A COMPARISON THEOREM VIA THE RICCI FLOW

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ABSTRACT. We prove a comparison theorem for the compact surfaces with negative Euler characteristic via the Ricci flow.

1. Introduction

A good way to understand geometry of manifolds is to compare general manifolds with the ones with constant curvature. There have been many such comparison results. Among them are Bishop and Guenther's volume comparison theorems, Cheeger-Yau's heat kernel comparison theorem, Faber-Krahn's comparison theorem, to name only a few (see [1], [7]). Those comparison theorems not only have great impact on geometry but also are particularly beautiful. For example, Faber-Krahn's theorem (see [1], [7]) states that in Euclidean space $\mathbb{R}^n$, the first Dirichlet eigenvalue of the Laplacian of a domain Ω is greater than or equal to that of a ball B as long as Ω and B have the equal volume. In this paper, we prove a new comparison theorem (Theorem 1.1) that may be considered as an analogue for closed surfaces. While Faber-Krahn's is on domains of flat space $\mathbb{R}^n$, ours is on curved surfaces. One can expect that the curvature enters to play, as showed in Theorem 1.1. Theorem 1.1 actually implies a sharp upper bound for the i^{th} eigenvalue. That is stated in Theorem 1.2.

We prove our results by using Hamilton's Ricci flow. The proofs are enlightened by Perelman's work [6] on the monotonic property of the first eigenvalue of the operator $-4\Delta + R$ under the Ricci flow. Perelman [6] used that and nondecreasing property of his entropy under the Ricci flow to rule out nontrivial steady or expanding breathers on compact manifolds, among other applications.

In this paper, we let M be a compact surface without boundary. For any Riemannian metric g on M, we let K_g be the Gauss curvature, κ_g the minimum of the Gauss curvature, $\mathsf{Vol}_g(M)$ the volume of M, $d\mu_g$ the volume element, Δ_g the Lalacian of the metric g. We have the following results.

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Theorem 1.1 (Comparison Theorem)**.** *Let M be a compact surface with Euler Characteristic $\chi < 0$, g any Riemannian metric on M, and $\tilde{g}$ a Riemannian metric on M that has constant Gauss curvature $K_{\tilde{g}}$ and that is in the same conformal class as g. If $\mathsf{Vol}_g(M) = \mathsf{Vol}_{\tilde{g}}(M)$, and if the i^{th} eigenvalue λ_g of the Lapacian Δ_g has C^1-dependence on metric g, then we have*

$$\frac{\lambda_g}{\kappa_g} \geq \frac{\lambda_{\tilde{g}}}{\kappa_{\tilde{g}}},$$

where the constant Gauss curvature for metric $\tilde{g}$ is $K_{\tilde{g}} = 2\pi\chi/\mathsf{Vol}_g(M)$.

Theorem 1.2 (Sharp Upper Bound)**.** *Let M be a compact surface with Euler Characteristic $\chi < 0$, g any Riemannian metric on M, and $\tilde{g}$ a Riemannian metric on M that has constant Gauss curvature $K_{\tilde{g}}$ and that is in the same conformal class as g. Let σ be a lower bound of the Gauss curvature K_g. If $\mathsf{Vol}_g(M) = \mathsf{Vol}_{\tilde{g}}(M)$, and if the i^{th} eigenvalue λ_g of the Lapacian Δ_g has C^1-dependence on metric g, then λ_g has an upper bound*

$$\lambda_g \leq \frac{\lambda_{\tilde{g}}}{\kappa_{\tilde{g}}}\,\sigma,$$

that is,

$$\lambda_g \leq \frac{\lambda_{\tilde{g}}}{2\pi\chi}\,\mathsf{Vol}_{\tilde{g}}(M)\,\sigma,$$

where the constant Gauss curvature for metric $\tilde{g}$ is $K_{\tilde{g}} = 2\pi\chi/\mathsf{Vol}_g(M)$.

In particular, we have

$$\lambda_g \leq \frac{\lambda_{\tilde{g}}}{\kappa_{\tilde{g}}}\,\kappa_g,$$

that is

$$\lambda_g \leq \frac{\lambda_{\tilde{g}}}{2\pi\chi}\,\mathsf{Vol}_{\tilde{g}}(M)\,\kappa_g,$$

for which, the equality holds for metric g with constant Gauss curvature.

2. Proofs

We prove Theorem 1.2 first, and prove Theorem 1.1 after that.

Proof of Theorem 1.2. We evolve metric g in the theorem by the normalized Ricci flow. Let $g(t)$ be the solution of the normalized Ricci flow

$$\frac{\partial}{\partial t} g(t) = (r - R)g(t), \tag{2.1}$$

with initial value

$$g(0) = g, \tag{2.2}$$

where R is the the scalar curvature of the metric $g(t)$, r the average of the scalar curvature

$$r = \int_M R d\mu \Big/ \int_M d\mu,$$

and $d\mu = d\mu_t$ the volume element of $g(t)$. It is known from [3], [4], see also [2], that $g(t)$ exists for all $t \geq 0$.

By (2.1) we have

$$\frac{d}{dt}(d\mu) = (r - R)d\mu$$

and

$$\frac{d}{dt}\mathsf{Vol}_{g(t)}(M) = \frac{d}{dt}\int_M 1 d\mu = \int_M (r - R)d\mu = 0.$$

So the volume $\mathsf{Vol}_{g(t)}(M)$ remains constant in t, that is

$$V =: \mathsf{Vol}_{g(t)}(M) = \mathsf{Vol}_{g(0)}(M) \quad \forall t \geq 0.$$

By the Gauss-Bonnet Theorem, we have

$$r = 4\pi\chi/V < 0. \tag{2.3}$$

Therefore r is a negative constant and the lower bounds of R are negative as well. It is known from [4], see also [2], that $g(t)$ converges exponentially in any C^k-norm to a smooth metric $g(\infty)$ that has constant Gauss curvature $r/2$.

$R/2$ is the Gauss curvature K of the metric $g(t)$. Let $\sigma < 0$ be a lower bound of $K|_{t=0} = R|_{t=0}/2$,

$$K|_{t=0} \geq \sigma. \tag{2.4}$$

Let λ be the i^{th} eigenvalue of the Laplacian Δ of metric $g(t)$, u the corresponding eigenfunction. Then we have

$$-\Delta u = \lambda u.$$

Take derivatives with respect to t,

$$-(\frac{\partial}{\partial t}\Delta)u - \Delta\frac{\partial}{\partial t}u = (\frac{d}{dt}\lambda)u + \lambda\frac{\partial}{\partial t}u.$$

Multiply the equation by u and integrate,

$$-\int u(\frac{\partial}{\partial t}\Delta)ud\mu - \int u\Delta\frac{\partial}{\partial t}ud\mu = (\frac{d}{dt}\lambda)\int u^2 d\mu + \lambda\int u\frac{\partial}{\partial t}ud\mu. \tag{2.5}$$

Standard calculations (Ch. [2]) show that

$$\frac{\partial}{\partial t}(\Delta_{g(t)}) = (R - r)\Delta_{g(t)}. \tag{2.6}$$

Now (2.5), (2.6) and the equation

$$-\int u\Delta\frac{\partial}{\partial t}u = -\int \Delta u\,\frac{\partial}{\partial t}u = \lambda\int u\frac{\partial}{\partial t}u,$$

imply that

$$\begin{aligned}\frac{d}{dt}\lambda &= -\int u(\frac{\partial}{\partial t}\Delta)ud\mu \Big/ \int u^2 d\mu \\ &= -\int u(R-r)\Delta u d\mu \Big/ \int u^2 d\mu \\ &= \lambda \int (R-r)u^2 d\mu \Big/ \int u^2 d\mu.\end{aligned}$$

Therefore we have

$$\lambda(t) = \lambda(0)\exp\Big\{\int_0^t \frac{\int_M (R-r)u^2 d\mu}{\int_M u^2 d\mu} d\tau\Big\}.$$

The scalar curvature R evolves by the equation

$$\frac{\partial}{\partial t}R = \Delta R + R(R-r).$$

Using the maximum principle to compare R with the solution

$$s(t) = \frac{r}{1-\left(1-\frac{r}{2\sigma}\right)e^{rt}}$$

of the initial value problem of ODE

$$\begin{cases} \frac{d}{dt}s = s(s-r), \\ s(0) = 2\sigma, \end{cases}$$

we get

$$R \geq \frac{r}{1-\left(1-\frac{r}{2\sigma}\right)e^{rt}} \qquad \forall x \in M, \forall t \in [0,\infty),$$

and

$$\begin{aligned}&\int_0^t \frac{\int_M \Big(R(\tau)-r\Big)u^2 d\mu}{\int_M u^2 d\mu} d\tau \\ &\geq \int_0^t \frac{\int_M \left(\frac{r}{1-\left(1-\frac{r}{2\sigma}\right)e^{r\tau}} - r\right)u^2 d\mu}{\int_M u^2 d\mu} d\tau \\ &= \ln \frac{\frac{r}{2\sigma}e^{rt}}{1-\left(1-\frac{r}{2\sigma}\right)e^{rt}} - rt.\end{aligned}$$

Therefore

$$\lambda(t) \geq \lambda(0)\frac{\frac{r}{2\sigma}e^{rt}}{1-\left(1-\frac{r}{2\sigma}\right)e^{rt}} \Big/ e^{rt} = \lambda(0)\frac{\frac{r}{2\sigma}}{1-\left(1-\frac{r}{2\sigma}\right)e^{rt}}.$$

Note that the eigenvalues of the Laplacian continuously depends on the metric. Letting $t \to \infty$, we have

$$\lambda(\infty) \geq \lambda(0)\frac{r}{2\sigma}, \tag{2.7}$$

where $\lambda(\infty)$ is the i^{th} eigenvalue of the Laplacian of the metric $g(\infty)$ that has constant Gauss curvature $r/2$.

Let $\tilde{g} = g(\infty)$. It is clear that $\kappa_{\tilde{g}} = r/2$, since Gauss curvature $K_{\tilde{g}}$ for metric $\tilde{g}$ is constant. Taking these with (2.2) into (2.7), we get the first estimate in the theorem, which with (2.3) yield the second estimate in the theorem

$$\lambda_g \leq \frac{\lambda_{\tilde{g}}}{2\pi\chi}\,\sigma \mathsf{Vol}_{\tilde{g}}(M).$$

By the Gauss-Bonnet Theorem, the constant Gauss curvature for metric $\tilde{g}$ is $r/2 = 2\pi\chi/\mathsf{Vol}_{\tilde{g}}(M) = 2\pi\chi/\mathsf{Vol}_g(M)$.

Letting $\sigma = \kappa_g$, we get the third and fourth estimates in the theorem from the first two. $\square$

Proof of Theorem 1.1. Letting $g(\infty) = \tilde{g}$, $\sigma = \kappa_g$ and $r/2 = \kappa_{\tilde{g}}$ in (2.7), with (2.2), we get

$$\lambda_g \leq \lambda_{\tilde{g}} \frac{\kappa_g}{\kappa_{\tilde{g}}}.$$

Therefore we have

$$\frac{\lambda_g}{\kappa_g} \geq \frac{\lambda_{\tilde{g}}}{\kappa_{\tilde{g}}}$$

The last estimate follows from the above one. $\square$

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References

[1] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, Orlando, Florida, 1984.

[2] B. Chow and D. Knopf, *The Ricci flow: An introduction*, Mathematical Surveys and Monographs, AMS, Providence, RI, 2004.

[3] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom., **17**(1982), no. 2, 255–306.

[4] R. Hamilton, *The Ricci flow on surfaces*, Mathematics and General Relativity (San Cruz, CA 1986), 237–262. Contemp. Math., **71**, Amer. Math. Soc., Providence, RI, 1988.

[5] Li Ma, *Eigenvalue Monotonicity for the Ricci-Hamilton Flow*, Annals of Global Analysis and Geometry, **337**(2006), no. 2, 435–441.

[6] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159.

[7] R. Schoen, and S. T. Yau, *Lecture Notes on Differential Geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, Vol 1, International Press, 1994.

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