TESTING BENSON'S REGULARITY CONJECTURE

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ABSTRACT. D. J. Benson conjectures that the Castelnuovo-Mumford regularity of the cohomology ring of a finite group is always zero. More generally he conjectures that there is always a *very strongly quasi-regular* system of parameters. Computer calculations show that the second conjecture holds for all groups of order less than 256.

1. INTRODUCTION

This paper is concerned with a conjecture of D. J. Benson [4] about the commutative algebra of group cohomology rings. There are several results relating the group structure of a finite group G to the commutative algebra of its cohomology ring $H^*(G) = H^*(G, k)$ with coefficients in a field of characteristic p. For the Krull dimension and depth we have the following inequalities, where S denotes a Sylow p-subgroup of G. Recall that the p-rank of G is the rank of the largest elementary abelian p-subgroup.

$$(1) \quad p\text{-rk}(Z(S)) \leq \text{depth}\, H^*(S) \leq \text{depth}\, H^*(G) \leq \dim H^*(G) = p\text{-rk}(G)\,.$$

See Evens' book [11] for proofs of the first inequality (Duflot's theorem) and the last one (due to Quillen). The second inequality is Theorem 2.1 of Benson's paper [2] and "must be well known". The third is automatic for finitely generated connected k-algebras. Note that the dimension and depth only depend on G, p: not on k. These inequalities motivate the following definitions.

Definition. Let G, p, k, S be as above. The group-theoretic defect $\text{gtD}_p(G)$, and the Cohen–Macaulay defect $\delta_p(G)$ are defined by

$$\text{gtD}(G) = p\text{-rk}(G) - p\text{-rk}(Z(S)) \quad \delta(G) = \dim H^*(G, k) - \text{depth}\, H^*(G, k)$$

It follows from Eqn. (1) that

$$(2) \quad 0 \leq \delta(G) \leq \text{gtD}(G) \qquad \text{gtD}(G) = \text{gtD}(S) \qquad \delta(G) \leq \delta(S)\,.$$

The term Cohen–Macaulay defect (sometimes deficiency) is already in use among workers in the field. To state Benson's conjectures we need some terminology.

Definition. Let p, k be as above. Let A be a graded commutative k-algebra which is both connected and finitely generated. Connected means that $A^0 = k$ and $A^{<0} = 0$. Let $\zeta_1, \ldots, \zeta_r$ be a system of homogeneous elements in $A^{>0}$, and set $n_i = |\zeta_i| > 0$.

a) The system is called a filter-regular system of parameters if multiplication by ζ_{i+1} has finite-dimensional kernel as an endomorphism of $A/(\zeta_1, \ldots, \zeta_i)$ for each $0 \leq i \leq r$, where $\zeta_{r+1} = 0$. Observe that a filter-regular system of parameters really is a system of parameters.

b) A very strongly quasi-regular system of parameters is a system which is a filter-regular system of parameters by virtue of the property that this kernel is restricted to degrees $\leq n_1 + \cdots + n_i + d_i$ for each $0 \leq i \leq r$, where $d_r = -r$ and $d_i = -i - 1$ for all $i < r$.

Theorem 1.1 (Benson). *Let G be a finite group, p a prime number and k a field of characteristic p.*

a) *The cohomology ring $A = H^*(G, k)$ does have filter-regular systems of parameters: the Dickson invariants (suitably interpreted) form one.*

b) *Either every filter-regular system of parameters in A is very strongly quasi-regular, or none are.*

c) *If the Cohen–Macaulay defect of G satisfies $\delta(G) \leq 2$ then there is a very stong quasi-regular system of parameters in A. In particular $\delta(G) \leq 2$ holds for all 267 groups of order 64.*

d) *If there is a very strongly quasi-regular system of parameters in A, then the Castelnuovo–Mumford regularity of A is zero.*

Proof. The main reference is Benson's paper [4]. Part a) is Coroll. 9.8 and Part b) is Coroll. 4.7(c), whereas Part d) follows from Coroll. 4.7(c) and Theorem 4.2. The first statement of Part c) is Theorem 1.5 of [4]; the second one was observed by Carlson [8, 10], who computed the cohomology ring of every group of order 64. The reader may find the tabulated data in [3, Appendix] useful.   □

Remark. A weaker version of the $\delta(G) = 2$ case of c) was also proved by Okuyama and Sasaki. It is a shame that their paper [20] appeared so late: I know that it had completed the refereeing process by the end of April 2001, but it had been superseded by the time it was finally published in 2004.

Conjecture 1.2 (Benson [4]). Let G be a finite group, p a prime number and k a field of characteristic p. The cohomology ring $H^*(G, k)$ has Castelnuovo–Mumford regularity zero.

Conjecture 1.3. Let G be a finite group, p a prime number and k a field of characteristic p. The cohomology ring $H^*(G, k)$ always contains a very strongly quasi-regular system of parameters.

Remark. Conjecture 1.2 is Benson's Conjecture 1.1. By Theorem 1.1 d) it is a weak form of Conjecture 1.3, which is only implicitly present in Benson's paper.

Kuhn has shown that Conjecture 1.2 has applications to the study of central essential cohomology [16].

The conjectures have been verified in two families of cases. Benson showed in [5] that if Conjecture 1.2 holds for H then it also holds for the wreath product $G = H \wr C_p$. And the second verification is the following theorem, the main result of the present paper.

Theorem 1.4. *Conjecture 1.3 holds for every group of order less than 256.*

Proof. By Theorem 1.1 c) a counterexample has to have $\delta(G) \geq 3$. By Proposition 2.1 the only groups of order less than 256 satisfying $\delta(G) \geq 3$ have order 128 and satisfy $\delta(G) = 3$. By Proposition 3.1 there are fourteen groups of order 128 with $\delta(G) = 3$, and each of these satisfies the conjecture. $\qquad\square$

2. REDUCTION TO THE CASE $|G| = 128$

Proposition 2.1. *Let G be a group of order less than* 256. *Then $\delta(G) \leq 3$; and if $\delta(G) = 3$ then $|G| = 128$.*

Proof. Let S be a Sylow p-subgroup of G. In view of the inequality $\delta(G) \leq \delta(S)$ and the restriction $|G| < 256$ it suffices to consider the case where G itself is a p-group.

So suppose G is a p-group with $\delta(G) \geq 3$. By Lemma 2.2 below it follows that $p = 2$, that $\delta(G) = 3$, and that either $|G| = 64$ or $|G| = 128$. But Carlson's computations [see Theorem 1.1c) above] show that $\delta(G) \leq 2$ if $|G| = 64$. $\qquad\square$

Lemma 2.2. *Let G be a finite group and p a prime number.*
  a) *If $\delta(G) \geq 3$ or more generally if $\mathrm{gtD}(G) \geq 3$ then p^5 divides the order of G. If $p = 2$ or $p = 3$ then p^6 must divide $|G|$.*
  b) *If $\delta(G) \geq 4$ or more generally if $\mathrm{gtD}(G) \geq 4$ then p^6 divides the order of G. If $p = 2$ or $p = 3$ then p^7 must divide $|G|$.*
  c) *If $p = 2$ and $\mathrm{gtD}(G) \geq 4$ then $|G|$ is divisible by* 256.

Proof. a): By Eqn. (2) we have $\delta(G) \leq \mathrm{gtD}(G)$. It is apparent from the definition that a finite group and its Sylow p-subgroups have the same group-theoretic defect. So it suffices to consider the case where G is a p-group and $\mathrm{gtD}(G) \geq 3$.

Every nontrivial p-group has a centre of rank at least one. So a p-group with $\mathrm{gtD} \geq 3$ must have a subgroup which is elementary abelian of rank 4. It must be nonabelian too, so the order must be at least p^5.

Suppose that there is such a group of order p^5. Then the centre is cyclic of order p, and there is an elementary abelian subgroup V of order p^4. This V is a maximal subgroup of a p-group and therefore normal. Let $a \in G$ lie outside V. Then $G = \langle a, V \rangle$ and the conjugation action of a on V must be nontrivial of order p. So the minimal polynomial of the action divides $X^p - 1 = (X - 1)^p$. This means that the action has a Jordan normal form with sole eigenvalue 1. The

eigenvectors in each Jordan block belong to the centre of G. So as the centre is cyclic there can only be one Jordan block, of size 4. But for $p = 2$ the size 3 Jordan block does not square to the identity, so there can be no blocks of size 3 or higher. Similarly there can be no size 4 block for $p = 3$, since it does not cube to the identity. So for $p = 2, 3$ there must be more than one Jordan block.

b): analogous.

c): We need to show that if $|G| = 128$ then $\mathrm{gtD}(G) \geq 4$ cannot happen. Such a group would need to contain an elementary abelian 2-group V satisfying the equation $p\text{-rk}(V) = 4 + p\text{-rk}(Z(G))$. The rank of the centre must be at least one, and if it is three or more then $V = G$ and therefore G is abelian, a contradiction. If the centre has rank two, then V has index two in G and we are in the same situation as in a), except now we want two Jordan blocks. But we need at least three, since V has rank six and only Jordan blocks of size one or two are allowed.

If the centre has rank one, then V has rank five and we may pick a subgroup H with $V \leq H \leq G$ and $[G : H] = [H : V] = 2$. If H is elementary abelian then we are back in the immediately preceding case of an order two action on a rank six elementary abelian. If H is not elementary abelian then the usual Jordan block considerations mean that C has rank at least 3, where C is the unique largest central elementary abelian of H. By applying the Jordan block considerations to the conjugation action of G/H on C, we see that $Z(G)$ must have rank at least two, a contradiction. $\qquad\square$

Remark 2.3. In fact there are only two groups of order 64 with $\mathrm{gtD} = 3$. Here are their numbers in the Hall–Senior list [15] and in the Small Groups Library [6]. Their defects are taken from the tables in [3, Appendix].

Small Group	Hall–Senior	$\delta(G)$
32	250	2
138	259	1

As we shall see below there are 14 groups of order 128 with $\delta(G) = 3$.

Remark 2.4. For $p \geq 5$ let G be the following semidirect product group of order p^5: there is a rank four elementary abelian on the bottom and a cyclic group of order p on top. The conjugation action is a size 4 Jordan block. This group has $\mathrm{gtD}(G) = 3$, since

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \binom{n}{3} \\ 0 & 1 & \binom{n}{1} & \binom{n}{2} \\ 0 & 0 & 1 & \binom{n}{1} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. The groups of order 128

Let G be a group of order 128. By Lemma 2.2c) we have $\mathrm{gtD}(G) \leq 3$.

Proposition 3.1. *Only* 57 *out of the* 2328 *groups of order* 128 *satisfy* $\mathrm{gtD}(G) = 3$. *Of these* 57 *groups,* 43 *satisfy* $\delta(G) \leq 2$. *The remaining* 14 *groups satisfy* $\delta(G) = 3$. *Each of these* 14 *groups of order* 128 *with* $\delta(G) = 3$ *satisfies Conjecture 1.3.*

According to the numbering of the Small Groups Library [6] *these fourteen groups are: numbers* 36, 48, 52, 194, 515, 551, 560, 561, 761, 780, 801, 813, 823 *and* 836.

Proof. By machine computation. Inspecting the Small Groups library, one sees that there are 57 groups with $\mathrm{gtD}(G) = 3$. See Appendix A for a discussion of how the p-rank is computed.

These 57 groups are listed in Table 1. The cohomology rings of these 57 groups were computed using an improved version of the author's cohomology program [14]. These cohomology rings may be viewed online [13]. The cohomology rings were calculated using Benson's test for completion [4, Thm 10.1].

Benson's test involves constructing a filter-regular system of parameters and determining in which degrees it is not strictly regular. This means that one automatically determines whether the group satisfies Conjecture 1.3 when one computes cohomology using Benson's test. The Cohen–Macaulay defect is another by-product of a computation based on Benson's test.

The value of $\delta(G)$ for each of the 57 groups is given in Table 1. The fourteen groups listed in the statement of the proposition are indeed the only ones with $\delta(G) = 3$. The computations showed that these 14 groups do satisfy the conjecture. □

Remark. The test is phrased in such a way that it is easy to implement. With one exception: it is not immediately clear how to construct a filter-regular system of parameters in low degrees. This point is discussed in the next section.

Remark. The computation that took the longest time was group number 836, one of the $\delta(G) = 3$ groups. Its cohomology ring has 65 generators and 1859 generators.

Remark. The distribution of these 57 groups by Cohen–Macaulay defect is as follows:

$\delta(G)$	0	1	2	3
#G	1	11	31	14

Remark. Some of these groups have been studied before. Groups 928 and 1578 are wreath products: $D_8 \wr 2$ and $2^3 \wr 2$ respectively. By the Carlson–Henn result [9] one has $\delta(D_8 \wr 2) = 1$ and $\delta(2^3 \wr 2) = 2$. Groups 850 of order 128 is a direct product of the form $G = H \times 2$, where H is group number 32 of order 64; and the same applies to group 1755 of order 128 and group 138 of order 64. It is immediate that $\delta(G) = \delta(H)$ for such groups, so Carlson's work guarantees $\delta(G) \leq 2$ for both these groups of order 128.

gp	K	d	r	δ		gp	K	d	r	δ
<u>36</u>	5	2	2	3		850	5	3	2	2
<u>48</u>	5	2	2	3		852	4	2	1	2
<u>52</u>	4	1	1	3		853	4	2	1	2
<u>194</u>	5	2	2	3		854	4	2	1	2
513	5	3	2	2		859	4	2	1	2
<u>515</u>	5	2	2	3		860	4	2	1	2
527	4	2	1	2		866	4	2	1	2
<u>551</u>	5	2	2	3		928	4	3	1	1
<u>560</u>	4	1	1	3		929	4	2	1	2
<u>561</u>	4	1	1	3		931	4	2	1	2
621	5	3	2	2		932	4	2	1	2
623	4	2	1	2		934	4	2	1	2
630	5	3	2	2		1578	6	4	3	2
635	4	2	1	2		1615	4	3	1	1
636	4	2	1	2		1620	4	2	1	2
642	4	2	1	2		1735	5	3	2	2
643	4	3	1	1		1751	4	2	1	2
645	4	3	1	1		1753	4	3	1	1
646	4	2	1	2		1755	5	4	2	1
740	4	2	1	2		1757	4	3	1	1
742	4	2	1	2		1758	4	3	1	1
753	5	3	2	2		1759	4	3	1	1
<u>761</u>	5	2	2	3		1800	4	2	1	2
764	4	2	1	2		2216	5	4	2	1
<u>780</u>	4	1	1	3		2222	5	3	2	2
<u>801</u>	4	1	1	3		2264	5	3	2	2
<u>813</u>	4	1	1	3		2317	4	3	1	1
<u>823</u>	4	1	1	3		2326	4	4	1	0
<u>836</u>	4	1	1	3						

TABLE 1. For each group of order 128 with $\mathrm{gtD}(G) = 3$, we give its number in the Small Groups library, the Krull dimension K and depth d of $H^*(G)$, the rank $r = K - 3$ of $Z(G)$ and the Cohen–Macaulay defect $\delta = K - d$. Underlined entries have $\delta = 3$. Notation based on that of [3, Appendix].

Group number 2326 is the extraspecial group 2_+^{1+6}; Quillen showed that its cohomology ring is Cohen–Macaulay [21]. Group number 931 is the Sylow 2-subgroup of the Mathieu groups M_{22} and M_{23}; its cohomology was studied by Adem and Milgram [1]. Group number 934 is the Sylow 2-subgroup of the Janko group J_2; its cohomology ring was calculated by Maginnis [17].

I am not aware of any previous cohomological investigations concerning the other two groups that I can name. One of these is group 932, the Sylow 2-subgroup of $G_2(3) : 2$. The other is group number 836, the Sylow 2-subgroup of one double cover of the Suzuki group $Sz(8)$. This group (number 836) turned out to have the most complicated cohomology ring in the study.

4. The weak rank-restriction condition

How does one construct a filter-regular system of parameters in a cohomology ring which is defined over the prime field $\mathbb{F}_p$ and also lies in low degree? An efficient implementation of Benson's test calls for an answer to this question.

Benson shows that the Dickson invariants (suitably interpreted) form a filter-regular system of parameters. This means a sequence of cohomology classes in $H^*(G)$ whose restrictions to the elementary abelian subgroups of G are (powers of) the appropriate Dickson invariants. Given information about restriction to subrings it is a straightforward task to compute classes with the appropriate restriction patterns. However the degrees involved can be large.

Definition. (c.f. [4, §8])   Let G be a p-group with p-rk$(G) = K$. Let $C = \Omega_1(Z(G))$. Homogeneous elements $\zeta_1, \ldots, \zeta_K \in H^*(G)$ satisfy the *weak rank restriction condition* if for each rank elementary abelian subgroup $V \geq C$ of G the following holds, where $s = p$-rk(V):

> The restrictions of $\zeta_1, \ldots, \zeta_s$ to V form a homogeneous system of parameters for $H^*(V)$; and the restrictions of $\zeta_{s+1}, \ldots, \zeta_K$ are zero.

Lemma 4.1. *If $\zeta_1, \ldots, \zeta_K \in H^*(G)$ satisfy the weak rank restriction condition then they constitute a filter-regular system of parameters.*

Proof. By a well known theorem of Quillen (see e.g. Evens' book [11]), $\zeta_1, \ldots, \zeta_K$ is a homogeneous system of parameters for $H^*(G)$. The proof of Theorem 9.6 of [4] applies just as well to parameters satisfying the weak rank restriction condition, because if E is an arbitrary rank i elementary abelian subgroup of G, then setting $V = \langle C, E \rangle$ one has $V \geq C$ and $C_G(V) = C_G(E)$, yet the rank of V is at least as large as the rank of E. So the restrictions of $\zeta_1, \ldots, \zeta_i$ to $H^*(C_G(E))$ do form a regular sequence, by the same argument based on the Broto–Henn approach to Duflot's theorem. □

Lemma 4.2. *Let G be a p-group with p-rk$(G) = K$ and p-rk$(Z(G)) = r$. Let $C = \Omega_1(Z(G))$. Suppose that homogeneous elements $\zeta_1, \ldots, \zeta_K \in H^*(G)$ satisfy the following conditions:*

a) *The restrictions of $\zeta_1, \ldots, \zeta_r$ to $H^*(C)$ form a regular sequence there; and*
b) *For each rank $r+s$ elementary abelian subgroup $V \geq C$ of G the restrictions of $\zeta_{r+s+1}, \ldots, \zeta_K$ to V are zero, and for $1 \leq i \leq s$ the restrictions of ζ_{r+i} to V is a power of the ith Dickson invariant in $H^*(V/C)$.*

Then $\zeta_1, \ldots, \zeta_K \in H^*(G)$ *is a filter-regular system of parameters for* $H^*(G)$. *Moreover such systems of parameters exist.*

Remark. By the ith Dickson invariant I mean the one which restricts nontrivially to dimension i subspaces, but has zero restriction to smaller subspaces. That is, if $i < j$ then the ith Dickson invariant lies in lower degree than the jth Dickson invariant.

Proof. Such classes clearly satisfy the weak rank restriction condition. The existence of $\zeta_1, \ldots, \zeta_r$ already follows from Evens' theorem that the cohomology ring of an arbitrary subgroup $H \leq G$ is a finitely generated module over the image of restriction from G. For the ζ_{r+i}: these are given by restrictions to each elementary abelian subgroup, and these restrictions satisfy the compatibility conditions that one expects from genuine restrictions, c.f. Quillen's work on the spectrum of a cohomology ring [22]. This means that – on raising these defining restrictions by sufficiently high pth powers – the ζ_{r+i} do indeed exist.   □

Remark. The point is that Lemma 4.2 is a recipe for constructing a filter-regular system of parameters. Recent work of Kuhn [16] in fact shows that one can choose the generators of $H^*(G)$ in such a way that $\zeta_1, \ldots, \zeta_r$ may be chosen from amongst these generators.

An additional saving follows from the fact that if $\zeta_1, \ldots, \zeta_K$ is a system of parameters and $\zeta_1, \ldots, \zeta_{K-1}$ is filter-regular, then the whole system is automatically filter-regular. This means that one can replace the ζ_K of Lemma 4.2 by any element that completes a system of parameters.

In earlier calculations, filter-regular parameters were constructed by hand on a trial and error basis. Subsequently most calculations were performed or re-performed using the parameter choice method of Lemma 4.2. In the worst case calculations this meant finishing the computation in degree 17, although the presentation was finished earlier.

5. THE a-INVARIANTS

Let k be a field and R a connected finitely presented graded commutative k-algebra. Let M be a finitely generated graded R-module, and $\mathfrak{m}$ the ideal in R of all elements in positive degree. The a-invariants of M are defined by

$$a_{\mathfrak{m}}^i(M) = \max\{m \mid H_{\mathfrak{m}}^{i,m}(M) \neq 0\},$$

with $a_{\mathfrak{m}}^i(M) = -\infty$ if $H_{\mathfrak{m}}^i(M) = 0$. One can then take

$$\text{Reg}(M) = \max_{i \geq 0}\{a_{\mathfrak{m}}^i(M) + i\}$$

as the definition of the Castelnuovo–Mumford regularity of M. Table 2 lists the a-invariants of $H^*(G)$ for the 57 groups of order 128 with $\text{gtD}(G) = 3$.

In order to calculate the a-invariants, one uses Lemma 4.3 of [4], which is based on methods of N. V. Trung. The lemma says that if $\zeta \in R^n$ is such that $\text{Ann}_M(\zeta)$

gp	K	$a_{\mathfrak{m}}^{K-3}$	$a_{\mathfrak{m}}^{K-2}$	$a_{\mathfrak{m}}^{K-1}$	$a_{\mathfrak{m}}^{K}$	gp	K	$a_{\mathfrak{m}}^{K-3}$	$a_{\mathfrak{m}}^{K-2}$	$a_{\mathfrak{m}}^{K-1}$	$a_{\mathfrak{m}}^{K}$
36	5	-5	-5	-5	-5	850	5	$-\infty$	-6	-5	-5
48	5	-5	-5	-5	-5	852	4	$-\infty$	-5	-4	-4
52	4	-4	-4	-4	-4	853	4	$-\infty$	-5	-4	-4
194	5	-6	-6	-5	-5	854	4	$-\infty$	-5	-4	-4
513	5	$-\infty$	-5	-5	-5	859	4	$-\infty$	-6	-4	-4
515	5	-5	-5	-5	-5	860	4	$-\infty$	-5	-4	-4
527	4	$-\infty$	-4	-4	-4	866	4	$-\infty$	-5	-4	-4
551	5	-6	-5	-5	-5	928	4	$-\infty$	$-\infty$	-4	-4
560	4	-6	-5	-4	-4	929	4	$-\infty$	-5	-4	-4
561	4	-5	-4	-4	-4	931	4	$-\infty$	-3	-4	-4
621	5	$-\infty$	-5	-5	-5	932	4	$-\infty$	-6	-4	-4
623	4	$-\infty$	-4	-4	-4	934	4	$-\infty$	-3	-5	-4
630	5	$-\infty$	-5	-5	-5	1578	6	$-\infty$	-6	-6	-6
635	4	$-\infty$	-4	-4	-4	1615	4	$-\infty$	$-\infty$	-4	-4
636	4	$-\infty$	-4	-4	-4	1620	4	$-\infty$	-4	-4	-4
642	4	$-\infty$	-4	-4	-4	1735	5	$-\infty$	-5	-5	-5
643	4	$-\infty$	-4	-4	-4	1751	4	$-\infty$	-6	-4	-4
645	4	$-\infty$	$-\infty$	-4	-4	1753	4	$-\infty$	$-\infty$	-4	-4
646	4	$-\infty$	-4	-4	-4	1755	5	$-\infty$	$-\infty$	-5	-5
740	4	$-\infty$	-4	-4	-4	1757	4	$-\infty$	$-\infty$	-4	-4
742	4	$-\infty$	-4	-4	-4	1758	4	$-\infty$	$-\infty$	-5	-4
753	5	$-\infty$	-5	-5	-5	1759	4	$-\infty$	$-\infty$	-4	-4
761	5	-5	-5	-5	-5	1800	4	$-\infty$	-4	-4	-4
764	4	$-\infty$	-5	-4	-4	2216	5	$-\infty$	$-\infty$	-5	-5
780	4	-6	-4	-4	-4	2222	5	$-\infty$	-6	-5	-5
801	4	-4	-4	-4	-4	2264	5	$-\infty$	-6	-5	-5
813	4	-4	-4	-4	-4	2317	4	$-\infty$	$-\infty$	-4	-4
823	4	-4	-4	-4	-4	2326	4	$-\infty$	$-\infty$	$-\infty$	-4
836	4	-4	-5	-4	-4						

TABLE 2. For each of the 57 groups of order 128 with $\mathrm{gtD}(G) = 3$, we give its number in the Small Groups library, the Krull dimension K and the last four a-invariants of $H^*(G)$. For $0 \le i < K - 3$ one has $a_{\mathfrak{m}}^i = -\infty$. The defect $\delta(G)$ is three if and only if $a_{\mathfrak{m}}^{K-3}$ is finite.

consists entirely of $\mathfrak{m}$-torsion, then

$$(3) \qquad a_{\mathfrak{m}}^{i+1}(M) + n \le a_{\mathfrak{m}}^i(M/\zeta M) \le \max(a_{\mathfrak{m}}^i(M), a_{\mathfrak{m}}^{i+1}(M) + n).$$

Replacing ζ by a suitable power if necessary, one can arrange for $a_{\mathfrak{m}}^{i+1}(M) + n \ge a_{\mathfrak{m}}^i(M)$ and therefore $a_{\mathfrak{m}}^{i+1}(M) = a_{\mathfrak{m}}^i(M/\zeta M) - n$. So the a-invariants of $H^*(G)$ may be computed recursively. To start the recursion we need $a_{\mathfrak{m}}^0(M)$, which is the

$\mathfrak{m}$-torsion: so if $\zeta \in R^n$ is such that $\mathrm{Ann}_M(\zeta)$ is finite dimensional, then $a^0_{\mathfrak{m}}(M)$ is the top dimension of $\mathrm{Ann}_M(\zeta)$.

Hence by starting from a filter-regular system of parameters and raising some of the parameters to higher powers if necessary, one may compute the a-invariants of $H^*(G)$ by just computing kernels. For a cohomology computation one chooses parameters in low degrees. So it is perhaps surprising that a survey of the author's computations of all 256 nonabelian groups of order 64 and some 61 groups of order 128 led to precisely one case where powers of the chosen parameters were necessary. This is the Sylow 2-subgroup of $L_3(4)$ which has a-invariants $-\infty, -\infty, -3, -5, -4$: a filter-regular system of parameters in degrees $4, 4, 2, 2$ led to kernels with top degrees $-\infty, -\infty, 5, 5, 8$, leading to problems with the calculation of $a^3_{\mathfrak{m}}$. Squaring the third parameter led to kernels with top degrees $-\infty, -\infty, 5, 7, 10$, which was sufficient to permit calculation of the a-invariants.

6. Excess and defect

Definition. As in the introduction let G be a finite group, p a prime number and k a field of characteristic p. We define the Duflot excess $e(G) = e_p(G)$ by

$$e_p(G) = \mathrm{depth}\, H^*(G, k) - p\text{-rk}(Z(S)).$$

The following inequalities follow immediately from this definition taken together with Equations (1) and (2).

$$(4) \qquad 0 \le e(G) \le \mathrm{gtD}(G) \qquad \delta(G) + e(G) = \mathrm{gtD}(G) \qquad e(G) \ge e(S).$$

Quillen showed that the extraspecial 2-group $G = 2^{1+2n}_+$ has Cohen–Macaulay cohomology [21]. So this group has $e(G) = n$ and $\delta(G) = 0$.

Now let p be an odd prime, and let G be the extraspecial p-group $G = p^{1+2n}_+$ of exponent p. With the single exception of the case $(p, n) = (3, 1)$, Minh proved [19] that this group has $\delta(G) = n$ and $e(G) = 0$. In the one exceptional case the cohomology ring is Cohen–Macaulay [18].

One good way to produce groups with small $e(G)/\delta(G)$ ratio satisfying Conjecture 1.2 is by iterating the wreath product construction. By passing from H to $H \wr C_p$ one multiplies the p-rank by p but increases the depth by one only [9].

Question 6.1. How (for large values of n) are the p-groups of order p^n distributed on the graph with $\delta(G)$ on the x-axis and $e(G)$ on the y-axis?

7. Outlook

To test the conjecture further we need to find more high defect groups.

There are 24 groups of order 3^6 with $\mathrm{gtD}(G) = 3$. The presence of essential classes in low degrees demonstrates that at least three of these groups have $\delta(G) = 3$. These groups are numbers 35, 56 and 67 in the Small Groups Library. There are essential classes in degrees 4, 2 and 4 respectively. Recall from Carlson's

paper [7] that the presence of essential classes means that depth $H^*(G, k) = p\text{-rk}Z(G)$ and therefore $\delta(G) = \text{gtD}(G)$.

Group number 299 of order 256 has $\text{gtD}(G) = 4$. The presence of an essential class in $H^3(G)$ means that $\delta(G) = 4$ too.

Appendix A. Computing the p-rank

The Small Groups library was accessed from GAP [12]. There is no built-in command in GAP that returns the p-rank of a given p-group. The simplest way to calculate it using existing commands would to be to generate the entire subgroup lattice and then filter out the elementary abelian subgroups. We chose instead to enumerate the conjugacy classes of elementary abelian subgroups using a straightforward if not particularly efficient inductive approach.

Let G be a p-group. If $|G|$ is small then it is feasible to list all the elements of the group. By testing each element one then obtains the list of all order p elements. A further element by element test yields all central elements of order p. This is one way to obtain the greatest central elementary abelian subgroup $\Omega_1(Z(G))$, denoted C in the paper. Of course, if one only wants $\Omega_1(Z(G))$ then it is quicker to make use of the function `IndependentGeneratorsOfAbelianGroup`.

Carlson [7] shows that one only needs the elementary abelians which contain $\Omega_1(Z(G))$. Given such an elementary abelian V of order p^d, one can list all the order p elements in $C_G(V)$ and so obtain all the order p^{d+1} elementary abelians containing V. This is the inductive step: the induction starts with $V = C$.

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