

CONNECTED SUMS WITH $\mathbb{H}P^n$ OR CaP^2 AND THE YAMABE INVARIANT

CHANYOUNG SUNG

ABSTRACT. Let M be a smooth closed $4k$ -manifold whose Yamabe invariant $Y(M)$ is nonpositive. We show that

$$Y(M \sharp_l \mathbb{H}P^k \sharp_m \overline{\mathbb{H}P^k}) = Y(M),$$

where l, m are nonnegative integers, and $\mathbb{H}P^k$ is the quaternionic projective space. When $k = 4$, we also have

$$Y(M \sharp_l CaP^2 \sharp_m \overline{CaP^2}) = Y(M),$$

where CaP^2 is the Cayley plane.

1. INTRODUCTION

The Yamabe invariant is an invariant of a smooth closed manifold defined using the scalar curvature. Let M be a closed smooth n -manifold. By the well-known solution of the Yamabe problem, each conformal class of a smooth Riemannian metric on M contains a so-called *Yamabe metric* which has constant scalar curvature. Moreover, letting

$$[g] = \{\varphi g \mid \varphi : M \rightarrow \mathbb{R}^+ \text{ is smooth}\}$$

be the conformal class of a Riemannian metric g , a Yamabe metric of $[g]$ actually realizes

$$Y(M, [g]) := \inf_{\tilde{g} \in [g]} \frac{\int_M s_{\tilde{g}} dV_{\tilde{g}}}{\left(\int_M dV_{\tilde{g}}\right)^{\frac{n-2}{n}}},$$

where $s_{\tilde{g}}$ and $dV_{\tilde{g}}$ respectively denote the scalar curvature and the volume element of \tilde{g} . The value $Y(M, [g])$, which is the value of the scalar curvature of a Yamabe metric with the total volume 1 is the *Yamabe constant* of the conformal class.

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In a quest of a “best” Yamabe metric or more ambitiously a “canonical” metric on M , one naturally takes the supremum of the Yamabe constants over the set of all conformal classes on M . This is possible, because by Aubin’s theorem [2], the Yamabe constant of any conformal class on any n -manifold is always bounded by that of the unit n -sphere $S^n(1) \subset \mathbb{R}^{n+1}$, which is $n(n-1)(\text{Vol}(S^n(1)))^{2/n}$.

The *Yamabe invariant* of M , $Y(M)$, is then defined as the supremum of the Yamabe constants over the set of all conformal classes on M . This supremum is not always attained, but if it is attained by a metric which is the unique Yamabe metric with total volume 1 in its conformal class, then the metric has to be an Einstein metric. ([1]) In general, one can hope a singular or degenerate Einstein metric leading to a kind of a “geometrization” from a maximizing sequence of Yamabe metrics. It is also noteworthy that the Yamabe invariant is a topological invariant of a closed manifold depending only on the smooth structure of the manifold.

The Yamabe invariant of a compact orientable surfaces is just $4\pi\chi(M)$ where $\chi(M)$ denotes the Euler characteristic of M by the Gauss-Bonnet theorem. In higher dimensions, it is not an easy task to compute the Yamabe invariant. Nevertheless recently there have been much progresses in dimension 3 and 4. In dimension 3, the geometrization by the Ricci flow gives a lot of answers, and in dimension 4, the Spin^c structure and the Dirac operator are keys for computing the Yamabe invariant. In particular, LeBrun [7, 8] showed that if M is a compact Kähler surface whose Kodaira dimension is not equal to $-\infty$, then

$$Y(M) = -4\sqrt{2}\pi\sqrt{(2\chi + 3\sigma)(\tilde{M})},$$

where σ denotes the signature and \tilde{M} is the minimal model of M . Now based on this evidence, one can ask if the blowing-up does not change the Yamabe invariant of a closed orientable 4-manifold with nonpositive Yamabe invariant, namely

Question 1.1. *Let M be a closed orientable 4-manifold with $Y(M) \leq 0$. Is there an orientation of M such that $Y(M \sharp l \overline{\mathbb{C}P^2}) = Y(M)$ for any integer $l > 0$? What about in higher dimensions?*

Further one can also ask whether the analogous statement holds true for the “quaternionic blow-up”, i.e. a connected sum with the quaternionic projective space $\overline{\mathbb{H}P^k}$ with reverse orientation, or even a connected sum with the Cayley plane $\overline{\mathbb{C}aP^2}$ with reverse orientation. More generally we prove :

Theorem 1.2. *Let M be a smooth closed $4k$ -manifold with $Y(M) \leq 0$. Then*

$$Y(M \sharp l \overline{\mathbb{H}P^k} \sharp m \overline{\mathbb{H}P^k}) = Y(M),$$

where l, m are nonnegative integers. When $k = 4$, we also have

$$Y(M \sharp l CaP^2 \sharp m \overline{CaP^2}) = Y(M).$$

2. PRELIMINARIES

A computationally useful formula for the Yamabe constant is

$$|Y(M, [g])| = \inf_{\tilde{g} \in [g]} \left(\int_M |s_{\tilde{g}}|^{\frac{n}{2}} d\mu_{\tilde{g}} \right)^{\frac{2}{n}},$$

where the infimum is attained only by a Yamabe metric. (For a proof, see [8, 14].) So when $Y(M, [g]) \leq 0$, this implies that

$$Y(M, [g]) = - \inf_{\tilde{g} \in [g]} \left(\int_M |s_{\tilde{g}}^-|^{\frac{n}{2}} d\mu_{\tilde{g}} \right)^{\frac{2}{n}},$$

where $s_{\tilde{g}}^-$ is defined as $\min\{s_{\tilde{g}}, 0\}$. Therefore when $Y(M) \leq 0$,

$$(2.1) \quad Y(M) = - \inf_g \left(\int_M |s_g|^{\frac{n}{2}} d\mu_g \right)^{\frac{2}{n}} = - \inf_g \left(\int_M |s_g^-|^{\frac{n}{2}} d\mu_g \right)^{\frac{2}{n}}.$$

Also essential is Kobayashi's connected sum formula [6, 13]

$$Y(M_1 \sharp M_2) \geq \begin{cases} -(|Y(M_1)|^{\frac{n}{2}} + |Y(M_2)|^{\frac{n}{2}})^{\frac{2}{n}} & \text{if } Y(M_i) \leq 0 \forall i \\ \min(Y(M_1), Y(M_2)) & \text{otherwise,} \end{cases}$$

which is in fact a special case of the surgery formula in codimension 3 or more ([11]).

We also need to know about the geometry and topology of $\mathbb{H}P^k$ and CaP^2 . Both have the homogeneous Einstein metric of positive scalar curvature unique up to constant and can be viewed as the mapping cones of the (generalized) Hopf fibrations

$$\pi_1 : S^{4k-1} \rightarrow \mathbb{H}P^{k-1}$$

with S^3 fibers and

$$\pi_2 : S^{15} \rightarrow S^8$$

with S^7 fibers respectively. Thus a connected sum with them or their orientation-reversed ones replaces a point with $\mathbb{H}P^{k-1}$ and S^8 respectively so that it deserves the name "blow-up".

These fibrations have the associated geometries of Riemannian submersion with totally geodesic fibers. In case of π_1 , S^{4k-1} and S^3 are endowed with the round metric of constant curvature 1, and $\mathbb{H}P^{k-1}$ is given the homogeneous Einstein metric with curvature ranging between 1 and 4. In case of π_2 , the total space and the fibers have the round metric of curvature 1, but the base has the round metric of curvature 4.

We will denote the round n -sphere with the metric of constant curvature $\frac{1}{a^2}$ by $S^n(a)$, i.e. the sphere of radius a in the Euclidean \mathbb{R}^{n+1} .

3. PROOF OF THEOREM

It's enough to prove for one connected sum. Let M' be $M \sharp \mathbb{H}P^k$ or $M \sharp \overline{\mathbb{H}P^k}$, and set $n = 4k$. First recall that $\mathbb{H}P^k$ admits a metric of positive scalar curvature meaning that $Y(\mathbb{H}P^k) > 0$. Thus by the connected sum formula,

$$Y(M') \geq Y(M).$$

The idea of the proof is to surger out an $\mathbb{H}P^{k-1}$ in M' by performing the Gromov-Lawson surgery ([4]) to get back M without decreasing the Yamabe constant much.

To prove by contradiction, let's assume

$$Y(M') > Y(M) + 2c > Y(M)$$

for a constant $c > 0$. Let g be an unit-volume Yamabe metric on M' such that

$$s_g \equiv Y(M', [g]) = Y(M) + 2c.$$

Let W be an $\mathbb{H}P^{k-1} \subset \mathbb{H}P^k$ embedded in M' . Take a δ -tubular neighborhood

$$N(\delta) = \{x \in M' \mid \text{dist}_g(x, W) < \delta\}$$

of W for $\delta > 0$. We will take δ small enough so that $N(\delta)$ is diffeomorphic to $\mathbb{H}P^k - \{\text{a point}\}$ and the boundary of $N(\delta)$ is diffeomorphic to S^{4k-1} .

First, we consider the case of $Y(M) = 0$ so that $s_g > 0$. We perform a Gromov-Lawson surgery described in [4, 12, 13] on $N(\delta)$ along W keeping the scalar curvature positive to get a cylindrical end isometric to

$$(S^{4k-1} \times [0, 1], \hat{g} + dt^2),$$

where (S^{4k-1}, \hat{g}) is a Riemannian submersion onto $(W, g_W = g|_W)$ with totally geodesic fibers isometric to $S^3(\varepsilon)$, the round 3-sphere of radius $\varepsilon \ll 1$. Here, the horizontal distribution is given by the connections on the normal bundle. By arranging ε sufficiently small, \hat{g} has positive scalar curvature.

Now let's take a homotopy

$$H_b(t) = \lambda(t)g_W + (1 - \lambda(t))g_{std}$$

of smooth metrics on W from g_W to the homogeneous Einstein metric g_{std} of $\mathbb{H}P^{k-1}$ with curvature ranging from 1 to 4, where $\lambda : [0, 1] \rightarrow [0, 1]$ is a smooth decreasing function with the property that it is 1 for t near 0 and 0 near 1. This induces a homotopy $H_1(t)$ of smooth metrics on S^{4k-1} through a Riemannian submersion with totally geodesic fibers $S^3(\varepsilon)$. And then we homotope the horizontal distribution to that of the Hopf fibration through a Riemannian submersion with totally geodesic fibers $S^3(\varepsilon)$. Let's denote this homotopy on S^{4k-1} be $H_2(t)$ for $t \in [1, 2]$. When ε is sufficiently small, $H_1(t) + dt^2$ and $H_2(t) + dt^2$ will give a metric of positive scalar curvature

on $S^{4k-1} \times [0, 2]$, because it is a Riemannian submersion with totally geodesic fibers onto $\mathbb{H}P^{k-1} \times [0, 2]$. We concatenate this part to the above one obtained from the Gromov-Lawson surgery to get a smooth metric with the boundary isometric to the squashed sphere S^{4k-1} coming from the Hopf fibration. Let's denote this metric on the boundary by h_ε for a later purpose.

We want to close it up by a $4k$ -ball B^{4k} equipped with a metric of positive scalar curvature. To construct such a metric we resort to the Gromov-Lawson surgery again. Take a sphere S^{4k} with any metric of positive scalar curvature and let p be any point on it. As before, we perform a Gromov-Lawson surgery in a sufficiently small neighborhood of p to get a $4k$ -ball with the positive scalar curvature and the cylindrical end isometric to $S^{4k-1}(\varepsilon') \times [0, 1]$ for a $\varepsilon' > 0$. And then we take a homothety of the whole thing by $\frac{1}{\varepsilon'}$ so that the boundary is isometric to the round sphere $(S^{4k-1}(1), h_1)$. In order to glue this to the above obtained part, we have to homotope the metric on the boundary. We take a homotopy

$$H_3(t) = \lambda(t)h_1 + (1 - \lambda(t))h_\varepsilon$$

for $t \in [0, 1]$.

Lemma 3.1. *The metric $H_3(t)$ on S^{4k-1} has positive scalar curvature for every $t \in [0, 1]$.*

Proof. Note that h_1 and h_ε differ only by the size of the Hopf fiber. So for each t , $H_3(t)$ also has the same Riemannian submersion structure with the fiber isometric to the round 3-sphere of radius $r(t) := \lambda(t) + (1 - \lambda(t))\varepsilon$. By the O'Neill's formula [3],

$$s_{H_3(t)} = \frac{1}{r^2(t)}s_f + s_b \circ \pi - r^2(t)|A|^2,$$

where s_f , s_b , and A denote the scalar curvature of the fiber and the base, and the integrability tensor for $t = 0$ respectively. Thus $s_{H_3(t)}$ is constant for each t and increases as t increases. From the fact that $s_{H_3(0)} \equiv (4k-1)(4k-2) > 0$, the result follows. \blacksquare

Nevertheless the metric $H_3(t) + dt^2$ on $S^{4k-1} \times [0, 1]$ may not have positive scalar curvature in general. But due to Gromov and Lawson's lemma in [4], for a sufficiently large constant $L > 0$, $H_3(\frac{t}{L}) + dt^2$ on $S^{4k-1} \times [0, L]$ has positive scalar curvature. Now we have a desired $4k$ -ball to be glued to the part made previously out of M' .

After the gluing, what we get is just M with a specially devised smooth metric which we denote by \bar{g} . Note that

$$s_{\bar{g}} > 0,$$

which is a contradiction to the fact that $Y(M) = 0$ so that M does not admit a metric of positive scalar curvature metric.

Secondly, in case of $Y(M) < 0$, we use a method suggested by C. LeBrun in [9]. We first take a conformal change $e^{2\varphi}g$ of (M', g) such that $\varphi \equiv 0$ outside $N(\delta)$ and the scalar curvature of $e^{2\varphi}g$ is positive on a much smaller neighborhood $N(\delta')$ of W .¹ Moreover one can arrange that it satisfies

$$-\left(\int_{M'} |s_{e^{2\varphi}g}^-|^{\frac{n}{2}} d\mu_{e^{2\varphi}g}\right)^{\frac{2}{n}} > -\left(\int_{M'} |s_g^-|^{\frac{n}{2}} d\mu_g\right)^{\frac{2}{n}} - \epsilon$$

for any $\epsilon > 0$.² Let's take $\epsilon < c$.

Now we perform a Gromov-Lawson surgery on $(N(\delta'), e^{2\varphi}g)$ keeping the positivity of the scalar curvature there. Then in the same way as above, we perform the homotopy process and close it up with a $4k$ -ball to get the final metric \bar{g} . Since this process makes the scalar curvature positive, we have

$$\begin{aligned} -\left(\int_M |s_{\bar{g}}^-|^{\frac{n}{2}} d\mu_{\bar{g}}\right)^{\frac{2}{n}} &= -\left(\int_{M'} |s_{e^{2\varphi}g}^-|^{\frac{n}{2}} d\mu_{e^{2\varphi}g}\right)^{\frac{2}{n}} \\ &> (Y(M) + 2c) - c \\ &> Y(M). \end{aligned}$$

This is a contradiction to the formula (2.1), and completes a proof for the $\mathbb{H}P^k$ case.

The case of CaP^2 can be proved in the same way using the fact that CaP^2 also admits a metric of positive scalar curvature, and is the mapping cone

¹One can take φ to be a smooth nonpositive function $\rho(r)$ such that $r(x) = \text{dist}_g(x, W)$ for $x \in M'$ and

$$\rho(r) = \begin{cases} -ar^2 & \text{for } r \leq \frac{\delta}{3} \\ 0 & \text{for } r \geq \delta \end{cases}$$

where $a > 0$ is a constant. Since φ takes the maximum on W , $d\varphi$ is identically zero at W . For $z \in W$ and a geodesic normal coordinate (x_1, \dots, x_n) around it such that $\frac{\partial}{\partial x_1}|_z, \dots, \frac{\partial}{\partial x_{n-4}}|_z$ are tangent to W ,

$$\begin{aligned} (\Delta_g \varphi)|_z &= \left(\frac{1}{\sqrt{|g|}} \partial_k (\sqrt{|g|} g^{kl} \partial_l \varphi)\right)|_z = (g^{kl} \partial_k \partial_l (-ar^2))|_z \\ &= -a \left(\sum_{i=n-3}^n \partial_i^2 (x_{n-3}^2 + \dots + x_n^2)\right)|_z = -8a, \end{aligned}$$

and hence

$$\begin{aligned} s_{e^{2\varphi}g}|_W &= (e^{-2\varphi}(s_g - 2(n-1)\Delta_g \varphi - (n-2)(n-1)|d\varphi|^2))|_W \\ &= s_g|_W + 16(n-1)a, \end{aligned}$$

which is positive for sufficiently large a .

²For this, one replaces the above φ with $\delta^2 \rho(\frac{r}{\delta})$ which is in fact equal to φ for $r \leq \frac{\delta}{3}$, and takes δ sufficiently small. Then as $\delta \rightarrow 0$, the scalar curvature is bounded below while the volume of $N(\delta)$ tends to zero.

of the (generalized) Hopf fibration $\pi : S^{15} \rightarrow S^8$ with S^7 fibers as explained in the previous section.

4. EXAMPLES AND FINAL REMARKS

Obviously the theorem is vacuous for the case of $\mathbb{H}P^1$ which is diffeomorphic to S^4 .

Example

Let H be a closed Hadarmard-Cartan manifold, i.e. one with a metric of nonpositive sectional curvature. By the well-known theorem of Gromov and Lawson [5] on the enlargeable manifolds, H cannot carry a metric with positive scalar curvature. Therefore $Y(H) \leq 0$. Applying our theorem to H , one has

$$Y(H \sharp l \mathbb{H}P^k \sharp m \overline{\mathbb{H}P^k}) = Y(H).$$

For a specific example, take $M = T^n \times H$, where T^n is an n -dimensional torus and H is as above, e.g. a product of closed real hyperbolic manifolds. Now since M has an obvious F -structure, its Yamabe invariant is actually 0 by collapsing the T^n -part. (Refer to Paternain and Petean [10].) Thus

$$Y(M \sharp l \mathbb{H}P^k \sharp m \overline{\mathbb{H}P^k}) = 0.$$

Similar examples can also be constructed for CaP^2 . ◇

Going back to the question 1.1 addressed in the introduction, our argument does not apply to the case of complex projective space $\mathbb{C}P^k$. We still have the fact that $\mathbb{C}P^k$ is the mapping cone of the Hopf fibration

$$\pi : S^{2k-1} \rightarrow \mathbb{C}P^{k-1}$$

with S^1 fibers. So the $\mathbb{C}P^{k-1}$ is embedded as a submanifold of codimension 2 which is one less for the Gromov-Lawson surgery to work. Moreover the statement corresponding to the theorem 1.2 can not be true at least in dimension 4. This is because of Wall's stabilization theorem [16]. Let M be a simply-connected closed smooth 4-manifold. Then there exists integers l, m such that

$$M \sharp l \mathbb{C}P^2 \sharp m \overline{\mathbb{C}P^2} = a \mathbb{C}P^2 \sharp b \overline{\mathbb{C}P^2},$$

where $a = l + \frac{1}{2}(b_2(M) + \sigma(M))$ and $b = m + \frac{1}{2}(b_2(M) - \sigma(M))$. But we know that $Y(a \mathbb{C}P^2 \sharp b \overline{\mathbb{C}P^2}) > 0$. Thus the Yamabe invariant changes drastically by taking connected sums with both $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$. We do not know yet whether the stabilization phenomenon of the Yamabe invariant is prevalent also in higher dimensions. But at least the question 1.1 is worth investigating in dimension both 4 and higher.

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DEPARTMENT OF MATHEMATICS AND INSTITUTE FOR MATHEMATICAL SCIENCES, KONKUK UNIVERSITY, 1 HWAYANG-DONG, GWANGJIN-GU, SEOUL, KOREA
E-mail address: cysung@kias.re.kr