follows,  $Q$  will denote an  $n$ -dimensional integrable distribution on  $M$  transversal to  $\mathcal{F}$  such that

- (i)  $\omega_\alpha(Y, Y') = 0$  for any  $Y, Y' \in \Gamma(Q)$  and for all  $\alpha \in \{1, \dots, k\}$ ,
- (ii)  $[X, Y] \in \Gamma(L_\alpha \oplus Q)$  for any  $X \in \Gamma(L_\alpha)$  and for any  $Y \in \Gamma(Q)$ .

Occasionally, we will denote by  $\mathcal{G}$  the foliation integral to  $Q$ .

The geometric interpretation of the condition (i) is that, for each  $\alpha \in \{1, \dots, k\}$  and for any  $x \in M$ ,  $Q_x$  is a Lagrangian subspace of the symplectic vector space  $(L_{\alpha_x} \oplus Q_x, \omega_{\alpha_x})$ . The condition (ii) is more technical; it will be essential for proving some preliminary results, like the following Lemma 3.2, and then for the generalization of the Hess's construction to the  $k$ -symplectic setting. Its geometric meaning is that for each fixed  $\alpha \in \{1, \dots, k\}$ , the subbundle  $L_\alpha \oplus Q$  is integrable, hence it defines a foliation whose leaves are symplectic manifolds with respect to the restriction of the  $k$ -symplectic form  $\omega_\alpha$  to the leaves. We also have that  $(L_\alpha, Q)$  is a bi-Lagrangian structure on the leaves of the foliation defined by  $L_\alpha \oplus Q$ .

A simple example of a  $k$ -symplectic manifold endowed with a transversal integrable distribution verifying (i) and (ii) is given by  $\mathbb{R}^{n(k+1)}$  with its standard  $k$ -symplectic structure given by Theorem 2.1 and taking as  $Q$  the distribution spanned by  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ .

We also remark that the splitting  $TM = L \oplus Q = L_1 \oplus \dots \oplus L_k \oplus Q$  induces a canonical isomorphism between  $Q$  and  $N\mathcal{F} := TM/L$ , the normal bundle to the foliation  $\mathcal{F}$ . In particular, it follows that  $Q^* = \text{ann}(L)$  and, arguing in the same way for the foliation  $\bigoplus_{\beta \neq \alpha} L_\beta \oplus Q$ , we get that  $L_\alpha^* = \text{ann}(\bigoplus_{\beta \neq \alpha} L_\beta \oplus Q)$ , for each  $\alpha \in \{1, \dots, k\}$ . Taking into account these remarks, we can prove the following preliminary lemmas:

**Lemma 3.1.** *Let  $X, X' \in \Gamma(L)$ . For each  $\alpha \in \{1, \dots, k\}$ , the map*

$$\varphi_\alpha^{XX'} : V \mapsto (\mathcal{L}_X i_{X'} \omega_\alpha)(V) = X(\omega_\alpha(X', V)) - \omega_\alpha(X', [X, V]),$$

*for any  $V \in \Gamma(TM)$ , belongs to  $Q^*$ .*

*Proof.* For any  $X'' \in \Gamma(L)$ ,  $(\mathcal{L}_X i_{X'} \omega_\alpha)(X'') = X(\omega_\alpha(X', X'')) - \omega_\alpha(X', [X, X'']) = 0$ , from which, since  $Q^* = \text{ann}(L)$ , we get the result.  $\square$

**Lemma 3.2.** *Let  $Y, Y' \in \Gamma(Q)$ . For each  $\alpha \in \{1, \dots, k\}$ , the map*

$$\psi_\alpha^{YY'} : V \mapsto (\mathcal{L}_Y i_{Y'} \omega_\alpha)(V) = Y(\omega_\alpha(Y', V)) - \omega_\alpha(Y', [Y, V]),$$

*for any  $V \in \Gamma(TM)$ , belongs to  $L_\alpha^*$ .*

*Proof.* Since  $L_\alpha^* = \text{ann}(\bigoplus_{\beta \neq \alpha} L_\beta \oplus Q)$ , we have to prove that  $(\mathcal{L}_Y i_{Y'} \omega_\alpha)(X) = 0$  and  $(\mathcal{L}_Y i_{Y'} \omega_\alpha)(Y'') = 0$  for any  $X \in \Gamma(L_\beta)$ ,  $\beta \neq \alpha$ , and for any  $Y'' \in \Gamma(Q)$ . Indeed,  $(\mathcal{L}_Y i_{Y'} \omega_\alpha)(X) = Y(\omega_\alpha(Y', X)) - \omega_\alpha(Y', [Y, X]) = 0$  by the definition of  $L_\beta$  and by (ii). Next,  $(\mathcal{L}_Y i_{Y'} \omega_\alpha)(Y'') = Y(\omega_\alpha(Y', Y'')) - \omega_\alpha(Y', [Y, Y'']) = 0$  by (i) and by the integrability of  $Q$ .  $\square$

**Theorem 3.3.** *Let  $(M, \omega_\alpha, \mathcal{F})$ ,  $\alpha \in \{1, \dots, k\}$ , be a  $k$ -symplectic manifold and let  $Q$  be an integrable distribution supplementary to  $T\mathcal{F}$  verifying the above conditions (i), (ii) and such that  $(i_1^*)^{-1}(\psi_1^{YY'}) = \dots = (i_k^*)^{-1}(\psi_k^{YY'})$  for any  $Y, Y' \in \Gamma(Q)$ , where  $\psi_1^{YY'}, \dots, \psi_k^{YY'}$  are the maps defined in Lemma 3.2. Then there exists a unique connection  $\nabla$  on  $M$  satisfying the following properties:*

- (1)  $\nabla \mathcal{F}_\alpha \subset \mathcal{F}_\alpha$  for each  $\alpha \in \{1, \dots, k\}$ , and  $\nabla Q \subset Q$ ,
- (2)  $\nabla \omega_1 = \dots = \nabla \omega_k = 0$ ,
- (3)  $T(X, Y) = 0$  for any  $X \in \Gamma(L)$  and for any  $Y \in \Gamma(Q)$ ,

where  $T$  denotes the torsion tensor field of  $\nabla$ .

*Proof.* According to the decomposition  $TM = L_1 \oplus \dots \oplus L_k \oplus Q$ , we define a connection  $\nabla^{L_\alpha}$  on each subbundle  $L_\alpha$ , a connection  $\nabla^Q$  on  $Q$  and then we take the sum of these connections for defining a global connection on  $M$ . Fix an  $\alpha \in \{1, \dots, k\}$ . We define  $\nabla_Y^{L_\alpha} X := [Y, X]_{L_\alpha}$  for any  $X \in \Gamma(L_\alpha)$  and  $Y \in \Gamma(Q)$ . Now we have to define  $\nabla_X^{L_\alpha} X'$  for  $X \in \Gamma(L)$ ,  $X' \in \Gamma(L_\alpha)$ . Since  $i_\alpha : L_\alpha \rightarrow Q^*$  is an isomorphism for any fixed  $X \in \Gamma(L)$ ,  $X' \in \Gamma(L_\alpha)$ , by Lemma 3.1, there exists a unique section  $H_\alpha(X, X') \in \Gamma(L_\alpha)$  such that  $i_\alpha(H_\alpha(X, X')) = \varphi_\alpha^{XX'}$ , that is  $\omega_\alpha(H_\alpha(X, X'), Y) = X(\omega_\alpha(X', Y)) - \omega_\alpha(X', [X, Y])$  for any  $Y \in \Gamma(Q)$ . We set  $\nabla_X^{L_\alpha} X' := H_\alpha(X, X') \in \Gamma(L_\alpha)$ . Now we define the connection  $\nabla^Q$ . For any  $X \in \Gamma(L)$  and  $Y \in \Gamma(Q)$  we put  $\nabla_X^Q Y := [X, Y]_Q$ . It remains to define  $\nabla_Y^Q Y'$  for  $Y, Y' \in \Gamma(Q)$ . The isomorphism  $i_\alpha : L_\alpha \rightarrow Q^*$  determines an isomorphism  $i_\alpha^*$  between  $Q$  and  $L_\alpha^*$  such that  $i_\alpha^*(Y)(X) = \omega_\alpha(Y, X)$ . Then, for any fixed  $Y, Y' \in \Gamma(Q)$ , by Lemma 3.2, there exists a unique section  $H_\alpha(Y, Y') \in \Gamma(Q)$  such that  $i_\alpha^*(H(Y, Y')) = \psi_\alpha^{YY'}$ , that is  $\omega_\alpha(H_\alpha(Y, Y'), X) = Y(\omega_\alpha(Y', X)) - \omega_\alpha(Y', [Y, X])$  for any  $X \in \Gamma(L_\alpha)$ . Moreover, our assumption ensures that  $H_1(Y, Y') = \dots = H_k(Y, Y') =: H(Y, Y')$ . We set  $\nabla_Y^Q Y' := H(Y, Y') \in \Gamma(Q)$ . Now we prove that  $\nabla^Q$  is a connection on  $Q$  and, for each  $\alpha \in \{1, \dots, k\}$ ,  $\nabla^{L_\alpha}$  is a connection on  $L_\alpha$ . For any  $X \in \Gamma(L)$ ,  $Y \in \Gamma(Q)$  and  $f \in C^\infty(M)$  we have

$$\begin{aligned} \nabla_{fX}^Q Y &= [fX, Y]_Q = f[X, Y]_Q - Y(f)X_Q = f[X, Y]_Q = f\nabla_X^Q Y, \\ \nabla_X^Q (fY) &= [X, fY]_Q = f[X, Y]_Q + X(f)Y_Q = f\nabla_X^Q Y + X(f)Y, \end{aligned}$$

and, for any  $X \in \Gamma(L_\alpha)$ ,  $Y, Y' \in \Gamma(Q)$ ,

$$\begin{aligned} \omega_\alpha(\nabla_{fY}^Q Y', X) &= \omega_\alpha(H_\alpha(fY, Y'), X) \\ &= fY(\omega_\alpha(Y', X)) - \omega_\alpha(Y', [fY, X]) \\ &= fY(\omega_\alpha(Y', X)) - f\omega_\alpha(Y', [Y, X]) + X(f)\omega_\alpha(Y', Y) \\ &= f\omega_\alpha(H_\alpha(Y, Y'), X) \\ &= \omega_\alpha(f\nabla_Y^Q Y', X), \end{aligned}$$

from which we get  $\nabla_{fY}^Q Y' = f\nabla_Y^Q Y'$ . Moreover,

$$\begin{aligned}\omega_\alpha(\nabla_Y^Q(fY'), X) &= \omega_\alpha(H(Y, fY'), X) \\ &= Y(\omega_\alpha(fY', X)) - \omega_\alpha(fY', [Y, X]) \\ &= fY(\omega_\alpha(Y', X)) + Y(f)\omega_\alpha(Y', X) - f\omega_\alpha(Y', [Y, X]) \\ &= f\omega_\alpha(H(Y, Y'), X) + Y(f)\omega_\alpha(Y', X) \\ &= \omega_\alpha(f\nabla_Y^Q Y' + Y(f)Y', X),\end{aligned}$$

from which we obtain  $\nabla_Y^Q(fY') = f\nabla_Y^Q Y' + Y(f)Y'$ . Now we prove that  $\nabla^{L_\alpha}$  is a connection on the subbundle  $L_\alpha$ , for each  $\alpha \in \{1, \dots, k\}$ . As before it is easy to show that  $\nabla_{fY}^{L_\alpha} X = f\nabla_Y^{L_\alpha} X$  and  $\nabla_Y^{L_\alpha}(fX) = f\nabla_Y^{L_\alpha} X + Y(f)X$  for any  $X \in \Gamma(L_\alpha)$  and  $Y \in \Gamma(Q)$ . Then for any  $X \in \Gamma(L)$ ,  $X' \in \Gamma(L_\alpha)$  and any  $Y \in \Gamma(Q)$

$$\begin{aligned}\omega_\alpha(\nabla_{fX}^{L_\alpha} X', Y) &= \omega_\alpha(H_\alpha(fX, X'), Y) \\ &= fX(\omega_\alpha(X', Y)) - \omega_\alpha(X', [fX, Y]) \\ &= fX(\omega_\alpha(X', Y)) - f\omega_\alpha(X', [X, Y]) + Y(f)\omega_\alpha(X', X) \\ &= f\omega_\alpha(H_\alpha(X, X'), Y) \\ &= \omega_\alpha(f\nabla_X^{L_\alpha} X', Y),\end{aligned}$$

from which we get  $\nabla_{fX}^{L_\alpha} X' = f\nabla_X^{L_\alpha} X'$ . Moreover,

$$\begin{aligned}\omega_\alpha(\nabla_X^{L_\alpha}(fX'), Y) &= \omega_\alpha(H_\alpha(X, fX'), Y) \\ &= X(\omega_\alpha(fX', Y)) - \omega_\alpha(fX', [X, Y]) \\ &= fX(\omega_\alpha(X', Y)) + X(f)\omega_\alpha(X', Y) - f\omega_\alpha(X', [X, Y]) \\ &= f\omega_\alpha(H_\alpha(X, X'), Y) + X(f)\omega_\alpha(X', Y) \\ &= \omega_\alpha(f\nabla_X^{L_\alpha} X' + X(f)X', Y)\end{aligned}$$

from which we get  $\nabla_X^{L_\alpha}(fX') = f\nabla_X^{L_\alpha} X' + X(f)X'$ . Therefore we can define a global connection on  $M$  putting, for any  $V, W \in \Gamma(TM)$ ,

$$(3.1) \quad \nabla_V W = \nabla_V^{L_1} W_{L_1} + \dots + \nabla_V^{L_k} W_{L_k} + \nabla_V^Q W_Q.$$

Now we prove that the connection  $\nabla$  satisfies (1)–(3). By construction  $\nabla$  preserves the distributions  $L_\alpha$  and  $Q$ . Then, by (1) we have that, obviously,  $(\nabla_V \omega_\alpha)(X, X') = 0$  for any  $X, X' \in \Gamma(L)$  and  $V \in \Gamma(TM)$ . For the same reason,  $(\nabla_V \omega_\alpha)(Y, Y') = 0$  for any  $Y, Y' \in \Gamma(Q)$  and  $V \in \Gamma(TM)$ . Now, let  $X \in \Gamma(L)$ ,  $X' \in \Gamma(L_\alpha)$  and  $Y \in \Gamma(Q)$ . Then

$$\begin{aligned}(\nabla_X \omega_\alpha)(X', Y) &= X(\omega_\alpha(X', Y)) - \omega_\alpha(H(X, X'), Y) - \omega_\alpha(X', [X, Y]_Q) \\ &= X(\omega_\alpha(X', Y)) - X(\omega_\alpha(X', Y)) + \omega_\alpha(X', [X, Y]) \\ &\quad - \omega_\alpha(X', [X, Y]) = 0.\end{aligned}$$

Moreover, for any  $\beta \neq \alpha$   $(\nabla_X \omega_\beta)(X', Y) = 0$  because  $\nabla_X X' \in \Gamma(L_\alpha)$ . Finally, for any  $X' \in \Gamma(L_\alpha)$  and  $Y, Y' \in \Gamma(Q)$ ,

$$\begin{aligned}(\nabla_Y \omega_\alpha)(X', Y') &= Y(\omega_\alpha(X', Y')) - \omega_\alpha([Y, X']_{L_\alpha}, Y') - \omega_\alpha(X', H(Y, Y')) \\ &= Y(\omega_\alpha(X', Y')) - \omega_\alpha([Y, X']_{L_\alpha}, Y') + Y(\omega_\alpha(Y', X')) \\ &\quad - \omega_\alpha(Y', [Y, X']) = 0.\end{aligned}$$

Thus we conclude that  $(\nabla_V \omega_\alpha)(X, Y) = 0$  for any  $X \in \Gamma(L_\alpha)$ ,  $Y \in \Gamma(Q)$  and  $V \in \Gamma(TM)$ . Analogously, one can compute for all the other cases, concluding that  $\nabla \omega_\alpha = 0$  for all  $\alpha \in \{1, \dots, k\}$ . Finally, for any  $X \in \Gamma(L_\alpha)$  and  $Y \in \Gamma(Q)$  we have  $T(X, Y) = [X, Y]_Q - [Y, X]_{L_\alpha} - [X, Y] = [X, Y]_{L_\alpha \oplus Q} - [X, Y] = 0$ , since by (ii)  $[X, Y] \in \Gamma(L_\alpha \oplus Q)$ . It remains to prove the uniqueness of this connection up to the properties (1)–(3). Let  $X \in \Gamma(L)$  and  $Y \in \Gamma(Q)$ . For any  $X' \in \Gamma(L)$  we have, by (1) and (3),  $\omega_\alpha(\nabla_X Y, X') = \omega_\alpha(\nabla_Y X + [X, Y], X') = \omega_\alpha([X, Y], X')$ , for all  $\alpha \in \{1, \dots, k\}$ , from which we get  $\nabla_X Y = [X, Y]_Q$ . Then, using (3) again, we obtain  $\nabla_Y X = [Y, X]_L$ . Moreover, for any  $X \in \Gamma(L)$ ,  $X' \in \Gamma(L_\alpha)$  and  $Y \in \Gamma(Q)$  by (2) we have  $\omega_\alpha(\nabla_X X', Y) = X(\omega_\alpha(X', Y)) - \omega_\alpha(X', \nabla_X Y) = X(\omega_\alpha(X', Y)) - \omega_\alpha(X', [X, Y]_Q) = X(\omega_\alpha(X', Y)) - \omega_\alpha(X', [X, Y]) = \omega_\alpha(H_\alpha(X, X'), Y)$ , from which, since  $\nabla_X X', H_\alpha(X, X') \in \Gamma(L_\alpha)$ , we get  $\nabla_X X' = H_\alpha(X, X')$ . Similarly, one can find that  $\nabla_Y Y' = H(Y, Y')$  for any  $Y, Y' \in \Gamma(Q)$ .  $\square$

**Proposition 3.4.** *The connection  $\nabla$  defined in Theorem 3.3 is torsion free along the leaves of the foliations  $\mathcal{F}$  and  $\mathcal{G}$ .*

*Proof.* Let  $X \in \Gamma(L_\beta)$  and  $X' \in \Gamma(L_\alpha)$  and assume that  $\alpha \neq \beta$ . We have  $T(X, X') = H_\alpha(X, X') - H_\beta(X', X) - [X, X'] \in \Gamma(L)$ . Then for any  $Y \in \Gamma(Q)$

$$\begin{aligned} \omega_\alpha(T(X, X'), Y) &= \omega_\alpha(H_\alpha(X, X') - [X, X'], Y) \\ &= X(\omega_\alpha(X', Y)) - \omega_\alpha(X', [X, Y]) - \omega_\alpha([X, X'], Y) \\ &= 3d\omega_\alpha(X, X', Y) = 0 \end{aligned}$$

since each  $\omega_\alpha$  is closed. Analogously,  $\omega_\alpha(T(X, X'), Y) = 0$ . Moreover, for each  $\gamma \neq \alpha, \beta$

$$\omega_\gamma(T(X, X'), Y) = -\omega_\gamma([X, X'], Y) = 3d\omega_\gamma(X, X', Y) = 0.$$

Then  $T(X, X') \in C(\omega_1) \cap \dots \cap C(\omega_k) = \{0\}$ . If  $X, X' \in \Gamma(L_\alpha)$ , we have  $T(X, X') = H_\alpha(X, X') - H_\alpha(X', X) - [X, X'] \in \Gamma(L_\alpha)$  and

$$\begin{aligned} \omega_\alpha(T(X, X'), Y) &= X(\omega_\alpha(X', Y)) - \omega_\alpha(X', [X, Y]) - X'(\omega_\alpha(X, Y)) \\ &\quad + \omega_\alpha(X, [X', Y]) - \omega_\alpha([X, X'], Y) \\ &= 3d\omega_\alpha(X, X', Y) = 0, \end{aligned}$$

hence  $T(X, X') = 0$ . Analogously, one can prove that  $T(Y, Y') = 0$  for any  $Y, Y' \in \Gamma(Q)$ .  $\square$

**Proposition 3.5.** *The curvature tensor field of the connection  $\nabla$  defined in Theorem 3.3 vanishes along the leaves of the foliations  $\mathcal{F}$  and  $\mathcal{G}$ .*

*Proof.* For any  $X, X' \in \Gamma(L)$  and  $Y \in \Gamma(Q)$ , using the integrability of  $L$ , we have

$$\begin{aligned} R_{X, X'} Y &= \nabla_X [X', Y]_Q - \nabla_{X'} [X, Y]_Q - \nabla_{[X, X']} Y \\ &= \nabla_X [X', Y]_Q - \nabla_{X'} [X, Y]_Q - [[X, X'], Y]_Q = 0 \end{aligned}$$

by the Jacobi identity. Then, for any  $X, X' \in \Gamma(L)$  and  $X'' \in \Gamma(L_\alpha)$  we have

$$(3.2) \quad R_{X, X'} X'' = H_\alpha(X, H_\alpha(X', X'')) - H_\alpha(X', H_\alpha(X, X'')) - H_\alpha([X, X'], X'')$$

Now, for any  $Y \in \Gamma(Q)$

$$\begin{aligned}\omega_\alpha(H_\alpha(X, H_\alpha(X', X'')), Y) &= X(\omega_\alpha(H_\alpha(X', X''), Y)) - \omega_\alpha(H_\alpha(X', X''), [X, Y]) \\ &= X(\omega_\alpha(H(X', X''), Y)) - \omega_\alpha(H(X', X''), [X, Y]) \\ &= X(X'(\omega_\alpha(X'', Y))) - X(\omega_\alpha(X'', [X', Y])) \\ &\quad - X'(\omega_\alpha(X'', [X', [X, Y]]) + \omega_\alpha(X'', [X', [X, Y]]),\end{aligned}$$

$$\begin{aligned}\omega_\alpha(H_\alpha(X', H_\alpha(X, X'')), Y) &= X'(\omega_\alpha(H_\alpha(X, X''), Y)) - \omega_\alpha(H_\alpha(X, X''), [X', Y]) \\ &= X'(\omega_\alpha(H(X, X''), Y)) - \omega_\alpha(H(X, X''), [X', Y]) \\ &= X'(X(\omega_\alpha(X'', Y))) - X'(\omega_\alpha(X'', [X, Y])) \\ &\quad - X(\omega_\alpha(X'', [X, [X', Y]])) + \omega_\alpha(X'', [X, [X', Y]])\end{aligned}$$

and

$$\omega_\alpha(H_\alpha([X, X'], X''), Y) = [X, X'](\omega_\alpha(X'', Y)) - \omega_\alpha(X'', [[X, X'], Y]).$$

Therefore

$$\begin{aligned}\omega_\alpha(R_{X, X'}X'', Y) &= [X, X'](\omega_\alpha(X'', Y)) + \omega_\alpha(X'', [X', [X, Y]]) - \omega_\alpha(X'', [X, [X', Y]]) \\ &\quad - [X, X'](\omega_\alpha(X'', Y)) + \omega_\alpha(X'', [[X, X'], Y]) \\ &= \omega_\alpha(X'', [[X, X'], Y] + [[X', Y], X] + [[Y, X], X']) = 0\end{aligned}$$

by the Jacobi identity. This shows that  $R_{X, X'} = 0$  for any  $X, X' \in \Gamma(L)$ . In the same way, one can prove the flatness along the leaves of the foliation defined by  $Q$ .  $\square$

**Corollary 3.6.** *The leaves of the foliations  $\mathcal{F}$  and  $\mathcal{G}$  admit a canonical flat affine structure.*

Now we give an interpretation of the connection stated in Theorem 3.3 in terms of some geometric structures which can be attached to a  $k$ -symplectic manifold. So let  $(M, \omega_\alpha, \mathcal{F})$ ,  $\alpha \in \{1, \dots, k\}$ , be a  $k$ -symplectic manifold and let  $Q$  be a distribution transversal to  $\mathcal{F}$  such that  $\omega_\alpha(Y, Y') = 0$  for any  $Y, Y' \in \Gamma(Q)$ . Assume that  $M$  admits a Riemannian metric  $g$  such that the distributions  $L_1, \dots, L_k, Q$  are mutually orthogonal. For each  $\alpha \in \{1, \dots, k\}$ , since  $\omega_\alpha$  is non-degenerate on  $L_\alpha \oplus Q$ , one can find a linear map  $A_\alpha : L_\alpha \oplus Q \rightarrow L_\alpha \oplus Q$  such that  $\omega_\alpha(X, Y) = g(X, A_\alpha Y)$ , for any  $X, Y \in \Gamma(L_\alpha \oplus Q)$ . The operator  $A_\alpha$ ,  $\alpha \in \{1, \dots, k\}$ , is skew-symmetric and  $A_\alpha A_\alpha^*$ ,  $\alpha \in \{1, \dots, k\}$ , is symmetric and positive definite, thus it diagonalizes with positive eigenvalues  $(\lambda_\alpha)_i$ ,  $i \in \{1, \dots, 2n\}$ ,  $A_\alpha A_\alpha^* = B_\alpha \text{diag}\{(\lambda_\alpha)_1, \dots, (\lambda_\alpha)_{2n}\} B_\alpha^{-1}$ . Set  $\sqrt{A_\alpha A_\alpha^*} := B_\alpha \text{diag}\{\sqrt{(\lambda_\alpha)_1}, \dots, \sqrt{(\lambda_\alpha)_{2n}}\} B_\alpha^{-1}$  which is also symmetric and positive definite. Set

$$J_\alpha := \begin{cases} (\sqrt{A_\alpha A_\alpha^*})^{-1} A_\alpha, & \text{on } L_\alpha \oplus Q; \\ 0, & \text{on } L_\beta, \beta \neq \alpha. \end{cases}$$

Then  $(J_1, \dots, J_k)$  is a family of endomorphisms of the tangent space satisfying

- (i)  $L_\alpha = \bigcap_{\beta \neq \alpha} \ker(J_\beta)$ ,
- (ii)  $J_\alpha^2 = -I$  on  $L_\alpha \oplus Q$  and  $J_\alpha L_\alpha = Q$ ,  $J_\alpha Q = L_\alpha$ ,
- (iii)  $\omega_\alpha(X, Y) = g(X, J_\alpha Y)$  for any  $X, Y \in \Gamma(TM)$ .

Note also that the Riemannian metric  $g$  satisfies  $g(J_\alpha X, J_\alpha Y) = g(X, Y)$  for each  $\alpha \in \{1, \dots, k\}$  and for any  $X, Y \in \Gamma(TM)$ . We call  $(J_1, \dots, J_k, g)$  a *compatible almost  $k$ -Kähler structure*. Now assume to be under the assumptions of Theorem 3.3. Note that for each  $\alpha \in \{1, \dots, k\}$ , the leaves of the foliation defined by  $L_\alpha \oplus Q$ , endowed with the

tensor fields induced by  $J_\alpha$ , are almost Kähler manifolds. Then we have that  $[J_\alpha, J_\alpha] = 0$  if and only if each leaf of the foliation  $L_\alpha \oplus Q$  is Kählerian. When  $[J_\alpha, J_\alpha] = 0$ , for each  $\alpha \in \{1, \dots, k\}$ , that is the leaves of all the foliations  $L_\alpha \oplus Q$  are Kähler manifolds, we say that  $(M, \omega_\alpha, \mathcal{F}, J_\alpha, g)$  is a  $k$ -Kähler manifold. Then we have the following result.

**Theorem 3.7.** *Let  $(M, \omega_\alpha, \mathcal{F}, J_\alpha, g)$ ,  $\alpha \in \{1, \dots, k\}$ , be a  $k$ -Kähler manifold. If the Levi-Civita connection  $\nabla^g$  preserves the distributions  $L_\alpha$ , then it preserves also  $Q$  and it coincides with the canonical connection  $\nabla$ .*

*Proof.* We show that the Levi-Civita connection  $\nabla^g$  satisfies the properties (1), (2), (3) which, according to Theorem 3.3, define uniquely the canonical connection  $\nabla$ . First of all we prove that  $\nabla^g$  preserves  $Q$ . Let  $Y \in \Gamma(Q)$ . Then, since  $\nabla^g g = 0$ , for any  $V \in \Gamma(TM)$  and  $X \in \Gamma(T\mathcal{F})$ , we have

$$0 = (\nabla_V^g g)(X, Y) = V(g(X, Y)) - g(\nabla_V^g X, Y) - g(X, \nabla_V^g Y) = -g(X, \nabla_V^g Y),$$

since  $\nabla^g \mathcal{F} \subset \mathcal{F}$ . Thus  $\nabla^g Q \subset Q$ . Finally we have to prove that  $\nabla^g \omega_\alpha = 0$ , for each  $\alpha \in \{1, \dots, k\}$ . We observe, firstly, that  $\nabla^g J_\alpha = 0$ , for each  $\alpha \in \{1, \dots, k\}$ . This is a consequence of the definition of  $J_\alpha$ , of the fact that the leaves of the foliation defined by  $L_\alpha \oplus Q$  are Kählerian manifolds, and of the above properties that  $\nabla^g L_\alpha \subset L_\alpha$  and  $\nabla^g Q \subset Q$ . Now we can prove that  $(\nabla_V^g \omega_\alpha)(W, W') = 0$ , for any  $V, W, W' \in \Gamma(TM)$ . This equality holds immediately for  $W, W' \in \Gamma(L)$  and for  $W, W' \in \Gamma(Q)$  because  $L$  and  $Q$  are preserved by  $\nabla^g$ . So it remains to show that  $(\nabla_V^g \omega_\alpha)(X, Y) = 0$ , for any  $X \in \Gamma(L)$  and  $Y \in \Gamma(Q)$ . In fact, since  $\nabla^g J_\alpha = 0$  and  $\nabla^g g = 0$ ,

$$\begin{aligned} (\nabla_V^g \omega_\alpha)(X, Y) &= V(g(X, J_\alpha Y)) - g(\nabla_V^g X, J_\alpha Y) - g(X, J_\alpha \nabla_V^g Y) \\ &= V(g(X, J_\alpha Y)) - g(\nabla_V^g X, J_\alpha Y) - g(X, \nabla_V^g J_\alpha Y) \\ &= (\nabla_V^g g)(X, J_\alpha Y) = 0. \end{aligned}$$

This concludes the proof.  $\square$

#### 4. APPLICATIONS

In this section we will examine some consequences of Theorem 3.3. It can be useful to find the connection defined in Theorem 3.3 in Darboux coordinates  $\{x_1, \dots, x_n, y_1, \dots, y_{kn}\}$  according to Theorem 2.1. There exist functions  $t_i^{\alpha j}$  such that  $Q = \text{span}\{X_1, \dots, X_n\}$ , where  $X_i := \frac{\partial}{\partial x_i} - \sum_{\alpha=1}^k \sum_{j=1}^n t_i^{\alpha j} \frac{\partial}{\partial y_{(\alpha-1)n+j}}$ . We put  $Y_{\alpha i} := \frac{\partial}{\partial y_{(\alpha-1)n+i}}$ . Then by a straightforward computation we have that

$$\begin{aligned} \nabla_{Y_{\alpha i}} Y_{\beta j} &= 0, \quad \nabla_{Y_{\alpha i}} X_j = 0, \\ \nabla_{X_i} Y_{\alpha j} &= \sum_{\beta=1}^k \sum_{h=1}^n \frac{\partial t_i^{\beta h}}{\partial y_{(\alpha-1)n+j}} Y_{\beta h}, \quad \nabla_{X_i} X_j = - \sum_{h=1}^n \frac{\partial t_i^{\alpha j}}{\partial y_{(\alpha-1)n+h}} X_h, \end{aligned}$$



where the functions  $t_i^{\alpha j}$  satisfy the conditions  $\frac{\partial t_i^{\alpha j}}{\partial y_{(\beta-1)n+h}} = 0$  for  $\alpha \neq \beta$ , and  $\frac{\partial t_i^{1j}}{\partial y_h} = \dots = \frac{\partial t_i^{kj}}{\partial y_{(k-1)n+h}}$ , for all  $i, j, h \in \{1, \dots, n\}$ . Moreover, the curvature is given by

$$(4.1) \quad R_{Y_{\alpha i}, Y_{\beta j}} = 0, \quad R_{X_i, X_j} = 0,$$

$$(4.2) \quad R_{Y_{\alpha i}, X_j} Y_{\beta h} = \sum_{\gamma=1}^k \sum_{l=1}^n \frac{\partial^2 t_j^{\gamma i}}{\partial y_{(\alpha-1)n+i} \partial y_{(\beta-1)n+h}} Y_{\gamma l},$$

$$(4.3) \quad R_{Y_{\alpha i}, X_j} X_k = - \sum_{l=1}^n \frac{\partial^2 t_i^{\alpha k}}{\partial y_{(\alpha-1)n+i} \partial y_{(\alpha-1)n+l}} X_l.$$

Then we have that the curvature 2-form of  $\nabla$  has the following very simple expression<sup>1</sup>:

$$\Omega = \sum \Omega_{\alpha i j} dx_i \wedge dy_{(\alpha-1)n+j}.$$

from which it follows that  $\Omega^h$  vanishes for  $h > n$ . Thus if  $f \in I^h(G)$  is an  $\text{ad}(G)$ -invariant polynomial of degree  $h$ , where  $G = Sp(k, n; \mathbb{R})$ , we have that  $f(\Omega) = 0$  for  $h = \text{deg}(f) > n$ . This proves the following result.

**Proposition 4.1.** *Under the assumptions of Theorem 3.3, we have that  $\text{Pont}^j(TM) = 0$  for all  $j > 2n$ , where  $\text{Pont}(TM)$  denotes the Pontryagin algebra of the bundle  $TM$ .*

Another strong consequence of Theorem 3.3 is the existence of an Ehresmann connection. We recall the concept of *Ehresmann connection* for foliations. Let  $(M, \mathcal{F})$  be a foliated manifold and  $D$  a distribution on  $M$  which is supplementary to the tangent bundle  $L$  of the foliation  $\mathcal{F}$  at every point. A *horizontal curve* is a piecewise smooth curve  $\beta : [0, b] \rightarrow M$ ,  $b \in \mathbb{R}$ , such that  $\beta'(t) \in D_{\beta(t)}$  for all  $t \in [0, b]$ . A *vertical curve* is a piecewise smooth curve  $\alpha : [0, a] \rightarrow M$ ,  $a \in \mathbb{R}$ , such that  $\alpha'(t) \in L_{\alpha(t)}$  for all  $t \in [0, a]$ , i.e. which lies entirely in one leaf of  $\mathcal{F}$ . A *rectangle* is a piecewise smooth map  $\sigma : [0, a] \times [0, b] \rightarrow M$  such that for every fixed  $s \in [0, b]$  the curve  $\sigma_s := \sigma|_{[0, a] \times \{s\}}$  is vertical and for every fixed  $t \in [0, a]$  the curve  $\sigma^t := \sigma|_{\{t\} \times [0, b]}$  is horizontal. The curves  $\sigma_0 = \sigma(\cdot, 0)$ ,  $\sigma_b = \sigma(\cdot, b)$ ,  $\sigma^0 = \sigma(0, \cdot)$  and  $\sigma^a = \sigma(a, \cdot)$  are called, respectively, the *initial vertical edge*, the *final vertical edge*, the *initial horizontal edge* and the *final horizontal edge* of  $\sigma$ . We say that the distribution  $D$  is an *Ehresmann connection* for  $\mathcal{F}$  if for every vertical curve  $\alpha$  and horizontal curve  $\beta$  with the same initial point, there exists a rectangle whose initial edges are  $\alpha$  and  $\beta$  (cf. [6]). This rectangle is unique and is called the *rectangle associated to  $\alpha$  and  $\beta$* . It is known ([5]) that every totally geodesic foliation of a complete Riemannian manifold admits an Ehresmann connection, namely the distribution orthogonal to the leaves of the foliation. Furthermore, by the duality Riemannian – totally geodesic, the orthogonal bundle to a Riemannian foliation is also an Ehresmann connection for this foliation.

Recall that given a foliated manifold  $(M, \mathcal{F})$  and a supplementary subbundle  $D$  to  $T\mathcal{F}$  (not necessarily an Ehresmann connection), any horizontal curve  $\tau : [0, 1] \rightarrow M$  defines a family of diffeomorphisms  $\{\varphi_t : V_0 \rightarrow V_t\}_{t \in [0, 1]}$  such that

- (1) each  $V_t$  is a neighborhood of  $\tau(t)$  in the leaf of  $\mathcal{F}$  through  $\tau(t)$ , for all  $t \in [0, 1]$ ,
- (2)  $\varphi_t(\tau(0)) = \tau(t)$  for all  $t \in [0, 1]$ ,
- (3) for any fixed  $p \in V_0$  the curve  $t \mapsto \varphi_t(p)$  is horizontal,

<sup>1</sup>Throughout all this work, if no confusion is feared, we identify forms on  $M$  with their lifts to principal bundle of linear frames  $LM$ .

(4)  $\varphi_0 : V_0 \rightarrow V_0$  is the identity map.

This family of diffeomorphisms is called an *element of holonomy* along  $\tau$  ([6]). It is shown in ([14]) and in ([5]) that an element of holonomy along  $\tau$  exists and is unique, in the sense that any two elements of holonomy must agree on some neighborhood of  $\tau(0)$  in the leaf through  $\tau(0)$ . When the leaves of  $\mathcal{F}$  have a geometric structure – such as a Riemannian metric or a linear connection – we say that  $D$  *preserves the geometry of the leaves* if the element of holonomy along any horizontal curve is a local isomorphism of the particular geometric structure.

Using the canonical connection which we have defined in § 3 we prove now the following result.

**Theorem 4.2.** *Let  $(M, \omega_\alpha, \mathcal{F})$ ,  $\alpha \in \{1, \dots, k\}$ , be a compact connected  $k$ -symplectic manifold and let  $Q$  be an integrable distribution transversal to  $\mathcal{F}$  and satisfying the assumptions of Theorem 3.3. If the leaves of  $\mathcal{F}$  are complete affine manifolds, then the distribution  $Q$  is an Ehresmann connection for  $\mathcal{F}$ . Furthermore, if the canonical connection  $\nabla$  on  $M$  induced by  $Q$  is everywhere flat, then the Ehresmann connection  $Q$  preserves  $\nabla$ .*

*Proof.* Let  $\alpha : [0, a] \rightarrow M$  and  $\beta : [0, b] \rightarrow M$  be, respectively, a vertical and a horizontal curve such that  $\alpha(0) = x = \beta(0)$ . We need to show that there exists a full rectangle  $\sigma : [0, a] \times [0, b] \rightarrow M$  whose initial vertical and horizontal edges are just  $\alpha$  and  $\beta$ , respectively. First we will show it under the further assumption that  $\alpha$  is a geodesic (with respect to the connection  $\nabla$ ). Fix an  $s \in [0, b]$ . We transport by parallelism the vector  $\alpha'(0)$  along the curve  $\beta$ , obtaining a vector  $v_s \in T_{\beta(s)}M$  which is in turn tangent to  $\mathcal{F}$  since the  $\nabla$ -parallel transport preserves the foliation  $\mathcal{F}$  (note also that the vector  $v_s$  does not depend on the curve because the curvature vanishes identically). Let  $\tau_s$  be the geodesic determined by the initial conditions  $\tau_s(0) = \beta(s)$  and  $\tau_s'(0) = v_s$ . Since the foliation  $\mathcal{F}$  is totally geodesic (with respect to  $\nabla$ ),  $\tau_s$  is a curve lying on the leaf  $\mathcal{L}_s$  of  $\mathcal{F}$  passing for  $\beta(s)$ , and the assumption on the completeness of  $\mathcal{L}_s$  implies that we can extend  $\tau_s$  for all the values of the parameter  $t$ . In this way we obtain a map  $\sigma : [0, a] \times [0, b] \rightarrow M$ , defined by  $\sigma(t, s) := \tau_s(t)$ , and it is easy to show that it is just the rectangle we are looking for. Now we have to prove the theorem dropping the assumption that the curve  $\alpha$  is a geodesic. Because  $M$  is compact and the leaves of  $\mathcal{F}$  are complete affine manifolds with respect to  $\nabla$ , we find  $\epsilon > 0$  such that for any  $x \in M$ , the  $\epsilon$ -ball  $B(x, \epsilon)$  is convex. As the leaves are totally geodesic, the  $\epsilon$ -balls  $B_{\mathcal{L}}(x, \epsilon)$  in any leaf  $\mathcal{L}$  coincide with the corresponding connected components of  $B(x, \epsilon) \cap \mathcal{L}$ . Therefore, for any  $x \in M$ , there exists  $\epsilon > 0$  such that the  $\epsilon$ -balls  $B_{\mathcal{L}}(x, \epsilon)$  are convex. Suppose now that  $\alpha : [0, a] \rightarrow M$  is a vertical curve contained in  $B_{\mathcal{L}}(x, \epsilon)$ , with  $x = \alpha(0)$ . Let  $\alpha_t$  denote the geodesic on  $\mathcal{L}$  joining  $x$  with  $\alpha(t)$ , for any fixed  $t \in [0, a]$ . Then we define

$$\sigma(t, s) := \sigma_{\alpha_t, \beta|_{[0, s]}}(t, s),$$

for any  $(t, s) \in [0, a] \times [0, b]$ , where  $\sigma_{\alpha_t, \beta|_{[0, s]}}$  denotes the rectangle associated to the curves  $\alpha_t$  and  $\beta|_{[0, s]}$ . By the first part of the proof,  $\sigma$  is just the rectangle whose initial edges are  $\alpha$  and  $\beta$ . Finally, if  $\alpha$  is *any* leaf curve on  $M$ , not necessarily contained in  $B_{\mathcal{L}}(x, \epsilon)$ , then we can always find a partition of  $[0, a]$ , say  $0 = t_0 < t_1 < \dots < t_m = a$ , with the property that, for any  $i \in \{0, \dots, m-1\}$ ,  $\alpha(t_i), \alpha(t_{i+1}) \in B(\alpha(t_i), \epsilon)$ . Let  $\sigma_{(0)}$  be the rectangle corresponding to  $\alpha|_{[0, t_1]}$  and  $\beta$ . The curve  $\beta_1 := \sigma_{(0)}|_{\{t_1\} \times [0, b]}$  is horizontal and  $\beta_1(0) = \alpha(t_1)$ , so we can find a rectangle  $\sigma_{(1)}$  whose edges are  $\alpha|_{[t_1, t_2]}$  and  $\beta_1$ . After  $m$  steps we have  $m$  rectangles  $\sigma_{(0)}, \sigma_{(1)}, \dots, \sigma_{(m-1)}$  and we can define

$\sigma := \sigma_{(0)} \cup \sigma_{(1)} \cup \cdots \cup \sigma_{(m-1)}$  obtaining the rectangle whose initial edges are  $\alpha$  and  $\beta$ . The last part of the statement follows directly from ([6, Proposition 5.3]).  $\square$

The existence of an Ehresmann connection implies strong consequences for the foliation. Many of them have been studied in ([6]), from which we have the following results.

**Corollary 4.3.** *Let  $(M, \omega_\alpha, \mathcal{F})$ ,  $\alpha \in \{1, \dots, k\}$ , be a  $k$ -symplectic manifold satisfying the assumptions of Theorem 4.2. Then the following statements hold:*

- (a) *Any two leaves of  $\mathcal{F}$  can be joined by a horizontal curve.*
- (b) *The universal covers of any two leaves of  $\mathcal{F}$  are isomorphic.*
- (c) *The universal cover  $\tilde{M}$  of  $M$  is topologically a product  $\tilde{\mathcal{L}} \times \tilde{Q}$ , where  $\tilde{\mathcal{L}}$  is the universal cover of the leaves of  $\mathcal{F}$  and  $\tilde{Q}$  the universal cover of the leaves of the foliation integral to  $Q$ .*

In general, to each leaf  $\mathcal{L}$  of a foliation admitting an Ehresmann connection  $D$ , it is attached a group  $H_D(\mathcal{L}, x)$ ,  $x \in \mathcal{L}$ , defined as follows ([6]). Let  $\Omega_x$  be the set of all horizontal curves  $\beta : [0, 1] \rightarrow M$  with starting point  $x$ . Then there is an action of the fundamental group  $\pi_1(\mathcal{L}, x)$  of  $\mathcal{L}$  on  $\Omega_x$  given in the following way: for any  $\delta = [\tau] \in \pi_1(\mathcal{L}, x)$  and for any  $\beta \in \Omega_x$ ,  $\tau \cdot \beta$  is the final horizontal edge of the rectangle corresponding to  $\tau$  and  $\beta$ . It can be proved that this definition does not depend on the vertical loop  $\tau$  in  $x$  representing  $\delta$ . Let  $K_D(\mathcal{L}, x) = \{\delta \in \pi_1(\mathcal{L}, x) : \tau \cdot \beta = \beta \text{ for all } \beta \in \Omega_x\}$ . Then  $K_D(\mathcal{L}, x)$  is a normal subgroup of  $\pi_1(\mathcal{L}, x)$  and we define

$$H_D(\mathcal{L}, x) := \pi_1(\mathcal{L}, x) / K_D(\mathcal{L}, x).$$

It is known that  $H_D(\mathcal{L}, x)$  does not depend on the Ehresmann connection  $D$ , thus it is an invariant of the foliation. Then we have the following result.

**Corollary 4.4.** *Let  $(M, \omega_\alpha, \mathcal{F})$ ,  $\alpha \in \{1, \dots, k\}$ , be a  $k$ -symplectic manifold satisfying Theorem 4.2. If  $\mathcal{F}$  has a compact leaf  $\mathcal{L}_0$  with finite  $H_D(\mathcal{L}_0, x_0)$ , then every leaf  $\mathcal{L}$  of  $\mathcal{F}$  is compact with finite  $H_D(\mathcal{L}, x)$ .*

*Proof.* It is a direct consequence of ([7, Theorem 1]).  $\square$

Another consequence of Theorem 4.2 is the following result.

**Corollary 4.5.** *Let  $(M, \omega_\alpha, \mathcal{F})$ ,  $\alpha \in \{1, \dots, k\}$ , be a  $k$ -symplectic manifold satisfying the assumptions of Theorem 4.2. Then  $\mathcal{F}$  has no vanishing cycles. Moreover, the homotopy groupoid of  $\mathcal{F}$  is a Hausdorff manifold.*

*Proof.* The assertions follow from ([25, Theorem 2]) and ([25, Corollary 2]).  $\square$

Now we study more deeply  $k$ -symplectic manifolds whose canonical connections are flat. From (4.1)–(4.3) it follows that the geometric interpretation of the flatness of  $\nabla$  is that the functions  $t_i^{\alpha_j}$  are leaf-wise affine. Usually this condition is expressed saying that  $Q$  is an *affine transversal distribution* for  $\mathcal{F}$  (see, for instance, [22], [23]). In the following theorem we give a normal form for flat  $k$ -symplectic manifolds:

**Theorem 4.6.** *Let  $(M, \omega_\alpha, \mathcal{F})$ ,  $\alpha \in \{1, \dots, k\}$ , be a  $k$ -symplectic manifold and  $Q$  a distribution satisfying the assumptions of Theorem 3.3. If the corresponding canonical connection  $\nabla$  is flat, then there exist local coordinates  $\{x_1, \dots, x_n, y_1, \dots, y_{kn}\}$  with respect to which each 2-form  $\omega_\alpha$  is given by*

$$(4.4) \quad \omega_\alpha = \sum_{i=1}^n dx_i \wedge dy_{(\alpha-1)n+i},$$

$\mathcal{F}$  is described by the equations  $\{x_1 = \text{const.}, \dots, x_n = \text{const.}\}$  and  $Q$  is spanned by  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ .

*Proof.* Let  $x \in M$  be a point and  $U \subset M$  a chart containing  $x$ . One can consider an adapted basis  $\{e_1, \dots, e_{n(k+1)}\}$  of  $T_x M$  such that, for each  $\alpha \in \{1, \dots, k\}$ ,  $\{e_{(\alpha-1)n+1}, \dots, e_{\alpha n}\}$  is a basis of  $L_{\alpha x}$ ,  $\{e_{kn+1}, \dots, e_{n(k+1)}\}$  is a basis of  $Q_x$ , and

$$(4.5) \quad \omega_\alpha (e_{(\beta-1)n+i}, e_{(\gamma-1)n+j}) = \omega_\alpha (e_{kn+i}, e_{kn+j}) = 0,$$

$$(4.6) \quad \omega_\alpha (e_{(\beta-1)n+i}, e_{kn+j}) = -\frac{1}{2} \delta_{\alpha\beta} \delta_{ij},$$

for all  $\alpha, \beta, \gamma \in \{1, \dots, k\}$ ,  $i, j \in \{1, \dots, n\}$ . For each  $l \in \{1, \dots, n(k+1)\}$  we define a vector field  $E_k$  on  $U$  by the  $\nabla$ -parallel transport along curves. More precisely, for any  $y \in U$  we consider a curve  $\gamma : [0, 1] \rightarrow U$  such that  $\gamma(0) = x$ ,  $\gamma(1) = y$  and define  $E_l(y) := \tau_\gamma(e_l)$ ,  $\tau_\gamma : T_x M \rightarrow T_y M$  being the parallel transport along  $\gamma$ . Note that  $E_l(y)$  does not depend on the curve joining  $x$  and  $y$ , since  $R \equiv 0$ . Thus we obtain  $n(k+1)$  vector fields on  $U$ ,  $E_1, \dots, E_{n(k+1)}$  such that, for each  $\alpha \in \{1, \dots, k\}$ ,  $i \in \{1, \dots, n\}$ ,  $E_{(\alpha-1)n+i} \in \Gamma(L_\alpha)$  and  $E_{kn+i} \in \Gamma(Q)$ , since the connection  $\nabla$  preserves the subbundles  $L_\alpha$  and  $Q$ . Moreover, by (4.5)–(4.6) we have for any  $y \in U$  and  $\alpha, \beta, \gamma \in \{1, \dots, k\}$ ,  $i, j \in \{1, \dots, n\}$

$$(4.7) \quad \omega_\alpha (E_{(\beta-1)n+i}, E_{(\gamma-1)n+j}) = \omega_\alpha (E_{kn+i}, E_{kn+j}) = 0,$$

$$(4.8) \quad \omega_\alpha (E_{(\beta-1)n+i}, E_{kn+j}) = -\frac{1}{2} \delta_{\alpha\beta} \delta_{ij}.$$

Indeed, for all  $l, m \in \{1, \dots, n(k+1)\}$ ,

$$\frac{d}{dt} \omega_\alpha (E_l(\gamma(t)), E_m(\gamma(t))) = \omega_\alpha (\nabla_{\gamma'} E_l, E_m) + \omega_\alpha (E_l, \nabla_{\gamma'} E_m) = 0$$

because  $\omega_\alpha$  is parallel with respect to  $\nabla$ . Thus  $\omega_{\alpha_x}(e_l, e_m) = \omega_{\alpha_y}(E_l(y), E_m(y))$ , for any  $y \in U$ . Note that, by construction, we have  $\nabla_{E_l} E_m = 0$  for all  $l, m \in \{1, \dots, n(k+1)\}$ . From this, Theorem 3.3 and Proposition 3.4 it follows that the vector fields  $E_1, \dots, E_{n(k+1)}$  commute each other. Therefore there exist local coordinates  $\{x_1, \dots, x_n, y_1, \dots, y_{kn}\}$ ,  $\alpha \in \{1, \dots, k\}$ , such that  $E_{(\alpha-1)n+i} = \frac{\partial}{\partial y_{\alpha i}}$  and  $E_{kn+j} = \frac{\partial}{\partial x_j}$ , for any  $i, j \in \{1, \dots, n\}$ . Note that by (4.7)–(4.8) we get that  $\omega_\alpha = \sum_{i=1}^n dx_i \wedge dy_{(\alpha-1)n+i}$ . Thus, with respect this coordinate system,

$$(i) \text{ each } L_\alpha \text{ is spanned by } \frac{\partial}{\partial y_{(\alpha-1)n+1}}, \dots, \frac{\partial}{\partial y_{\alpha n}},$$

$$(ii) \text{ } Q \text{ is spanned by } \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n},$$

$$(iii) \text{ the } k\text{-symplectic forms } \omega_\alpha \text{ are given by } \omega_\alpha = \sum_{i=1}^n dx_i \wedge dy_{(\alpha-1)n+i}.$$

This proves the assertion.  $\square$

*Remark 4.7.* Theorem 4.6 should be compared with Theorem 2.1. It should be remarked that according to Theorem 2.1 there always exist local coordinates  $\{x_1, \dots, x_n, y_1, \dots, y_{kn}\}$  verifying (4.4) and such that the foliation  $\mathcal{F}$  is locally given by the equations  $\{x_1 = \text{const.}, \dots, x_n = \text{const.}\}$ . On the other hand, by the general theory of foliations there always exists local coordinates  $\{x'_1, \dots, x'_n, y'_1, \dots, y'_{kn}\}$  with respect to which the foliation defined by  $Q$  is described by the equations  $\{y'_1 = \text{const.}, \dots, y'_{kn} = \text{const.}\}$ . In general these two types of coordinate systems do not coincide. Theorem 4.6 just states that a sufficient condition for this is expressed by the flatness of the canonical connection. Note that this condition is also necessary, as easily it follows from (4.1)–(4.3).

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