

# Branching integrals and Casselman phenomenon

YURI A. NERETIN<sup>1</sup>

To Mark Iosifovich Graev in his 85 birthday

Let  $G$  be a real semisimple Lie group,  $K$  its maximal complex subgroup, and  $G_{\mathbb{C}}$  its complexification. It is known that all the  $K$ -finite matrix elements on  $G$  admit holomorphic continuation to branching functions on  $G_{\mathbb{C}}$  having singularities at the a prescribed divisor. We propose a geometric explanation of this phenomenon.

## 1 Introduction

**1.1. Casselman theorem.** Let  $G$  be a real semisimple Lie group, let  $K$  be the maximal compact subgroup. Let  $G_{\mathbb{C}}$  be the complexification of  $G$ .

Let  $\rho$  be an infinite-dimensional irreducible representation of  $G$  in a complete separable locally convex space  $W$ <sup>2</sup>. Recall that a vector  $w \in W$  is  $K$ -finite if the orbit  $\rho(G)v$  spans a finite dimensional subspace in  $W$ .<sup>3</sup>

A  $K$ -finite matrix element is a function on  $G$  of the form

$$f(g) = \ell(\rho(g)v)$$

where  $v$  is a  $K$ -finite vector in  $W$  and  $\ell$  is a  $K$ -finite linear functional, i.e., a  $K$ -finite element of the dual representation.

**Theorem 1.1**<sup>4</sup> *There is an (explicit) complex submanifold  $\Delta \subset G_{\mathbb{C}}$  of codimension 1 such that each  $K$ -finite matrix element of  $G$  admits a continuation to an analytic multi-valued branching function on  $G_{\mathbb{C}} \setminus \Delta$ .*

EXAMPLE. Let  $G = \mathrm{SL}(2, \mathbb{R})$  be the group of real matrices  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , whose determinant = 1. Then  $K = \mathrm{SO}(2)$  consists of matrices  $\begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}$ ,

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<sup>2</sup>the case of unitary representations in Hilbert spaces is sufficiently non-trivial.

<sup>3</sup>Let us rephrase the definition. We restrict  $\rho$  to the subgroup  $K$  and decompose the restriction into a direct sum  $\sum V_i$  of finite-dimensional representations of  $K$ . Finite sums of the form  $\sum_{v_j \in V_j} v_j$  are precisely all the  $K$ -finite vectors.

<sup>4</sup>Theorem was obtained in famous preprints of W.Casselman on the Subrepresentation Theorem. Unfortunately, these works are unavailable for author; however they are included to the paper of W.Casselman and Dr.Milicic [1]. There are (at least) two known proofs; the original proof is based on properties of system of partial differential equations for matrix elements [1], also the theorem can be reduced to properties of Heckman–Opdam hypergeometric functions [2] by a simple trick [3].

where  $\varphi \in \mathbb{R}$ ; the group  $G_{\mathbb{C}}$  is the group of complex  $2 \times 2$  matrices with determinant = 1. The submanifold  $\Delta \subset \mathrm{SL}(2, \mathbb{C})$  is a union of the following four manifolds

$$a = 0, \quad b = 0, \quad c = 0, \quad d = 0 \quad (1.1)$$

Indeed, in this case, there exists a canonical  $K$ -eigenbasis. All the matrix elements in this basis are Gauss hypergeometric functions of the form

$${}_2F_1(\alpha, \beta; \gamma; \theta), \quad \text{where } \theta = \frac{ad}{bc}$$

where the indices  $\alpha, \beta, \gamma$  depend on parameters of a representation and numbers of basis elements (see [6]).

Branching points of  ${}_2F_1$  are  $\theta = 0, 1, \infty$ . Since  $ad - bc = 1$ , only  $\theta = 0$  and  $\theta = \infty$  are admissible; this implies (1.1).  $\square$

Thus a representation  $\rho$  of a real semisimple group admits a continuation to an analytic matrix-valued function on  $G_{\mathbb{C}}$  having singularities at  $\Delta$ . This fact seems to be strange if we look to explicit constructions of representations.

Our purpose is to clarify this phenomenon and to find a direct geometric construction of the analytic continuation. We achieve this aim for a certain special case (namely, for principal maximally degenerate series of  $\mathrm{SL}(n, \mathbb{R})$ , see Section 2) and formulate a general conjecture (Section 3). It seems that our explanation (a reduction to the 'Thom isotopy Theorem'), see [4], [5] is trivial. However, as far as I know it is not known for experts in the representation theory.

Section 4 is informal and contains a brief exposition of various phenomena related to analytic continuations of representations. Addendum contains a general discussion of holomorphic continuations of representations.

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## 2 Isotopy of cycles

**2.1. Principal degenerate series for groups  $\mathrm{SL}(n, \mathbb{R})$ .** Let  $G = \mathrm{SL}(n, \mathbb{R})$  be the group of all real matrices with determinant = 1. The maximal compact subgroup  $K = \mathrm{SO}(n)$  is the group of all real orthogonal matrices.

Denote by  $\mathbb{R}\mathbb{P}^{n-1} \subset \mathbb{C}\mathbb{P}^{n-1}$  the real and complex projective spaces; recall that the manifold  $\mathbb{R}\mathbb{P}^{n-1}$  is orientable iff  $n$  is even.

Denote by  $d\omega$  the  $\mathrm{SO}(n)$ -invariant Lebesgue measure on  $\mathbb{R}\mathbb{P}^{n-1}$ , let  $d(\omega g)$  be its pushforward under the map  $g$ , denote by

$$J(g, x) := \frac{d\omega g}{d\omega}$$

the Jacobian of a transformation  $g$  at a point  $x$ .

Fix  $\alpha \in \mathbb{C}$ . Define a representation  $T_\alpha(g)$  of the group  $\mathrm{SL}(n, \mathbb{R})$  in the space  $C^\infty(\mathbb{R}\mathbb{P}^{n-1})$  by the formula

$$T_\alpha(g)f(x) = f(xg)J(g, x)^\alpha$$

The representations  $T_\alpha$  are called *representations of principal degenerate series*. If  $\alpha \in \frac{1}{2} + i\mathbb{R}$ , then this representation is unitary in  $L^2(\mathbb{R}\mathbb{P}^{n-1})$ .

**2.2. Discriminant submanifold  $\Delta$ .** Denote by  $g^t$  the transpose of a matrix  $g$ . Denote by  $\Delta$  the submanifold in  $\mathrm{SL}(n, \mathbb{C})$  consisting of matrices  $g$  such that the equation

$$\det(gg^t - \lambda) = 0$$

has a multiple root.

We wish to construct a continuation of the function  $g \mapsto T_\alpha(g)$  to a multi-valued function on  $\mathrm{SL}(n, \mathbb{C}) \setminus \Delta$ .

For simplicity, we assume  $n$  is even.<sup>5</sup>

**2.3. Invariant measure.** Denote by  $x_1 : x_2 : \dots : x_n$  the homogeneous coordinates in a projective space. The  $\mathrm{SO}(n)$ -invariant  $(n-1)$ -form on  $\mathbb{R}\mathbb{P}^{n-1}$  is given by

$$d\omega(x) = \left(\sum_j x_j^2\right)^{-n/2} \sum_j (-1)^j x_j dx_1 \dots \widehat{dx_j} \dots dx_n$$

This expression can be regarded as a meromorphic  $(n-1)$ -form on  $\mathbb{C}\mathbb{P}^{n-1}$  having a pole on the quadric

$$Q(x) := \sum x_j^2 = 0$$

Now we can treat the Jacobian  $J(g, x)$  as a meromorphic function on  $\mathbb{C}\mathbb{P}^{n-1}$  having a zero at the quadric  $Q(x) = 0$  and a pole on the shifted quadric  $Q(gx) = 0$ .

**2.4.  $K$ -finite functions.** The following functions span the space of  $K$ -finite functions on  $\mathbb{R}\mathbb{P}^{n-1}$ :

$$f(x) = \frac{\prod x_j^{k_j}}{(\sum x_j^2)^{\sum k_j/2}}, \quad \text{where } \sum k_j \text{ is even}$$

Evidently, they have singularities at the quadric  $Q(x) = 0$  mentioned above.

**2.5.  $K$ -finite matrix elements.**  $K$ -finite matrix elements are given by formula

$$\{f_1, f_2\} = \int_{\mathbb{R}\mathbb{P}^{n-1}} f_1(x)f_2(xg)J(g, x)^\alpha d\omega(x) \quad (2.1)$$

The integrand is a holomorphic form on  $\mathbb{C}\mathbb{P}^{n-1}$  of the maximal degree ramified over quadrics  $Q(x) = 0$ ,  $Q(xg) = 0$ . Denote by  $\mathfrak{U} = \mathfrak{U}[g]$  the complement to

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<sup>5</sup>If  $n$  is odd, then we must replace the integrand in (2.1) by a form on two sheet covering of  $\mathbb{C}\mathbb{P}^{n-1} \setminus \mathbb{R}\mathbb{P}^{n-1}$ . Also we must replace the cycle  $\mathbb{R}\mathbb{P}^{n-1}$  by its two-sheet covering.

these quadrics. Therefore locally in  $\mathfrak{U}$  the integrand is a closed  $(n - 1)$ -form. Hence we can replace  $\mathbb{R}\mathbb{P}^{n-1}$  by an arbitrary isotopic cycle  $C$  in  $\mathfrak{U}$ .

**2.6. Reduction to Pham Theorem.** Now let  $g(s)$  be a path in  $\mathrm{SL}(n, \mathbb{C})$  starting in  $\mathrm{SL}(n, \mathbb{R})$ . For each  $s$  one has a pair  $Q(x) = 0$ ,  $Q(x \cdot g(s)) = 0$  of quadrics and the corresponding complement  $\mathfrak{U}(g(s))$ .

*Is it possible to construct an isotopy  $C(s)$  of the cycle  $\mathbb{R}\mathbb{P}^{n-1}$  such that  $C(s) \subset \mathfrak{U}(g(s))$  for all  $s$ ?*

Now recall the following Pham theorem (see F.Pham [4]), V.A.Vasiliev [5]).

**Theorem 2.1** *Let  $R_1(s), \dots, R_l(s)$  be nonsingular complex hypersurfaces in  $\mathbb{C}\mathbb{P}^k$  depending on a parameter. Assume that  $R_j$  are transversal (at all points for all values of the parameter  $s$ ). Then each cycle in the complement to  $\cup R_j(s)$  admits an isotopy according the parameter.*

### 2.7. Transversality of quadrics.

**Lemma 2.2** *Let  $A, B$  be non-degenerate symmetric matrices. Assume that all the roots of the characteristic equation*

$$\det(A - \lambda B) = 0$$

*are pairwise distinct. Then quadrics  $\sum a_{ij}x_ix_j = 0$  and  $\sum b_{ij}x_ix_j = 0$  are transversal.*

By the Weierstrass theorem such pair of quadrics can be reduced to

$$\sum \lambda_j x_j^2 = 0, \quad \sum x_j^2 = 0 \tag{2.2}$$

where  $\lambda_j$  are the roots of the characteristic equation. If they are not transversal at a point  $x$ , then rank of the matrix

$$\begin{pmatrix} \lambda_1 x_1 & \dots & \lambda_n x_n \\ x_1 & \dots & x_n \end{pmatrix}$$

is 1. Therefore

$$(\lambda_i - \lambda_j)x_ix_j = 0 \quad \text{for all } i, j \tag{2.3}$$

The system (2.3), (2.2) is inconsistent.  $\square$

**2.8. Last step.** In our case, the matrices of quadratic forms are  $gg^t$  and 1. Therefore, by the virtue of the Pham Theorem a desired isotopy of the cycle  $\mathbb{R}\mathbb{P}^{n-1}$  exists.

## 3 General case

By the Subrepresentation Theorem, all the irreducible representations of a semisimple group  $G$  are subrepresentations of the principal (generally, non-unitary) series. Therefore, it suffices to construct analytic continuations for representations of the principal series.

For definiteness, we discuss the spherical principal series of the group  $G = \mathrm{SL}(n, \mathbb{R})$ .

**3.1. Spherical principal series for  $G = \mathrm{SL}(n, \mathbb{R})$ .** Denote by  $\mathrm{Fl}(\mathbb{R}^n)$  the space of all complete flags of subspaces

$$\mathcal{W} : 0 \subset W_1 \subset \cdots \subset W_{n-1} \subset \mathbb{R}^n$$

in  $\mathbb{R}^n$ ; here  $\dim W_k = k$ . By  $\mathrm{Gr}_k(\mathbb{R}^n)$  we denote the Grassmannian of all  $k$ -dimensional subspaces in  $\mathbb{R}^n$ . By  $\gamma_k$  we denote the natural projection  $\mathrm{Fl}(\mathbb{R}^n) \rightarrow \mathrm{Gr}_k(\mathbb{R}^n)$ .

By  $\omega_k$  we denote the  $\mathrm{SO}(n)$ -invariant measure on  $\mathrm{Gr}_k(\mathbb{R}^n)$ . For  $g \in \mathrm{GL}(n, \mathbb{R})$  we denote by  $J_k(g, V)$  the Jacobian of the transformation  $V \mapsto Vg$  of  $\mathrm{Gr}_k(\mathbb{R}^n)$ ,

$$J_k(g, V) = \frac{d\omega_k(Vg)}{d\omega_k(V)}$$

Fix  $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{C}$ . The representation  $T_\alpha$  of the spherical principal series of the group  $\mathrm{SL}(n, \mathbb{R})$  acts in the space  $C^\infty(\mathrm{Fl}(\mathbb{R}^n))$  by the formula

$$T_\alpha(g)f(\mathcal{W}) = f(\mathcal{W} \cdot g) \prod_{k=1}^{n-1} J_k(g, \gamma_k(\mathcal{W}))^{\alpha_k}$$

**3.2. Singularities.** Consider the symmetric bilinear form in  $\mathbb{C}^n$  given by

$$B(x, y) = \sum x_j y_j$$

By  $L_k \subset \mathrm{Gr}_k(\mathbb{C}^n)$  we denote the set of all the  $k$ -dimensional subspaces, where the form  $B$  is degenerate<sup>6</sup>. By  $\mathcal{L} \subset \mathrm{Fl}(\mathbb{C}^n)$  we denote the set of all the flags  $W_1 \subset \cdots \subset W_{n-1}$ , where  $W_k \in L_k$  for some  $k$ .

In fact, all the  $K$ -finite functions on  $\mathrm{Fl}(\mathbb{R}^n)$  admit analytic continuations to  $\mathrm{Fl}(\mathbb{C}^n) \setminus \mathcal{L}$  (a singularity on  $\mathcal{L}$  is a pole or two-sheet branching).

**3.3. A conjecture.**

**Conjecture 3.1** *Let  $\gamma(t)$  be a path on  $\mathrm{GL}(n, \mathbb{C})$  avoiding the discriminant submanifold  $\Delta$ , let  $\gamma(0) \in \mathrm{SL}(n, \mathbb{R})$ . Then there is an isotopy  $C(t)$  of the cycle  $\mathrm{Fl}(\mathbb{R}^n)$  in the space  $\mathrm{Fl}(\mathbb{C}^n)$  avoiding the submanifolds  $\mathcal{L}$  and  $\mathcal{L} \cdot g(s)$*

Such isotopy produces an analytic continuation of representations of principal series of  $\mathrm{SL}(n, \mathbb{R})$ .

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<sup>6</sup>Equivalently, we can consider all the  $(k-1)$ -dimensional subspaces in  $\mathbb{C}\mathbb{P}^{n-1}$  tangent to the quadric  $\sum x_j^2 = 0$ .

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## Addendum. Survey of holomorphic continuations of representations

Let  $G$  be a connected linear Lie group, i.e., a Lie group admitting an embedding to a matrix group. Denote by  $G_{\mathbb{C}}$  its complexification. Let  $\rho$  be an irreducible representation of  $G$  (in a Frechet space). We are interested in the following problems:

- Is it possible to extend  $\rho$  holomorphically to  $G_{\mathbb{C}}$ ?
  - Is it possible to extend  $\rho$  holomorphically to an open domain  $U \subset G_{\mathbb{C}}$ .
- See, also, [16], Section 1.5.

**A.1. Weyl trick.** Let  $\rho$  be a finite dimensional representation of a semisimple Lie group  $G$ . Then  $\rho$  admits the holomorphic continuation to the group  $G_{\mathbb{C}}$ .

**A.2. Why the Weyl trick does not survive for infinite-dimensional unitary representations?** Let  $G$  be a noncompact Lie group, let  $\rho$  be its irreducible faithful unitary representation. Let  $X$  be a noncentral element of the Lie algebra  $\mathfrak{g}$ . It is more-or-less obvious that the operator  $\rho(X)$  is unbounded<sup>7</sup>.

Then, for  $t, s \in \mathbb{R}$ ,

$$\rho(\exp(t + is)X) = \exp(isX) \exp(tX)$$

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<sup>7</sup>I propose a collection of arguments that can be combined to a proof in various way. First, this is valid for the two-dimensional non-Abelian algebra (and therefore this is valid for elements of  $\mathfrak{g}$  that can be included to such a subalgebra). Second, this is valid for nilpotent algebras, we apply Kirillov’s monomial induction theorem, see [8], then the Lie algebra acts by first order differential operators. which are obviously unbounded. Third, for we know that for semisimple Lie groups  $K$ -spectra are always unbounded. Fourth, elements  $X$  with bounded  $\rho(X)$  form an ideal. In fifth, we can consider its normalizer

Since  $iX$  is self-adjoint, then  $\exp(tX)$  is unitary; on the other hand  $\exp(isX)$  have to be unbounded for all positive  $s$  or for all negative  $s$  (and usually it is unbounded for all  $s$ ).

However, this argument does not removes completely an idea of holomorphic continuation, since it remain two following logical possibilities

- a holomorphic extension exists in spite of the unboundedness of operators.
- If a specter of  $X$  is contained on a positive half-line, then  $\exp(tX)$  is defined for negative  $t$ . We can hope to construct something from elements of this kind.

The second variant is realized for Olshanski semigroups, see below, the first variant is general, this follows from the Nelson Theorem.

**A.3. Nelson's paper.** In 1959 E.Nelson [13] proved that each unitary irreducible representation  $\rho$  of a real Lie group  $G$  has a dense set of analytic vectors. This implies that  $\rho$  can be extended analytically to a sufficiently small neighborhood of  $G$  in  $G_{\mathbb{C}}$ .

However, usually this continuation can be done in a constructive way as it is explained below (see also [16], Section 1.5.).

**A.4. Induced representations.** First, we recall a definition of induced representations.

Consider a Lie group  $G$  and its closed connected subgroup  $H$ . Let  $\rho$  be a representation of  $H$  in a *finite-dimensional* complex space  $V$ .

These data allow to construct canonically a vector bundle (*skew product*) over  $G/H$  with a fiber  $V$ . Recall a construction (see, for instance, [8]). Consider the direct product  $G \times V$  and the equivalence relation

$$(g, v) \sim (gh^{-1}, \rho(h)v), \quad \text{where } g \in G, v \in V, h \in H$$

Denote by  $R = G \times_V H$  the quotient space. The standard map  $G \rightarrow G/H$  determines a map  $R \rightarrow G/H$  (we simply forget  $v$ ). A fiber can be (noncanonically) identified with  $V$ .

Next, the group  $G$  acts on  $G \times V$  by transformations

$$(g, v) \mapsto (rg, v), \quad \text{where } g \in G, v \in V, r \in G$$

This action induces the action of  $G$  on  $G \times_H V$ , the last action commutes with the projection  $G \times_H V \rightarrow G/H$ .

Therefore  $G$  acts in the space of sections of the bundle  $G \times_H V \rightarrow G/H$  (because graph of a section is a subset in the total space; the group  $G$  simply move subsets). *Induced representation*  $\pi = \text{Ind}_H^G(\rho)$  is the representation of  $G$  in a space of sections of the bundle  $G \times_H V \rightarrow G/H$ .

The most important example are principal series, which were partially discussed above.

Our definition is not satisfactory since rather often it is necessary to specify the space of sections (for instance, smooth functions,  $L^2$ -functions, distributions, etc.). This discussion is far beyond our purposes, for the moment let us consider the space  $C^\infty[G/H; \rho]$  of smooth sections.

**A.5. Analytic continuation of induced representations.** Here we discuss some heuristic arguments (see [16], Section 1.5). Their actual usage depends on explicit situation.

Denote by  $H_{[\mathbb{C}]}$  the complexification of the group  $H$  inside  $G_{\mathbb{C}}$ .<sup>8</sup>

The (finite-dimensional) representation  $\rho$  admits a holomorphic extension to a representation of the universal covering group of  $\tilde{H}_{[\mathbb{C}]}$  of  $H_{[\mathbb{C}]}$  in the space  $V$ . For a moment, let us require two assumptions<sup>9</sup>

- $H_{[\mathbb{C}]}$  is closed in  $G_{\mathbb{C}}$ .
- $\rho$  is a linear representation of  $H_{[\mathbb{C}]}$ .

Under these assumptions, the same construction of skew-product produces the bundle  $(G_{\mathbb{C}}) \times_{H_{\mathbb{C}}} V \rightarrow G_{\mathbb{C}}/H_{\mathbb{C}}$ . Moreover,

$$G \times_H V \subset (G_{\mathbb{C}}) \times_{H_{\mathbb{C}}} V$$

Now let us agree with the next assumption<sup>10</sup>. *Let the space of holomorphic sections of  $(G_{\mathbb{C}}) \times_{H_{\mathbb{C}}} V$  be dense in  $C^\infty$  on  $G/H$ .* Then we get a holomorphic continuation of the induced representation  $\pi$  to the whole complex group  $G_{\mathbb{C}}$ . More precisely, we slightly reduce the space of representation, but 'formulae' determining a representation are the same.

This variant is realized for all the nilpotent Lie groups.

**A.6. Nilpotent Lie groups.**

EXAMPLE. Let  $a, b, c$  range in  $\mathbb{R}$ , Consider operators  $T(a, b, c)$  in  $L^2(\mathbb{R})$  given by

$$T(a, b, c)f(x) = f(x + a)e^{ibx+c}$$

They form a 3-dimensional group, namely the Heisenberg group. Now let  $a, b, c$  range in  $\mathbb{C}$  and  $f$  ranges in the space  $Hol(\mathbb{C})$  of entire functions. Then the same formula determines a representation of the complex Heisenberg group in  $Hol(\mathbb{C})$ . After this operation, the space of representation was completely changed. However it is easy to invent a dense subspace in  $L^2(\mathbb{R})$  consisting of holomorphic functions and invariant with respect to all the operators  $T(a, b, c)$ .  $\square$

Now, let  $G$  be a nilpotent Lie group. By the Kirillov Theorem [7], each unitary representation of a nilpotent Lie group  $G$  is induced from an one-dimensional representation of a subgroup  $H$ ; the manifolds  $G/H$  are equivalent to standard spaces  $\mathbb{R}^m$ . Therefore,  $G_{\mathbb{C}}/H_{\mathbb{C}}$  is the standard complex space  $\mathbb{C}^m$ , and we get a representation of  $G_{\mathbb{C}}$  in the space of entire functions.

However, the following R.Goodman–G.L.Litvinov Theorem (R.Goodman [4], G.L.Litvinov [11], [12]) is more delicate.

**Theorem.** *Let  $\rho$  be an irreducible unitary representation of a nilpotent group  $G$  in a Hilbert space  $W$ . There exists a (noncanonical) dense subspace  $Y$  with*

<sup>8</sup>For elements  $X$  of the Lie algebra of  $H$  we consider the subgroup in  $G_{\mathbb{C}}$  spanned by all  $\exp((t + is)X)$ .

<sup>9</sup>The second assumption is very restrictive. It is not hold for the parabolic induction.

<sup>10</sup>If it is not hold, then we go to subsection A.7, where all the assumptions are omitted.



its own Frechet topology and holomorphic representation  $\tilde{\rho}$  of  $G$  in the space  $W$  coinciding with  $\rho$  on  $G$ .

Let us explain how to produce a subspace  $Y$ . Let  $\rho$  be a unitary representation of a nilpotent group  $G$  in a space  $W$ . Let  $f$  be an entire function on  $G_{\mathbb{C}}$  (it is specified below). Consider the operator

$$\rho(f) = \int_G f(g)\rho(g) dg$$

Let  $r \in G_{\mathbb{C}}$ . We write formally

$$\rho(r)\rho(f) = \int_G f(g)\rho(rg) dg = \int_G f(gr^{-1})\rho(g)$$

Assume that for each  $r \in G_{\mathbb{C}}$  the function  $\gamma_r(g) := f(gr^{-1})$  is integrable on  $G$ . Under this condition we can define operators

$$\rho(r) : \{\text{Image of } \rho(f)\} \rightarrow W$$

as just now.

In fact, we need a subspace  $Z$  in  $L^1(G)$  consisting of entire functions and invariant with respect to complex shifts. To be sure that

$$Y := \cup_{f \in Z} \text{Im}(\rho(f)) \subset W$$

is dense, we need in a sequence of positive  $f_j \in Z$  converging to  $\delta$ -functions; then  $\rho(f_j)v$  converges to  $v$  for all  $v \in W$ . In what follows we describe a simple trick that allows to construct many functions  $f$  and a subspace  $Z$ .

First, let  $G = T_n$  be the unipotent upper triangular subgroup of order  $n$ . Let  $t_{ij}$ ,  $i < j$ , be the natural coordinates on  $T_n$ . Write them in the order

$$t_{12}, t_{23}, t_{34}, \dots, t_{(n-1)n}, t_{13}, t_{24}, \dots, t_{(n-2)n}, t_{14}, \dots$$

and re-denote these coordinates by  $x_1, x_2, x_3, \dots$ . In this coordinates, the right shift  $g \mapsto gr^{-1}$  is an affine transformation of the form

$$(x_1, x_2, x_3, \dots) \mapsto (x_1 + a_1, x_2 + a_2 + b_{21}x_1, x_3 + a_3 + b_{31}x_1 + b_{32}x_2, \dots)$$

Now we can choose a desired function  $f$  in the form

$$f(x) = \exp\left\{-\sum p_j x_j^{2k_j}\right\}, \quad \text{where } p_j > 0 \text{ and } k_1 > k_2 > \dots > 0 \text{ are integers}$$

For an arbitrary nilpotent  $G$  we apply the Ado theorem (in fact, the standard proof, see [17]) produces a polynomial embedding of  $G$  to some  $T_n$  with very large  $n$ ).

**A.7. Local holomorphic continuations.** Now let  $G \supset H$  be arbitrary.

The construction of a skew product survives locally. It determines a holomorphic bundle on a (noncanonical) neighborhood  $U \subset G_{\mathbb{C}}/H_{\mathbb{C}}$  of  $G/H$ . Denote

by  $\mathcal{A}(U)$  the space of holomorphic sections of this bundle. Let  $r \in G_{\mathbb{C}}$  satisfies  $r \cdot G/H \subset U$ . The  $r$  induces an operator  $\pi(r) : \mathcal{A}(U) \rightarrow C^\infty$  and we obtain an approach to local analytic continuation induced representation.

**A.8. Local analytic continuations for semisimple groups.** For definiteness, consider  $G = \mathrm{SL}(n, \mathbb{R})$ . Denote by  $P$  be the minimal parabolic (i.e.,  $P$  is the group of upper triangular matrices), Then  $G/P$  is the flag space mentioned above. Next,  $G_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$ , and  $P_{[\mathbb{C}]}$  is the group of complex upper-triangular matrices of the form

$$B = \begin{pmatrix} b_{11} & b_{12} & \dots \\ 0 & b_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \quad a_{ii} \neq 0, \text{ other } a_{ij} \text{ are arbitrary.}$$

Evidently, the group  $P_{\mathbb{C}}$  is not simply connected.

Fix  $s_j \in \mathbb{C}$  and consider the one-dimensional character

$$\chi_s(B) := \prod_{j=1}^n b_{jj}^{s_j}$$

of  $P$ . The function  $\chi_s$  is defined on  $P_{\mathbb{C}}$  only locally, however this is sufficient for the arguments of the previous subsection.

This kind of arguments can be easily applied to an arbitrary representation of a nondegenerate principal series. Keeping in the mind Subquotient Theorem, we easily get the following statement.

**Observation 3.1** *Let  $G$  be a semisimple Lie group. Then there are (non-canonical) open sets  $U_1 \subset U_2 \subset G_{\mathbb{C}}$  containing  $G$  such that  $U_2 \supset U_1 \cdot U_1$  and the following property holds. Let  $\rho$  be an irreducible representation of  $G$  in a Frechet space  $W$ . Then there is a (noncanonical) dense subspace  $Y \subset W$  (equipped with its own Frechet topology) and an operator-valued holomorphic function  $\tilde{\rho} : U_2 \rightarrow \mathrm{Hom}(Y, W)$  such that  $\tilde{\rho} = \rho$  on  $G$  and*

$$\tilde{\rho}(g_1)\tilde{\rho}(g_2)y = \tilde{\rho}(g_1g_2)y \quad \text{for } g_1, g_2 \in U_1, y \in Y$$

Certainly, the operators  $\tilde{\rho}(g)$  are unbounded in the topology of  $\mathrm{Hom}(W, W)$ .

**A.9. Crown.** D.N.Akhiezer and S.G.Gindikin (see [1]) constructed a certain explicit domain  $\mathcal{A} \subset G_{\mathbb{C}}$  ('crown') to which all the spherical functions of a real semisimple group  $G$  can be extended. Also the crown is a domain of holomorphy of all irreducible representations of  $G$ , see B.Krotz, R.Stanton, [9], [10].

Relation of their constructions with our previous considerations are not completely clear.

**A.10. Olshanski semigroup.** In all the previous examples, operators of holomorphic continuation are unbounded in the initial topology. There is an important exception.

Unitary highest weight representations of a semisimple Lie group  $G$  admit holomorphic continuations to a certain subsemigroup  $\Gamma \subset G_{\mathbb{C}}$  (M.I.Graev [6], G.I.Olshanski [19]). Since this situation is well understood, we omit further discussion, see also [18].

**A.11. Infinite-dimensional groups.** Induction (in different variants) is the main tool of construction of representations of Lie groups.

For infinite-dimensional groups the induction exists<sup>11</sup> but it is a secondary tool (however, the algebraic variant of induction is important for infinite dimensional Lie algebras). A more effective instrument are symplectic and orthogonal spinors.

Let us realize the standard real orthogonal group  $O(2n)$  as a group of  $(n+n) \times (n+n)$  complex matrices  $g$  having the structure

$$g = \begin{pmatrix} \Phi & \Psi \\ \overline{\Psi} & \overline{\Phi} \end{pmatrix}$$

that are orthogonal in the following sense

$$g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g^t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Actually we do the following. Consider the space  $\mathbb{R}^{2n}$  equipped with a standard basis  $e_1, \dots, e_n, e_{n+1}, \dots, e_{2n}$ . Then we pass to the space  $\mathbb{C}^{2n}$  and write matrices of real orthogonal operators in the basis

$$e_1 + ie_{n+1}, e_2 + ie_{n+2}, \dots, e_n + ie_{2n}, e_1 - ie_{n+1}, e_2 - ie_{n+2}, \dots, e_n - ie_{2n}$$

Next, we put  $n = \infty$ . Denote by  $OU(2\infty)$  the group of all the bounded matrices of the same structure satisfying an additional condition:  $\Psi$  is a Hilbert–Schmidt matrix (i.e., the sum  $\sum |\psi_{kl}|^2$  of squares of matrix elements is finite).

By the well-known theorem of F.A.Berezin [2], [3] and D.Shale–W.Stinespring [20], the spinor representation is well-defined on the group  $OU(2\infty)$ .

Numerous infinite dimensional groups  $G$  can be embedded in a natural way to  $OU(2\infty)$ , after this we can restrict the spinor representation to  $G$ . For instance, for the loop group  $C^\infty(S^1, SO(2n))$  we consider the natural action in the space  $L^2(S^1, \mathbb{R}^{2n})$  and define the operator of the complex structure in this space via Hilbert transform (see, for instance, [16]). Applying the spinor representation, we get the so-called basic representation of the loop group. For production of other highest weight representations we apply restrictions and tensoring.

However, the group  $OU(2\infty)$  admits a complexification  $OGL(2\infty, \mathbb{C})$  consisting of complex matrices  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  that are orthogonal in the same sense with Hilbert–Schmidt blocks  $B$  and  $C$ . The spinor representation of  $OU(2\infty)$  admits a holomorphic continuation to the complex group  $OGL(2\infty)$ , see [14],

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<sup>11</sup>If a group acts on the space with measure, then it acts in  $L^2$ .

[16], the operators of continuation are unbounded in the initial topology, but are bounded on a certain dense Frechet subspace equipped with its own topology.

This produces highest weight representation of complex loop groups as free byproducts (see another approach in R.Goodman, N.Wallach [5]).

More interesting phenomenon arises for the group Diff of diffeomorphisms of circle, in this case the analytic continuation exists in spite of nonexistence  $\text{Diff}_{\mathbb{C}}$ , see [15].

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Math.Dept., University of Vienna,  
 Nordbergstrasse, 15, Vienna, Austria  
 &  
 Institute for Theoretical and Experimental Physics,  
 Bolshaya Cheremushkinskaya, 25, Moscow 117259, Russia  
 e-mail: [neretin@iccme.ru](mailto:neretin@iccme.ru)  
 URL:[www.mat.univie.ac.at/~neretin](http://www.mat.univie.ac.at/~neretin)