

Surgery, Yamabe invariant, and Seiberg-Witten theory

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Abstract

By using the gluing formula of the Seiberg-Witten invariant, we compute the Yamabe invariant $Y(X)$ of 4-manifolds X obtained by performing surgeries along points, circles or tori on compact Kähler surfaces. For instance, if M is a compact Kähler surface of nonnegative Kodaira dimension, and N is a smooth closed oriented 4-manifold with $b_2^+(N) = 0$ and $Y(N) \geq 0$, then we show that

$$Y(M\#N) = Y(M).$$

1 Introduction

The Yamabe invariant is a real-valued invariant of a smooth closed manifold defined using the scalar curvature. It somehow measures how much the negative scalar curvature is inevitable, and it can be used as a means to get to a canonical metric on a given manifold.

Let M be a closed smooth n -manifold. In any conformal class

$$[g] = \{\varphi g \mid \varphi : M \rightarrow \mathbb{R}^+ \text{ is smooth}\},$$

there exists a smooth Riemannian metric of constant scalar curvature, so-called *Yamabe metric*, realizing the minimum of the normalized total scalar

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curvature

$$\inf_{\tilde{g} \in [g]} \frac{\int_M s_{\tilde{g}} dV_{\tilde{g}}}{\left(\int_M dV_{\tilde{g}}\right)^{\frac{n-2}{n}}},$$

where $s_{\tilde{g}}$ and $dV_{\tilde{g}}$ respectively denote the scalar curvature and the volume element of \tilde{g} . That minimum value is called the *Yamabe constant* of the conformal class, and denoted as $Y(M, [g])$. Then the *Yamabe invariant* is defined as the supremum of the Yamabe constants over the set of all conformal classes on M , and one can hope for a canonical metric as a limit of such a maximizing sequence.

The Yamabe invariant of a compact orientable surfaces is $4\pi\chi(M)$ where $\chi(M)$ denotes the Euler characteristic of M by the Gauss-Bonnet theorem. In general, it is not quite easy to exactly compute the Yamabe invariant. Recently much progress has been made in low dimensions. In dimension 3, the geometrization by the Ricci flow gave many answers, and in dimension 4, the Spin^c structure and the Dirac operator have been remarkable tools for computing the Yamabe invariant. LeBrun [8, 9, 10] used the Seberg-Witten theory to show that if M is a compact Kähler surface whose Kodaira dimension $\kappa(M)$ is not equal to $-\infty$, then

$$Y(M) = -4\sqrt{2}\pi\sqrt{(2\chi + 3\tau)(\tilde{M})},$$

where τ denotes the signature and \tilde{M} is the minimal model of M , and for $\mathbb{C}P^2$,

$$Y(\mathbb{C}P^2) = 12\sqrt{2}\pi.$$

In particular, note that if $\kappa(M) = 0$ or 1, $Y(M) = 0$.

One notes that the blow-up does not change the Yamabe invariant of Kähler surfaces and may ask:

Question 1.1 *Let M be a smooth closed orientable 4-manifold with $Y(M) \leq 0$. Is there an orientation of M such that $Y(M \sharp m \overline{\mathbb{C}P^2}) = Y(M)$ for any integer $m > 0$? What about connected sums or surgeries along circles with 4-manifolds with negative-definite intersection form and nonnegative Yamabe invariant?*

In this article, we will show :

Theorem 1.2 *Let M be a closed Kähler surface of $\kappa(M) \geq 0$ (with $b_2^+(M) > 1$ if $\kappa(M) > 0$), and N be a smooth closed oriented 4-manifold with $b_2^+(N) = 0$ and $Y(N) \geq 0$. Then*

$$Y(M\#N) = Y(M).$$

More generally, we prove the case of the surgery along circles.

Definition 1 *Let M_1 and M_2 be smooth n -manifolds with embedded k -spheres c_1 and c_2 respectively, where the normal bundles are trivial. A surgery of M_1 and M_2 along c_i 's are defined as the result of deleting tubular neighborhood of each c_i and gluing the remainders by identifying two boundaries $S^k \times S^{n-k-1}$ using a diffeomorphism of S^k and the reflection map of S^{n-k-1} . When M_2 is not specified, it means a surgery with S^n .*

Theorem 1.3 *Let M be a closed Kähler surface with $\kappa(M) \geq 0$ and $b_2^+(M) > 1$, and N_i for $i = 1, \dots, m$ be smooth closed oriented 4-manifolds with $b_2^+(N_i) = 0$, $b_1(N_i) \geq 1$, and $Y(N_i) \geq 0$. Suppose that $c_i \subset N_i$ is an embedded circle nontrivial in $H_1(N_i, \mathbb{R})$ for $i = 1, \dots, m$. If \tilde{M} is a manifold obtained from M by performing a surgery with $\cup_{i=1}^m N_i$ along $\cup_{i=1}^m c_i$, then*

$$Y(\tilde{M}) = Y(M).$$

Note that the surgery on M with $(S^1 \times S^3)\#N$ along a null-homotopic circle in M and a circle representing $[S^1] \times \{\text{pt}\} \in H_1(S^1 \times S^3, \mathbb{Z})$ is $M\#N$.

When $b_1(N_i) = 0$, theorem 1.3 is no longer true in general. For example, take a closed non-spin simply-connected Kähler surface M with $\kappa(M) \geq 0$ and $b_2^+(M) > 1$. Let $N_i = S^4$, and c_i be any embedded circle in N_i for all i . Performing a surgery around null-homotopic circles in M with $\cup_{i=1}^m N_i$, we get \tilde{M} which is just $M\#(\#_{i=1}^m (S^2 \times S^2))$. By applying Wall's results [26, 27], it is diffeomorphic to $M\#(\#_{i=1}^m (\mathbb{C}P^2\#\overline{\mathbb{C}P^2}))$ which again becomes diffeomorphic to $a \mathbb{C}P^2\# a \overline{\mathbb{C}P^2}$ where $a = m + \frac{1}{2}(b_2(M) + \tau(M))$, if m is sufficiently large. But

$$Y(a \mathbb{C}P^2\# a \overline{\mathbb{C}P^2}) > 0 \geq Y(M).$$

We also give a different proof of the following result proved by Gursky and LeBrun in [6]:

Theorem 1.4 *Let N be a smooth closed oriented 4-manifold satisfying $b_2(N) = 0$ and $Y(N) \geq 12\sqrt{2}\pi (= Y(\mathbb{C}P^2))$. Then*

$$Y(\mathbb{C}P^2\#N) = Y(\mathbb{C}P^2).$$

For surgeries of codimension less than 3, in general the Yamabe invariant changes drastically after a surgery. But some surgeries along T^2 in 4-manifolds do preserve the Yamabe invariant.

We introduce some well-known different types of surgeries in 4-manifolds. Suppose that a smooth 4-manifold M contains a homologically essential torus T^2 with self-intersection zero. Deleting a tubular neighborhood $T^2 \times D^2$ of T^2 and gluing back using a diffeomorphism φ of the boundary T^3 , we get a new smooth 4-manifold M_φ called a generalized logarithmic transform of M .

Now suppose that two smooth 4-manifolds M_1 and M_2 each contain an embedded closed surface F with self-intersection zero. Deleting a tubular neighborhood $F \times D^2$ in each and gluing the remaining parts along the boundary $F \times S^1$ using a diffeomorphism of F and the complex conjugation map of S^1 , we get a fiber sum of M_1 and M_2 . When it is performed along two embedded surfaces in M , we call it an internal fiber sum of M .

Finally a knot surgery manifold for a knot $K \subset S^3$ with the knot exterior $E(K)$ is a smooth 4-manifold obtained by gluing $M \setminus (T^2 \times D^2)$ and $S^1 \times E(K)$ along the boundary T^3 in such a way that the homology class $[\text{pt} \times \partial D^2]$ is identified with $[\text{pt} \times \lambda]$ where λ is a longitude of K . Then a knot surgery of M is the same as the fiber sum of M with $S^1 \times M_K$ along the torus $S^1 \times m \subset S^1 \times M_K$, where m is a meridian circle to K and M_K is the 3-manifold obtained by performing 0-framed surgery on K .

Now let M be a closed Kähler surface of Kodaira dimension equal to 0 or 1 with $b_2^+(M) > 1$. It is known that M admits a T -structure defined by Cheeger and Gromov [3]. (For an explicit construction, see Paternain and Petean [17].) The existence of a T -structure implies that the manifold admits a sequence of Riemannian metrics with volume form converging to zero uniformly while the sectional curvatures are bounded below, so that the Yamabe invariant must be nonnegative. ([17]) Let \tilde{M} be the manifold obtained from M by a generalized logarithmic transform or an internal fiber sum or a fiber sum with $S^1 \times N$ where N is a closed oriented 3-manifold with nonzero Seiberg-Witten invariant along an embedded T^2 which is a regular orbit of the above T -structure. Then \tilde{M} has an obvious induced T -structure, and if \tilde{M} also has a nontrivial Seiberg-Witten invariant, we immediately get

$$Y(\tilde{M}) = Y(M) = 0.$$

It is interesting to note that these phenomena also appear in some cases of Kodaira dimension 2 as follows:

Theorem 1.5 *Let $M = \Sigma_1 \times \Sigma_2$ be a product of two Riemann surfaces of genus > 1 , and $\alpha_1, \dots, \alpha_m$ and β_1, \dots, β_m be non-intersecting homologically-essential circles embedded in Σ_1 and Σ_2 respectively.*

Suppose that X_k for $k = 1, \dots, \mu$ where $\mu \leq m$ is a closed oriented 3-manifold with $b_1(X_k) \geq 1$ and nonzero Seiberg-Witten invariant in a chamber, and c_k for $k = 1, \dots, \mu$ is an embedded circle in X_k representing a non-torsion generator of $H_1(X_k, \mathbb{Z})$.

Let \tilde{M} be a manifold obtained from M by performing on $\cup_{i=1}^m \alpha_i \times \beta_i$ an internal fiber sum or a fiber sum with $\cup_{k=1}^{\mu} S^1 \times X_k$ around $\cup_{k=1}^{\mu} S^1 \times c_k$. Then

$$Y(\tilde{M}) = Y(M).$$

Corollary 1.6 *Let each M_i for $i = 1, \dots, l$ be a product of two Riemann surfaces of genus > 1 , and T_1, \dots, T_m be tori embedded in $\cup_{i=1}^l M_i$ as above. Let \tilde{M} be a manifold obtained from $\cup_{i=1}^l M_i$ by performing on $\cup_{i=1}^m T_i$ an internal fiber sum or a fiber sum with $S^1 \times X_k$'s as above. Then*

$$Y(\tilde{M}) = -\left(\sum_{i=1}^l |Y(M_i)|^2\right)^{\frac{1}{2}}.$$

It is left as a further question whether the above theorems still hold true for any homologically essential tori.

2 Basic formulae of Yamabe invariant

When $Y(M) \leq 0$, it can be written as a very nice form:

$$Y(M) = -\inf_g \left(\int_M |s_g|^{\frac{n}{2}} d\mu_g \right)^{\frac{2}{n}} = -\inf_g \left(\int_M |s_g^-|^{\frac{n}{2}} d\mu_g \right)^{\frac{2}{n}},$$

where $s_g^- = \min(s_g, 0)$. (For a proof, see [10, 22].)

Another practical formula is the gluing formula of the Yamabe invariant under the surgery.

Theorem 2.1 (Kobayashi [7], Petean and Yun [18]) *Let M_1, M_2 be smooth closed manifolds of dimension $n \geq 3$. Suppose that an $(n - q)$ -dimensional smooth closed (possibly disconnected) manifold W embeds into*

both M_1 and M_2 with isomorphic normal bundle. Assume $q \geq 3$. Let M be any manifold obtained by gluing M_1 and M_2 along W . Then

$$Y(M) \geq \begin{cases} -(|Y(M_1)|^{n/2} + |Y(M_2)|^{n/2})^{2/n} & \text{if } Y(M_i) \leq 0 \forall i \\ \min(Y(M_1), Y(M_2)) & \text{if } Y(M_1) \cdot Y(M_2) \leq 0 \\ \min(Y(M_1), Y(M_2)) & \text{if } Y(M_i) \geq 0 \forall i \text{ and } q = n \end{cases}$$

A nontrivial estimation of the Yamabe invariant on 4-manifolds comes from the Seiberg-Witten theory.

Theorem 2.2 (LeBrun [8, 9]) *Let (M, g) be a smooth closed oriented Riemannian 4-manifold with $b_2^+(M) \geq 1$. Let \mathfrak{s} be a Spin^c structure on M with first chern class $c_1(\mathfrak{s})$. Suppose that Seiberg-Witten invariant of \mathfrak{s} is nontrivial in a chamber. Then*

$$Y(M, [g]) \leq \frac{|4\pi c_1(\mathfrak{s}) \cup [\omega]|}{\sqrt{[\omega]^2/2}}$$

where ω is nonzero and self-dual harmonic with respect to g . If the Seiberg-Witten invariant of \mathfrak{s} is nontrivial for any small perturbation, then

$$Y(M, [g]) \leq -4\sqrt{2}\pi \|c_1^+(\mathfrak{s})\|_{L^2}$$

where c_1^+ denotes the self-dual harmonic part of $c_1(\mathfrak{s})$.

3 Computation of Seiberg-Witten invariant

Let M be a smooth closed oriented Riemannian 4-manifold and P be its orthonormal frame bundle which is a principal $SO(4)$ bundle. Consider oriented \mathbb{R}^3 -vector bundles \wedge_+^2 and \wedge_-^2 consisting of self-dual 2-forms and anti-self-dual 2-forms respectively. Let's let P_1 and P_2 be associated $SO(3)$ frame bundles. Unless M is spin, it is impossible to lift these to principal $SU(2)$ bundles. Instead there always exists the \mathbb{Z}_2 -lift, a principal $U(2) = SU(2) \otimes_{\mathbb{Z}_2} U(1)$ bundle, of a $SO(3) \oplus U(1)$ bundle, when the $U(1)$ bundle on the bottom, denoted by L , has first chern class equal to $w_2(TM)$ modulo 2. We call this lifting a Spin^c structure on M .

Let W_+ and W_- be \mathbb{C}^2 -vector bundles associated to the above-obtained principal $U(2)$ bundles. One can define a connection ∇_A on them by lifting the Levi-Civita connection and a $U(1)$ connection A on L . Then the

Dirac operator $D_A : \Gamma(W_+) \rightarrow \Gamma(W_-)$ is defined as the composition of $\nabla_A : \Gamma(W_+) \rightarrow T^*M \otimes \Gamma(W_+)$ and the Clifford multiplication.

For a section Φ of W_+ , (perturbed) Seiberg-Witten equations of (A, Φ) is given by

$$\begin{cases} D_A \Phi = 0 \\ F_A^+ + \mu = \Phi \otimes \Phi^* - \frac{|\Phi|^2}{2} \text{Id}, \end{cases}$$

where F_A^+ is the self-dual part of the curvature dA of A , and a purely imaginary self-dual 2-form μ is a perturbation term, and finally the identification of both sides in the second equation comes from the Clifford action.

Now we review the Seiberg-Witten invariant as defined by Ozsváth and Szabó [19]. Suppose $b_2^+(M) > 0$, and let \mathfrak{s} be a Spin^c structure on M . The configuration space \mathfrak{B} of the Seiberg-Witten equations is given by

$$(\mathcal{A}(W_+) \times \Gamma(W_+)) / \text{Map}(M, S^1),$$

where $\mathcal{A}(W_+)$ is the space of connections on $L = \det(W_+)$ and is identified with $\Omega^1(M; i\mathbb{R})$, and $\text{Map}(M, S^1)$ is the group of gauge transformations. Since $\Gamma(W_+)$ is contractible, \mathfrak{B} is homotopy-equivalent to $T^{b_1(M)} = \frac{H^1(M; \mathbb{R})}{H^1(M; \mathbb{Z})}$. The irreducible configuration space \mathfrak{B}^* is

$$(\mathcal{A}(W_+) \times (\Gamma(W_+) - \{0\})) / \text{Map}(M, S^1),$$

and it is homotopy-equivalent to $\mathbb{C}P^\infty \times \frac{H^1(M; \mathbb{R})}{H^1(M; \mathbb{Z})}$ so that

$$H^*(\mathfrak{B}^*; \mathbb{Z}) \simeq \mathbb{Z}[U] \otimes \wedge^* H^1(M; \mathbb{Z}).$$

Defining the graded algebra $\mathbb{A}(M)$ over \mathbb{Z} by

$$\mathbb{Z}[H_0(M; \mathbb{Z})] \otimes \wedge^* H_1(M; \mathbb{Z})$$

with $H_0(M; \mathbb{Z})$ grading two and $H_1(M; \mathbb{Z})$ grading one, we have an obvious isomorphism

$$\mu : \mathbb{A}(M) \xrightarrow{\sim} H^*(\mathfrak{B}^*; \mathbb{Z})$$

such that μ maps the positive generator of $H_0(M; \mathbb{Z})$ to U . Note that the μ map restricted to a subset $H_1(M; \mathbb{Z}) \otimes \mathbb{Z}$ is given by $\text{Hol}_c^*(d\theta)|_{\mathfrak{B}^*}$ for $c \in H_1(M; \mathbb{Z})$, where $\text{Hol}_c : \mathfrak{B} \rightarrow S^1$ is the holonomy map around c .

Then the Seiberg-Witten invariant $SW_{M, \mathfrak{s}}$ is a function

$$SW_{M, \mathfrak{s}} : \mathbb{A}(M) \rightarrow \mathbb{Z}$$

$$a \mapsto \langle \mathfrak{M}_{M,\mathfrak{s}}, \mu(a) \rangle,$$

where $\mathfrak{M}_{M,\mathfrak{s}} \subset \mathfrak{B}$ is the moduli space, i.e. the solution space modulo gauge transformations of the Seiberg-Witten equations of (M, \mathfrak{s}) . It turns out that $SW_{M,\mathfrak{s}}$ is independent of the Riemannian metric and a generic perturbation, if $b_2^+(M) > 1$. (When $b_2^+(M) = 1$, it may depend on the choice of the chamber.) For a noncompact M with cylindrical-end metric, we can do the same job by considering solutions with finite energy. Here, the energy of a solution $(A(t), \Phi(t))$ in temporal gauge on the cylinder $T := \partial M \times [0, \infty]$ is defined as

$$\|A'(t)\|_{L^2(T)}^2 + \|\Phi'(t)\|_{L^2(T)}^2,$$

where the temporal gauge means that A has no temporal component dt .

We denote $\mathfrak{M}_{M,\mathfrak{s}}^{irr} := \mathfrak{M}_{M,\mathfrak{s}} \cap \mathfrak{B}^*$ and $\mathfrak{M}_{M,\mathfrak{s}}^{red} := \mathfrak{M}_{M,\mathfrak{s}} - \mathfrak{M}_{M,\mathfrak{s}}^{irr}$. It is also useful to define the Seiberg-Witten series of M to be the element of the group ring $\mathbb{Z}[H_2(M, \mathbb{Z})]$ given by

$$\overline{SW}_M := \sum_{\mathfrak{s}} SW_{M,\mathfrak{s}} \underbrace{(1 \otimes \cdots \otimes 1)}_{d(\mathfrak{s})/2} PD(c_1(\mathfrak{s})),$$

where $d(\mathfrak{s}) := \dim_{\mathbb{R}} \mathfrak{M}_{M,\mathfrak{s}}$, PD denotes the Poincaré-dual, and \mathfrak{s} runs over all isomorphism classes of Spin^c structures on M with even $d(\mathfrak{s})$.

For more details about the Seiberg-Witten theory and the gluing of the moduli spaces, the readers are referred to [12, 11, 13, 15, 16, 23, 24].

Before stating the theorem, we note the following lemma.

Lemma 3.1 *Let N be a smooth closed oriented 4-manifold with negative intersection form Q . Then there exists a Spin^c structure \mathfrak{s}' on N satisfying $c_1^2(\mathfrak{s}') = -b_2(N)$.*

Proof. By the Donaldson's theorem, Q is diagonalizable over \mathbb{Z} . (Although the original Donaldson's theorem [4] is stated for simply-connected ones, a simple Mayer-Vietoris argument can be applied for this generalization.) Let $\{\alpha_1, \dots, \alpha_{b_2(N)}\}$ be a basis for $H^2(N, \mathbb{Z}) \otimes \mathbb{Q}$ diagonalizing Q .

We have to show that there exists $x \in H^2(N, \mathbb{Z})$ satisfying that $Q(x, x) = -b_2(N)$, and x is characteristic, i.e. $Q(x, \alpha) \equiv Q(\alpha, \alpha) \pmod{2}$ for any $\alpha \in H^2(N, \mathbb{Z})$. It is easy to check that $x = \sum_{i=1}^{b_2(N)} \pm \alpha_i$ do the job. \blacksquare

Theorem 3.2 *Let M and N be smooth closed oriented 4-manifolds such that $b_2^+(M) > 0$, $b_2^+(N) = 0$, and $b_1(N) \geq 1$. Let $c \subset N$ be an embedded circle nontrivial in $H_1(N, \mathbb{R})$ and \tilde{M} be a manifold obtained by performing a surgery on M with N along c .*

If $\tilde{\mathfrak{s}}$ is the Spin^c structure on \tilde{M} obtained by gluing a Spin^c structure \mathfrak{s} on M and a Spin^c structure \mathfrak{s}' on N satisfying $c_1^2(\mathfrak{s}') = -b_2(N)$, then

$$SW_{\tilde{M}, \tilde{\mathfrak{s}}}(a \cdot [d_1] \cdots [d_{b_1(N)-1}]) = \pm SW_{M, \mathfrak{s}}(a)$$

for $a \in \mathbb{A}(M)$, where $[d_1], \dots, [d_{b_1(N)-1}]$ along with $r[c]$ for some $r \in \mathbb{Q}$ form a basis for the torsion-free part of $H_1(N, \mathbb{Z})$.

Proof. By removing a tubular neighborhood $S^1 \times D^3$ around the circle where the surgery is performed, we construct \hat{M} and \hat{N} with cylindrical end modeled on $S^1 \times S^2$ with a standard metric of positive scalar curvature which we denote by Y . For Y with the trivial Spin^c structure, the moduli space is the set $\chi(Y)$ of flat connections modulo gauge transformations of the trivial Spin^c structure, which is diffeomorphic to S^1 .

On $S^1 \times D^3$ we put a metric of positive scalar curvature with the same cylindrical-end, and see that its moduli space with the trivial Spin^c structure is also the set $\chi(S^1 \times D^3)$ of flat connections modulo gauge transformations of the trivial Spin^c structure, which is unobstructed. In an obvious way, $\chi(S^1 \times D^3)$ is diffeomorphic to $\chi(Y)$. From $b_2^+(\hat{M}) > 0$, $\mathfrak{M}_{\hat{M}, \mathfrak{s}} = \mathfrak{M}_{\hat{M}, \mathfrak{s}}^{irr}$ and it is unobstructed by using a generic exponentially-decaying perturbation.

Let $\hat{\mathcal{G}}$ be the gauge transformations on \hat{M} . (Note that any gauge transformations on Y extend to $S^1 \times D^3$ and \hat{N} . We will denote such extensions of $\hat{\mathcal{G}}$ also by $\hat{\mathcal{G}}$ by abuse of notation.) Letting $\hat{\chi}(Y)$ be the set of equivalence classes of flat connections on Y modulo $\hat{\mathcal{G}}$, $\hat{\chi}(Y)$ is a covering of $\chi(Y)$ with fiber $H^1(Y, \mathbb{Z})/H^1(\hat{M}, \mathbb{Z})$. Similarly we define $\hat{\chi}(S^1 \times D^3)$ and $\mathfrak{M}_{\hat{N}, \mathfrak{s}'}$. (In fact, $\hat{\chi}(S^1 \times D^3) = \hat{\chi}(Y)$.) Since the asymptotic map

$$(\partial_\infty, \partial_\infty) : \mathfrak{M}_{\hat{M}, \mathfrak{s}} \times \hat{\chi}(S^1 \times D^3) \rightarrow \hat{\chi}(Y) \times \hat{\chi}(Y)$$

is transversal to the diagonal $\Delta \subset \hat{\chi}(Y) \times \hat{\chi}(Y)$, $\mathfrak{M}_{\hat{M}, \mathfrak{s}}$ is diffeomorphic to the fibred product, i.e

$$\mathfrak{M}_{\hat{M}, \mathfrak{s}} \simeq (\partial_\infty, \partial_\infty)^{-1} \Delta = \mathfrak{M}_{\hat{M}, \mathfrak{s}} \times_{\hat{\chi}(Y)} \hat{\chi}(S^1 \times D^3) \simeq \mathfrak{M}_{\hat{M}, \mathfrak{s}}.$$

For \hat{N} part, first $\mathfrak{M}_{\hat{N}, \mathfrak{s}'}^{irr}$ is unobstructed by a generic perturbation, but reducible part is nontrivial because \hat{N} does not have a metric of positive scalar

curvature in general. Importantly, $\widehat{\mathfrak{M}}_{\hat{N}, \mathfrak{s}'}^{red}$ is non-empty for any perturbation, because $b_2^+(N) = 0$.

Lemma 3.3 *When $b_1(N) \leq 1$, by a generic exponentially-decaying perturbation, $\widehat{\mathfrak{M}}_{\hat{N}, \mathfrak{s}'}^{red}$ is unobstructed for the gluing with $\mathfrak{M}_{\hat{M}, \mathfrak{s}}$.*

Proof. We will follow Vidussi's method [25]. Recall the deformation complex of appropriate weighted Sobolev spaces :

$$0 \rightarrow \Omega_\delta^0(\hat{N}, i\mathbb{R}) \rightarrow \Omega_\delta^1(\hat{N}, i\mathbb{R}) \times \Gamma_\delta(W_+) \rightarrow \Omega_{\delta,+}^2(\hat{N}, i\mathbb{R}) \times \Gamma_\delta(W_-) \rightarrow 0$$

and the Kuranishi model near a reducible solution $(A, 0)$:

$$H^1(\hat{N}, Y; i\mathbb{R}) \times H^1(Y, i\mathbb{R}) \times \ker D_A \rightarrow H^1(Y, i\mathbb{R})/H^1(\hat{N}, i\mathbb{R}) \times \text{coker } D_A.$$

The virtual dimension of the moduli space is

$$2 \text{ind}_{\mathbb{C}} D_A + b_1(\hat{N}) = \frac{1}{4}(c_{\hat{N}} - \tau(\hat{N})) - \eta_B(0) + b_1(\hat{N})$$

where $c_{\hat{N}} = -\frac{1}{4\pi^2} \int_{\hat{N}} F_A \wedge F_A$, τ is the signature, and $\eta_B(0)$ is the eta invariant of the Dirac operator associated with the asymptotic limit B of A . From our assumption $c_{\hat{N}} = c_1^2(\mathfrak{s}') = \tau(\hat{N})$, and the $\eta_B(0)$ vanishes for Y with a standard metric. (see [14].) Therefore the virtual dimension is $b_1(\hat{N})$. For the surjectivity in the above Kuranishi picture, we only need to show $\text{coker } D_A = 0$ for a generic exponentially-decaying perturbation. Since the index is zero, it's equivalent to showing $\ker D_A = 0$.

Letting $d^+\nu \in \Omega_{\delta,+}^2(\hat{N}, i\mathbb{R})$ be a perturbation term (Recall $b_2^+(\hat{N}) = 0$.), $F_{A+\nu}^+ = d^+\nu$ and $(A + \nu, 0)$ is a reducible solution for the perturbed Seiberg-Witten equations. Suppose there exists a nonzero Φ satisfying $D_{A+\nu}\Phi = 0$. Consider a smooth map

$$F : \widehat{\mathfrak{M}}_{\hat{N}, \mathfrak{s}'}^{red} \times (\Gamma_\delta(W_+) - \{0\}) \times \Omega_\delta^1(\hat{N}, i\mathbb{R}) \rightarrow \Gamma_\delta(W_+)$$

$$(A, \Phi, \nu) \mapsto D_{A+\nu}\Phi.$$

Since the differential DF is surjective, $F^{-1}(0)$ is a smooth manifold. Applying the Sard-Smale theorem to the projection map π_3 onto the third factor, for a second category subset of ν , $F^{-1}(0) \cap \pi_3^{-1}(\nu)$ is a smooth manifold of dimension $b_1(\hat{N}) + 2 \text{ind}_{\mathbb{C}} D_{A+\nu} = b_1(\hat{N}) \leq 1$. On the other

hand, as $D_{A+\nu}$ is \mathbb{C} -linear, the real dimension of the kernel of $D_{A+\nu}$ must be greater than or equal to 2 unless it is empty. By this contradiction, our claim is proved. \blacksquare

We first consider the case of $b_1(N) = 1$, in which $\hat{\mathfrak{M}}_{\hat{N},s'}^{red}$ is diffeomorphic to $\hat{\chi}(Y)$, and $\hat{\mathfrak{M}}_{\hat{N},s'}^{irr}$ is zero-dimensional by the dimension formula. Chop off \hat{M} and \hat{N} at $Y \times \{t\}$ for $t \gg 1$ and glue them along the boundary to get \tilde{M} . Then

$$\begin{aligned} \mathfrak{M}_{\tilde{M},\tilde{s}} &\simeq (\mathfrak{M}_{\hat{M},s}(\ast) \times_{\hat{\chi}(Y)} (\hat{\mathfrak{M}}_{\hat{N},s'}^{irr} \times S^1)) / S^1 \cup (\mathfrak{M}_{\hat{M},s} \times_{\hat{\chi}(Y)} \hat{\mathfrak{M}}_{\hat{N},s'}^{red}) \\ &= (\mathfrak{M}_{\hat{M},s}(\ast) \times_{\hat{\chi}(Y)} \hat{\mathfrak{M}}_{\hat{N},s'}^{irr}) \cup \mathfrak{M}_{\hat{M},s}, \end{aligned}$$

where $\mathfrak{M}_{\hat{M},s}(\ast)$ is the based moduli space, i.e. the solution space modulo based gauge transformations which are equal to 1 at a fixed point. As is well-known, the Seiberg-Witten invariant vanishes on $\mathfrak{M}_{\hat{M},s}(\ast)$, because the μ cocycles are pulled back from $\mathfrak{M}_{\hat{M},s}$. Therefore the Seiberg-Witten invariant for $\mathfrak{M}_{\tilde{M},\tilde{s}}$ is obtained from the evaluation on $\mathfrak{M}_{\hat{M},s}$ to give our desired formula.

Now, we turn to the case when $b_1(N) \geq 2$. Because of the obstruction issue, we first kill d_i 's by the surgery, and glue with M , and finally revive the d_i 's by the (inverse) surgery. Let N' be the manifold obtained from N by the surgery around $d_1, \dots, d_{b_1(N)-1}$ with $b_1(N) - 1$ copies of S^4 . Then clearly $b_1(N') = 1$ and moreover :

Lemma 3.4 *Let $U := N - V$ and $V := \cup_{i=1}^{b_1(N)-1} S^1 \times D^3$ which is a tubular neighborhood of $\cup_{i=1}^{b_1(N)-1} d_i$. Then $H_2(N, \mathbb{Z}) \simeq H_2(U, \mathbb{Z}) \simeq H_2(N', \mathbb{Z})$ with isomorphic intersection pairing, where both isomorphisms are induced by the obvious inclusions.*

Proof. This can be seen in the Mayer-Vietoris sequence. First for $(U, V, N = U \cup V)$,

$$H_2(\partial U) \xrightarrow{i_*} H_2(U) \oplus H_2(V) \xrightarrow{\varphi} H_2(N) \rightarrow H_1(\partial U) \rightarrow H_1(U) \oplus H_1(V).$$

Since φ is surjective, which is because $H_1(\partial U)$ injects into $H_1(U)$, it is enough to show that $i_* = 0$. Obviously $H_2(V) = 0$, and to prove that $i_*(H_2(\partial U)) =$

$0 \in H_2(U)$, consider the following commutative diagram of exact sequences :

$$\begin{array}{ccccc} H_3(U, \partial U) & \xrightarrow{\partial_*} & H_2(\partial U) & \xrightarrow{i_*} & H_2(U) \\ PD \downarrow & & \downarrow PD & & \downarrow PD \\ H^1(U) & \xrightarrow{i^*} & H^1(\partial U) & \xrightarrow{\partial^*} & H^2(U, \partial U). \end{array}$$

Since c_i 's are non-torsion in N , $H^1(\partial U) = i^*(H^1(U))$, and hence it gets mapped to zero by ∂^* .

The case for $(U, \cup_{i=1}^m D^2 \times S^2, N')$ is similar. In this case, i_* maps $H_2(\partial U)$ isomorphically onto $H_2(\cup_{i=1}^m D^2 \times S^2)$. \blacksquare

We perform the surgery on M with N' around c to get M' . Let $\tilde{\mathfrak{s}}$ be the resulting Spin^c structure on M' . (We abused the notation, because it is basically the same as $\tilde{\mathfrak{s}}$ on \tilde{M} .) Since $b_2^+(N') = 0$, we can apply the previous process to get

$$SW_{M', \tilde{\mathfrak{s}}}(a) = SW_{M, \mathfrak{s}}(a)$$

for $a \in \mathbb{A}(M)$.

In order to get \tilde{M} , we perform an (inverse) surgery on M' around two spheres which are the cores of the added $D^2 \times S^2$'s in the surgery around c_i 's. Those two spheres are homologically trivial, and we can apply Ozsváth and Szabó's theorem [19],

$$SW_{\tilde{M}, \tilde{\mathfrak{s}}}(a \cdot [d_1] \cdots [d_{b_1(N)-1}]) = \pm SW_{M', \tilde{\mathfrak{s}}}(a)$$

for $a \in \mathbb{A}(M)$. This completes the proof. \blacksquare

Theorem 3.5 *Let M and N be smooth closed oriented 4-manifolds such that $b_2^+(M) > 0$ and $b_2^+(N) = 0$. Suppose $\tilde{\mathfrak{s}}$ is the Spin^c structure on $M \# N$ obtained by gluing a Spin^c structure \mathfrak{s} on M and a Spin^c structure \mathfrak{s}' on N satisfying $c_1^2(\mathfrak{s}') = -b_2(N)$. Then*

$$SW_{M \# N, \tilde{\mathfrak{s}}}(a \cdot [d_1] \cdots [d_{b_1(N)}]) = \pm SW_{M, \mathfrak{s}}(a)$$

for $a \in \mathbb{A}(M)$, where $[d_1], \dots, [d_{b_1(N)}]$ form a basis for the torsion-free part of $H_1(N, \mathbb{Z})$.

Proof. This is an immediate corollary of the previous theorem, because $M \# N$ is the same as the manifold obtained from M by a surgery with $(S^1 \times S^3) \# N$ along a circle representing $[S^1] \times \{\text{pt}\} \in H_1(S^1 \times S^3, \mathbb{Z})$. \blacksquare

4 Proof of Theorem 1.2

Just for simplicity, we may assume that M is minimal, because $\overline{\mathbb{C}P^2}$'s can be absorbed into N . By the gluing formula of the theorem 2.1,

$$Y(\tilde{M}) \geq Y(M).$$

To obtain the reverse inequality, the computations in the previous section allows us to apply LeBrun's theorem 2.2. Let \mathfrak{s} be the Spin^c structure on M induced by the canonical line bundle, which has nonzero Seiberg-Witten invariant for a chamber.

Let's first consider the case of $\kappa(M) = 0$. Recall that M is finitely covered by T^4 or K3 surfaces which we denote by X . To the contrary, suppose there exists a metric of positive scalar curvature on $M\#N$. Then so does $X\#N\#\cdots\#N$ where the number of copies of N is the order of the covering map from X to M . Since $b_2^+(X) \geq 2$, the Seiberg-Witten invariant of the obvious Spin^c structure $\tilde{\mathfrak{s}}$ on $X\#N\#\cdots\#N$ is well-defined independently of the chamber and nonzero by theorem 3.5. This means that it cannot admit a metric of positive scalar curvature, which is a contradiction.

Now let's consider the case when $\kappa(M) > 0$. Let $c_1(\mathfrak{s}) + E$ be the first chern class of $\tilde{\mathfrak{s}}$ on $M\#N$ as in theorem 3.5, where E comes from N . For any metric g on \tilde{M}

$$\begin{aligned} ((c_1(\mathfrak{s}) \pm E)^+)^2 &= (c_1(\mathfrak{s})^+ \pm E^+)^2 \\ &= (c_1(\mathfrak{s})^+)^2 \pm 2c_1(\mathfrak{s})^+ \cdot E^+ + (E^+)^2 \\ &\geq (c_1(\mathfrak{s})^+)^2 \pm 2c_1(\mathfrak{s})^+ \cdot E^+. \end{aligned}$$

Thus at least one of $((c_1(\mathfrak{s}) + E)^+)^2$ and $((c_1(\mathfrak{s}) - E)^+)^2$ should be greater than or equal to $(c_1(\mathfrak{s})^+)^2$. Say $((c_1(\mathfrak{s}) + E)^+)^2 \geq (c_1(\mathfrak{s})^+)^2$. By applying the second inequality of the theorem 2.2, we get

$$\begin{aligned} Y(\tilde{M}, [g]) &\leq -4\sqrt{2}\pi \|(c_1(\mathfrak{s}) + E)^+\|_{L^2} \\ &\leq -4\sqrt{2}\pi \|c_1^+(\mathfrak{s})\|_{L^2} \\ &\leq -4\sqrt{2}\pi \sqrt{c_1^2(\mathfrak{s})} = Y(M), \end{aligned}$$

completing the proof.

5 Proof of Theorem 1.3

Again for simplicity's sake, we may assume that M is minimal, because any embedded circle in $X \# \overline{\mathbb{C}P^2} \# \cdots \# \overline{\mathbb{C}P^2}$ can be moved to $X - D^4$ by an isotopy, where D^4 is the 4-ball in which the connected sums with $\overline{\mathbb{C}P^2}$'s are done, and $\overline{\mathbb{C}P^2}$'s can be absorbed into N_1 . Then the proof is the same as before.

Remark In case that $\kappa(M) = 0$ and $b_2^+(M) = 1$, if the surgery is done along the circle which is trivially covered by the covering map from X to M , then we can lift up the surgery downstairs and use the previous argument in the connected sum case to obtain the same result. \square

6 Proof of Theorem 1.4

Again by the gluing formula of the theorem 2.1, it is immediate that

$$Y(\mathbb{C}P^2 \# N) \geq Y(\mathbb{C}P^2).$$

For the reverse inequality, let \mathfrak{s} be the Spin^c structure on $\mathbb{C}P^2$ induced by the canonical line bundle, and $[\omega]$ be a nonzero element of $H^2(\mathbb{C}P^2; \mathbb{Z})$. Recall that the Seiberg-Witten invariant of $(\mathbb{C}P^2, \mathfrak{s})$ for a perturbation $t\omega$ with $|t| \gg 1$ is nonzero for either $t > 0$ or $t < 0$. By the theorem 3.5, so is $(\mathbb{C}P^2 \# N, \tilde{\mathfrak{s}})$. Therefore the first inequality of theorem 2.2 applies, and the right hand side of the inequality is

$$\frac{|4\pi c_1 \cup [\omega]|}{\sqrt{[\omega]^2/2}} = \frac{|4\pi(3H \cdot tH)|}{\sqrt{(tH \cdot tH)/2}} = 12\sqrt{2}\pi,$$

where H denotes the hyperplane class of $\mathbb{C}P^2$. This completes the proof.

7 Proof of Theorem 1.5 and Corollary 1.6

Let's first consider the case of the theorem 1.5. Recall that M admits a Kähler-Einstein metric so that

$$Y(M) = -4\sqrt{2}\pi \sqrt{c_1^2(\mathfrak{s})},$$

where \mathfrak{s} is the Spin^c structure on M given by the canonical line bundle. By the adjunction formula, $c_1(\mathfrak{s})$ vanishes on each torus $T_j := \alpha_j \times \beta_j$.

To apply the product formula of the Seiberg-Witten series, we check if the so-called "admissibility" condition in [16] is satisfied. Let's denote $M - (\cup_{j=1}^m T_j \times D^2)$ by M' and the inclusion map $\partial M' \hookrightarrow M'$ by i . Let γ_j be $\{\text{pt}\} \times \partial D^2 \subset T_j \times \partial D^2$. There are two non-obvious things to check: $i_*[\gamma_j] = 0 \in H_1(M', \mathbb{Z})$ for all j , and the cokernel of $i^* : H^1(M', \mathbb{Z}) \rightarrow H^1(\partial M', \mathbb{Z})$ is freely generated by the Poincaré-duals of $[T_j]$'s in $\partial M'$.

For the first one, consider the following commutative diagram of exact sequences :

$$\begin{array}{ccccc} H_2(M', \partial M') & \xrightarrow{\partial_*} & H_1(\partial M') & \xrightarrow{i_*} & H_1(M') \\ PD \downarrow & & \downarrow PD & & \downarrow PD \\ H^2(M') & \xrightarrow{i^*} & H^2(\partial M') & \xrightarrow{\partial^*} & H^3(M', \partial M'). \end{array}$$

It's enough to show that $PD([\gamma_j])$ belongs to the image of i^* . This is because $PD([\gamma_j]) \in H^2(\partial M')$ which is the dual of $[T_j] \times \{\text{pt}\} \in H_2(\partial M')$ actually comes from $H^2(M)$ via pull-back.

For the second one, we need the following commutative diagram of exact sequences :

$$\begin{array}{ccccc} H_3(M', \partial M') & \xrightarrow{\partial_*} & H_2(\partial M') & \xrightarrow{i_*} & H_2(M') \\ PD \downarrow & & \downarrow PD & & \downarrow PD \\ H^1(M') & \xrightarrow{i^*} & H^1(\partial M') & \xrightarrow{\partial^*} & H^2(M', \partial M'). \end{array}$$

By using the above result $i_*[\gamma_j] = 0$, $i_*([\alpha_j] \times [\gamma_j])$ and $i_*([\beta_j] \times [\gamma_j])$ are all zero in $H_2(M')$. But $i_*([\alpha_j] \times [\beta_j])$ is nonzero because it is nonzero even in $H_2(M)$. Thus the cokernel of i^* is freely generated by $PD([\alpha_j] \times [\beta_j])$'s.

In the same way, these two properties also hold for $X'_k := S^1 \times X_k - (S^1 \times c_k)$ for all k . (Here we need the condition $[c_k] \equiv \pm 1 \in H_1(X_k, \mathbb{R})$.)

Note that by using the Mayer-Vietoris argument, it follows from $i_*[\gamma_j] = 0$ that $H_2(M')$ is mapped isomorphically into $H_2(M)$ by the inclusion, and likewise for X_k 's.

Recall that the Seiberg-Witten series of M is given by

$$\overline{SW}_M = [\Sigma_1]^{\chi(\Sigma_2)} [\Sigma_2]^{\chi(\Sigma_1)} + (-1)^{\frac{\chi(\Sigma_1)\chi(\Sigma_2)}{4}} [\Sigma_1]^{-\chi(\Sigma_2)} [\Sigma_2]^{-\chi(\Sigma_1)}$$

(see [12].), and the Seiberg-Witten invariant of $S^1 \times X_k$ is the same as that of X_k with its basic classes coming from X_k via the pull-back.(see [1].) Now applying the product formula for the Seiberg-Witten series [16],

$$\begin{aligned} \overline{SW}_{\tilde{M}} &= (\overline{SW}_{M'} \prod_{k=1}^{\mu} \overline{SW}_{X'_k})|_{\varphi_*} \\ &= (\overline{SW}_M \prod_{j=1}^m ([T_j]^{-1} - [T_j]) \prod_{k=1}^{\mu} \overline{SW}_{S^1 \times X_k} ([S^1 \times c_k]^{-1} - [S^1 \times c_k]))|_{\varphi_*}, \end{aligned}$$

where $|_{\varphi_*}$ denotes the identification in the homology induced by the gluing map of the fiber sum construction, and if $b_1(X_k) = 1$, we mean the Seiberg-Witten Series for a chamber.

Now taking \mathfrak{s}' with nonzero Seiberg-Witten invariant from $S^1 \times X_k$ parts and gluing with \mathfrak{s} , we obtain a Spin^c structure $\tilde{\mathfrak{s}}$ with nonzero Seiberg-Witten invariant on \tilde{M} such that $c_1^2(\tilde{\mathfrak{s}}) = c_1^2(\mathfrak{s})$, because

$$c_1^2(\mathfrak{s}') = \langle c_1(\mathfrak{s}), c_1(\mathfrak{s}') \rangle = \langle c_1(\mathfrak{s}), T_j \rangle = \langle c_1(\mathfrak{s}'), T_j \rangle = [T_j] \cdot [T_k] = 0 \quad \forall j, k.$$

This enables us to apply the second inequality of the theorem 2.2 and to get

$$Y(\tilde{M}) \leq -4\sqrt{2}\pi \sqrt{c_1^2(\mathfrak{s})} = Y(M).$$

To show the reverse inequality, we need to construct a Riemannian metric on \tilde{M} whose Yamabe constant is arbitrarily close to $Y(M)$. Let's take a maximal subset of $\{\alpha_1, \dots, \alpha_m\}$, any two elements of which are mutually non-isotopic, and may assume that it is $\{\alpha_1, \dots, \alpha_{m'}\}$ for $m' \leq m$ by renaming. In the same way, we define $\{\beta_1, \dots, \beta_{m''}\}$. Let g_1 be a complete metric of constant curvature -1 on $\hat{\Sigma}_1 := \Sigma_1 - \cup_{j=1}^{m'} \alpha_j$. It is well-known that the metric near the infinity is the cusp metric, i.e. $dt^2 + e^{-2t}g_{S^1}, t \in [a, \infty)$, where g_{S^1} is the metric on the circle of radius 1. At each cusp, we cut it at $t = b$ for $b \gg 1$ and glue a cylinder with a metric $dt^2 + e^{-2b}g_{S^1}, t \in [b, b+1]$ along $\{b\} \times S^1$. Then the resulting metric is only C^0 , so to obtain a nearby smooth metric, take a smooth decreasing convex function $\rho : [b-1, b] \rightarrow [0, 1]$ such that $\rho \equiv e^{-t}$ near $b-1$, and $\rho \equiv e^{-b}$ near b . Then $dt^2 + \rho^2 g_{S^1}$ is a smooth metric with curvature ranging from -1 to 0 , and we glue the corresponding cylindrical ends along the boundary to get back Σ_1 with a metric \tilde{g}_1 parameterized by $b \gg 1$. In the same fashion, we construct \tilde{g}_2 on

Σ_2 parameterized by $b \gg 1$, using a complete metric g_2 of constant curvature -1 on $\hat{\Sigma}_2 := \Sigma_2 - \cup_{j=1}^{m''} \beta_j$.

In $(M, \tilde{g}_1 + \tilde{g}_2)$, we can find a δ -neighborhood $N_j = \{x \in M | \text{dist}(x, T_j) \leq \delta\}$ for all $j = 1, \dots, m$ such that they are mutually disjoint for some $\delta > 0$ when b and c are sufficiently large. Note that N_j are all isometric to the product $e^{-2b}g_{S^1} + e^{-2b}g_{S^1} + g_{D^2(\delta)}$ where $g_{D^2(\delta)}$ is the flat metric on the disk of radius δ , and δ can remain constant if we take b further larger. For the fiber sum, we bend $g_{D^2(\delta)}$ on $D^2(\frac{\delta}{2}) - D^2(\frac{\delta}{3})$ to a metric like a horn with a cylindrical end $dt^2 + g_{S^1(\frac{\delta}{3})}, t \in [0, \frac{\delta}{4}]$, where $g_{S^1(\frac{\delta}{3})}$ is the metric on the circle of radius $\frac{\delta}{3}$.

For the $S^1 \times X_k - (S^1 \times c_k)$ part, we first consider a finite-volume complete metric h on $X_k - c_k$ such that h near the infinity is of the form $e^{-2t}g_{S^1} + dt^2 + g_{S^1(\frac{\delta}{3})}$, where $e^{-2t}g_{S^1}$ is the metric in the c_k direction. Then applying the cutoff procedure as before, we change h into a metric \tilde{h} parameterized by b with a cylindrical end $e^{-2b}g_{S^1} + dt^2 + g_{S^1(\frac{\delta}{3})}, t \in [0, \frac{\delta}{4}]$. Surely the volume and curvature of \tilde{h} is bounded independently of $b > 0$. We finally take the metric $e^{-2b}g_{S^1} + \tilde{h}$ on $S^1 \times (X_k - (S^1 \times c_k))$.

We now perform the fiber sum to get a metric \tilde{g} on \tilde{M} . The important thing is that if we take b sufficiently large, the volume of the gluing region and the parts from $S^1 \times X_k$'s is made arbitrarily small with its curvature bounded. Thus applying the Gauss-Bonnet theorem for complete finite-volume hyperbolic surfaces, we have that for any $\epsilon > 0$, there exists \tilde{g} such that

$$\begin{aligned} -\left(\int_{\tilde{M}} s_{\tilde{g}}^2 d\mu_{\tilde{g}}\right)^{\frac{1}{2}} &\geq -\left(\int_{\hat{\Sigma}_2 \times \hat{\Sigma}_2} s_{g_1+g_2}^2 d\mu_{g_1+g_2}\right)^{\frac{1}{2}} - \epsilon \\ &= -2(4\pi\chi(\hat{\Sigma}_1)4\pi\chi(\hat{\Sigma}_2))^{\frac{1}{2}} - \epsilon \\ &= -2(4\pi\chi(\Sigma_1)4\pi\chi(\Sigma_1))^{\frac{1}{2}} - \epsilon \\ &= Y(M) - \epsilon, \end{aligned}$$

which is our desired inequality.

The case of Corollary 1.6 goes exactly the same. What we need is Kobayashi's formula [7] on the Yamabe invariant of the disjoint union by which

$$Y(M_1 \cup \dots \cup M_l) = -\left(\sum_{i=1}^l |Y(M_i)|^2\right)^{\frac{1}{2}}$$

for $Y(M_i) \leq 0 \quad \forall i$.

Remark As mentioned in the introduction, a knot surgery is a special case of the above construction. M_K has the same homology as $S^1 \times S^2$ with $[m]$ generating $H_1(M_K, \mathbb{Z})$, and

$$\overline{SW}_{S^1 \times M_K} = \frac{\Delta_K([T]^2)}{([T]^{-1} - [T])^2},$$

where T denotes $S^1 \times m$, and Δ_K is the symmetrized Alexander polynomial of a knot K . (For a proof, see [5] and [16].)

It is an interesting question whether a knot surgery on $M = \Sigma_1 \times \Sigma_2$ as above does not change the homeomorphism class of M . \square

8 Examples

Let M be a Kähler surface of nonnegative Kodaira dimension, and N_i be an S^1 bundle over a rational homology 3-sphere for $i = 1, \dots, m$. Then

$$Y(M \# N_1 \# \dots \# N_m) = Y(M).$$

Also we can perform surgeries with a product of S^1 with a rational homology 3-sphere along $S^1 \times \{\text{pt}\}$ to get the same result.

For $\mathbb{C}P^2$ case, presently we don't have many examples but

$$Y(\mathbb{C}P^2 \#_{i=1}^m (S^1 \times S^3)) = 12\sqrt{2}\pi.$$

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