ISOPERIMETRIC REGIONS IN SPHERICAL CONES AND YAMABE CONSTANTS OF $M \times S^1$

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Abstract. We study isoperimetric regions on Riemannian manifolds of the form $(M^n \times (0,\pi), \sin^2(t) g + dt^2)$ where g is a metric of positive Ricci curvature $\geq n-1$. When g is an Einstein metric we use this to compute the Yamabe constant of $(M \times \mathbb{R}, g + dt^2)$ and so to obtain lower bounds for the Yamabe invariant of $M \times S^1$.

1. Introduction

Given a closed Riemannian manifold (M, g) we consider the conformal class of the metric g, $[g]$. The Yamabe constant of $[g]$, $Y(M,[g])$, is the infimum of the normalized total scalar curvature functional on the conformal class. Namely,

$$Y(M,[g]) = \inf_{h \in [g]} \frac{\int \mathbf{s}_h \; dvol(h)}{(Vol(M,h))^{\frac{n-2}{n}}},$$

where $\mathbf{s}_h$ denotes the scalar curvature of the metric h and $dvol(h)$ its volume element.

If one writes metrics conformal to g as $h = f^{4/(n-2)}\; g$, one obtains the expression

$$Y(M,[g]) = \inf_{f \in C^\infty(M)} \frac{\int (\; 4a_n \|\nabla f\|_g^2 + f^2 \mathbf{s}_g \;) \; dvol(g)}{\|f\|_{p_n}^2},$$

where $a_n = 4(n-1)/(n-2)$ and $p_n = 2n/(n-2)$. It is a fundamental result on the subject that the infimum is actually achieved ([24, 23, 3, 21]). The functions f achieving the infimum are called *Yamabe functions* and the corresponding metrics $f^{4/(n-2)}\; g$ are called *Yamabe metrics*. Since the critical points of the total scalar curvature functional restricted to a conformal class of metrics are precisely the metrics of constant scalar curvature in the conformal class, Yamabe metrics are metrics of constant scalar curvature.

It is well known that by considering functions supported in a small normal neighborhood of a point one can prove that $Y(M^n,[g]) \leq Y(S^n,[g_0])$, where g_0 is the round metric of radius one on the sphere and (M^n, g) is any closed n-dimensional Riemannian manifold ([3]). We will use the notation $Y_n = Y(S^n,[g_0])$ and $V_n = Vol(S^n, g_0)$. Therefore $Y_n = n(n-1)V_n^{\frac{2}{n}}$.

Then one defines the *Yamabe invariant* of a closed manifold M [11, 22] as

$$Y(M) = \sup_g Y(M,[g]) \leq Y_n.$$

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It follows that $Y(M)$ is positive if and only if M admits a metric of positive scalar curvature. Moreover, the sign of $Y(M)$ determines the technical difficulties in understanding the invariant. When the Yamabe constant of a conformal class is non-positive there is a unique metric (up to multiplication by a positive constant) of constant scalar curvature in the conformal class and if g is any metric in the conformal class, the Yamabe constant is bounded from below by $(\inf_M \mathbf{s}_g)\ (Vol(M,g))^{2/n}$. This can be used for instance to study the behavior of the invariant under surgery and so to obtain information using cobordism theory [19, 18, 6]. Note also that in the non-positive case the Yamabe invariant coincides with Perelman's invariant [2]. The previous estimate is no longer true in the positive case, but one does get a lower bound in the case of positive Ricci curvature by a theorem of S. Ilias: if $Ricci(g) \geq \lambda g$ ($\lambda > 0$) then $Y(M,[g]) \geq n\lambda(Vol(M,g))^{2/n}$ ([9]). Then in order to use this inequality to find lower bounds on the Yamabe invariant of a closed manifold M one would try to maximize the volume of the manifold under some positive lower bound of the Ricci curvature. Namely, if one denotes $\mathbf{Rv}(M) = \sup\{Vol(M,g) : Ricci(g) \geq (n-1)g\}$ then one gets $Y(M) \geq n(n-1)(\mathbf{Rv}(M))^{2/n}$ (one should define $\mathbf{Rv}(M) = 0$ if M does not admit a metric of positive Ricci curvature). Very little is known about the invariant $\mathbf{Rv}(M)$. Of course, Bishop's inequality tells us that for any n-dimensional closed manifold $\mathbf{Rv}(M^n) \leq \mathbf{Rv}(S^n)$ (which is of course attained by the volume of the metric of constant sectional curvature 1). Moreover, G. Perelman [17] proved that there is a constant $\delta = \delta_n > 0$ such that if $\mathbf{Rv}(M) \geq \mathbf{Rv}(S^n) - \delta_n$ then M is homeomorphic to S^n. Beyond this, results on $\mathbf{Rv}(M)$ have been obtained by computing Yamabe invariants, so for instance $\mathbf{Rv}(\mathbf{CP}^2) = 2\pi^2$ (achieved by the Fubini-Study metric as shown by C. LeBrun [12] and M. Gursky and C. LeBrun [10]) and $\mathbf{Rv}(\mathbf{RP}^3) = \pi^2$ (achieved by the metric of constant sectional curvature as shown by H. Bray and A. Neves [7]).

Of course, there is no hope to apply the previous comments directly when the fundamental group of M is infinite. Nevertheless it seems that even in this case the Yamabe invariant is realized by conformal classes of metrics which maximize volume with a fixed positive lower bound on the Ricci curvature "in certain sense". The standard example is $S^{n-1} \times S^1$. The fact that $Y(S^n \times S^1) = Y_{n+1}$ is one of the first things we learned about the Yamabe invariant [11, 22]. One way to see this is as follows: first one notes that $\lim_{T\to\infty} Y(S^n \times S^1, [g_0 + T^2 dt^2]) = Y(S^n \times \mathbb{R}, [g_0 + dt^2])$ [1] (the Yamabe constant for a non-compact Riemannian manifold is computed as the infimum of the Yamabe functional over compactly supported functions). But the Yamabe function for $g_0 + dt^2$ is precisely the conformal factor between $S^n \times \mathbb{R}$ and $S^{n+1} - \{S, N\}$. Therefore one can think of $Y(S^n \times S^1) = Y_{n+1}$ as realized by the positive Einstein metric on $S^{n+1} - \{S, N\}$. We will see in this article that a similar situation occurs for any closed positive Einstein manifold (M, g) (although we only get the lower bound for the invariant).

Let (N, h) be a closed Riemannian manifold. An *isoperimetric region* is an open subset U with boundary ∂U such that ∂U minimizes area among hypersurfaces bounding a region of volume $Vol(U)$. Given any positive number s, $s < Vol(N, h)$, there exists an isoperimetric region of volume s. Its boundary is a stable constant mean curvature hypersurface with some singularities of codimension at least 7. Of course one does not need a closed Riemannian manifold to consider isoperimetric regions, apriori one only needs to be able to compute volumes of open subsets and areas of hypersurfaces. One defines the *isoperimetric function* of (N,h) as $I_h : (0,1) \to \mathbb{R} > 0$ by

$$I_h(\beta) = \inf\{Vol(\partial U)/Vol(N,h) : Vol(U,h) = \beta Vol(N,h)\},$$

where $Vol(\partial U)$ is measured with the Riemannian metric induced by h (on the non-singular part of ∂U).

Given a closed Riemannian manifold (M,g) we will call the *spherical cone* on M the space X obtained collapsing $M \times \{0\}$ and $M \times \{\pi\}$ in $M \times [0,\pi]$ to points S and N (the vertices) with the metric $\mathbf{g} = \sin^2(t)g + dt^2$ (which is a Riemannian metric on $X - \{S,N\}$). Now if $Ricci(g) \geq (n-1)g$ one can see that $Ricci(\mathbf{g}) \geq n\mathbf{g}$. One should compare this with the Euclidean cones considered by F. Morgan and M. Ritoré in [16]: $\hat{g} = t^2 g + dt^2$ for which $Ricci(g) \geq (n-1)g$ implies that $Ricci(\hat{g}) \geq 0$. The importance of these spherical cones for the study of Yamabe constants is that if one takes out the vertices the corresponding (non-complete) Riemannian manifold is conformal to $M \times \mathbb{R}$. But using the (warped product version) of the Ros Product Theorem [20, Proposition 3.6] (see [15, Section 3]) and the Levy-Gromov isoperimetric inequality [8] one can understand isoperimetric regions in these spherical cones. Namely,

Theorem 1.1. *Let (M^n, g) be a compact manifold with Ricci curvature $Ricci(g) \geq (n-1)g$. Let $(X, \mathbf{g})$ be its spherical cone. Then geodesic balls around any of the vertices are isoperimetric.*

But now, since the spherical cone over (M,g) is conformal to $(M \times \mathbb{R}, g + dt^2)$ we can use the previous result and symmetrization of a function with respect to the geodesic balls centered at a vertex to prove:

Theorem 1.2. *Let (M,g) be a closed Riemannian manifold of positive Ricci curvature, $Ricci(g) \geq (n-1)g$ and volume V. Then*

$$Y(M \times \mathbb{R}, [g + dt^2]) \geq (V/V_n)^{\frac{2}{n+1}} \; Y_{n+1}.$$

As we mentioned before one of the differences between the positive and non-positive cases in the study of the Yamabe constant is the non-uniqueness of constant scalar curvature metrics on a conformal class with positive Yamabe constant. And the simplest family of examples of non-uniqueness comes from Riemannian products. If (M,g) and (N^n,h) are closed Riemannian manifolds of constant scalar curvature and $\mathbf{s}_g$ is positive then for small $\delta > 0$, $\delta g + h$ is a constant scalar curvature metric on $M \times N$ which cannot be a Yamabe metric. If (M,g) is Einstein and $Y(M) = Y(M,[g])$ it seems reasonable that $Y(M \times N) = \lim_{\delta \to 0} Y(M \times N, [\delta g + h])$. Moreover as it is shown in [1]

$$\lim Y(M \times N, [\delta g + h]) = Y(M \times \mathbb{R}^n, [g + dt^2]).$$

The only case which is well understood is when $M = S^n$ and $N = S^1$. Here every Yamabe function is a function of the S^1-factor [22] and the Yamabe function for $(S^n \times \mathbb{R}, g_0 + dt^2)$ is the factor which makes $S^n \times \mathbb{R}$ conformal to $S^{n+1} - \{S, N\}$. It seems possible that under certain conditions on (M, g) the Yamabe functions of $(M \times \mathbb{R}^n, g + dt^2)$ depend only on the second variable. The best case scenario would be that this is true if g is a Yamabe metric but it seems more attainable the case when g is Einstein. It is a corollary to the previous theorem that this is actually true in the case $n = 1$. Namely, using the notation (as in [1]) $Y_N(M \times N, g + h)$ to denote the infimum of the $(g + h)$-Yamabe functional restricted to functions of the N-factor we have:

Corollary 1.3. *Let (M^n, g) be a closed positive Einstein manifold with Ricci curvature Ricci$(g) = (n-1)g$. Then*

$$Y(M \times \mathbb{R}, [g + dt^2]) = Y_{\mathbb{R}}(M \times \mathbb{R}, g + dt^2) = \left(\frac{V}{V_n}\right)^{\frac{2}{n+1}} Y_{n+1}.$$

As $Y(M \times \mathbb{R}, [g + dt^2]) = \lim_{T\to\infty} Y(M \times S^1, [g + T dt^2])$ it also follows from Theorem 1.2 that:

Corollary 1.4. *If (M^n, g) is a closed Einstein manifold with Ricci$(g) = (n-1)g$ and volume V then*

$$Y(M \times S^1) \geq (V/V_n)^{\frac{2}{n+1}} Y_{n+1}.$$

So for example using the product metric we get

$$Y(S^2 \times S^2 \times S^1) \geq \left(\frac{2}{3}\right)^{(2/5)} Y_5$$

and using the Fubini-Study metric we get

$$Y(\mathbf{CP}^2 \times S^1) \geq \left(\frac{3}{4}\right)^{(2/5)} Y_5.$$

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2. Isoperimetric regions in spherical cones

As we mentioned in the introduction, the isoperimetric problem for spherical cones (over manifolds with Ricci curvature $\geq n - 1$) is understood using the Levy-Gromov isoperimetric inequality (to compare the isoperimetric functions of M and of S^n) and the Ros Product Theorem for warped products (to compare then the isoperimetric functions of the spherical cone over M to the isoperimetric function of S^{n+1}). See for example section 3 of [15] (in particular **3.2** and the remark after it). For the reader familiar with isoperimetric problems, this should be enough to understand Theorem 1.1. In this section, for the convenience of the reader, we will give a brief outline on these issues. We will mostly discuss and follow section 3 of [20] and ideas in [16, 13] which we think might be useful in dealing with other problems arising from the study of Yamabe constants.

Let (M^n, g) be a closed Riemannian manifold of volume V and Ricci curvature $Ricci(g) \geq (n-1)g$. We will consider $(X^{n+1}, \mathbf{g})$ where as a topological space X is the suspension of M ($X = M \times [0, \pi]$ with $M \times \{0\}$ and $M \times \{\pi\}$ identified to points S and N) and $\mathbf{g} = \sin^2(t)\ g\ + dt^2$. Of course X is not a manifold (except when M is S^n) and $\mathbf{g}$ is a Riemannian metric only on $X - \{S, N\}$.

The following is a standard result in geometric measure theory.

Theorem : For any positive number $r < Vol(x)$ there exists an isoperimetric open subset U of X of volume r. Moreover ∂U is a smooth stable constant mean curvature hypersurface of X except for a singular piece $\partial_1 U$ which consists of (possibly) S, N, and a subset of codimension at least 7.

Let us call $\partial_0 U$ the regular part of ∂U, $\partial_0 U = \partial U - \partial_1 U$. Let X_t, $t \in (-\varepsilon, \varepsilon)$, be a variation of $\partial_0 U$ such that the volume of the enclosed region U_t remains constant. Let $\lambda(t)$ be the area of X_t. Then $\lambda'(0) = 0$ and $\lambda''(0) \geq 0$. The first condition is satisfied by hypersurfaces of constant mean curvature and the ones satisfying the second condition are called *stable*. If N denotes a normal vector field to the hypersurface then variations are obtained by picking a function h with compact support on $\partial_0 U$ and moving $\partial_0 U$ in the direction of $h\ N$. Then we have that if the mean of h on $\partial_0 U$ is 0 then $\lambda_h'(0) = 0$ $\lambda_h''(0) \geq 0$. This last condition is written as

$$Q(h,h) = -\int_{\partial_0 U} h(\Delta h + (Ricci(N,N) + \sigma^2)h) dvol(\partial_0 U) \geq 0.$$

Here we consider $\partial_0 U$ as a Riemannian manifold (with the induced metric) and use the corresponding Laplacian and volume element. σ^2 is the square of the norm of the second fundamental form. This was worked out by J. L. Barbosa, M. do Carmo and J. Eschenburg in [4, 5]. As we said before, the function h should apriori have compact support in $\partial_0 U$ but as shown by F. Morgan and M. Ritoré [16, Lemma 3.3] it is enough that h is bounded and $h \in L^2(\partial_0 U)$. This is important in order to study stable constant mean curvature surfaces on a space like X because X admits what is called a *conformal* vector field $V = \sin(t)\partial/\partial t$ and the function h one wants to consider is $h = div(V - \mathbf{g}(V,N)\ N)$ where N is the unit normal to the hypersurface (and then h is the divergence of the tangencial part of V). This has been used for instance in [13, 16] to classify stable constant mean curvature hypersurfaces in Riemannian manifolds with a conformal vector field. When the hypersurface is smooth this function h has mean 0 by the divergence theorem and one can apply the stability condition. But when the hypersurface has singularities one would apriori need the function h to have compact support on the regular part. This was done by F. Morgan and M. Ritoré in [16, Lemma 3.3].

We want to prove that the geodesic balls around S are isoperimetric. One could try to apply the techniques of Morgan and Ritoré in [16] and see that they are the only stable constant mean curvature hypersurfaces in X. This should be possible, and actually it might be necessary to deal with isoperimetric regions of more general singular spaces that appear naturally in the study of Yamabe constants of Riemannian products. But in this case we will instead take a more direct approach using the Levy-Gromov isoperimetric inequality [8] and Ros Product Theorem [20].

The sketch of the proof is as follows: First one has to note that geodesic balls centered at the vertices *produce* the same isoperimetric function as the one of the round sphere. Therefore to prove that geodesic balls around the vertices are isoperimetric is equivalent to prove that the isoperimetric function of $\mathbf{g}$ is bounded from below by the isoperimetric function of g_0. To do this, given any open subset U of X one considers its symmetrization $U^s \subset S^{n+1}$, so the the *slices* of U^s are geodesic balls with the same normalized volumes as the slices of U. Then by the Levy-Gromov isoperimetric inequality we can compare the normalized areas of the boundaries of the slices. We have to prove that the normalized area of ∂U^s is at most the normalized area of ∂U. This follows from the warped product version of [20, Proposition 3.6]. We will give an outline following Ros' proof for the Riemannian product case. We will use the notion of Minkowski content. This is the bulk of the proof and we will divide it into Lemma 2.1, Lemma 2.2 and Lemma 2.3.

Proof of Theorem 1.1 : Let $U \subset X$ be a closed subset. For any $t \in (0, \pi)$ let

$$U_t = U \cap (M \times \{t\}).$$

Fix any point $E \in S^n$ and let $(U^s)_t$ be the geodesic ball centered at E with volume

$$Vol((U^s)_t, g_0) = \frac{V_n}{V}\ Vol(U_t, g).$$

(recall that $V = Vol(M, g)$ and $V_n = Vol(S^n, g_0)$). Let $U^s \subset S^{n+1}$ be the corresponding subset (i.e. we consider $S^{n+1} - \{S, N\}$ as $S^n \times (0, \pi)$ and U^s is such that $U^s \cap (S^n \times \{t\}) =(U^s)_t$. One might add S and/or N to make U^s closed and connected). Note that one can write $(U^s)_t = (U_t)^s = U^s_t$ as long as there is no confusion (or no difference) on whether we are considering it as a subset of S^n or as a subset of S^{n+1}.

Now

$$Vol(U) = \int_0^{\pi} \sin^n(t)\ Vol(U_t, g)\ dt$$

$$= \frac{V}{V_n} \int_0^{\pi} \sin^n(t)\ Vol((U^s)_t, g_0)\ dt = \frac{V}{V_n} Vol(U^s, g_0).$$

Also if $B(r) = M \times [0, r]$ (the geodesic ball of radius r centered at the vertex at 0) then

$$Vol(B(r)) = \int_0^r \sin^n(t) V dt = \frac{V}{V_n} \int_0^r \sin^n(t) V_n dt = \frac{V}{V_n} Vol(B_0(r)) \quad (1)$$

where $B_0(r)$ is the geodesic ball of radius r in the round sphere. And

$$Vol(\partial B(r)) = \sin^n(r) V = \frac{V}{V_n} Vol(\partial B_0(r)) \qquad (2).$$

Formulas (1) and (2) tell us that the geodesic balls around the vertices in X produce the same isoperimetric function as the round metric g_0. Therefore given any open subset $U \subset X$ we want to compare the area of ∂U with the area of the boundary of the geodesic ball in S^{n+1} with the same normalized volume as U.

Given a closed set W let $B(W,r)$ be the set of points at distance at most r from W. Then one considers the *Minkowski content* of W,

$$\mu^+(W) = \liminf \frac{Vol(B(W,r)) - Vol(W)}{r}.$$

If W is a smooth submanifold with boundary then $\mu^+(W) = Vol(\partial W)$. And this is still true if the boundary has singularities of codimension ≥ 2 (and finite codimension 1 Hausdorff measure).

The Riemannian measure on (S^n, g_0), normalized to be a probability measure is what is called a *model measure*: if D^t, $t \in (0,1)$ is the family of geodesic balls (with volume $Vol(D^t) = t$) centered at some fixed point then they are isoperimetric regions which are ordered by volume and such that for any t, $B(D^t, r) = D^{t'}$ for some t'. See [20, Section 3.2]. The following result follows directly from the Levy-Gromov isoperimetric inequality [8, Appendix C] and [20, Proposition 3.5] (see the lemma in [14, page 77] for a more elementary proof and point of view on [20, Proposition 3.5]).

Lemma 2.1. *: Let (M,g) be a closed Riemannian manifold of volume V and Ricci curvature $Ricci(g) \geq (n-1)g$. For any nonempty closed subset $\Omega \subset M$ and any $r \geq 0$ if B_Ω is a geodesic ball in (S^n, g_0) with volume $Vol(B_\Omega) = (V_n/V) Vol(\Omega)$ then $Vol(B(B_\Omega, r)) \leq (V_n/V) Vol(B(\Omega, r))$.*

Proof. Given any closed Riemannian manifold (M,g), dividing the Riemannian measure by the volume one obtains a probability measure which we will denote μ_g. As we said before, the round metric on the sphere gives a model measure μ_{g_0}. On the other hand the Levy-Gromov isoperimetric inequality [8] says that $I_{\mu_g} \geq I_{\mu_{g_0}}$. The definition of B_Ω says that $\mu_g(\Omega) = \mu_{g_0}(B_\Omega)$ and what we want to prove is that $\mu_g(B(\Omega,r)) \geq \mu_{g_0}(B(B_\Omega, r))$. Therefore the statement of the lemma is precisely [20, Proposition 3.5].

□

Fix a positive constant λ. Note that the previous lemma remains unchanged if we replace g and g_0 by λg and λg_0: the correspondence $\Omega \to B_\Omega$ is the same and $\mu_{\lambda g} = \mu_g$.

Lemma 2.2. *For any $t_0 \in (0,\pi)$ $B((U^s)_{t_0}, r) \subset (B(U_{t_0}, r))^s$.*

Proof. First note that the distance from a point $(x,t) \in X$ to a vertex depends only on t and not on x (or even on X). Therefore if r is greater than the distance δ between t_0 and 0 or π then both sets in the lemma will contain a geodesic ball of radius $r-\delta$ around the corresponding vertex.

Also observe that the distance between points (x,t_0) and (y,t) depends only on the distance between x and y (and t, t_0, and the function in the warped product, which in this case is sin) but not on x, y or X. In particular for any t so that $|t-t_0| < r$, $(B((U^s)_{t_0}, r))_t$ is a geodesic ball.

We have to prove that for any t

$$(B((U^s)_{t_0}, r))_t \subset ((B(U_{t_0}, r))^s)_t.$$

But since they are both geodesic balls centered at the same point it is enough to prove that the volume of the subset on the left is less than or equal to the volume of the subset on the right. By the definition of symmetrization the normalized volume of $((B(U_{t_0}, r))^s)_t$ is equal to the normalized volume of $(B(U_{t_0}, r))_t$. But from the previous comment there exist $\rho > 0$ such that, considered as subsets of M,

$$(B(U_{t_0}, r))_t = B(U_{t_0}, \rho)$$

and, as subsets of S^n,

$$(B((U^s)_{t_0}, r))_t = B(U^s_{t_0}, \rho).$$

The lemma then follows from Lemma 2.1 (and the comments after it).

□

Now for any closed subset $U \subset X$ let B_U be a geodesic ball in (S^{n+1}, g_0) with volume $Vol(B_U, g_0) = (V_n/V) Vol(U, \mathbf{g})$. Since geodesic balls in round spheres are isoperimetric (and $Vol(B_U, g_0) = Vol(U^s, g_0)$) it follows that $Vol(\partial B_U) \leq \mu^+(U^s)$.

Lemma 2.3. *Given any closed set $U \subset X$, $\mu^+(U) \geq (V/V_n) Vol(\partial B_U)$.*

Proof. Since $(B(U,r))^s$ is closed and $B(U^s, r)$ is the closure of $\cup_{t\in(0,\pi)} B(U^s_t, r)$ we have from the previous lemma that

$$B(U^s, r) \subset (B(U,r))^s.$$

Then

$$Vol(\partial B_U) \leq \mu^+(U^s) = \liminf \frac{Vol(B(U^s,r)) - Vol(U^s)}{r}$$

$$\leq \liminf \frac{Vol((B(U,r))^s) - Vol(U^s)}{r}$$

$$= (V_n/V) \liminf \frac{Vol(B(U,r)) - Vol(U)}{r} = (V_n/V)\mu^+(U)$$

and the lemma follows.

□

Now if we let B_U^M be a geodesic ball around a vertex in X with volume

$$Vol(B_U^M,\mathbf{g}) = Vol(U,\mathbf{g}) = \frac{V}{V_n} Vol(B_U, g_0)$$

then it follows from (1) and (2) in the beginning of the proof that

$$Vol(\partial B_U^M,\mathbf{g}) = \frac{V}{V_n} Vol(\partial B_U, g_0).$$

and so by Lemma 2.3

$$Vol(\partial B_U^M,\mathbf{g}) \leq \mu^+(U)$$

and Theorem 1.1 is proved.

□

3. The Yamabe constant of $M \times \mathbb{R}$

Now assume that g is a metric of positive Ricci curvature, $Ricci(g) \geq (n-1)g$ on M and consider as before the spherical cone $(X,\mathbf{g})$ with $\mathbf{g} = \sin^2(t) g + dt^2$. By a direct computation the sectional curvature of $\mathbf{g}$ is given by:

$$K_{\mathbf{g}}(v_i,v_j) = \frac{K_g(v_i,v_j) - \cos^2(t)}{\sin^2(t)}$$

$$K_{\mathbf{g}}(v_i, \partial/\partial t) = 1,$$

for a g-orthonormal basis $\{v_1,...,v_n\}$. And the Ricci curvature is given by:

$$Ricci(\mathbf{g})(v_i, \partial/\partial t) = 0$$

$$Ricci(\mathbf{g})(v_i,v_j) = Ricci(g)(v_i,v_j) - (n-1)\cos^2(t)\delta_i^j + \sin^2(t)\delta_i^j$$

$$Ricci(\mathbf{g})(\partial_t,\partial_t) = n.$$

Therefore by picking $\{v_1,...,v_n\}$ which diagonalizes $Ricci(g)$ one easily sees that if $Ricci(g) \geq (n-1)g$ then $Ricci(\mathbf{g}) \geq n\mathbf{g}$. Moreover, if g is an Einstein metric with Einstein constant $n-1$ the $\mathbf{g}$ is Einstein with Einstein constant n.

Let us recall that for non-compact Riemannian manifolds one defines the Yamabe constant of a metric as the infimum of the Yamabe functional of the metric over smooth compactly supported functions (or functions in L_1^2, of course). So for instance if g is a Riemannian metric on the closed manifold M then

$$Y(M\times\mathbb{R},[g+dt^2]) = \inf_{f\in C_0^\infty(M\times\mathbb{R})} \frac{\int_{M\times\mathbb{R}} \left(a_{n+1}\|\nabla f\|^2 + \mathbf{s}_g\ f^2 \right) dvol(g+dt^2)}{\|f\|_{p_{n+1}}^2}.$$

Proof of Theorem 1.2 : We have a closed Riemannian manifold (M^n, g) such that $Ricci(g) \geq (n-1)g$. Let $f_0(t) = \cosh^{-2}(t)$ and consider the diffeomorphism

$$H : M \times (0,\pi) \to M \times \mathbb{R}$$

given by $H(x,t) = (x, h_0(t))$, where $h_0 : (0,\pi) \to \mathbb{R}$ is the diffeomorphism defined by $h_0(t) = cosh^{-1}((\sin(t))^{-1})$ on $[\pi/2, \pi)$ and $h_0(t) = -h_0(\pi/2 - t)$ if $t \in (0, \pi/2)$.

By a direct computation $H^*(f_0(g + dt^2)) = \mathbf{g} = \sin^2(t) g + dt^2$ on $M \times (0,\pi)$.

Therefore by conformal invariance if we call $g_{f_0} = f_0(g + dt^2)$

$$Y(M \times \mathbb{R}, [g + dt^2]) = \inf_{f \in C_0^\infty(M \times \mathbb{R})} \frac{\int_{M \times \mathbb{R}} \left(a_{n+1} \|\nabla f\|^2_{g+dt^2} + \mathbf{s}_g f^2 \right) \, dvol(g + dt^2)}{\|f\|^2_{p_{n+1}}}$$

$$= \inf_{f \in C_0^\infty(M \times \mathbb{R})} \frac{\int_{M \times \mathbb{R}} \left(a_{n+1} \|\nabla f\|^2_{g_{f_0}} + \mathbf{s}_{g_{f_0}} f^2 \right) \, dvol(g_{f_0})}{\|f\|^2_{p_{n+1}}}$$

$$= \inf_{f \in C_0^\infty(M \times (0,\pi))} \frac{\int_{M \times (0,\pi)} \left(a_{n+1} \|\nabla f\|^2_{\mathbf{g}} + \mathbf{s}_{\mathbf{g}} f^2 \right) \, dvol(\mathbf{g})}{\|f\|^2_{p_{n+1}}} = Y(M \times (0,\pi), [\mathbf{g}]).$$

Now, as we showed in the previous section, $Ricci(\mathbf{g}) \geq n$. Therefore $\mathbf{s}_{\mathbf{g}} \geq n(n+1)$. So we get

$$Y(M \times \mathbb{R}, [g + dt^2]) \geq \inf_{f \in C_0^\infty(M \times (0,\pi))} \frac{\int_{M \times (0,\pi)} \left(a_{n+1} \|\nabla f\|^2_{\mathbf{g}} + n(n+1) f^2 \right) \, dvol(\mathbf{g})}{\|f\|^2_{p_{n+1}}}.$$

To compute the infimum one needs to consider only non-negative functions. Now for any non-negative function $f \in C_0^\infty(M \times (0,\pi)\ )$ consider its symmetrization $f_* : X \to \mathbb{R}_{\geq 0}$ defined by $f_*(S) = \sup f$ and $f_*(x,t) = s$ if and only if $Vol(B(S,t), \mathbf{g}) = Vol(\{f > s\}, \mathbf{g})$ (i.e. f_* is a non-increasing function of t and $Vol(\{f_* > s\}) = Vol(\{f > s\})$ for any s). It is inmediate that the L^q-norms of f_* and f are the same for any q. Also, by the coarea formula

$$\int \|\nabla f\|^2_{\mathbf{g}} = \int_0^\infty \left(\int_{f^{-1}(t)} \|\nabla f\|_{\mathbf{g}} d\sigma_t \right) dt.$$

$$\geq \int_0^\infty (\mu(f^{-1}(t)))^2 \left(\int_{f^{-1}(t)} \|\nabla f\|^{-1}_{\mathbf{g}} d\sigma_t \right)^{-1} dt$$

by Hölder's inequality, where $d\sigma_t$ is the measure induced by $\mathbf{g}$ on $\{f^{-1}(t)\}$. But

$$\int_{f^{-1}(t)} \|\nabla f\|^{-1}_{\mathbf{g}} d\sigma_t = -\frac{d}{dt}(\mu\{f > t\})$$

$$= -\frac{d}{dt}(\mu\{f_* > t\}) = \int_{f_*^{-1}(t)} \|\nabla f_*\|_{\mathbf{g}}^{-1} d\sigma_t$$

and since $f^{-1}(t) = \partial\{f > t\}$ by Theorem 1.1 we have $\mu(f^{-1}(t)) \geq \mu(f_*^{-1}(t))$. Therefore

$$\int_0^\infty (\mu(f^{-1}(t)))^2 \left(\int_{f^{-1}(t)} \|\nabla f\|_{\mathbf{g}}^{-1} d\sigma_t \right)^{-1} dt$$

$$\geq \int_0^\infty (\mu(f_*^{-1}(t)))^2 \left(\int_{f_*^{-1}(t)} \|\nabla f_*\|_{\mathbf{g}}^{-1} d\sigma_t \right)^{-1} dt$$

(and since $\|\nabla f_*\|_{\mathbf{g}}$ is constant along $f_*^{-1}(t)$)

$$= \int_0^\infty \mu(f_*^{-1}(t)) \|\nabla f_*\|_{\mathbf{g}} \; dt$$

$$= \int_0^\infty \left(\int_{f_*^{-1}(t)} \|\nabla f_*\|_{\mathbf{g}} d\sigma_t \right) dt = \int \|\nabla f_*\|_{\mathbf{g}}^2.$$

Considering S^{n+1} as the spherical cone over S^n we have the function $f_*^0 : S^{n+1} \to \mathbb{R}_{\geq 0}$ which corresponds to f_*.

Then for all s

$$Vol(\{f_*^0 > s\}) = \left(\frac{V_n}{V} \right) \; Vol(\{f_* > s\},$$

and so for any q,

$$\int (f_*^0)^q dvol(g_0) = \left(\frac{V_n}{V} \right) \int (f_*)^q dvol(\mathbf{g}).$$

Also for any $s \in (0, \pi)$

$$\mu((f_*^0)^{-1}(s)) = \frac{V_n}{V} \mu(f_*^{-1}(s)),$$

and since $\|\nabla f_*^0\|_{g_0} = \|\nabla f_*\|_{\mathbf{g}}$ we have

$$\int \|\nabla f_*^0\|_{g_0}^2 = \frac{V_n}{V} \int \|\nabla f_*\|_{\mathbf{g}}^2.$$

We obtain

$$Y(M \times \mathbb{R}, [g + dt^2]) \geq \inf_{f \in C_0^\infty(M \times (0,\pi))} \frac{\int_{M \times (0,\pi)} a_{n+1} \|\nabla f\|_{\mathbf{g}}^2 + n(n+1) f^2 \; dvol(\mathbf{g})}{\|f\|_{p_{n+1}}^2}$$

$$\geq \inf_{f \in C_0^\infty(M \times (0,\pi))} \frac{\int_{M \times (0,\pi)} a_{n+1} \|\nabla f_*\|_{\mathbf{g}}^2 + n(n+1) f_*^2 \; dvol(\mathbf{g})}{\|f_*\|_{p_{n+1}}^2}$$

$$= \left(\frac{V}{V_n}\right)^{1-(2/p_{n+1})} \inf_{f\in C_0^\infty(M\times(0,\pi))} \frac{\int_{M\times(0,\pi)} a_{n+1}\|\nabla f_*^0\|_{g_0}^2 + n(n+1){f_*^0}^2 dvol(g_0)}{\|f_*^0\|_{p_{n+1}}^2}$$

$$\geq \left(\frac{V}{V_n}\right)^{2/(n+1)} Y_{n+1}$$

This finishes the proof of Theorem 1.2.

□

Proof of Corollary 1.3 : Note that if $\mathbf{s}_g$ is constant $Y_{\mathbb{R}}(M\times\mathbb{R}, g+dt^2)$ only depends on $\mathbf{s}_g$ and $V = Vol(M,g)$ Actually,

$$Y_{\mathbb{R}}(M\times\mathbb{R}, g+dt^2) = \inf_{f\in C_0^\infty(\mathbb{R})} \frac{\int_{\mathbb{R}} \ a_{n+1}\|\nabla f\|_{dt^2}^2 V + \mathbf{s}_g V f^2 \ dt^2}{(\int_{\mathbb{R}} f^p)^{2/p} \ V^{2/p}}$$

$$= V^{1-(2/p)} \inf_{f\in C_0^\infty(\mathbb{R})} \frac{\int_{\mathbb{R}} \ a_{n+1}\|\nabla f\|_{dt^2}^2 + \mathbf{s}_g f^2 \ dt^2}{(\int_{\mathbb{R}} f^p)^{2/p}}.$$

But as we said

$$\inf_{f\in C_0^\infty(\mathbb{R})} \frac{\int_{\mathbb{R}} \ a_{n+1}\|\nabla f\|_{dt^2}^2 + \mathbf{s}_g f^2 \ dt^2}{(\int_{\mathbb{R}} f^p)^{2/p}}$$

is independent of (M,g) and it is known to be equal to $Y_{n+1}V_n^{-2/(n+1)}$. Corollary 1.3 then follows directly from Theorem 1.2.

□

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