

Cup products in Hopf cyclic cohomology via cyclic modules I

Bahram Rangipour

Department of Mathematics and Statistics,
University of New Brunswick,
Fredericton, NB, CANADA, E3B 5A3

Email address: bahram@unb.ca

Abstract

This is the first one in a series of two papers on the continuation of our study in cup products in Hopf cyclic cohomology. In this note we construct cyclic cocycles of algebras out of Hopf cyclic cocycles of algebras and coalgebras. In the next paper we consider producing Hopf cyclic cocycle from “equivariant” Hopf cyclic cocycles. Our approach in both situations is based on (co)cyclic modules and bi(co)cyclic modules together with Eilenberg-Zilber theorem which is different from the old definition of cup products defined in [11] via traces and cotraces on DG algebras and coalgebras.

1 Introduction and some preliminaries

Hopf cyclic cohomology was invented by Connes and Moscovici as part of their fundamental work of computing the class of the index of the hypoelliptic signature operator [4]. The decidedly nontrivial idea was to show that the index cocyclic is in the range of a characteristic map.

Hopf cyclic cohomology was vastly generalized to study Hopf-(co)module (co)algebras and coefficients (partially in [12] and comprehensively in [9, 8]); later on Hopf cyclic cohomology was generalized to encompass the category of bialgebra-(co)module (co)algebras [10]. In [9] it was conjectured that any characteristic map as above is just a component of a cup product in Hopf

cyclic cohomology. In [11] the author and M. Khalkhali proved the existence of the cup products defined via traces and cotraces over DG algebras and coalgebras. As an intermediate step, characteristic maps via higher traces of Crainic [5] and Gorokhovsky [7] can be thought of as cup products with trivial coefficients. In [3] it is beautifully disclosed how the idea of cup product is applicable in case that (or even in general) the algebra under question possesses no invariant trace; as a replacement one takes advantage of an invariant cyclic cocycle to realize Hopf cyclic cocycles as cyclic cocycles on the algebra.

In these notes, we define and study the above cup products in a very straightforward way by using a method we learned from [14]. The fact that the equivariant property of cocycles yields that the produced cocycle to be well-defined on the convolution and crossed product algebra is nontrivial. This prompts us to do more research on this cup products [15]. This products also has to be analyzed from the category and representation theory point of view. The merit of our definitions is their simplicity and lack of dependence on the algebra or coalgebra structures, since we use a Hopf twisting map the whole procedure should work to a great extent for arbitrary twisting maps. As one knows the cyclic cohomology of Hopf algebras is defined as the cyclic cohomology of a canonical cocyclic module associated to the Hopf algebra; one uses this fact to produce Hopf cyclic cocycles by exploiting “equivariant” Hopf cyclic cocycles [15].

There are at least eight kinds of cup products defined on Hopf cyclic cohomology but only two of them so far are applied in NCG, the reason could be their lack of classic and/or geometric counterparts. For the first product, one starts with Hopf cyclic cocycles over an algebra and a coalgebra with coefficients in a SAYD module. To define the cup product one needs the coalgebra to act on the algebra. The next step is to construct, via a twisting map, the cup product as cocycle over the convolution algebra. But one knows that any cocycle over the convolution algebra is automatically a cocycle over the algebra. This cup product generalizes the characteristic map of Connes-Moscovici [4]. Ingredients for the second cup product are cyclic cocycles on a module algebra over a Hopf algebra and on a comodule algebra over the same Hopf algebra. Out of the two Hopf cyclic cocycles one produces a cyclic cocycle over the crossed product algebra. This cup product generalizes the ordinary cup product in cyclic cohomology of algebras as defined by Connes

[1].

For the reader convenience, we briefly recall the definition of Hopf cyclic cohomology of coalgebras and algebras under the symmetry of Hopf algebras and with coefficients in stable anti Yetter-Drinfeld (SAYD) modules [4, 8, 9]. In this note \mathcal{H} is a Hopf algebra, $\mu, \eta, \Delta, \varepsilon$, and S be its product, unite, co-product, counit and antipode, which is also supposed invertible, respectively. We use the Sweedler's notation for coproduct, i.e., $\Delta(h) = h_{(1)} \otimes h_{(2)}$. Let C be a \mathcal{H} -module coalgebra, that is a coalgebra endowed with an action, say from left, of \mathcal{H} such that its comultiplication and counit are \mathcal{H} -linear, i.e.,

$$\Delta(hc) = h_{(1)}c_{(1)} \otimes h_{(2)}c_{(2)}, \quad \varepsilon(hc) = \varepsilon(h)\varepsilon(c). \quad (1.1)$$

As the coefficients in this cohomology theory the notion of SAYD module is defined in [8] and recalled as follows. It is said that a right module M which is also a left comodule is a right-left SAYD module over a Hopf algebra \mathcal{H} if it satisfies the following conditions for any $h \in \mathcal{H}$, and $m \in M$.

$$m_{\langle \bar{0} \rangle} m_{\langle \bar{-1} \rangle} = m \quad (1.2)$$

$$(mh)_{\langle \bar{-1} \rangle} \otimes (mh)_{\langle \bar{0} \rangle} = S(h_{(3)})m_{\langle \bar{-1} \rangle} h_{(1)} \otimes m_{\langle \bar{0} \rangle} h_{(2)}, \quad (1.3)$$

where the coaction of \mathcal{H} is denoted by $\Delta_M(m) = m_{\langle \bar{-1} \rangle} \otimes m_{\langle \bar{0} \rangle}$.

Having the datum (\mathcal{H}, C, M) , one defines [9] cocyclic module $\{C_{\mathcal{H}}^n(C, M), \partial_i, \sigma_j, \tau\}_{n \geq 0}$ as follows.

$$\mathcal{C}^n := C_{\mathcal{H}}^n(C, M) = M \otimes_{\mathcal{H}} C^{\otimes n+1}, \quad n \geq 0,$$

with the following cocyclic structure,

$$\partial_i : \mathcal{C}^n \rightarrow \mathcal{C}^{n+1}, \quad 0 \leq i \leq n+1 \quad (1.4)$$

$$\sigma_j : \mathcal{C}^n \rightarrow \mathcal{C}^{n-1}, \quad 0 \leq j \leq n-1, \quad (1.5)$$

$$\tau : \mathcal{C}^n \rightarrow \mathcal{C}^n, \quad (1.6)$$

defined explicitly as follows, where we abbreviate $\tilde{c} = c^0 \otimes \dots \otimes c^n$,

$$\partial_i(m \otimes_{\mathcal{H}} \tilde{c}) = m \otimes_{\mathcal{H}} c^0 \otimes \dots \otimes \Delta(c_i) \otimes \dots \otimes c^n, \quad (1.7)$$

$$\partial_{n+1}(m \otimes_{\mathcal{H}} \tilde{c}) = m_{\langle \bar{0} \rangle} \otimes_{\mathcal{H}} c^0_{(2)} \otimes c^1 \otimes \dots \otimes c^n \otimes m_{\langle \bar{-1} \rangle} c^0_{(1)}, \quad (1.8)$$

$$\sigma_i(m \otimes_{\mathcal{H}} \tilde{c}) = m \otimes_{\mathcal{H}} c^0 \otimes \dots \otimes \epsilon(c^{i+1}) \otimes \dots \otimes c^n, \quad (1.9)$$

$$\tau(m \otimes_{\mathcal{H}} \tilde{c}) = m_{\langle \bar{0} \rangle} \otimes_{\mathcal{H}} c^1 \otimes \dots \otimes c^n \otimes m_{\langle \bar{-1} \rangle} c^0. \quad (1.10)$$

It is checked [9] that ∂_i, σ_j , and τ satisfy the following identities, which are recalled from [1] as the definition of cocyclic module.

$$\partial_j \partial_i = \partial_i \partial_{j-1}, \quad i < j, \quad \sigma_j \sigma_i = \sigma_i \sigma_{j+1}, \quad i \leq j \quad (1.11)$$

$$\sigma_j \partial_i = \begin{cases} \partial_i \sigma_{j-1} & i < j \\ 1_n & \text{if } i = j \text{ or } i = j + 1 \\ \partial_{i-1} \sigma_j & i > j + 1; \end{cases} \quad (1.12)$$

$$\tau_n \partial_i = \partial_{i-1} \tau_{n-1}, \quad 1 \leq i \leq n, \quad \tau_n \partial_0 = \partial_n \quad (1.13)$$

$$\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1}, \quad 1 \leq i \leq n, \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2 \quad (1.14)$$

$$\tau_n^{n+1} = 1_n. \quad (1.15)$$

As the motivating example of the above theory one recovers the cyclic complex of a Hopf algebra \mathcal{H} endowed with a modular pair in involution (MPI) (δ, σ) , which we recall it here from [4]. The character δ is an algebra map $\mathcal{H} \rightarrow \mathbb{C}$, and the group-like element $\sigma \in \mathcal{H}$ is a coalgebra map $\mathbb{C} \rightarrow \mathcal{H}$, i.e. $\sigma := \sigma(1)$ satisfies $\Delta(\sigma) = \sigma \otimes \sigma$. The pair (δ, σ) is called MPI if $\delta(\sigma) = 1$, and $\tilde{S}_\delta = Ad\sigma$, where the twisted antipode \tilde{S}_δ is defined by

$$\tilde{S}_\delta(h) = (\delta * S)(h) = \delta(h_{(1)})S(h_{(2)}). \quad (1.16)$$

One knows that \mathcal{H} is left \mathcal{H} -module coalgebra via left multiplication. On the other hand if one lets $M = {}^\sigma \mathbb{C}_\delta$ to be the ground field \mathbb{C} endowed with the left \mathcal{H} coaction via σ and right \mathcal{H} action via the character δ , then its checked [9] that (δ, σ) is a MPI if and only if ${}^\sigma \mathbb{C}_\delta$ is a SAYD. Thanks to the multiplication and the antipode of \mathcal{H} one identifies $C_{\mathcal{H}}(\mathcal{H}, M)$ with $M \otimes \mathcal{H}^{\otimes n}$ via the following map,

$$\mathcal{I} : M \otimes_{\mathcal{H}} \mathcal{H}^{\otimes(n+1)} \rightarrow M \otimes \mathcal{H}^{\otimes n}, \quad (1.17)$$

$$\mathcal{I}(m \otimes_{\mathcal{H}} h^0 \otimes \dots \otimes h^n) = mh^0_{(1)} \otimes S(h_{(2)}) \cdot (h^1 \otimes \dots \otimes h^n). \quad (1.18)$$

As a result one simplifies ∂_i, σ_j , and τ , in this case, and recovers the original definition of the cyclic cohomology of Hopf algebras [4].

$$\partial_0(h^1 \otimes \dots \otimes h^{n-1}) = 1 \otimes h^1 \otimes \dots \otimes h^{n-1}, \quad (1.19)$$

$$\partial_j(h^1 \otimes \dots \otimes h^{n-1}) = h^1 \otimes \dots \otimes \Delta h^j \otimes \dots \otimes h^{n-1}, \quad 1 \leq j \leq n-1$$

$$\partial_n(h^1 \otimes \dots \otimes h^{n-1}) = h^1 \otimes \dots \otimes h^{n-1} \otimes \sigma,$$

$$\sigma_i(h^1 \otimes \dots \otimes h^{n+1}) = h^1 \otimes \dots \otimes \varepsilon(h^{i+1}) \otimes \dots \otimes h^{n+1}, \quad 0 \leq i \leq n,$$

$$\tau_n(h^1 \otimes \dots \otimes h^n) = (\Delta^{n-1} \tilde{S}(h^1)) \cdot h^2 \otimes \dots \otimes h^n \otimes \sigma.$$

Similarly an algebra which is \mathcal{H} -module and its algebra structure is \mathcal{H} -linear is called \mathcal{H} -module algebra. Let A be a \mathcal{H} -module algebra and then one endows $M \otimes A^{\otimes n+1}$ with the diagonal action of \mathcal{H} and forms $C_{\mathcal{H}}^n(A, M) = \text{Hom}_{\mathcal{H}}(M \otimes A^{\otimes n+1}, \mathbb{C})$ as the space of \mathcal{H} -linear maps. It is checked that the following defines a cocyclic module structure on $C^n(A, M)$.

$$\begin{aligned} (\partial_i \varphi)(m \otimes \tilde{a}) &= \varphi(m \otimes a^0 \otimes \dots \otimes a^i a^{i+1} \otimes \dots \otimes a^{n+1}), & 0 \leq i < n, \\ (\partial_{n+1} \varphi)(m \otimes \tilde{a}) &= \varphi(m_{\langle \bar{0} \rangle} \otimes (S^{-1}(m_{\langle \bar{-1} \rangle}) a^{n+1}) a^0 \otimes a^1 \otimes \dots \otimes a^n), \\ (\sigma_i \varphi)(m \otimes \tilde{a}) &= \varphi(m \otimes a^0 \otimes \dots \otimes a^i \otimes 1 \otimes \dots \otimes a^{n-1}), & 0 \leq i \leq n-1, \\ (\tau \varphi)(m \otimes \tilde{a}) &= \varphi(m_{\langle \bar{0} \rangle} \otimes (S^{-1}(m_{\langle \bar{-1} \rangle}) a^n) \otimes a^0 \otimes \dots \otimes a^{n-1}), \end{aligned}$$

The cyclic cohomology of this cocyclic module is denoted by $HC_{\mathcal{H}}^*(A, M)$.

An algebra is called \mathcal{H} -comodule coalgebra if it is a \mathcal{H} comodule and its coalgebra structure are \mathcal{H} colinear. Similar to the other case, one defines ${}^{\mathcal{H}}C^n(A, M)$ to be the space of all colinear maps from $A^{\otimes n+1}$ to M . One checks that the following defines a cocyclic module structure on ${}^{\mathcal{H}}C^n(A, M)$.

$$\begin{aligned} (\partial_i \varphi)(\tilde{a}) &= \varphi(a^0 \otimes \dots \otimes a^i a^{i+1} \otimes \dots \otimes a^{n+1}), & 0 \leq i < n, \\ (\partial_{n+1} \varphi)(\tilde{a}) &= \varphi(a^{n+1}_{\langle 0 \rangle} a^0 \otimes a^1 \dots \otimes a^{n-1} \otimes a^n) a^{n+1}_{\langle -1 \rangle}, \\ (\sigma_i \varphi)(\tilde{a}) &= \varphi(a^0 \otimes \dots \otimes a^i \otimes 1 \otimes \dots \otimes a^{n-1}), & 0 \leq i \leq n-1, \\ (\tau \varphi)(a^0 \otimes \dots \otimes a^n) &= \varphi(a^n_{\langle 0 \rangle} \otimes a^0 \otimes \dots \otimes a^{n-1} \otimes a^{n-1}) a^n_{\langle -1 \rangle}. \end{aligned}$$

The cyclic cohomology of this cocyclic module is denoted by ${}^{\mathcal{H}}HC^*(A, M)$. For completeness, we record below the bi-complex

$$(CC^{*,*}(C, \mathcal{H}, M), b, B)$$

that computes the Hopf cyclic cohomology of a coalgebra C with coefficients in a SAYD module M under the symmetry of a Hopf algebra \mathcal{H} :

$$CC^{p,q}(C, \mathcal{H}; M) = \begin{cases} C_{\mathcal{H}}^{q-p}(C, M), & q \geq p, \\ 0, & q < p, \end{cases}$$

the operator

$$b : C_{\mathcal{H}}^n(C, M) \rightarrow C_{\mathcal{H}}^{n+1}(C, M), \quad b = \sum_{i=0}^{n+1} (-1)^i \partial_i$$

and the B -operator $B : C_{\mathcal{H}}^n(C, M) \rightarrow C_{\mathcal{H}}^{n-1}(C, M)$ is defined by the formula

$$B = A \circ B_0, \quad n \geq 0,$$

where

$$B_0 = \sigma_{n-1} \tau (1 - (-1)^n \tau)$$

and

$$A = 1 + \lambda + \dots + \lambda^n, \quad \text{with} \quad \lambda = (-1)^{n-1} \tau_n.$$

The groups $\{HC^n(\mathcal{H}; \delta, \sigma)\}_{n \in \mathbb{N}}$ are computed from the first quadrant total complex $(TC^*(\mathcal{H}; \delta, \sigma), b + B)$,

$$TC^n(\mathcal{H}; \delta, \sigma) = \sum_{k=0}^n CC^{k, n-k}(\mathcal{H}; \delta, \sigma),$$

and the periodic groups $\{HP^i(\mathcal{H}; \delta, \sigma)\}_{i \in \mathbb{Z}/2}$ are computed from the full total complex $(TP^*(\mathcal{H}; \delta, \sigma), b + B)$,

$$TP^i(\mathcal{H}; \delta, \sigma) = \sum_{k \in \mathbb{Z}} CC^{k, i-k}(\mathcal{H}; \delta, \sigma).$$

Let $(C^n, \delta_i, \sigma_i, \tau_n)$ and $(C^m, \delta_i, \sigma_i, \tau_n)$ be two cocyclic objects in the category of vector spaces. Their product is the cocyclic object $((C \times C')^n, \delta_i, \sigma_i, \tau_n)$ with $(C \times C')^n = C^n \otimes C'^n$ and $\delta_i = \delta_i \otimes \delta_i$, $\sigma_i = \sigma_i \otimes \sigma_i$ and $\tau_n = \tau_n \otimes \tau_n$. Their tensor product is the bicocyclic module $C \otimes C'$ defined by $(C \otimes C')^{m, n} = C^m \otimes C'^n$ with horizontal and vertical structure borrowed from C and C' respectively. Eilenberg-Zilber states that the Cyclic cyclic cohomology of mixed complexes $C \times C'$ and $Tot(C \otimes C')$ are the same via the shuffle map [13].

2 Module algebras paired with module coalgebras

Let \mathcal{H} be a Hopf algebra, A be a \mathcal{H} -module algebra and (δ, σ) be a modular pair in involution on \mathcal{H} . Connes and Moscovici [4] showed that the following defines a map of cocyclic modules

$$\begin{aligned} \chi : \mathcal{H}_{(\delta, \sigma)}^{\natural} &\rightarrow C^*(A), \\ \chi(h^1 \otimes \dots \otimes h^n)(a^0 \otimes \dots \otimes a^n) &= \tau(a^0 h^1(a^1) \dots h^n(a^n)). \end{aligned} \tag{2.1}$$

Here $\tau : A \rightarrow \mathbb{C}$ is a δ -invariant σ -trace, i.e. for all a, b and h

$$\tau(ha) = \delta(h)\tau(a), \quad (2.2)$$

$$\tau(ab) = \tau(b\sigma a). \quad (2.3)$$

The above map then induces the following characteristic map on the level of cohomologies:

$$\chi : HC_{(\delta, \sigma)}^n(\mathcal{H}) \rightarrow HC^n(A). \quad (2.4)$$

Hopf cyclic cohomology and SAYD (stable anti-Yetter-Drinfeld) modules, generalize cyclic cohomology of Hopf algebras and MPI (modular pair in involution), respectively. Now a δ -invariant σ -trace is exactly a closed cyclic cocycle in $C_{\mathcal{H}}^0(A, {}^\sigma \mathbb{C}_\delta)$. These facts prompted us in [9] to conjecture that there should exist a generalization of characteristic map as a pairing between Hopf cyclic cohomology of module algebras and module coalgebras:

$$HC_{\mathcal{H}}^n(A, M) \otimes HC_{\mathcal{H}}^m(C, M) \rightarrow HC^{n+m}(A, M), \quad (2.5)$$

where M is a left-right SAYD module over H and C is a \mathcal{H} module coalgebra acting on A in the sense that there is a map

$$C \otimes A \rightarrow A, \quad (2.6)$$

such that for any $h \in \mathcal{H}$, any $c \in C$ and any $a, b \in A$ one has

$$(hc)a = h(ca) \quad (2.7)$$

$$c(ab) = (c_{(1)}a)(c_{(2)}b) \quad (2.8)$$

$$c(1) = \epsilon(c)1 \quad (2.9)$$

Although there is a proof of the above conjecture in [11], we would like to give a more direct proof based on theory of cyclic modules instead of traces on DG algebras. The advantage of this new view is not only its simplicity but also its efficiency which enables one to use the precise expression of these cup products as it is shown in [15].

One constructs a very useful convolution algebra $B = \text{Hom}_{\mathcal{H}}(C, A)$, which is the algebra of all \mathcal{H} -linear maps from A to C . The unit of this algebra is given by $\eta \circ \epsilon$, where $\eta : \mathbb{C} \rightarrow A$ is the unit of A . The multiplication of $f, g \in B$ is given by

$$(f * g)(c) = f(c_{(1)})g(c_{(2)}) \quad (2.10)$$

Now consider two cocyclic modules

$$(C_{\mathcal{H}}^*(A, M), \delta_i, \sigma_j, t), \quad \text{and} \quad (C_{\mathcal{H}}^*(C, M), d_i, s_j, \tau)$$

and let us make a new bicocyclic module by just tensoring these two ones. The bicocyclic module has in bidegree (p, q)

$$C^{p,q} := \text{Hom}_{\mathcal{H}}(M \otimes A^{\otimes p+1}, \mathbb{C}) \otimes (M \otimes_{\mathcal{H}} C^{\otimes q+1}),$$

with horizontal structure $\vec{\partial}_i = d_i \otimes \text{Id}$, $\vec{\sigma}_j = s_j \otimes \text{Id}$, and $\vec{\tau} = t \otimes \text{Id}$ and vertical structure $\uparrow \partial_i = \delta_i \otimes \text{Id}$, $\uparrow \sigma_j = \sigma_j \otimes \text{Id}$, and $\uparrow \tau = \tau \otimes \text{Id}$. Obviously $(C^{n,m}, \vec{\partial}, \vec{\sigma}, \vec{\tau}, \uparrow \partial, \uparrow \sigma, \uparrow \tau)$ defines a bicocyclic module, where $\otimes := \otimes$ for which we use it to distinguish between $\text{Hom}_{\mathcal{H}}(M \otimes A^{\otimes p+1}, \mathbb{C})$ and $(M \otimes_{\mathcal{H}} C^{\otimes q+1})$.

Now let us define the map

$$\begin{aligned} \Psi_c : C^{n,n} &\rightarrow \text{Hom}(B^{\otimes n+1}, \mathbb{C}), \\ \Psi_c(\phi \otimes m \otimes c^0 \otimes \dots \otimes c^n)(f^0 \otimes \dots \otimes f^n) &= \phi(m \otimes f^0(c^0) \otimes \dots \otimes f^n(c^n)), \end{aligned} \quad (2.11)$$

which is obviously well defined due to the facts that f is \mathcal{H} -linear, ϕ is equivariant and (2.7) holds.

Proposition 2.1. *The map Ψ_c defines a cyclic map between the diagonal of $C^{*,*}$ and the cocyclic module $C^*(B)$.*

Proof. We have to show that Ψ commutes with the cyclic structures of the two sides. Indeed we just check it for the first face and cyclic operator and leave the rest to the reader. We have

$$\begin{aligned} \Psi_c(\vec{\partial}_0 \uparrow \partial_0(\phi \otimes m \otimes c^0 \otimes \dots \otimes c^n))(f^0 \otimes \dots \otimes f^{n+1}) &= \\ \Psi_c(d_0 \phi \otimes \delta_0(m \otimes c^0 \otimes \dots \otimes c^n))(f^0 \otimes \dots \otimes f^{n+1}) &= \\ \phi(m \otimes f^0(c^0_{(1)}) f^1(c^0_{(2)}) \otimes f^2(c^1) \otimes \dots \otimes f^{n+1}(c^n)) &= \\ \phi(m \otimes (f^0 * f^1)(c^0) \otimes f^2(c^1) \otimes \dots \otimes f^{n+1}(c^n)) &= \\ (d_0 \Psi_c(\phi \otimes m \otimes \tilde{c}))(f^0 \otimes \dots \otimes f^{n+1}). \end{aligned}$$

$$\begin{aligned}
& \Psi_c(\vec{\tau} \uparrow \tau(\phi \otimes m \otimes c^0 \otimes \dots \otimes c^n))(a^0 \otimes \dots \otimes a^n) = \\
& \Psi_c(t\phi \otimes \tau(m \otimes c^0 \otimes \dots \otimes c^n))(f^0 \otimes \dots \otimes f^n) = \\
& t\phi(m_{\langle \bar{0} \rangle} f^0(c^1) \otimes \dots \otimes f^{n-1}(c^n)) \otimes m_{\langle \bar{-1} \rangle} f^n(c^0)) = \\
& \phi(m_{\langle \bar{0} \rangle} \otimes S^{-1}(m_{\langle \bar{-1} \rangle}) m_{\langle \bar{-2} \rangle} f^n(c^0) \otimes f^0(c^1) \otimes \dots \otimes f^{n-1}(c^n)) = \\
& \phi(m \otimes f^n(c^0) \otimes f^0(c^1) \otimes \dots \otimes f^{n-1}(c^n)) = \\
& (t\Psi_c(\phi \otimes m \otimes c^0 \otimes \dots \otimes c^n))(f^0 \otimes \dots \otimes f^n).
\end{aligned}$$

□

One can take advantage of properties (2.7), (2.8), and (2.9) to prove that there exists a natural unital algebra map $\natural : A \rightarrow \text{Hom}_{\mathcal{H}}(A, C)$, explicitly defined by $\natural(a)(c) = c(a)$. As a result, one obtains a cyclic map $\natural : C^*(B, \mathbb{C}) \rightarrow C^*(A, \mathbb{C})$. One then composes \natural with Ψ_c to get a cyclic map

$$\Psi = \natural \circ \Psi_c : C^*(\mathfrak{D}(C^{*,*})) \rightarrow C^*(A, \mathbb{C}).$$

Let \mathcal{K} be a sub Hopf algebra of \mathcal{H} . Although A is a \mathcal{H} -module algebra, the coalgebra $\mathcal{C} = C(\mathcal{H}, \mathcal{K}) = \mathcal{H} \otimes_{\mathcal{K}} \mathbb{C}$ does not inherit this property from \mathcal{H} since the action of \mathcal{C} on A is not well-defined. One cures this problem by letting \mathcal{C} acts on the invariant sub algebra of A under the action of \mathcal{K} . Let

$$A^{\mathcal{K}} = \{a \in A \mid ka = \varepsilon(k)a\}. \quad (2.12)$$

One checks that the action of \mathcal{C} on $A^{\mathcal{K}}$ is well defined and satisfies (2.7), (2.8), and (2.9). One notes that it is not possible to write the map (2.11) for the case $A^{\mathcal{K}}$ and \mathcal{C} because $A^{\mathcal{K}}$ is not a \mathcal{H} -module algebra. Instead one writes the invariant form of (2.11) as follows. Let us introduce

$$C_r^{p,q} = C_{\mathcal{H}}^p(A, M) \otimes C_{\mathcal{H}}^q(\mathcal{H}, \mathcal{K}; M)$$

with its standard cyclic structure. Then one has a cyclic map

$$\begin{aligned}
\Psi_r : \mathfrak{D}(C_r^{*,*}) & \rightarrow C^*(A_{\mathcal{K}}), \\
\Psi_r(\phi \otimes m \otimes c^0 \otimes c^1 \otimes \dots \otimes c^n) & (a^0 \otimes \dots \otimes a^n) = \\
\phi(m \otimes c^0(a^0) \otimes \dots \otimes c^n(a^n)). &
\end{aligned} \quad (2.13)$$

It is shown that the above map defines a cyclic map. Note that it does not land in $C^*(A)$.

Corollary 2.2. *The map Ψ induces the following maps on cyclic cohomologies:*

$$\Psi : HC^n(\mathfrak{D}(C^{*,*})) \rightarrow HC^n(A) \quad (2.14)$$

$$\Psi_c : HC^n(\mathfrak{D}(C_c^{*,*})) \rightarrow HC^n(\text{Hom}_{\mathcal{H}}(C, A)) \quad (2.15)$$

$$\Psi_r : HC^n(\mathfrak{D}(C_r^{*,*})) \rightarrow HC^n(A_{\mathcal{K}}). \quad (2.16)$$

Now composing Ψ , Ψ_c , and Ψ_r with the corresponding Alexander-Whitney map AW one obtains the following cup products:

$$\cup = \Psi \circ AW : HC_{\mathcal{H}}^p(A, M) \otimes HC_{\mathcal{H}}^q(C, M) \rightarrow HC^{p+q}(A),$$

$$\cup = \Psi_c \circ AW : HC_{\mathcal{H}}^p(A, M) \otimes HC_{\mathcal{H}}^q(C, M) \rightarrow HC^{p+q}(\text{Hom}_{\mathcal{H}}(C, A)),$$

$$\cup = \Psi_r \circ AW : HC_{\mathcal{H}}^p(A, M) \otimes HC_{\mathcal{H}}^q(\mathcal{H}, \mathcal{K}; M) \rightarrow HC^{p+q}(A^{\mathcal{K}}).$$

Proposition 2.3. *The above cup product is precisely given by the following formula in the level of Hochschild cohomology.*

$$\begin{aligned} \cup : C_{\mathcal{H}}^p(A, M) \otimes C_{\mathcal{H}}^q(C, M) &\rightarrow C^{p+q}(A), \\ (\phi \cup (m \otimes c^0 \otimes \dots \otimes c^q))(a^0 \otimes \dots \otimes a^{p+q}) &= \\ \phi(m \otimes c^0_{(p+1)}(a^0)c^1(a^1) \dots c^q(a^q) \otimes c^0_{(1)}(a^{q+1}) \otimes \dots \otimes c^0_{(p)}(a^{p+q})) & \end{aligned}$$

Proof. By composing the AW map given in [13] with Ψ one obtains the above formula. \square

3 Module algebras paired with comodule algebras

Let \mathcal{H} be a Hopf algebra, A a left \mathcal{H} -module algebra, B a left \mathcal{H} -comodule algebra and M be a right-left SAYD module over \mathcal{H} . One constructs a crossed product algebra whose underlying vector space is $A \otimes B$ with the $1 \bowtie 1$ as its unit and the following multiplication:

$$(a \bowtie b)(a' \bowtie b') = ab_{<-1>}(a') \bowtie b_{<0>}b' \quad (3.1)$$

Now consider the two cocyclic modules

$$(C_{\mathcal{H}}^*(A, M), \delta_i, \sigma_j, t), \quad \text{and} \quad ({}^{\mathcal{H}}C^*(B, M), d_i, s_j, \tau)$$

introduced in [9] and let us make a new bicocyclic module by just tensoring these two ones. Its (p, q) component $C^{p,q}$ is given by

$$\text{Hom}_{\mathcal{H}}(M \otimes A^{\otimes p+1}, \mathbb{C}) \otimes^{\mathcal{H}} \text{Hom}(B^{\otimes q+1}, M),$$

with horizontal structure $\vec{\partial}_i = d_i \otimes \text{Id}$, $\vec{\sigma}_j = s_j \otimes \text{Id}$, and $\vec{\tau} = t \otimes \text{Id}$ and vertical structure $\uparrow \partial_i = \delta_i \otimes \text{Id}$, $\uparrow \sigma_j = \sigma_j \otimes \text{Id}$, and $\uparrow \tau = \tau \otimes \text{Id}$. Obviously $(C^{n,m}, \vec{\partial}, \vec{\sigma}, \vec{\tau}, \uparrow \partial, \uparrow \sigma, \uparrow \tau)$ defines a bicocyclic module. Now let us define the following map

$$\Psi : C^{n,n} \rightarrow \text{Hom}((A \rtimes B)^{\otimes n+1}, \mathbb{C}), \quad (3.2)$$

$$\begin{aligned} & \Psi(\phi \otimes \psi)(a^0 \rtimes b^0 \otimes \dots \otimes a^n \rtimes b^n) = \\ & \phi(\psi(b^0_{\langle 0 \rangle} \otimes \dots \otimes b^n_{\langle 0 \rangle}) \otimes S^{-1}(b^0_{\langle -1 \rangle} \dots b^n_{\langle -1 \rangle}) a^0 \otimes \dots \otimes S^{-1}(b^n_{\langle -n-1 \rangle}) a^n). \end{aligned}$$

Proposition 3.1. *The map Ψ defines a cyclic map between the diagonal of $C^{*,*}$ and the cocyclic module $C^*(A \rtimes B)$.*

Proof. We have to show that Ψ commutes with the cyclic structures. We shall check it for the first face operator and the cyclic operator and leave the rest to the reader.

$$\begin{aligned} & \Psi(\vec{\partial}_0 \uparrow \partial_0(\phi \otimes \psi))(a^0 \rtimes b^0 \otimes \dots \otimes a^{n+1} \rtimes b^{n+1}) = \\ & \Psi(d_0 \phi \otimes \delta_0 \psi)(a^0 \rtimes b^0 \otimes \dots \otimes a^{n+1} \rtimes b^{n+1}) = \\ & \delta_0 \phi(\delta_0 \psi(b^0_{\langle 0 \rangle} \otimes \dots \otimes b^{n+1}_{\langle 0 \rangle}) \otimes S^{-1}(b^0_{\langle -1 \rangle} \dots b^{n+1}_{\langle -1 \rangle}) a^0 \otimes \dots \\ & \quad \dots \otimes S^{-1}(b^{n+1}_{\langle -n-2 \rangle}) a^{n+1}) = \\ & \phi(\psi(b^0_{\langle 0 \rangle} b^1_{\langle 0 \rangle} \otimes \dots \otimes b^{n+1}_{\langle 0 \rangle}) \otimes \\ & \quad \otimes S^{-1}(b^0_{\langle -1 \rangle} \dots b^{n+1}_{\langle -1 \rangle}) a^0 S^{-1}(b^1_{\langle -2 \rangle} \dots b^{n+1}_{\langle -2 \rangle}) a^1 \otimes \dots \\ & \quad \dots \otimes S^{-1}(b^{n+1}_{\langle -n-2 \rangle}) a^{n+1}) = \\ & \phi(\psi(b^0_{\langle 0 \rangle} b^1_{\langle 0 \rangle} \otimes \dots \otimes b^{n+1}_{\langle 0 \rangle}) \otimes \\ & \quad \otimes S^{-1}(b^0_{\langle -1 \rangle} \dots b^{n+1}_{\langle -1 \rangle}) a^0 S^{-1}(b^1_{\langle -2 \rangle} \dots b^{n+1}_{\langle -2 \rangle}) a^1 \otimes \dots \\ & \quad \dots \otimes S^{-1}(b^{n+1}_{\langle -n-2 \rangle}) a^{n+1}) = \\ & \phi(\psi(b^0_{\langle 0 \rangle} b^1_{\langle 0 \rangle} \otimes \dots \otimes b^{n+1}_{\langle 0 \rangle}) \otimes \\ & \quad \otimes S^{-1}(b^0_{\langle -1 \rangle} b^1_{\langle -1 \rangle} \dots b^{n+1}_{\langle -1 \rangle}) (a^0 b^0_{\langle -2 \rangle} a^1) \otimes \dots \\ & \quad \dots \otimes S^{-1}(b^{n+1}_{\langle -n-1 \rangle}) a^{n+1}) = \\ & \Psi(\phi \otimes \psi)(a^0 b^0_{\langle -1 \rangle} a^1 \rtimes b^0_{\langle 0 \rangle} b^1 \otimes a^2 \rtimes b^2 \otimes \dots \otimes a^{n+1} \rtimes b^{n+1}) = \\ & d_0 \Psi(\phi \otimes \psi)(a^0 \rtimes b^0 \otimes \dots \otimes a^{n+1} \rtimes b^{n+1}). \end{aligned}$$

Using the facts that ϕ is \mathcal{H} equivariant, ψ is \mathcal{H} colinear and M is SAYD one has:

$$\begin{aligned}
& \Psi(\vec{\tau} \uparrow \tau(\phi \otimes \psi))(a^0 \rtimes b^0 \otimes \dots \otimes a^n \rtimes b^n) = \\
& \Psi(t\phi \otimes \tau\psi)(a^0 \rtimes b^0 \otimes \dots \otimes a^n \rtimes b^n) = \\
& t\phi(\tau\psi(b^0_{\langle 0 \rangle} \otimes \dots \otimes b^n_{\langle 0 \rangle})) \otimes S^{-1}(b^0_{\langle 1 \rangle} \dots b^n_{\langle -1 \rangle}) a^0 \otimes \dots \otimes S^{-1}(b^n_{\langle -n-1 \rangle}) a^n = \\
& t\phi(\psi(b^n_{\langle 0 \rangle} \otimes b^0_{\langle 0 \rangle} \otimes \dots \otimes b^{n-1}_{\langle 0 \rangle}) b^n_{\langle -1 \rangle} \otimes \\
& \quad \otimes S^{-1}(b^0_{\langle 1 \rangle} \dots b^{n-1}_{\langle -1 \rangle} b^n_{\langle -2 \rangle}) a^0 \otimes \dots \otimes S^{-1}(b^n_{\langle -n-2 \rangle}) a^n) = \\
& \phi([\psi(b^n_{\langle 0 \rangle} \otimes b^0_{\langle 0 \rangle} \otimes \dots \otimes b^{n-1}_{\langle 0 \rangle}) b^n_{\langle -1 \rangle}]_{\langle 0 \rangle} \otimes \\
& S^{-1}([\psi(b^n_{\langle 0 \rangle} \otimes b^0_{\langle 0 \rangle} \otimes \dots \otimes b^{n-1}_{\langle 0 \rangle}) b^n_{\langle -1 \rangle}]_{\langle -1 \rangle} (S^{-1}(b^n_{\langle -n-2 \rangle}) a^n) \otimes \\
& \quad \otimes S^{-1}(b^0_{\langle 1 \rangle} \dots b^{n-1}_{\langle -1 \rangle} b^n_{\langle -2 \rangle}) a^0 \otimes \dots \otimes S^{-1}(b^{n-1}_{\langle -n+1 \rangle} b^n_{\langle -n-1 \rangle}) a^{n-1}) = \\
& \phi(\psi(b^n_{\langle 0 \rangle} \otimes b^0_{\langle 0 \rangle} \otimes \dots \otimes b^{n-1}_{\langle 0 \rangle}) b^n_{\langle -3 \rangle} \otimes \\
& S^{-1}(S(b^n_{\langle -2 \rangle}) b^n_{\langle -1 \rangle} b^0_{\langle -1 \rangle} \dots b^{n-1}_{\langle -1 \rangle}) b^n_{\langle -4 \rangle}) (S^{-1}(b^n_{\langle -n-2 \rangle}) a^n) \otimes \\
& \quad \otimes S^{-1}(b^0_{\langle 1 \rangle} \dots b^{n-1}_{\langle -1 \rangle} b^n_{\langle -5 \rangle}) a^0 \otimes \dots \otimes S^{-1}(b^{n-1}_{\langle -n+1 \rangle} b^n_{\langle -n-4 \rangle}) a^{n-1}) = \\
& \phi(\psi(b^n_{\langle 0 \rangle} \otimes b^0_{\langle 0 \rangle} \otimes \dots \otimes b^{n-1}_{\langle 0 \rangle}) b^n_{\langle -1 \rangle} \otimes \\
& S^{-1}(b^0_{\langle -1 \rangle} \dots b^{n-1}_{\langle -1 \rangle}) b^n_{\langle -2 \rangle}) (S^{-1}(b^n_{\langle -n-2 \rangle}) a^n) \otimes \\
& \quad \otimes S^{-1}(b^0_{\langle 1 \rangle} \dots b^{n-1}_{\langle -1 \rangle} b^n_{\langle -3 \rangle}) a^0 \otimes \dots \otimes S^{-1}(b^{n-1}_{\langle -n+1 \rangle} b^n_{\langle -n-2 \rangle}) a^{n-1}) = \\
& \phi(\psi(b^n_{\langle 0 \rangle} \otimes b^0_{\langle 0 \rangle} \otimes \dots \otimes b^{n-1}_{\langle 0 \rangle}) \otimes S^{-1}(b^n_{\langle -1 \rangle} b^0_{\langle -1 \rangle} \dots b^{n-1}_{\langle -1 \rangle}) a^n) \otimes \\
& \quad \otimes S^{-1}(b^0_{\langle 1 \rangle} \dots b^{n-1}_{\langle -1 \rangle}) a^0 \otimes \dots \otimes S^{-1}(b^{n-1}_{\langle -n+1 \rangle}) a^{n-1}) = \\
& \Psi(\phi \otimes \psi)(a^n \rtimes b^n \otimes a^0 \rtimes b^0 \otimes \dots \otimes a^{n-1} \rtimes b^{n-1}) = \\
& t\Psi(\phi \otimes \psi)(a^0 \rtimes b^0 \otimes \dots \otimes a^n \rtimes b^n).
\end{aligned}$$

□

Corollary 3.2. *The map Ψ defined in (3.2) induces a map on cyclic cohomologies:*

$$\Psi : HC^n(\mathfrak{D}(C^{*,*})) \rightarrow HC^n(A \rtimes B). \quad (3.3)$$

Now by composing Ψ with the corresponding map AW map one proves the existence of the following map:

$$\cup = \Psi \circ AW : HC_H^p(A, M) \otimes {}^{\mathcal{H}}HC^q(B, M) \rightarrow HC^{p+q}(A \rtimes B). \quad (3.4)$$

One uses the formula of AW map [13] to find the following expression for the above cup product.

Proposition 3.3. *The above cup product has the following formula in the level of Hochschild cohomology.*

$$\begin{aligned} \phi \cup \psi(a^0 \bowtie b^0 \otimes \dots \otimes a^{p+q} \bowtie b^{p+q}) = \\ \phi(\psi(b^{q+1}_{\langle 0 \rangle} \dots b^{p+q}_{\langle 0 \rangle} b^0_{\langle 0 \rangle} \otimes b^1_{\langle 0 \rangle} \otimes \dots \otimes b^q_{\langle 0 \rangle})_{\langle -1 \rangle} \otimes \\ S^{-1}(b^0_{\langle -1 \rangle} \dots b^q_{\langle -1 \rangle}) a^0 \dots S^{-1}(b^q_{\langle -q-1 \rangle}) a^q \otimes a^{q+1} \otimes b^{q+1}_{\langle -p-1 \rangle} a^{q+2} \otimes \dots \otimes \\ b^{q+1}_{\langle -1 \rangle} \dots b^{p+q-1}_{\langle -1 \rangle} a^{p+q}. \end{aligned}$$

Example 3.4. Let G be a discrete group acting by unital automorphisms on an algebra A and let k be a field of characteristic zero. In [?], the Hopf cyclic cohomology groups of the Hopf algebra $H = kG$ were computed in terms of group cohomology with trivial coefficients:

$${}^{kG}HC^p(kG, k) = \bigoplus_{i \geq 0} H^{p-2i}(G, k).$$

The cohomology groups $HC^q_{kG}(A, k)$ are easily seen to be the cohomology of the subcomplex of *invariant cyclic cochains* on A :

$$\varphi(ga_0, ga_1, \dots, ga_n) = \varphi(a_0, a_1, \dots, a_n),$$

for all $g \in G$ and $a_i \in A$. We denote this cohomology theory by $HC^q_G(A)$. We have thus a pairing

$$H^p(G) \otimes HC^q_G(A) \longrightarrow HC^{p+q}(A \rtimes G).$$

4 Cup product via traces

In this section we derive some formulas for cup products defined in [11]. Let us briefly recall it here. Let A be a left \mathcal{H} module algebra, B a left \mathcal{H} -comodule algebra and M a SAYD module on \mathcal{H} . Let also ΩA be a DG \mathcal{H} -module algebra over A and ΓB be a DG \mathcal{H} -comodule algebra over B . We recall that a closed M -trace on ΩA is a linear map $\int : M \otimes \Omega A \rightarrow \mathbb{C}$ such that

$$\int (h_{(1)} m \otimes h_{(2)} \omega) = \epsilon(h) \int (m \otimes \omega), \quad (4.1)$$

$$\int (m \otimes d\omega) = 0, \quad (4.2)$$

$$\int (m \otimes \omega^1 \otimes \omega^2) = (-1)^{\deg(\omega^1)\deg(\omega^2)} \int (m_{\langle \bar{0} \rangle} \otimes S^{-1}(m_{\langle -1 \rangle}) \omega^2 \omega^1). \quad (4.3)$$

Similarly a closed M -trace on ΓB is defined as a linear map $\int : \Gamma B \rightarrow M$ such that,

$$\left(\int \gamma\right)_{\langle -1 \rangle} \otimes \left(\int \gamma\right)_{\langle 0 \rangle} = \gamma_{\langle -1 \rangle} \otimes \int(\gamma_{\langle 0 \rangle}), \quad (4.4)$$

$$\int(d\gamma) = 0, \quad (4.5)$$

$$\int(\gamma^1 \gamma^2) = \int(\gamma^2_{\langle 0 \rangle} \gamma^1) \gamma^2_{\langle -1 \rangle}. \quad (4.6)$$

One identifies closed cyclic cocycles $\phi \in C_{\mathcal{H}}^p(A, M)$ and $\psi \in {}^{\mathcal{H}} HC^q(B, M)$ with closed M -traces on $\Omega(A)$ and $\Gamma(B)$, the universal \mathcal{H} -module DG algebra and \mathcal{H} -comodule algebra respectively, as follows:

$$\int_{\phi} m \otimes a^0 da^1 \dots da^n = \phi(m \otimes a^0 \otimes \dots \otimes a^n) \quad (4.7)$$

$$\int_{\psi} (b^0 db^1 \dots db^n) = \psi(b^0 \otimes \dots \otimes b^n) \quad (4.8)$$

Then one forms a DG algebra over $A \rtimes B$ as the crossed product of $\Omega(A)$ and $\Gamma(B)$, which we denote it by $\Omega(A) \rtimes \Gamma(B)$. For any two closed M -traces \int_1 and \int_2 on $\Omega(A)$ and $\Gamma(B)$ one defines [11] the closed trace $\int_1 \cup \int_2$ on $\Omega(A) \rtimes \Gamma(B)$ by

$$\left(\int_1 \cup \int_2\right)(\omega \rtimes \eta) = \int_1 \left(\int_2(\omega) \otimes \eta\right), \quad (4.9)$$

and hence the cup product of two cyclic cocycle is defined by

$$\begin{aligned} (\phi \cup \psi)(a^0 \rtimes b^0 \otimes a^1 \rtimes b^1 \otimes \dots \otimes a^{p+q} \rtimes b^{p+q}) = \\ \left(\int_{\phi} \cup \int_{\psi}\right)(a^0 \rtimes b^0 d(a^1 \rtimes b^1) \dots d(a^{p+q} \rtimes b^{p+q})). \end{aligned}$$

Now we want to derive a formula for the above cup product. To this end we need to know the (p, q) component of the form

$$\theta^n = a^0 \rtimes b^0 d(a^1 \rtimes b^1) \dots d(a^n \rtimes b^n).$$

For a permutation $\sigma \in Sh(q, p)$ we use $\bar{\sigma}(i) = \sigma(i) - 1$, and $\hat{\sigma}(i) = \sigma(i) + 1$, and define the following $p + q$ form

$$\begin{aligned} \theta_\sigma^n &= a^0 (b^0 \dots b^{n-1})_{\langle -n \rangle} a^1 \dots (b^{\bar{\sigma}(q+1)-1} \dots b^{n-1})_{\langle -n+\sigma(q+1)-2 \rangle} a^{\bar{\sigma}(q+1)} \\ &d((b^{\bar{\sigma}(q+1)} \dots b^{n-1})_{\langle -n+\bar{\sigma}(q+1) \rangle} a^{\sigma(q+1)} \dots (b^{\bar{\sigma}(q+2)-1} \dots b^{n-1})_{\langle -n+\sigma(q+2)-2 \rangle} a^{\bar{\sigma}(q+2)}) \dots \\ &\dots d((b^{\bar{\sigma}(n)} \dots b^{n-1})_{\langle -n+\bar{\sigma}(n) \rangle} a^{\sigma(n)} \dots b^{n-1} \langle -1 \rangle a^n) \bowtie \\ &b^0_{\langle 0 \rangle} \dots b^{\bar{\sigma}(1)}_{\langle 0 \rangle} d(b^{\sigma(1)}_{\langle 0 \rangle} \dots b^{\bar{\sigma}(2)}_{\langle 0 \rangle}) \dots d(b^{\sigma(q)}_{\langle 0 \rangle} \dots b^{n-1} \langle 0 \rangle b^n). \end{aligned}$$

Lemma 4.1. *The (q, p) th component of the above form θ^{p+q} is given by the following formula.*

$$\sum_{\sigma \in Sh(q, p)} (-1)^\sigma \theta_\sigma^{p+q}.$$

Proof. We prove it by induction. Obviously it is true for $(p, q) = (0, 0)$. Assume that the lemma is true for all (p, q) such that $p + q = n$, we prove it for all (p, q) that $p + q = n + 1$.

The (p, q) th component of $a^0 \bowtie b^0 d(a^1 \bowtie b^1) \dots d(a^{p+q} \bowtie b^{p+q})$ is $\theta(da^{p+q} \bowtie b^{p+q}) + \theta'(a^{p+q} \bowtie db^{p+q})$, where θ and θ' are $(p-1, q)$ th, and $(p, q-1)$ th component of $a^0 \bowtie b^0 d(a^1 \bowtie b^1) \dots d(a^{p+q-1} \bowtie b^{p+q-1})$ respectively. Now let $\mu \in Sh(q, p-1)$, one observes that

$$\begin{aligned} &\sum_{\sigma \in Sh(q-1, p)} (-1)^\sigma \theta_\sigma^{p+q-1} (a^{p+q} \bowtie db^{p+q}) + \sum_{\sigma \in Sh(q, p-1)} (-1)^\sigma \theta_\sigma^{p+q-1} (da^{p+q} \bowtie b^{p+q}) = \\ &\sum_{\sigma \in Sh(q, p)} \sum_{\sigma(q)=p+q} (-1)^\sigma \theta_\sigma^{p+q} + \sum_{\sigma \in Sh(q, p)} \sum_{\sigma(p+q)=p+q} (-1)^\sigma \theta_\sigma^{p+q} = \\ &\sum_{\sigma \in Sh(q, p)} (-1)^\sigma \theta_\sigma^{p+q}. \end{aligned}$$

□

As a result one has the following formula for cup product via traces.

Proposition 4.2. *Let $\phi \in C_{\mathcal{H}}^p(A, M)$ and $\psi \in {}^{\mathcal{H}}C^q(B, M)$ respectively be two Hopf cyclic cocycles on A and B with coefficients in a SAYD module M . Then $\phi \cup \psi \in C^{p+q}(A \rtimes B)$ is a cyclic cocycle and is precisely given by the*

following formula

$$\begin{aligned}
(\phi \cup \psi)(a^0 \succleftarrow b^0 \otimes a^1 \succleftarrow b^1 \otimes \dots \otimes a^{p+q} \succleftarrow b^{p+q}) = \\
\sum_{\sigma \in Sh(q,p)} (-1)^\sigma \partial_{\bar{\sigma}(q)} \dots \partial_{\bar{\sigma}(1)} \phi(\partial_{\bar{\sigma}(q+p)} \dots \partial_{\bar{\sigma}(q+1)} \psi(b^0_{\langle 0 \rangle} \otimes \dots \otimes b^{p+q-1}_{\langle 0 \rangle} b^{p+q}) \otimes \\
a^0 \otimes b^0_{\langle -p-q \rangle} a^1 \otimes \dots \otimes b^0_{\langle -1 \rangle} \dots b^{p+q-1}_{\langle -1 \rangle} a^{p+q}).
\end{aligned}$$

Similarly by following [11], one uses cotraces on DG coalgebras to defined a cup product that generalizes characteristic map in Hopf cyclic cohomology. Indeed let C be a \mathcal{H} -module coalgebra and A be a \mathcal{H} -module algebra satisfying the conditions (2.6)...(2.9). Let also $\phi \in C_{\mathcal{H}}^q(A, M)$ and $x := m \otimes_{\mathcal{H}} c^0 \otimes \dots \otimes c^p \in C_{\mathcal{H}}^p(C, M)$ be Hopf cyclic cocycles. Then one gets the following cyclic cocycle $x \cup \phi$ on A by

$$\begin{aligned}
x \cup \phi(a_0 \otimes a_1 \otimes \dots \otimes a_{p+q}) = \\
\sum_{\sigma \in Sh(q,p)} (-1)^\sigma \partial_{\bar{\sigma}(q)} \dots \partial_{\bar{\sigma}(1)} \phi(\partial_{\bar{\sigma}(q+p)} \dots \partial_{\bar{\sigma}(q+1)} x(a_0 \otimes \dots \otimes a_{p+q})),
\end{aligned}$$

where $(m \otimes c^0 \otimes \dots \otimes c^n)(a^0 \otimes \dots \otimes a^n) := m \otimes c^0(a_0) \otimes \dots \otimes c^n(a_n)$.

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