

On Cross-Ratio Distortion and Schwarz Derivative

A. Teplinsky*

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Abstract

The asymptotical estimates for the cross-ratio distortion with respect to a smooth monotone function of one variable in terms of its Schwarz derivative are established.

1 Introduction, Definitions and Results

Though the concept of cross-ratio of four consecutively connected segments has its origin in elementary geometry, the question about good estimates on cross-ratio distortion with respect to a smooth function of one variable arose in connection with studies in one-dimensional dynamics. Such tools were developed and applied first to a great success in [1] to the case of critical circle maps and in [2] to the case of unimodal interval maps. They also played an important role in [3] (although dated before [2], it refers to both [1] and a preprint version of [2]). However natural it would seem to apply such tools to circle diffeomorphisms, it was not done before the very recent works [4, 5], where some of the classical results of Herman's theory [6, 7, 8, 9] were re-proven and even strengthened due, in part, namely to a thorough investigation of the asymptotic expansions for cross-ratio distortion. The aim of this short paper is to prove optimal asymptotic estimates for cross-ratio distortion for both smooth and holomorphic cases without referring to one-dimensional dynamics, just in the elementary calculus framework. (Moreover, the only tool from the calculus we use is the Taylor's formula with the remainder term in asymptotic form.)

Let us start with the definitions. It is more convenient to talk about ratios and cross-ratios of points rather than segments.

The *ratio* of three pairwise distinct points x_1, x_2, x_3 is

$$R(x_1, x_2, x_3) = \frac{x_1 - x_2}{x_2 - x_3},$$

and the *ratio distortion* of those points with respect to the function f is

$$D(x_1, x_2, x_3; f) = \frac{R(f(x_1), f(x_2), f(x_3))}{R(x_1, x_2, x_3)} = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \cdot \frac{f(x_2) - f(x_3)}{x_2 - x_3}.$$

The *cross-ratio* of four pairwise distinct points x_1, x_2, x_3, x_4 is

$$\text{Cr}(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_2 - x_3)(x_4 - x_1)},$$

*Institute of Mathematics, Kiev, Ukraine

whereas the *cross-ratio distortion* of those points with respect to f is

$$\begin{aligned} \text{Dist}(x_1, x_2, x_3, x_4; f) &= \frac{\text{Cr}(f(x_1), f(x_2), f(x_3), f(x_4))}{\text{Cr}(x_1, x_2, x_3, x_4)} \\ &= \frac{f(x_1) - f(x_2)}{x_1 - x_2} \cdot \frac{f(x_2) - f(x_3)}{x_2 - x_3} \cdot \frac{f(x_3) - f(x_4)}{x_3 - x_4} \cdot \frac{f(x_4) - f(x_1)}{x_4 - x_1}. \end{aligned}$$

If the function f is differentiable and its first derivative does not have zeros, then both ratio and cross-ratio distortions are defined for not pairwise distinct points as well. Namely, these distortions can be defined as the appropriate limits, or just by formally substituting $f'(a)$ for $\frac{f(a)-f(a)}{a-a}$ in the definitions above. It is obvious that either $x_1 = x_3$ or $x_2 = x_4$ implies $\text{Dist}(x_1, x_2, x_3, x_4; f) = 1$.

As we find, the leading terms in the asymptotic expansion for cross-ratio distortion are directly related to the expression called ‘Schwarz derivative’ that manifests itself in many considerations of one-dimensional real and complex dynamics. The *Schwarz derivative*, or *Schwarzian*, of a three times differentiable function f at a point x is given by

$$\mathcal{S}f(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left(\frac{f''(x)}{f'(x)} \right)^2$$

as soon as that $f'(x) \neq 0$. The connection between cross-ratio distortion and Schwarzian becomes evident if one considers the two well-known facts about linear-fractional functions (a.k.a. ‘Moebius transformations’): on one hand, f is fractional-linear on $[A, B]$ if and only if $\mathcal{S}f \equiv 0$ on $[A, B]$; on the other, f is fractional-linear on $[A, B]$ if and only if the cross-ratio distortion of any four points from $[A, B]$ with respect to f is equal to 1. Thus both Schwarzian and cross-ratio distortion in a sense measure how far is the function f from being fractional-linear. This is similar to the relation between the second derivative, ratio distortion and non-linearity of a function. (A review of elementary facts known about cross-ratios and Schwarzians can be found in [2].)

Now we are ready to formulate our results. They are presented in a series of four estimates related to different degrees of smoothness: the first one applies to the case of smoothness C^2 and higher, the second one to C^3 and higher, the third one to C^4 and higher, and the last one to the holomorphic case. Let us remind that a domain $\Omega \subset \mathbb{C}$ is called *quasiconvex* if there exists a constant $\Lambda \geq 1$ such that for any two points $a, b \in \Omega$ there exists a simple curve connecting them such that its length does not exceed $\Lambda|a - b|$.

Note, that all the implicit constants, which are presented throughout this paper in the form of $\mathcal{O}(\cdot)$, depend on the function f and its segment of definition $[A, B]$ only (in the smooth case) or on the function F and a chosen compact subset of its domain of definition Ω only (in the holomorphic case). For a (finite) set M , by $\text{diam}M$ we denote its diameter, i.e. the greatest distance between its points.

Theorem 1. *Let $f \in C^r([A, B])$, and f' does not have zeroes on $[A, B]$. Consider four arbitrary points $x_1, x_2, x_3, x_4 \in [A, B]$ and denote $\Delta = \text{diam}\{x_1, x_2, x_3, x_4\}$. The stated below asymptotic estimates hold true.*

In the case of $r = 2 + \alpha$, $\alpha \in [0, 1]$, $\Delta \neq 0$:

$$\text{Dist}(x_1, x_2, x_3, x_4; f) = 1 + (x_1 - x_3)(x_2 - x_4)\mathcal{O}(\Delta^{\alpha-1}). \quad (1)$$

In the case of $r = 3 + \beta$, $\beta \in [0, 1]$:

$$\text{Dist}(x_1, x_2, x_3, x_4; f) = 1 + (x_1 - x_3)(x_2 - x_4) \left(\frac{1}{6} \mathcal{S}f(\theta) + \mathcal{O}(\Delta^\beta) \right) \quad (2)$$

with arbitrary $\theta \in [\min\{x_1, x_2, x_3, x_4\}, \max\{x_1, x_2, x_3, x_4\}]$.

In the case of $r = 4 + \gamma$, $\gamma \in [0, 1]$:

$$\begin{aligned} \text{Dist}(x_1, x_2, x_3, x_4; f) &= 1 + (x_1 - x_3)(x_2 - x_4) \\ &\times \left(\frac{1}{24} (\mathcal{S}f(x_1) + \mathcal{S}f(x_2) + \mathcal{S}f(x_3) + \mathcal{S}f(x_4)) + \mathcal{O}(\Delta^{1+\gamma}) \right). \end{aligned} \quad (3)$$

Let F be a holomorphic function defined on a quasiconvex domain $\Omega \subset \mathbb{C}$ such that F' does not have zeroes in Ω . Uniformly on compact subsets of Ω , the following asymptotic estimate holds true:

$$\begin{aligned} \text{Dist}(z_1, z_2, z_3, z_4; F) &= 1 + (z_1 - z_3)(z_2 - z_4) \\ &\times \left(\frac{1}{24} (\mathcal{S}F(z_1) + \mathcal{S}F(z_2) + \mathcal{S}F(z_3) + \mathcal{S}F(z_4)) + \mathcal{O}(\Delta^2) \right), \end{aligned} \quad (4)$$

where in this case $\Delta = \text{diam}\{z_1, z_2, z_3, z_4\}$.

Remark 1. We wish to stress it straight away that the leading terms in this asymptotic expansion are not too hard to derive by themselves, whereas the proof that the remainder term for $f \in C^r$ is $(x_1 - x_3)(x_2 - x_4)\mathcal{O}(\Delta^{r-3})$ rather than just $\mathcal{O}(\Delta^{r-1})$ is far from obvious (and it is clear that the distances $|x_1 - x_3|$ and $|x_2 - x_4|$ can be much smaller than Δ). A similar remark applies to the holomorphic case.

2 Proof of Theorem 1

Here we will consider the case $f \in C^{4+\gamma}([A, B])$, $\gamma \in [0, 1]$, and prove the estimate (3). As it will become evident, the proofs of (1), (2) and (4) follow the same lines with very slight modifications.

Let us introduce notations $\phi_k = \frac{f^{(k+1)}(\theta)}{(k+1)!f'(\theta)}$ and $d_i = x_i - \theta$. Let x_1, x_2, θ be arbitrary points from the segment $[A, B]$. It is easy to derive from the Taylor's expansions for $f(x_1)$ $f(x_2)$ with respect to the reference point θ that

$$\frac{f(x_1) - f(x_2)}{f'(\theta)(x_1 - x_2)} = 1 + P_1 + P_2 + P_3 + \mathcal{O}((\text{diam}\{x_1, x_2, \theta\})^{3+\gamma}), \quad (5)$$

where $P_k = \phi_k \frac{d_1^{k+1} - d_2^{k+1}}{x_1 - x_2} = \phi_k \sum_{j=0}^k d_1^j d_2^{k-j}$, $k \in \{1, 2, 3\}$, are the symmetric polynomials of degree k with respect to d_1 and d_2 .

Before we start the actual proof, let us show a way that produces the leading terms of the asymptotic expansion straight away, although does not give the optimal estimate. Using the expansion $\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + \mathcal{O}(t^4)$, we achieve

$$\begin{aligned} \ln \frac{f(x_1) - f(x_2)}{f'(\theta)(x_1 - x_2)} &= P_1 + \left[P_2 - \frac{1}{2} P_1^2 \right] \\ &+ \left[P_3 - P_1 P_2 + \frac{1}{3} P_1^3 \right] + \mathcal{O}((\text{diam}\{x_1, x_2, \theta\})^{3+\gamma}) \end{aligned} \quad (6)$$

(here and in what follows, in square brackets we group up terms of the same order). Now, if one would simply calculate $\ln \text{Dist}(x_1, x_2, x_3, x_4; f)$ as the sum of the four expressions

$$\ln \frac{f(x_1) - f(x_2)}{f'(\theta)(x_1 - x_2)} - \ln \frac{f(x_2) - f(x_3)}{f'(\theta)(x_2 - x_3)} + \ln \frac{f(x_3) - f(x_4)}{f'(\theta)(x_3 - x_4)} - \ln \frac{f(x_4) - f(x_1)}{f'(\theta)(x_4 - x_1)},$$

substituting the corresponding variants of the expansion (6), then after appropriate transformations the formula

$$\begin{aligned} \ln \text{Dist}(x_1, x_2, x_3, x_4; f) &= (x_1 - x_3)(x_2 - x_4) \\ &\times \frac{1}{24} (\mathcal{S}f(x_1) + \mathcal{S}f(x_2) + \mathcal{S}f(x_3) + \mathcal{S}f(x_4)) + \mathcal{O}(\Delta^{3+\gamma}) \end{aligned}$$

will be obtained. However, the remainder term in it is not what we are looking for. The optimal estimate (3) cannot be proven in such a direct way, and so we shall take a roundabout path in order to extract the multiple $(x_1 - x_3)(x_2 - x_4)$ from that remainder term.

Lemma 1. *The following exact equalities take place:*

$$(x_2 - x_3)(D(x_1, x_2, x_3; f) - 1) = (x_1 - x_3)(D(x_2, x_1, x_3; f) - 1)D(x_1, x_3, x_2; f); \quad (7)$$

$$\begin{aligned} &(x_2 - x_3)(x_1 - x_4)(\text{Dist}(x_1, x_2, x_3, x_4; f) - 1) \\ &= (x_1 - x_3)(x_2 - x_4)(\text{Dist}(x_2, x_1, x_3, x_4; f) - 1)\text{Dist}(x_1, x_3, x_2, x_4; f). \end{aligned} \quad (8)$$

Proof. We will prove both (7) and (8) under the condition that x_1, x_2, x_3, x_4 are pairwise distinct. The cases, when some of those points coincide, are very easy to check directly or can be reached from the pairwise distinct case by appropriate limit transitions.

One can see that $R(x_3, x_1, x_2) + R(x_3, x_2, x_1) = -1$;
also $R(f(x_3), f(x_1), f(x_2)) + R(f(x_3), f(x_2), f(x_1)) = -1$. Hence,

$$\frac{x_2 - x_3}{x_1 - x_2} - \frac{f(x_2) - f(x_3)}{f(x_1) - f(x_2)} = -\frac{x_1 - x_3}{x_2 - x_1} + \frac{f(x_1) - f(x_3)}{f(x_2) - f(x_1)},$$

which implies

$$(f(x_2) - f(x_3))(D(x_1, x_2, x_3; f) - 1) = (f(x_1) - f(x_3))(D(x_2, x_1, x_3; f) - 1).$$

The latter formula is easily transformed into the equality (7).

Since $(x_2 - x_3)(x_4 - x_1) - (x_1 - x_3)(x_4 - x_2) = (x_1 - x_2)(x_3 - x_4)$, we have $\text{Cr}(x_2, x_3, x_4, x_1) + \text{Cr}(x_1, x_3, x_4, x_2) = 1$; also $\text{Cr}(f(x_2), f(x_3), f(x_4), f(x_1)) + \text{Cr}(f(x_1), f(x_3), f(x_4), f(x_2)) = 1$. Hence,

$$\begin{aligned} &\frac{(x_2 - x_3)(x_4 - x_1)}{(x_1 - x_2)(x_3 - x_4)} - \frac{(f(x_2) - f(x_3))(f(x_4) - f(x_1))}{(f(x_1) - f(x_2))(f(x_3) - f(x_4))} \\ &= -\frac{(x_1 - x_3)(x_4 - x_2)}{(x_2 - x_1)(x_3 - x_4)} + \frac{(f(x_1) - f(x_3))(f(x_4) - f(x_2))}{(f(x_2) - f(x_1))(f(x_3) - f(x_4))}, \end{aligned}$$

and therefore

$$\begin{aligned} &(f(x_2) - f(x_3))(f(x_4) - f(x_1))(\text{Dist}(x_1, x_2, x_3, x_4; f) - 1) \\ &= (f(x_1) - f(x_3))(f(x_4) - f(x_2))(\text{Dist}(x_2, x_1, x_3, x_4; f) - 1), \end{aligned}$$

which is easy to transform into the equality (8).

Lemma 1 is proven.

Consider the expression

$$\begin{aligned} Q(\theta, x_1, x_2, x_3) = & \phi_1 + [\phi_2(d_1 + d_2 + d_3) - \phi_1^2(d_2 + d_3)] \\ & + [\phi_3(d_1^2 + d_2^2 + d_3^2 + d_1d_2 + d_2d_3 + d_3d_1) \\ & - \phi_1\phi_2((d_2^2 + d_2d_3 + d_3^2) + (d_2 + d_3)(d_1 + d_2 + d_3)) + \phi_1^3(d_2 + d_3)^2], \end{aligned}$$

which in the sequel we will denote simply as Q_{123} .

Proposition 1. *Let $f \in C^{4+\gamma}([A, B])$, $\gamma \in [0, 1]$, and $f' > 0$. For any four points $x_1, x_2, x_3, \theta \in [A, B]$ the following asymptotic estimate takes place:*

$$D(x_1, x_2, x_3; f) = 1 + (x_1 - x_3)(Q_{123} + \mathcal{O}(\Delta_\theta^{2+\gamma})), \quad (9)$$

where $\Delta_\theta = \text{diam}\{x_1, x_2, x_3, \theta\}$.

Remark 2. An arbitrary choice of θ in Proposition 1 makes that form of the asymptotic estimate the most general, giving an opportunity to produce different variants of the estimate (9) for different specific θ (in particular, one can consider the variants with $\theta = x_1$, $\theta = x_2$ or $\theta = x_3$).

First, let us prove the following lemma concerning the dependence of Q_{123} on θ .

Lemma 2. *Let $x_1, x_2, x_3, \theta, \tilde{\theta} \in [A, B]$, and $\tilde{Q}_{123} = Q(\tilde{\theta}, x_1, x_2, x_3)$. The following asymptotic estimate takes place: $\tilde{Q}_{123} - Q_{123} = \mathcal{O}(|\delta|^{2+\gamma})$, where $\delta = \tilde{\theta} - \theta$.*

Proof. Let us find the partial asymptotic expansions for $\tilde{\phi}_k = \frac{f^{(k+1)}(\tilde{\theta})}{(k+1)!f'(\tilde{\theta})}$ in terms of ϕ_k with respect to the powers of δ . In the case of $k = 1$ we write

$$\tilde{\phi}_1 = \frac{1}{2} \frac{f''(\tilde{\theta})/f'(\theta)}{f'(\tilde{\theta})/f'(\theta)} = \frac{\phi_1 + 3\phi_2\delta + 6\phi_3\delta^2 + \mathcal{O}(|\delta|^{2+\gamma})}{1 + 2\phi_1\delta + 3\phi_2\delta^2 + \mathcal{O}(|\delta|^3)}, \quad (10)$$

which implies (in view of the expansion $\frac{1}{1+t} = 1 - t + t^2 + \mathcal{O}(t^3)$ and after noticing that the absolute value of the denominator in (10) is confined between two positive constants)

$$\tilde{\phi}_1 = \phi_1 + [3\phi_2 - 2\phi_1^2]\delta + [6\phi_3 - 9\phi_2\phi_1 + 4\phi_1^3]\delta^2 + \mathcal{O}(|\delta|^{2+\gamma}).$$

Similarly obtain

$$\tilde{\phi}_2 = \frac{1}{2} \frac{f'''(\tilde{\theta})/f'(\theta)}{f'(\tilde{\theta})/f'(\theta)} = \frac{\phi_2 + 4\phi_3\delta + \mathcal{O}(|\delta|^{1+\gamma})}{1 + 2\phi_1\delta + \mathcal{O}(|\delta|^2)} = \phi_2 + [4\phi_3 - 2\phi_3\phi_2]\delta + \mathcal{O}(|\delta|^{1+\gamma})$$

and, finally, $\tilde{\phi}_3 = \phi_3 + \mathcal{O}(|\delta|^\gamma)$.

Now, substitute the derived expressions together with $\tilde{d}_i = x_i - \tilde{\theta} = d_i - \delta$, $i \in \{1, 2, 3\}$, into \tilde{Q}_{123} , subtract Q_{123} , and after transformations get the estimate of the lemma. Lemma 2 is proven.

Proof of Proposition 1. According to Lemma 2, it is enough to prove the estimate (9) for any single point $\theta \in [\min\{x_1, x_2, x_3\}, \max\{x_1, x_2, x_3\}]$, and that will imply that (9) is true for each $\theta \in [A, B]$. However, we will not specify the choice of θ in this proof, imposing only the condition

$\theta \in [\min\{x_1, x_2, x_3\}, \max\{x_1, x_2, x_3\}]$. (A constructivist reader is welcome to assume $\theta = x_1$, although that will not simplify the expressions.) This condition implies $\Delta_\theta = \text{diam}\{x_1, x_2, x_3\}$, which we will denote by Δ_{123} during this proof.

It follows from the definition of ratio distortion that

$$D(x_1, x_2, x_3; f) = 1 + \frac{c_{12} - c_{23}}{1 + c_{23}}, \quad (11)$$

where $c_{12} = \frac{f(x_1) - f(x_2)}{f'(\theta)(x_1 - x_2)} - 1$, $c_{23} = \frac{f(x_2) - f(x_3)}{f'(\theta)(x_2 - x_3)} - 1$. According to (5), we have

$$c_{12} = \phi_1(d_1 + d_2) + \phi_2(d_1^2 + d_1d_2 + d_2^2) + \phi_3(d_1^3 + d_1^2d_2 + d_1d_2^2 + d_2^3) + \mathcal{O}(\Delta_{123}^{3+\gamma}),$$

$$c_{23} = \phi_1(d_2 + d_3) + \phi_2(d_2^2 + d_2d_3 + d_3^2) + \phi_3(d_2^3 + d_2^2d_3 + d_2d_3^2 + d_3^3) + \mathcal{O}(\Delta_{123}^{3+\gamma}).$$

Substitute these expressions into (11) in view of $\frac{1}{1+t} = 1 - t + t^2 + \mathcal{O}(t^3)$ (noticing that the absolute value of the denominator $1 + c_{23}$ is confined between two positive constants again) and after transformations get

$$D(x_1, x_2, x_3; f) = 1 + (x_1 - x_3)Q_{123} + \mathcal{O}(\Delta_{123}^{3+\gamma}). \quad (12)$$

The estimate (12) implies (9) in the case when the points θ and x_2 lie between the points x_1 and x_3 (so that $\Delta_\theta = \Delta_{123} = |x_1 - x_3|$). Thus, in that case the lemma is proven.

Now suppose that θ and x_1 lie between x_2 and x_3 , so that $\Delta_\theta = \Delta_{123} = |x_2 - x_3|$. Having transposed the points in (12) as necessary, we obtain

$$D(x_2, x_1, x_3; f) = 1 + (x_2 - x_3)(Q_{213} + \mathcal{O}(\Delta_{123}^{2+\gamma})),$$

$$D(x_1, x_3, x_2; f) = 1 + (x_1 - x_2)Q_{132} + \mathcal{O}(\Delta_{123}^{3+\gamma}),$$

where Q_{213} and Q_{132} are obtained of Q_{123} by corresponding transpositions of variables d_1 , d_2 and d_3 . Using the equality (7), we get

$$\begin{aligned} D(x_1, x_2, x_3; f) &= 1 + (x_1 - x_3) \\ &\times (Q_{213} + \mathcal{O}(\Delta_{123}^{2+\gamma}))(1 + (d_1 - d_2)Q_{132} + \mathcal{O}(\Delta_{123}^{3+\gamma})). \end{aligned} \quad (13)$$

It is easy to calculate that

$$Q_{123} - Q_{213} = (d_1 - d_2)(\phi_1^2 + [2\phi_2\phi_1(d_1 + d_2 + d_3) - \phi_1^3(d_1 + d_2 + 2d_3)]),$$

$$Q_{213}Q_{132} = \phi_1^2 + [2\phi_2\phi_1(d_1 + d_2 + d_3) - \phi_1^3(d_1 + d_2 + 2d_3)] + \mathcal{O}(\Delta_{123}^2),$$

so (13) implies (9) indeed.

The case when θ and x_3 lie between x_1 and x_2 , is done similarly. Proposition 1 is proven.

Proof of (3). Let $\theta \in [\min\{x_1, x_2, x_3, x_4\}, \max\{x_1, x_2, x_3, x_4\}]$. Using the definitions of D and Dist and Proposition 1, we get

$$\begin{aligned} \text{Dist}(x_1, x_2, x_3, x_4; f) &= D(x_1, x_2, x_3; f) \cdot D(x_3, x_4, x_1; f) \\ &= (1 + (x_1 - x_3)(S_{123} + \mathcal{O}(\Delta^{2+\gamma}))) (1 + (x_3 - x_1)(S_{341} + \mathcal{O}(\Delta^{2+\gamma}))) \\ &= 1 + (x_1 - x_3)(S_{123} - S_{341} - (x_1 - x_3)S_{123}S_{341} + \mathcal{O}(\Delta^{2+\gamma})). \end{aligned}$$

Simple transformations show that

$$\begin{aligned} & S_{123} - S_{341} - (d_1 - d_3)S_{123}S_{341} \\ &= (d_2 - d_4)((\phi_2 - \phi_1^2) + (\phi_3 - 2\phi_2\phi_1 + \phi_1^3)(d_1 + d_2 + d_3 + d_4)) + \mathcal{O}(\Delta^3). \end{aligned}$$

It is time to notice that $\phi_2 - \phi_1^2 = \frac{1}{6}\mathcal{S}f(\theta)$, $\phi_3 - 2\phi_2\phi_1 + \phi_1^3 = \frac{1}{24}(\mathcal{S}f)'(\theta)$, and $\mathcal{S}f(\theta) + (\mathcal{S}f)'(\theta)d_i = \mathcal{S}f(x_i) + \mathcal{O}(|d_i|^{1+\gamma})$ for $i \in \{1, 2, 3, 4\}$, so that we finally obtain

$$\begin{aligned} \text{Dist}(x_1, x_2, x_3, x_4; f) &= 1 + (x_1 - x_3) \\ &\times \left((x_2 - x_4) \frac{1}{24} \sum_{i=1}^4 \mathcal{S}f(x_i) + \mathcal{O}(\Delta^{2+\gamma}) \right). \end{aligned} \quad (14)$$

The role of (14) in this proof is similar to the role of (12) in the proof of Proposition 1. Namely, in the case when x_1 and x_3 lie between x_2 and x_4 we have $\Delta = |x_2 - x_4|$, and hence (14) implies (3). Thus, in that case the theorem is proven. Notice, that if x_2 and x_4 lie between x_1 and x_3 , then the theorem is proven as well due to the symmetry $\text{Dist}(x_1, x_2, x_3, x_4; f) = \text{Dist}(x_2, x_1, x_4, x_3; f)$.

Now suppose that x_2 and x_3 lie between x_1 and x_4 , so that $\Delta = |x_1 - x_4|$. Obvious transpositions of points in (14) lead to

$$\text{Dist}(x_2, x_1, x_3, x_4; f) = 1 + (x_2 - x_3)(x_1 - x_4) \left(\frac{1}{24} \sum_{i=1}^4 \mathcal{S}f(x_i) + \mathcal{O}(\Delta^{1+\gamma}) \right),$$

$$\text{Dist}(x_1, x_3, x_2, x_4; f) = 1 + \mathcal{O}(\Delta^2),$$

and (3) follows from the equality (8). Thus the theorem is proven in this case, too. By symmetry, it is proven also for the case when x_1 and x_4 lie between x_2 and x_3 .

Finally, the case of x_1 and x_2 lying between x_3 and x_4 (and the symmetric one, with x_3 and x_4 between x_1 and x_2) is considered similarly. Thus (3) is proven.

It is quite obvious now that the proofs of (1) and (2) are easily obtained from the proof of (3) by cutting off all the derived partial asymptotic expansions at appropriate lower-order terms.

It is also not hard to check that (4) is proven by following the lines of the proof of (3) with $\gamma = 1$ in appropriate settings. All the statements of the form “ a lies between b and c ” are to be replaced with “ $\text{diam}\{a, b, c\} = |b - c|$ ”, whereas for “ $b \in [\min M, \max M]$ ” for a finite set M one has to substitute “ $\text{diam}(\{b\} \cup M) = \text{diam}M$ ”.

Theorem 1 is proven.

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