Pfaffian Systems from Twistor Fibrations[*]

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Abstract

Canonical twistor fibrations lead to Pfaffian systems by means of their superhorizontal distribution. The aim of this note is to identify explicitly the Pfaffian systems of five or less variables that arise in this way in terms of the classification given in [1].

1 Introduction

A Pfaffian system on an n-dimensional differentiable manifold is a collection of 1-forms which are linearly independent at each point. Geometrically this system gives rise to a subbundle of the tangent space, that is a distribution. Conversely a distribution induces an equivalence class of Pfaffian systems (see [1],[3],[6] or [9]). For $n \leq 5$ such systems were investigated in detail by E. Cartan.

On the total space of any canonical twistor fibration (see [2] or [5]), there is a natural "superhorizontal" distribution, which is important in the theory of harmonic maps. The total space is a flag manifold, and a corresponding Pfaffian system on the big cell of the flag manifold can be constructed by using Lie-theoretic local coordinates to express integral curves of the distribution. The equations for these holomorphic curves are solved explicitly in [4].

In this paper we identify explicitly all Pfaffian systems with $n \leq 5$ which arise from twistor fibrations in this way, in terms of the known classification. It is of interest to consider the significance of these twistor Pfaffian systems, and we discuss briefly one aspect here, namely the Lie algebra of infinitesimal symmetries. Cartan observed that the Lie algebra of the exceptional group G_2 can be characterized as the symmetry algebra of such a system. However, all the other systems with $n \leq 5$ arising from twistor fibrations have infinite-dimensional symmetry algebras. In the simplest example, we present an explanation for this phenomenon.

The paper is organized as follows. In section 2 we describe the basic definitions and theorems concerning Pfaffian systems. In addition, the classification of low dimensional Pfaffian systems from [1] is reproduced at the end of

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this section (tables 1-3). Section 3 reviews some facts from structure theory of Lie algebras as well as the concept of twistor fibration. The construction of Pfaffian systems from twistor fibrations is explained in section 4. Subsections 4.1 - 4.3 contain detailed computations for three representative examples: $\pi : F_3(\mathbb{C}) \to \mathbb{C}P^2$, $\pi : SO(5)/U(1) \times SO(2) \to SO(5)/SO(2) \times SO(3)$ and $\pi : G_2/U(2) \to G_2/SO(4)$. The complete list of results is given in table 4. Finally in the last section we discuss the Lie algebra of infinitesimal symmetries.

2 Pfaffian systems

In this section we will review some basic concepts and facts on Pfaffian systems theory (see [3]). Let M be an $n-$dimensional differentiable manifold. Let TM and T^*M denote the tangent and cotangent bundle respectively.

In the following $\Omega^k(M)$ will denote the space of all differentiable k forms on M. We have $\Omega^k(M) = \Gamma(\bigwedge^k T^*M)$, where $\Gamma(\bigwedge^k T^*M)$ denotes the space of sections of $\bigwedge^k T^*M$.

A (local) *Pfaffian system* of rank r is a set:

$$S = \{\omega_1 \ldots \omega_r\} \qquad \omega_i \in \Omega^1(M)|_U.$$

where the r differential 1-forms are defined on an open subset U of M and are required to be linearly independent at each point.

Given a Pfaffian system S, the vector subbundle D of $TM|_U$ with $n-r$ dimensional fibers $D_p = \bigcap_{i=1}^r \ker \omega_i(p)$ for all $\omega_i \in S$, $p \in U \subset M$ is a *distribution* on U. Conversely, a distribution give rise to an equivalence class of Pfaffian systems, since the annihilator subbundle of D, with fibers

$$D_p^\perp = \{\omega \in T_p^*M|\ \omega(X) = 0\ \forall X \in D_p\},$$

is locally spanned by r linearly independent 1-forms *i.e.* $D_p^\perp = \text{span}\{(\omega_1)_p \ldots (\omega_r)_p\} \subset T_p^*M$. The ideal of $\Gamma(\bigwedge^* T^*M)$ generated by the system is:

$$\mathcal{I}(D) = \bigoplus_{k=1}^n \mathcal{I}^k(D) = \Big\{ \sum_{i=1}^r \alpha_i \wedge \omega_i \mid \omega_i \in S,\ \alpha_i \in \Omega^*(M)\Big\}.$$

with $\mathcal{I}^k(D) = \{\omega \in \Omega^k(M) \mid \omega(X_1 \ldots X_k) = 0 \quad \forall X_i \in D\}$ for $k \geq 1$.

At a point $p \in M$ the *characteristic space* of $\mathcal{I}(D)$ is defined as:

$$\mathcal{A}(\mathcal{I}(D))_p = \{X_p \in T_pM \mid X_p \lrcorner \mathcal{I}(D)_p \subset \mathcal{I}(D)_p\}$$
$$= \{X_p \in T_pM \mid \omega_i(X_p) = 0,\ X_p \lrcorner d\omega_i \equiv 0 \mod S,\ i = 1 \ldots r\}.$$

Its annihilator $C(\mathcal{I}(D))_p := \mathcal{A}(\mathcal{I}(D))_p^\perp \subset T_p^*M$, called the *retracting* subspace at p, is the smallest subspace of T_p^*M such that $\bigwedge^*(C(\mathcal{I}(D))$ contains a finite set S of elements generating $\mathcal{I}(D)$ as an ideal. The dimension of $C(\mathcal{I}(D))_p$ is called the *class* of S at p.

A *symmetry* of S is a local biholomorphism ϕ on U which preserves D (or S), i.e. $\phi_*(D) = D$ (or $\phi^*(S) = S$). An *infinitesimal symmetry* is a vector field X over U which generates a local one parameter transformation group of symmetries of S. Namely X satisfies: $[X, \Gamma(D)] \subseteq \Gamma(D)$, or equivalently: $L_X \omega_i \equiv 0$ mod S with $\omega_i \in S$. The set of all infinitesimal symmetries of S is denoted $L(S)$.

Then the characteristic space $\mathcal{A}(\mathcal{I}(D)) = \bigcup_{p \in U} \mathcal{A}(\mathcal{I}(D))_p$ of $\mathcal{I}(D)$ can be described (*cf.* [6]) in terms of infinitesimal symmetries as:

$$\mathcal{A}(\mathcal{I}(D)) = L(S) \cap \Gamma(D). \tag{1}$$

A submanifold N of M is called an *integral manifold* of S (or D) if $T_p N \subseteq D_p$ for any $p \in N$. Equivalently an integrable manifold is given by an immersion $i : N \longmapsto M$ such that $i^*(\alpha) = 0$ for all $\alpha \in \mathcal{I}(D)$.

A Pfaffian system S (or the distribution D) is said to be *completely integrable* if there exists an integrable manifold of dimension $dim(M) - rank(S)$, passing through every point.

The integrability of Pfaffian systems can be characterized in various ways:

Theorem 1 (Frobenius)**.** *Let M be a differentiable manifold. Let $S = \{\omega_i\}_{i=1}^r$ by a Pfaffian system of rank r with induced $n - r$ dimensional distribution D defined over $U \subset M$. Let $\mathcal{I}(D)$ be the ideal generated by the system as above. Then the following are equivalent:*

i) *S (or D) is completely integrable.*

ii) *D is involutive:* $[\Gamma(D), \Gamma(D)] \subset \Gamma(D)$.

iii) *$\mathcal{I}(D)$ is a differential ideal: $d\mathcal{I}(D) \subset \mathcal{I}(D)$, i.e. $d\omega_i = \sum_{j=1}^r \alpha_j^i \wedge \omega_j$ for $\omega_j \in S$ and $\alpha_j^i \in \Omega^k(M)|_U$. Briefly we write: $d\omega_i \equiv 0 \mod S$.*

iv) *Locally there exists a coordinate system $(y_1 \ldots y_n)$ such that $\mathcal{I}(D)$ is generated by $(dy_1 \ldots dy_r)$.*

v) *$rank_p(S) = class_p(S)$ for every $p \in U \subset M$. In this case $\mathcal{A}(\mathcal{I}(D)) = D$ and $C(\mathcal{I}(D)) = S$.*

A natural problem in Pfaffian systems theory is the classification of systems with constant rank, which requires the study of invariants. The class and the rank are fundamental invariants that entirely determine the local classification of completely integrable systems by the Frobenius theorem.

For a nonintegrable Pfaffian system S of rank r and corresponding distribution D, we define the subbundle D' which has fibers spanned by all elements of $T_p M$ of the form $X_p + [Y_p, Z_p]$ with $X, Y, Z \in \Gamma(D)$, *i.e.*

$$D' = \Gamma(D) + [\Gamma(D), \Gamma(D)].$$

Notice that $D_p \subset D'_p$. A representative S' of the equivalence class of Pfaffian systems corresponding to this distribution is called (with abuse of notation) the

derived system of S. This leads to the notion of *higher derived systems*. The k-th derived system S^k is defined successively by:

$$S^{k+1} = (S^k)', \quad S^0 := S.$$

It follows that the rank and the class of S^k are invariants of such a Pfaffian systems. S is called *regular* if all derived systems are of constant rank. For a regular system S, there exists $\mu \in \mathbb{Z}_{\geq 0}$ such that:

$$S^{\mu+1} = S^\mu \subset \cdots \subset S^k \subset S^{k-1} \subset \cdots \subset S'' \subset S' \subset S.$$

S^μ is always 0 or completely integrable; It is the smallest completely integrable subsystem of S. (See [10],[1]).

The basic idea to achieve the classification of nonintegrable Pfaffian systems of constant rank is to look for completely integrable subsystems of S and use the defined invariants to determine a local model of the system.

In [1] the equivalence classes of low dimensional ($n \leq 5$) Pfaffian systems of constant rank and class is presented by giving local models of those systems. Nevertheless this is a partial classification, since for the cases $S_5^{11}(f)$ and $S_5^{15}(f)$ the local equivalence classes are not specified. For the sake of reference this classification is reproduced in the following tables.

Table 1. Pfaffian systems of dimension 3.

Pfaffian system S	rank(S)	class(S)
$S_3^1 = \{dz_1\}$	1	1
$S_3^2 = \{dz_1 + z_2 dz_3\}$	1	3
$S_3^3 = \{dz_1, \ dz_2\}$	2	2
$S_3^4 = \{dz_1, \ dz_2, \ dz_3\}$	3	3

Table 2. Pfaffian systems of dimension 4.

Pfaffian system S	rank(S)	class(S)	rank(S')	class(S')
$S_4^1 = \{dz_1\}$	1	1		
$S_4^2 = \{dz_1, \ dz_2, \ dz_3\}$	3	3		
$S_4^3 = \{dz_1, \ dz_2, \ dz_3, \ dz_4\}$	4	4		
$S_4^4 = \{dz_1 + z_2 dz_3\}$	1	3		
$S_4^5 = \{dz_1, \ dz_2\}$	2	2		
$S_4^6 = \{dz_1, \ dz_2 + z_3 dz_4\}$	2	4	1	1
$S_4^7 = \{dz_1 + z_2 dz_3, \ dz_2 + z_4 dz_3\}$	2	4	1	3

Pfaffian system S	rank(S)	class(S)	rank(S')	class(S')	rank(S'')	class(S'')
$S_5^1 = \{dz_1\}$	1	1				
$S_5^2 = \{dz_1 + z_2 dz_3\}$	1	3				
$S_5^3 = \{dz_1 + z_2 dz_3 + z_4 dz_5\}$	1	5				
$S_5^4 = \{dz_1,\ dz_2,\ dz_3,\ dz_4, dz_5\}$	5	5				
$S_5^5 = \{dz_1,\ dz_2,\ dz_3,\ dz_4\}$	4	4				
$S_5^6 = \{dz_1,\ dz_2,\ dz_3\}$	3	3				
$S_5^7 = \{dz_1,\ dz_2,\ dz_3 + z_4 dz_5\}$	3	5	2	2		
$S_5^8 = \{dz_1,\ dz_2 + z_3 dz_4,\ dz_3 + z_5 dz_4\}$	3	5	2	4	1	1
$S_5^9 = \{dz_1 + z_2 dz_3,\ dz_2 + z_4 dz_3,\ dz_3 + z_5 dz_4\}$	3	5	2	4	1	3
$S_5^{10} = \{dz_1 + z_2 dz_3,\ dz_2 + z_4 dz_3,\ dz_4 + z_5 dz_3\}$	3	5	2	4	1	3
$S_5^{11}(f) = \left\{ dz_1 + z_3 dz_4,\ dz_2 + z_5 dz_3 + f dz_4,\ dz_3 + \frac{\partial f}{\partial z_5} dz_4 \right\}$	3	5	2	5		
$S_5^{12} = \{dz_1,\ dz_2\}$	2	2				
$S_5^{13} = \{dz_1,\ dz_2 + z_3 dz_4\}$	2	4	1	1		
$S_5^{14} = \{dz_1 + z_2 dz_3,\ dz_2 + z_4 dz_3\}$	2	4	1	3		
$S_5^{15}(f) = \{dz_1 + z_3 dz_4,\ dz_2 + z_5 dz_3 + f dz_4\}$	2	5				

Table 3. Pfaffian systems of dimension 5 .

3   Twistor fibrations

The aim of this section is to review the twistor fibration concept. We start by recalling some facts from the structure theory of Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra of a compact semisimple Lie group with Cartan subalgebra $\mathfrak{t}$. Let $\mathfrak{g}^{\mathbb{C}}$ be the complexification of $\mathfrak{g}$, and $\mathfrak{t}^{\mathbb{C}}$ that of $\mathfrak{t}$.

A functional $\alpha \in (\mathfrak{t}^{\mathbb{C}})^*$ is called *a root* of $\mathfrak{g}^{\mathbb{C}}$ (with respect to $\mathfrak{t}^{\mathbb{C}}$) if $\mathfrak{g}_\alpha \neq \{0\}$, where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^{\mathbb{C}} \mid \mathrm{ad}(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{t}^{\mathbb{C}}\}$ is the *root space* of α. The Lie algebra $\mathfrak{g}^{\mathbb{C}}$ has then a root decomposition into the Cartan subalgebra and the root spaces:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \Big(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \Big).$$

Here Δ denotes the set of roots of $\mathfrak{g}^{\mathbb{C}}$. This is a finite subset of $(\mathfrak{t}^{\mathbb{C}})^*$.

It is also necessary to recall the existence of a subset $\Pi = \{\alpha_1 \ldots \alpha_r\}$ ($r = \dim_{\mathbb{C}} \mathfrak{t}^{\mathbb{C}}$) of Δ such that every root can be expressed uniquely as $\alpha = \sum_{i=1}^{l} n_i \alpha_i$ where the n_i are integers, either all nonnegative (α is then a *positive* root) or all non-positive (α is a *negative* root). Such a set is called *set of simple roots* for Δ. Each subset $I = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ of Π defines on Δ a *height* function n_I by $n_I(\alpha) = \sum_{i \in I} n_i$.

According to this any parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$ can be expressed as:

$$\mathfrak{p}_I = \mathfrak{t}^{\mathbb{C}} \oplus \Big(\bigoplus_{i \leq 0} \mathfrak{g}_i^I \Big), \quad \text{with} \quad \mathfrak{g}_i^I = \bigoplus_{n_I(\alpha)=i} \mathfrak{g}_\alpha.$$

Given a parabolic subalgebra $\mathfrak{p}_I$, there exists a unique element $\xi \in \mathfrak{t}^{\mathbb{C}}$ such that $\alpha(\xi) = \sqrt{-1}n_I(\alpha)$. *i.e.*

$$\bigoplus_{\alpha(\xi)=\sqrt{-1}i} \mathfrak{g}_\alpha = \mathfrak{g}_i^I.$$

This element is called the *canonical element* of $\mathfrak{p}_I$. This element defines an involutive automorphism $\tau_\xi = \mathrm{Ad}(\exp(\pi\xi))$ of $\mathfrak{g}$ (the complex linear extension to $\mathfrak{g}^{\mathbb{C}}$) called the *canonical involution* for $\mathfrak{p}_I$.

The facts mentioned above are intimately related with the concept of *generalized flag manifolds*, which are homogeneous spaces of the form G/H where G is a compact Lie group and H is the centralizer of a torus in G. We have a natural isomorphism $G^{\mathbb{C}}/P_I \cong G/H$ where $G^{\mathbb{C}}$ is the complexification of G and $P_I = \{g \in G^{\mathbb{C}} \mid \mathrm{Ad}(g)\mathfrak{p}_I \subseteq \mathfrak{p}_I\}$ is a parabolic subgroup of $G^{\mathbb{C}}$.

Let $G/H \cong G^{\mathbb{C}}/P_I$ be a generalized flag manifold and let ξ be the canonical element of the Lie algebra $\mathfrak{p}_I$ of P_I with canonical automorphism τ_ξ. Then the following is true:

- $G^{\mathbb{C}}/P_I \cong \mathrm{Ad}(G)\xi$.

- $H = P_I \cap G$.

- For $g = \exp \pi \xi$ define $\sigma_g = R_{g^{-1}} \circ L_g$. If K is a closed subgroup of $G^{\mathbb{C}}$ such that $H \subseteq K \subseteq G$, $G^0_{\sigma_g} \subseteq K \subseteq G_{\sigma_g}$, then G/K is a *symmetric space*. Here G_{σ_g} denotes the set of fixed points of σ_g and $G^0_{\sigma_g}$ its identity component.

Finally we have:

Definition 1. *The natural map* $\pi_\xi : G/H \longmapsto G/K$ *is called the canonical* (Burstall-Rawnsley) *twistor fibration associated to the generalized flag manifold* $G/H \cong G^{\mathbb{C}}/P_I$.

For details we refer to [5], [2] and [7].

4 Pfaffian systems from twistor fibrations

Now we are in a position to describe how Pfaffian systems and twistor fibrations are related. The key is to consider a certain subbundle of the tangent bundle of the generalized flag manifold of the twistor fibration.

Let $G^{\mathbb{C}}/P_I$ be a generalized flag manifold. Recall that as an homogeneous space its tangent bundle is isomorphic to $G^{\mathbb{C}} \times_{P_I} (\mathfrak{g}^{\mathbb{C}}/\mathfrak{p}_I)$. An explicit isomorphism $f : G^{\mathbb{C}} \times_{P_I} (\mathfrak{g}^{\mathbb{C}}/\mathfrak{p}_I) \to T G^{\mathbb{C}}/P_I$ is given by:

$$f([(g, [Y])]) = \frac{d}{ds} g \circ \exp sY P_I \Big|_{s=0}. \tag{2}$$

Definition 2. *The subbundle SH of* $T G^{\mathbb{C}}/P_I$ *which corresponds to*

$$G^{\mathbb{C}} \times_{P_I} \left((\mathfrak{p}_I \oplus \mathfrak{g}_1^I)/\mathfrak{p}_I \right)$$

is called the **superhorizontal distribution**.

This concept arises from the study of harmonic maps that come from twistor fibrations.

We shall construct a particular Pfaffian system corresponding to the super-horizontal distribution by expressing integral curves of the distribution in Lie theoretic local coordinates. The equations for integral curves of this distribution are solved expliclitly in [4], (see page 561).

It is well known that local coordinates for the generalized flag manifold, $G^{\mathbb{C}}/P_I$ are provided by the "*big cell*" $[\exp \bigoplus_{i>0} \mathfrak{g}_i^I] \cong \mathbb{C}^m$. Locally therefore we can construct any complex integral curve $\Phi : \mathbb{C} \longmapsto G^{\mathbb{C}}/P_I$ by giving a holomorphic map $C : \mathbb{C} \longmapsto \bigoplus_{i>0} \mathfrak{g}_i^I$ and setting $\Phi = \rho(\exp C)$, where ρ denotes the projection $G^{\mathbb{C}} \longmapsto G^{\mathbb{C}}/P_I$.

The condition for this integral curve to be tangent to the superhorizontal distribution, *i.e.* $\mathrm{d}\Phi\left(\frac{\partial}{\partial z}\right) \in SH$, is that:

$$\left((\exp C)^{-1}(\exp C)'\right)_{\mathfrak{g}_i^I} = 0 \qquad (i \geq 2) \tag{3}$$

In other words, $(\exp -C)(\exp C)'$ should have zero component in $\bigoplus_{i \geq 2} \mathfrak{g}_i^I$.

Using the formula for the derivative of the exponential map one has:

$$
\begin{aligned}
(\exp -C)(\exp C)' &= \frac{I - e^{-\mathrm{ad}C}}{\mathrm{ad}C} C' \\
&= C' - \frac{1}{2!}\mathrm{ad}C C' + \frac{1}{3!}(\mathrm{ad}C)^2 C' - \cdots \\
&= C' - \frac{1}{2}[C, C'] + \frac{1}{6}[C, [C, C']] - \cdots
\end{aligned}
$$

Applying this to (3), the resultant expression determines n differential equations in m functions, $C_i' = F(C_1, \ldots, C_{i-1})$ $(i \geq 2)$ for an integral curve of the superhorizontal distribution, where $n = \sum_{i \geq 2} \dim \mathfrak{g}_i^I$ and $m = \sum_{i \geq 1} \dim \mathfrak{g}_i^I$. These equations can be written in local coordinates corresponding to $\bigoplus_{i > 0} \mathfrak{g}_i^I$ as 1-forms of m-variables giving in this way a Pfaffian system S on $\mathbb{C}^m$ with $\mathrm{rank}(S) = n$.

Proposition 1. *The Pfaffian system S just obtained satisfies the following properties:*

- *$class(S) = \sum_{i \geq 1} \dim \mathfrak{g}_i^I = m$.*

- *The first derived system S' of S corresponds to:*

$$G^{\mathbb{C}} \times_{P_I} \left((\mathfrak{p}_I \oplus \mathfrak{g}_1^I \oplus \mathfrak{g}_2^I)/\mathfrak{p}_I\right).$$

Proof. Notice first that $\dim \mathcal{A}(\mathcal{I}(SH)) = 0$ is equivalent to $\mathcal{A}(\mathcal{I}(SH)) = L(S) \cap \Gamma(SH) = 0$, *i.e.* if $X \in \Gamma(SH)$ satisfies

$$[X, \Gamma(SH)] \subseteq \Gamma(SH), \tag{4}$$

then $X = 0$. We show that this is the case. Take $X \in SH$ satisfying (4). Then X is expressed as $X = [(g, [Z])]$ with $g \in G^{\mathbb{C}}$ and $[Z] \in \mathfrak{g}_1^I/\mathfrak{p}_I$. Now under the isomorphism f, the condition (4) is equivalent to having $Z \in \mathfrak{g}_1^I$ such that $[Z, \mathfrak{g}_1^I] \in \mathfrak{g}_1^I$. But since $[\mathfrak{g}_i^I, \mathfrak{g}_j^I] \in \mathfrak{g}_{i+j}^I$ we must have $Z = 0$.

Similarly, $SH' = \ker(S')$ is spanned by:

$$\Gamma(SH) + [\Gamma(SH), \Gamma(SH)],$$

which under the isomorphism f is equivalent to:

$$
\begin{aligned}
SH' &\cong G^{\mathbb{C}} \times_{P_I} \left(\mathfrak{p}_I \oplus (\mathfrak{g}_1^I + [\mathfrak{g}_1^I, \mathfrak{g}_1^I])/\mathfrak{p}_I\right) \\
&\cong G^{\mathbb{C}} \times_{P_I} \left(\mathfrak{p}_I \oplus \mathfrak{g}_1^I \oplus \mathfrak{g}_2^I/\mathfrak{p}_I\right).
\end{aligned}
$$

$\square$

Now we concentrate entirely on the case when $m \leq 5$ and compute all Pfaffian systems that arises from twistor fibrations of semisimple Lie algebras by using the above construction and identifying the obtained system – after a suitable change of local coordinates – with a model in the classification of low dimensional Pfaffian system mentioned in section 1 (tables 1, 2 and 3).

Theorem (2) collects these results in a detailed list. Since the proof is purely computational, in order to illustrate how the calculation is carried out we present three representative examples:

4.1 $\quad \pi : F_3(\mathbb{C}) \to \mathbb{C}P^2$.

Let $G = SU_3$, then $G^{\mathbb{C}} = SL_3\mathbb{C}$ with Lie algebra $\mathfrak{sl}_3\mathbb{C}$ consisting of all endomorphisms of $\mathbb{C}^3$ with zero trace. Let $\mathfrak{t}^{\mathbb{C}} = \{H \in \mathfrak{sl}_3\mathbb{C} \mid H = \mathrm{diag}(a_1, a_2, a_3)\}$ be the corresponding Cartan subalgebra.

The set of roots is then given by: $\Delta = \{\pm(r_i - r_j) \mid 1 \leq i, j \leq 3 \ i \neq j\}$ where r_i denotes the linear functional $r_i : \mathfrak{t}^{\mathbb{C}} \to \mathbb{C}$ defined by $r_i(H) = a_i$. In this case the set of simple roots is $\Pi = \{\alpha_1 = r_1 - r_2, \alpha_2 = r_2 - r_3\}$ and the corresponding root spaces are:

$$\mathfrak{g}_{\alpha_1} = \mathbb{C} \cdot E_{12} \ , \ \mathfrak{g}_{\alpha_2} = \mathbb{C} \cdot E_{23} \ , \ \mathfrak{g}_{\alpha_1 + \alpha_2} = \mathbb{C} \cdot E_{13} \tag{5}$$

where E_{ij} stands for the matrix whose (i, j) entry is 1 and all others 0.

If we choose $I = \Pi$, then:

$$\mathfrak{g}_1^I = \left\{ \begin{pmatrix} 0 & w_1 & 0 \\ 0 & 0 & w_2 \\ 0 & 0 & 0 \end{pmatrix} \ \middle| \ w_1, w_2 \in \mathbb{C} \right\} \ , \ \mathfrak{g}_2^I = \left\{ \begin{pmatrix} 0 & 0 & w_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ \middle| \ w_3 \in \mathbb{C} \right\}.$$

and the corresponding parabolic subalgebra is:

$$\mathfrak{p}_I = \mathfrak{t}^{\mathbb{C}} \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{-\alpha_2} \oplus \mathfrak{g}_{-\alpha_3} = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \in \mathfrak{sl}_3\mathbb{C} \right\}$$

with parabolic subgroup:

$$P_I = \{g \in G^{\mathbb{C}} \mid Ad(g)\mathfrak{p}_I \subseteq \mathfrak{p}_I\} = \left\{ \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix} \in SL_3\mathbb{C} \right\} \tag{6}$$

Since $H = P_I \cap G \cong S(U_1 \times U_1 \times U_1)$ the generalized flag manifold G/H is $SU_3/S(U_1 \times U_1 \times U_1) \cong F_3(\mathbb{C})$.

The canonical element is calculated as follows: Taking $\xi \in \mathfrak{t}^{\mathbb{C}}$ as

$$\xi = \mathrm{diag}(\sqrt{-1}a_1, \sqrt{-1}a_2, \sqrt{-1}a_3)$$

with $a_i \in \mathbb{R}$, $\sum a_i = 0$, since $\alpha_i(\xi) = \sqrt{-1}n_I(\alpha_i)$ we get $\alpha_1(\xi) = \alpha_2(\xi) = 1$ and $\alpha_3(\xi) = 2$, which implies $a_1 = 1, a_2 = 0, a_3 = -1$.

We have $g = \exp \pi \xi = \mathrm{diag}(e^{\sqrt{-1}\pi}, 1, e^{-\sqrt{-1}\pi})$ and therefore the set of fixed points of σ_g is:

$$G^{\sigma_g} = \left\{ \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix} \in SU_3 \mid a_{22} \in U_1 \ , \ \begin{pmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{pmatrix} \in U_2 \right\}$$

$$\cong S(U_1 \times U_2).$$

Here since $H \subset G^{\sigma_g}$, setting $K = G^{\sigma_g}$ we can write:

$$G/K \cong SU_3/S(U_1 \times U_2) \cong \mathbb{C}P^2.$$

Thus in this case the twistor fibration is precisely $\pi : F_3(\mathbb{C}) \to \mathbb{C}P^2$.

Now, the integral curve is constructed defining the map $C : \mathbb{C} \to \bigoplus_{i=1}^{2} \mathfrak{g}_i^I$ as:

$$C(z) = \begin{pmatrix} 0 & a(z) & c(z) \\ 0 & 0 & b(z) \\ 0 & 0 & 0 \end{pmatrix}.$$

Since

$$\mathfrak{g}_1^I \oplus \mathfrak{g}_2^I = \left\{ \begin{pmatrix} 0 & w_1 & w_3 \\ 0 & 0 & w_2 \\ 0 & 0 & 0 \end{pmatrix} \ \middle| \ w_1, w_2, w_3 \in \mathbb{C} \right\}$$

we have:

$$\exp(-C)(\exp C)' = \begin{pmatrix} 0 & a'(z) & c'(z) - \frac{1}{2}a(z)b'(z) + \frac{1}{2}a'(z)b(z) \\ 0 & 0 & b'(z) \\ 0 & 0 & 0 \end{pmatrix}$$

After applying the condition (3) we obtain the following ordinary differential equation:

$$c'(z) - \frac{1}{2}a(z)b'(z) + \frac{1}{2}a'(z)b(z) = 0.$$

In terms of local coordinates of $\exp \bigoplus_{i=0}^{2} \mathfrak{g}_i^I$ this becomes

$$dw_3 - \frac{1}{2}w_1 dw_2 + \frac{1}{2}w_2 dw_1 = 0.$$

Since

$$dw_3 - \frac{1}{2}w_1 dw_2 + \frac{1}{2}w_2 dw_1 = d(w_3 - \frac{1}{2}w_1 w_2) + w_2 dw_1,$$

the change of coordinates

$$(z_1, z_2, z_3) = (w_3 - \frac{1}{2}w_1 w_2, w_2, w_1)$$

identifies this Pfaffian system with:

$$S_3^2 = \{dz_1 + z_2 dz_3\}.$$

4.2 $\quad \pi : SO(5)/U(1) \times SO(2) \to SO(5)/SO(2) \times SO(3)$.

Let $G = SO(5)$. By definition the corresponding Lie algebra is given by:

$$\mathfrak{so}_5^{\mathbb{C}} = \{X \in Hom(\mathbb{C}^5, \mathbb{C}^5) \mid (Xv, w) + (v, Xw) = 0\}.$$

After choosing a basis:

$$v_1 = e_1 + \sqrt{-1}e_4, \quad v_2 = e_2 + \sqrt{-1}e_5, \quad v_3 = e_3,$$
$$v_4 = e_2 - \sqrt{-1}e_5, \quad v_5 = e_1 - \sqrt{-1}e_4.$$

where $\{e_i\}_{i=1}^5$ denotes the canonical basis of $\mathbb{C}^5$, a suitable matrix representation is obtained:

$$\mathfrak{so}(5, \mathbb{C}) = \left\{ \begin{pmatrix} r_1 & x & t_1 & y & 0 \\ \tilde{x} & r_2 & t_2 & 0 & -y \\ 2\tilde{t_1} & 2\tilde{t_2} & 0 & -2t_2 & -2t_1 \\ \tilde{y} & 0 & -\tilde{t_2} & -r_2 & -x \\ 0 & -\tilde{y} & -\tilde{t_1} & -\tilde{x} & -r_1 \end{pmatrix} \middle| \; r_1, r_2, x, y, t_1, t_2, \tilde{x}, \tilde{y}, \tilde{t_1}, \tilde{t_2} \in \mathbb{C} \right\}.$$

In this case the set of roots is: $\Delta = \{\pm(r_i \pm r_j), \pm r_k \mid 1 \leq i, j, k \leq 2, i \neq j\}$ with simple roots $\Pi = \{\alpha_1 = r_1 - r_2, \alpha_2 = r_2\}$, and the root spaces are:

$$\mathfrak{g}_{\alpha_1} = \mathbb{C} \cdot (E_{12} - E_{45}), \quad \mathfrak{g}_{\alpha_2} = \mathbb{C} \cdot (E_{23} - 2E_{34}),$$
$$\mathfrak{g}_{\alpha_1 + \alpha_2} = \mathbb{C} \cdot (E_{13} - 2E_{35}), \quad \mathfrak{g}_{\alpha_1 + 2\alpha_2} = \mathbb{C} \cdot (E_{14} - E_{25}).$$

Setting $I = \Pi$, we have:

$$\mathfrak{g}_1^I = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2}, \quad \mathfrak{g}_2^I = \mathfrak{g}_{\alpha_1 + \alpha_2}, \quad \mathfrak{g}_3^I = \mathfrak{g}_{\alpha_1 + 2\alpha_2}.$$

The map $C : \mathbb{C} \to \bigoplus_{i>0} \mathfrak{g}_i^I$ can be defined as:

$$C(z) = \begin{pmatrix} 0 & x(z) & t_1(z) & y(z) & 0 \\ 0 & 0 & t_2(z) & 0 & -y(z) \\ 0 & 0 & 0 & -2t_2(z) & -2t_1(z) \\ 0 & 0 & 0 & 0 & -x(z) \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In the same way as above, calculating $(\exp -C)(\exp C)'$ under the condition (3), we obtain the following system of differential equations:

$$t_1' - \frac{1}{2}xt_2' + \frac{1}{2}t_2 x' = 0, \quad y' + t_1 t_2' - t_2 t_1' + \frac{1}{3}t_2 x t_2' - \frac{1}{3}t_2^2 x' = 0.$$

In terms of local coordinates these are: $dt_1 - \frac{1}{2}xdt_2 + \frac{1}{2}t_2 dx = 0$ and $dy + t_1 dt_2 - t_2 dt_1 + \frac{1}{3}t_2 x dt_2 - \frac{1}{3}t_2^2 dx = 0$. The associated Pfaffian system $S = \{\omega_1, \omega_2\}$ is given by:

$$\omega_1 := dt_1 - \frac{1}{2}xdt_2 + \frac{1}{2}t_2 dx,$$
$$\omega_2 := dy + t_1 dt_2 - t_2 dt_1 + \frac{1}{3}t_2 x dt_2 - \frac{1}{3}t_2^2 dx.$$

Notice that S is not completely integrable and that the derived system is $S' = \{\omega_2\}$ since $d\omega_2 \wedge \omega_1 \wedge \omega_2 = 0$. Moreover, since $d\omega_2 \wedge \omega_2 \neq 0$ we have class$(S') = 3$ and the system must be equivalent to S_4^7. We shall find an explicit change of coordinates. First, ω_2 can be expressed as:

$$\omega_2 = d(y - t_1 t_2 - \frac{1}{3} x t_2^2) + (2t_1 + x t_2) dt_2,$$

to obtain:

$$(z_1, z_2, z_3) = (y - t_1 t_2 - \frac{1}{3} x t_2^2, \quad 2t_1 + x t_2, \quad t_2).$$

On the other hand:

$$\omega_1 = \frac{1}{2}(d(2t_1 + x t_2) + (-2x) dt_2)$$

hence the change of coordinates:

$$(z_1, z_2, z_3, z_4) = (y - t_1 t_2 - \frac{1}{3} x t_2^2, \quad 2t_1 + x t_2, \quad t_2, \quad -2x)$$

shows that the Pfaffian system is :

$$S_4^7 = \{dz_1 + z_2 dz_3, \quad dz_2 + z_4 dz_3\}.$$

4.3 $\pi : G_2/U(2) \to G_2/SO(4)$.

In this example we consider $G_2 = \{X \in SO(7, \mathbb{R}) \mid X(v \times w) = Xv \times Xw\}$, with Lie algebra:

$$\mathfrak{g}_2^{\mathbb{C}} = \{X \in \mathfrak{so}_7^{\mathbb{C}} \mid X(v \times w) = Xv \times w + v \times Xw\}.$$

As for the last example, in order to obtain a matrix representation for $SO(7)$ we choose a basis:

$$v_1 = e_1 + \sqrt{-1}e_5, \quad v_2 = e_2 + \sqrt{-1}e_6, \quad v_3 = e_3 + \sqrt{-1}e_7, \quad v_4 = e_4,$$
$$v_5 = e_3 - \sqrt{-1}e_7, \quad v_6 = e_2 - \sqrt{-1}e_6, \quad v_7 = e_1 - \sqrt{-1}.$$

Then under this representation any element of $\mathfrak{g}_2^{\mathbb{C}}$ can be expressed as:

$$\begin{pmatrix} r_2 + r_3 & x_1 & x_2 & y_3 & y_2 & y_1 & 0 \\ \tilde{x}_1 & r_2 & x_3 & -x_2 & y_3 & 0 & -y_1 \\ \tilde{x}_2 & \tilde{x}_3 & r_3 & x_1 & 0 & -y_3 & -y_2 \\ 2\tilde{y}_2 & -2\tilde{x}_2 & 2\tilde{x}_1 & 0 & -2x_1 & 2x_2 & -2y_3 \\ \tilde{y}_2 & \tilde{y}_3 & 0 & -\tilde{x}_1 & -r_3 & -x_3 & -x_2 \\ \tilde{y}_1 & 0 & -\tilde{y}_3 & \tilde{x}_2 & -\tilde{x}_3 & -r_2 & -x_1 \\ 0 & -\tilde{y}_1 & -\tilde{y}_2 & -\tilde{y}_3 & -\tilde{x}_2 & -\tilde{x}_1 & -r_2 - r_3 \end{pmatrix}$$

In this case the set of roots is:

$$\Delta = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)\}$$

with simple roots $\Pi = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = r_3$ and $\alpha_2 = r_2 - r_3$. Here the root spaces are:

$$\mathfrak{g}_{\alpha_1} = \mathbb{C} \cdot (E_{12} + E_{34} - 2E_{45} - E_{67}),$$
$$\mathfrak{g}_{\alpha_2} = \mathbb{C} \cdot (E_{23} - E_{56}),$$
$$\mathfrak{g}_{\alpha_1+\alpha_2} = \mathbb{C} \cdot (E_{13} - E_{24} + 2E_{46} - E_{57}),$$
$$\mathfrak{g}_{2\alpha_1+\alpha_2} = \mathbb{C} \cdot (E_{14} + E_{25} - E_{36} - 2E_{47}),$$
$$\mathfrak{g}_{3\alpha_1+\alpha_2} = \mathbb{C} \cdot (E_{15} - E_{37}),$$
$$\mathfrak{g}_{3\alpha_1+2\alpha_2} = \mathbb{C} \cdot (E_{16} - E_{27}).$$

Taking $I = \{\alpha_1\}$ we have :

$$\mathfrak{g}_1^I = \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1+\alpha_2}, \quad \mathfrak{g}_2^I = \mathfrak{g}_{2\alpha_1+\alpha_2}, \quad \mathfrak{g}_3^I = \mathfrak{g}_{3\alpha_1+\alpha_2} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2}.$$

Thus the map $C : \mathbb{C} \to \bigoplus_{i>0} \mathfrak{g}_i^I$ takes the form:

$$C(z) = \begin{pmatrix} 0 & x_1(z) & x_2(z) & y_3(z) & y_2(z) & y_1(z) & 0 \\ 0 & 0 & 0 & -x_2(z) & y_3(z) & 0 & -y_1(z) \\ 0 & 0 & 0 & x_1(z) & 0 & -y_3(z) & -y_2(z) \\ 0 & 0 & 0 & 0 & -2x_1(z) & 2x_2(z) & -2y_3(z) \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_2(z) \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_1(z) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This time the condition of the integral curve to the superhorizontal distribution gives the following system of differential equations:

$$y_3' - x_2 x_1' + x_1 x_2' = 0,$$
$$y_2' - \frac{3}{2}x_1 y_3' + \frac{3}{2}y_3 x_1' - x_1^2 x_2' + x_1 x_2 x_1' = 0,$$
$$y_1' + \frac{3}{2}x_2 y_3' - \frac{3}{2}y_3 x_2' + x_1 x_2 x_2' - x_2^2 x_1' = 0.$$

After expressing them in terms of local coordinates we set

$$\omega_1 := dy_3 - x_2 dx_1 + x_1 dx_2,$$
$$\omega_2 := dy_2 - \frac{3}{2}x_1 dy_3 + \frac{3}{2}y_3 dx_1 - x_1^2 dx_2 + x_1 x_2 dx_1,$$
$$\omega_3 := dy_1 + \frac{3}{2}x_2 dy_3 - \frac{3}{2}y_3 dx_2 + x_1 x_2 dx_2 - x_2^2 dx_1.$$

Then the induced Pfaffian system is $S = \{\omega_1, \omega_2, \omega_3\}$ which is not completely integrable. Now, for $i = 2, 3$ we have that:

$$d\omega_i \wedge \omega_1 \wedge \omega_2 \wedge \omega_3 = 0.$$

Therefore, the derived system is $S' = \{\omega_2, \omega_3\}$. Moreover since $d\omega_2 = -3\omega_1 \wedge dx_2$ and $d\omega_3 = 3\omega_1 \wedge dx_1$ we have class$(S') = 5$, and then the system S is identified to be $S_5^{11}(f)$.

Now ω_3 can be written as:

$$\omega_3 = d(y_2 - \frac{3}{2}x_1 y_3 - x_1^2 x_2) + (3y_3 + 3x_1 x_2)dx_1$$

hence:

$$(w_1, w_3, w_4) = (y_2 - \frac{3}{2}x_1 y_3 - x_1^2 x_2, \quad 3y_3 + 3x_1 x_2, \quad x_1)$$

In the same way:

$$\omega_2 = d(y_1 - \frac{3}{2}x_2 y_3 - x_1 x_2^2) + (x_2)d(3y_3 + 3x_1 x_2) - 3x_2^2 dx_1,$$

$$\omega_1 = \frac{1}{3}(d(3y_3 + 3x_1 x_2) - 6x_2 dx_1).$$

implies:

$$(w_1, w_2, w_3, w_4, w_5) =$$
$$(y_2 - \frac{3}{2}x_1 y_3 - x_1^2 x_2, \quad y_1 - \frac{3}{2}x_2 y_3 - x_1 x_2^2, \quad 3y_3 + 3x_1 x_2, \quad x_1, \quad x_2).$$

Therefore we must have $S_5^{11}(f)$ with $f = -3w_5^2$. But in general, the change of coordinates corresponding to $S_5^{11}(c \cdot w_5^2)$ is:

$$(z_1, z_2, z_3, z_4, z_5) = (w_1, w_2, \frac{1}{\sqrt[3]{c}}w_3, \sqrt[3]{c}w_4, \sqrt[3]{c}w_5).$$

Thus we can identify our Pfaffian system to be of the form $S_5^{11}(z_5^2)$. Explicitly this can be expressed as

$$S_5^{11}(z_5^2) = \{dz_1 + z_3 dz_4, \quad dz_2 + z_5 dz_3 + z_5^2 dz_4, \quad dz_3 + 2z_5 dz_4\}.$$

under the change of coordinates:

$$z_1 = y_2 - \frac{3}{2}x_1 y_3 - x_1^2 x_2,$$

$$z_2 = y_1 - \frac{3}{2}x_2 y_3 - x_1 x_2^2,$$

$$z_3 = -\frac{3}{\sqrt[3]{3}}y_3 - \frac{3}{\sqrt[3]{3}}x_1 x_2,$$

$$z_4 = -\sqrt[3]{3}x_1,$$

$$z_5 = -\sqrt[3]{3}x_2.$$

After the examples finally we are in position to state the main theorem of this section:

Theorem 2. *The Pfaffian systems of at most five variables that arise from superhorizontal distributions of twistor fibrations of semisimple Lie algebras are given explicitly in table 4.*

Proof. By direct calculation.

$\square$

Table 4. Low dimensional Pfaffian systems that arise from twistor fibrations.

Lie group	decomposition of $T_{1,0}G/H$	Pfaffian system
SU_3	$\underbrace{\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{\alpha+\beta}}^{\mathfrak{g}_2^I}$	$S_3^2 = \{dz_1 + z_2 dz_3\}$ $(w_3 - \frac{1}{2}w_1 w_2, w_2, w_1)$
SU_4	$\underbrace{\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\beta+\gamma}}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{\alpha+\beta+\gamma}}^{\mathfrak{g}_2^I}$	$S_5^{15}(0) = \{dz_1 + z_3 dz_4, dz_2 + z_5 dz_3\}$ $(w_5 + \frac{1}{2}w_1 w_3, w_4 - \frac{1}{2}w_1 w_2, w_1, -w_3, w_2)$
	$\underbrace{\mathfrak{g}_\beta \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_{\alpha+\beta}}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{\beta+\gamma} \oplus \mathfrak{g}_{\alpha+\beta+\gamma}}^{\mathfrak{g}_2^I}$	$S_5^{15}(0) = \{dz_1 + z_3 dz_4, dz_2 + z_5 dz_3\}$ $(w_5 - \frac{1}{2}w_2 w_3, w_4 + \frac{1}{2}w_1 w_2, w_2, w_3, -w_1)$
	$\underbrace{\mathfrak{g}_\alpha \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{\beta+\gamma}}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{\alpha+\beta+\gamma}}^{\mathfrak{g}_2^I}$	$S_5^3 = \{dz_1 + z_2 dz_3 + z_4 dz_5\}$ $(w_5 - \frac{1}{2}w_1 w_4 - \frac{1}{2}w_2 w_3, w_4, w_1, w_2, w_3)$
SO_5	$\underbrace{\mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta}}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{\alpha+2\beta}}^{\mathfrak{g}_2^I}$	$S_3^2 = \{dz_1 + z_2 dz_3\}$ $(w_3 - w_1 w_2, 2w_2, w_1)$
	$\underbrace{\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{\alpha+\beta}}^{\mathfrak{g}_2^I} \bigoplus \underbrace{\mathfrak{g}_{\alpha+2\beta}}^{\mathfrak{g}_3^I}$	$S_4^7 = \{dz_1 + z_2 dz_3, dz_2 + z_4 dz_3\}$ $(w_4 - w_2 w_3 - \frac{1}{3}w_1 w_2^2, 2w_3 + w_1 w_2, w_2, -2w_1)$
SO_6	$\underbrace{\mathfrak{g}_\beta \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{\alpha+\gamma}}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{\alpha+\beta+\gamma}}^{\mathfrak{g}_2^I}$	$S_5^3 = \{dz_1 + z_2 dz_3 + z_4 dz_5\}$ $(w_5 - \frac{1}{2}w_2 w_3 - \frac{1}{2}w_1 w_4, w_3, w_2, w_4, w_1)$
	$\underbrace{\mathfrak{g}_\alpha \oplus \mathfrak{g}_\gamma \oplus \mathfrak{g}_{\alpha+\beta}}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{\alpha+\gamma} \oplus \mathfrak{g}_{\alpha+\beta+\gamma}}^{\mathfrak{g}_2^I}$	$S_5^{15}(0) = \{dz_1 + z_3 dz_4, dz_2 + z_5 dz_3\}$ $(w_4 + \frac{1}{2}w_1 w_2, w_5 - \frac{1}{2}w_2 w_3, -w_1, w_2, w_3)$
	$\underbrace{\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\gamma}}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{\alpha+\beta+\gamma}}^{\mathfrak{g}_2^I}$	$S_5^{15}(0) = \{dz_1 + z_3 dz_4, dz_2 + z_5 dz_3\}$ $(w_4 + \frac{1}{2}w_1 w_2, w_5 - \frac{1}{2}w_2 w_3, -w_1, w_2, w_3)$
Sp_2	$\underbrace{\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta}}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{2\alpha+\beta}}^{\mathfrak{g}_2^I}$	$S_3^2 = \{dz_1 + z_2 dz_3\}$ $(w_3 - w_1 w_2, 2w_2, w_1)$
	$\underbrace{\mathfrak{g}_\alpha \oplus \mathfrak{g}_\beta}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{\alpha+\beta}}^{\mathfrak{g}_2^I} \bigoplus \underbrace{\mathfrak{g}_{\alpha+2\beta}}^{\mathfrak{g}_3^I}$	$S_4^7 = \{dz_1 + z_2 dz_3, dz_2 + z_4 dz_3\}$ $(w_4 - w_1 w_3 + \frac{1}{3}w_1^2 w_2, 2w_3 - w_1 w_2, w_1, 2w_2)$
Sp_3	$\underbrace{\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{\alpha+\beta+\gamma} \oplus \mathfrak{g}_{\alpha+2\beta+\gamma}}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{2\alpha+2\beta+\gamma}}^{\mathfrak{g}_2^I}$	$S_5^3 = \{dz_1 + z_2 dz_3 + z_4 dz_5\}$ $(w_5 - w_1 w_4 - w_2 w_3, 2w_4, w_1, 2w_3, w_2)$
G_2	$\underbrace{\mathfrak{g}_\alpha \oplus \mathfrak{g}_{\alpha+\beta}}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{2\alpha+\beta}}^{\mathfrak{g}_2^I} \bigoplus \underbrace{\mathfrak{g}_{3\alpha+\beta} \oplus \mathfrak{g}_{3\alpha+2\beta}}^{\mathfrak{g}_3^I}$	$S_5^{11}(z_5^2) = \{dz_1 + z_3 dz_4, dz_2 + z_5 dz_3 + z_5^2 dz_4, dz_3 + 2z_5 dz_4\}$ $(w_4 - \frac{3}{2}w_1 w_3 - w_1^2 w_2, w_5 - \frac{3}{2}w_2 w_3 - w_1 w_2^2, -\frac{3}{\sqrt[3]{3}}w_3 - \frac{3}{\sqrt[3]{3}}w_1 w_2, -\sqrt[3]{3}w_1, -\sqrt[3]{3}w_2)$
	$\underbrace{\mathfrak{g}_\beta \oplus \mathfrak{g}_{\alpha+\beta} \oplus \mathfrak{g}_{2\alpha+\beta} \oplus \mathfrak{g}_{3\alpha+\beta}}^{\mathfrak{g}_1^I} \bigoplus \underbrace{\mathfrak{g}_{3\alpha+2\beta}}^{\mathfrak{g}_2^I}$	$S_5^3 = \{dz_1 + z_2 dz_3 + z_4 dz_5\}$ $(w_5 - \frac{3}{2}w_2 w_3 - \frac{1}{2}w_1 w_4, 3w_2, w_3, w_4, w_1)$

Remark: Since $\mathfrak{so}_5$ is isomorphic to $\mathfrak{sp}_2$ and $\mathfrak{so}_6$ to $\mathfrak{su}_4$, the corresponding Pfaffian systems must be equivalent. However since in each case the details of the calculation are different and for the sake of confirmation their description is also included in the above list.

5   Application: infinitesimal symmetries.

The remainder of this note presents an example of how the twistor Pfaffian systems discussed above are related to the corresponding Lie algebra of infinitesimal symmetries, which in general is infinite dimensional.

It is well known that $\mathfrak{g}^{\mathbb{C}} \subseteq L(S)$, because the superhorizontal distribution is a $G^{\mathbb{C}}-$invariant distribution, *i.e.* $G^{\mathbb{C}}$ acts on $G^{\mathbb{C}}/P_I$ by left translations preserving SH. More precisely, for any $X \in \mathfrak{g}^{\mathbb{C}}$ the infinitesimal symmetries of SH are given in a canonical way by the vector fields:

$$X^*_{[g]} = \frac{d}{dt}[\exp(tX)g]|_{t=0}, \tag{7}$$

with $[g] \in B$, the big cell of $G^{\mathbb{C}}/P_I$.

Secondly, E. Cartan observed (see [8]) that the Pfaffian system $S_5^{11}(z_5^2)$ on $\mathbb{C}^5$ has the property $L(S_5^{11}(z_5^2)) \cong \mathfrak{g}_2^{\mathbb{C}}$ and therefore $\dim L(S_5^{11}(z_5^2))_{\mathbb{C}} = 14$.

More generally, K. Yamaguchi in [10] considered examples of regular differential systems, which turn out to agree with the Pfaffian systems arising from twistor fibrations. For any such system S, the main result of [10] asserts that $L(S) \cong \mathfrak{g}^{\mathbb{C}}$ except for the following three cases:

(1) $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g}^I_{-1} \oplus \mathfrak{g}^I_0 \oplus \mathfrak{g}^I_1$.

(2) $\mathfrak{g}^{\mathbb{C}} = \bigoplus_{i=-2}^{2} \mathfrak{g}^I_i$ (If $\dim \mathfrak{g}^I_{-2} = \dim \mathfrak{g}^I_2 = 1$).

(3) $\mathfrak{g}^{\mathbb{C}}$ is a Lie algebra of type A_l such that $I = \{\alpha_1, \alpha_m\}$, or type C_l such that $I = \{\alpha_1, \alpha_l\}$. $(1 < m < l)$.

Notice that all the examples presented in table 4 belong to one of these categories with the exception of $S_5^{11}(z_5^2)$, the Cartan case.

The fact that the Pfaffian system S_3^2 on $\mathbb{C}^3$ originates from *two different* twistor fibrations (see table 4) give us a natural explanation of the fact that $\dim L(S_3^2) = \infty$. In fact, as we have seen in 3.1, the twistor fibration

$$\pi : SU_3/S(U_1 \times U_1 \times U_1) \cong F_3(\mathbb{C}) \to SU_3/S(U_1 \times U_2) \cong \mathbb{C}P^2,$$

with big cell $\mathbb{C}^3$ has a superhorizontal distribution generated by $\{\partial_1, \ \partial_2 + w_1\partial_3\}$ and therefore a Pfaffian system $S = \{dw_3 - w_1 dw_2\}$, which is equivalent to S_3^2 under the change of coordinates: $(z_1, z_2, z_3) = (w_3, -w_1, w_2)$. By computing (7)

for a basis of the lie algebra $\mathfrak{sl}_3\mathbb{C}$, a set of 8 vector fields is obtained. The Lie algebra $\mathcal{L}'$ spanned by this set is a subalgebra of the infinitesimal symmetries of S_3^2. Similarly, for the fibration

$$\pi : Sp_2/Sp_1 \times U_1 \cong \mathbb{C}P^3 \to Sp_2/Sp_1 \times Sp_1 \cong S^4,$$

we have $SH = \langle \{\partial_1 - w_2\partial_3, \partial_2 + w_1\partial_3\} \rangle$ and $S = \{dw_3 + w_2dw_1 - w_1dw_2\}$ which under the change of coordinates $(z_1, z_2, z_3) = (w_3 - w_1w_2, 2w_2, w_1)$ also corresponds to S_3^2. In the same way as above, we can construct a Lie subalgebra $\mathcal{L}''$ of $L(S_3^2)$ spanned by 10 vector fields by means of (7).

By direct calculation we find that the Lie algebra $\langle \mathcal{L}', \mathcal{L}'' \rangle$ generated by $\mathcal{L}'$ and $\mathcal{L}''$ is an *infinite dimensional* Lie subalgebra of $L(S_3^2)$. In a future paper we shall discuss further ramifications of this observation.

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