

Hopf algebras and characters of classical groups

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Abstract. Schur functions provide an integral basis of the ring of symmetric functions. It is shown that this ring has a natural Hopf algebra structure by identifying the appropriate product, coproduct, unit, counit and antipode, and their properties. Characters of covariant tensor irreducible representations of the classical groups $GL(n)$, $O(n)$ and $Sp(n)$ are then expressed in terms of Schur functions, and the Hopf algebra is exploited in the determination of group-subgroup branching rules and the decomposition of tensor products. The analysis is carried out in terms of n -independent universal characters. The corresponding rings, $CharGL$, $CharO$ and $CharSp$, of universal characters each have their own natural Hopf algebra structure. The appropriate product, coproduct, unit, counit and antipode are identified in each case.

1. Introduction

The aim here is to provide a uniform setting for dealing with characters of finite-dimensional irreducible representations of the classical groups $GL(n)$, $O(n)$ and $Sp(n)$. More specifically, the intention is to determine certain group-subgroup branching rules and formulae for the decomposition of tensor products of irreducible representations of all of these groups by using Schur functions and the Hopf algebra of these symmetric functions, as described most recently in [1, 2]

2. The Hopf algebra of symmetric functions

Let $x = (x_1, x_2, \dots, x_n)$ be a sequence of n indeterminates and let $\Lambda^{(n)} = \mathbb{Z}[x]^{S_n}$ be the ring of polynomial symmetric functions of the indeterminates x_1, x_2, \dots, x_n . This ring may be graded by the total degree, d , of such polynomials, so that we may write $\Lambda^{(n)} = \bigoplus_d \Lambda_d^{(n)}$. An integral basis of $\Lambda_d^{(n)}$ is provided by the Schur functions [3, 4]

$$s_\lambda(x) = \frac{|x_i^{\lambda_j+n-j}|}{|x_i^{n-j}|}, \quad (1)$$

specified by partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of weight $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_n = d$ and length $\ell(\lambda) = p \leq n$, so that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p > 0$ with $\lambda_i = 0$ for any $i > p$.

Within the ring $\Lambda^{(n)}$, products of Schur functions decompose as follows:

$$s_\lambda(x) s_\mu(x) = \sum_\nu c_{\lambda\mu}^\nu s_\nu(x), \quad (2)$$

where the coefficients $c_{\lambda\mu}^{\nu}$ are known as Littlewood-Richardson coefficients. They are all non-negative integers and may be evaluated by means of the Littlewood-Richardson rule [3, 4].

If we move to a sequence $x = (x_1, x_2, \dots)$ of countably many independent indeterminates, then for all partitions λ of weight $|\lambda| = d$ with d finite, there exists a universal Schur function $s_{\lambda}(x)$ of $x = (x_1, x_2, \dots)$ [4] such that for all finite n we have $s_{\lambda}(x_1, x_2, \dots, x_n, 0, 0, \dots) = s_{\lambda}(x_1, x_2, \dots, x_n) \in \Lambda_d^{(n)}$. This stability property enables us to define the ring Λ of symmetric functions. This is the ring generated by $s_{\lambda}(x)$ for all partitions λ . Within this ring Λ , the multiplication rule is again given by (2), and still within Λ , skew Schur functions [3, 4] are defined by:

$$s_{\nu/\lambda}(x) = \sum_{\mu} c_{\lambda\mu}^{\nu} s_{\mu}(x). \quad (3)$$

Each partition λ defines a Young diagram F^{λ} whose successive rows lengths are the parts of λ , and whose successive column lengths are the parts of the conjugate partition, denoted here by λ' . Then for all $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ Cauchy's formula and its inverse take the form [3, 4]:

$$J(x, y) = \prod_{i,a} (1 - x_i y_a)^{-1} = \sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y); \quad (4)$$

$$I(x, y) = \prod_{i,a} (1 - x_i y_a) = \sum_{\lambda} (-1)^{|\lambda|} s_{\lambda}(x) s_{\lambda'}(y). \quad (5)$$

It follows that

$$\begin{aligned} J(x, y) I(x, y) &= \sum_{\lambda} s_{\lambda}(x) s_{\lambda}(y) \sum_{\mu} (-1)^{|\mu|} s_{\mu}(x) s_{\mu'}(y) \\ &= \sum_{\lambda, \mu, \nu} c_{\lambda\mu}^{\nu} s_{\nu}(x) (-1)^{|\mu|} s_{\lambda}(y) s_{\mu'}(y) = \sum_{\mu, \nu} s_{\nu}(x) (-1)^{|\mu|} s_{\nu/\mu}(y) s_{\mu'}(y). \end{aligned}$$

However $J(x, y) I(x, y) = 1 = s_0(x) s_0(y)$, so that by comparing coefficients of $s_{\nu}(x)$ we have the Schur function identity:

$$\sum_{\mu} (-1)^{|\mu|} s_{\nu/\mu}(y) s_{\mu'}(y) = \delta_{\nu,0} s_0(y), \quad (6)$$

for all y , where the sum is to be taken over all partitions μ .

The ring Λ of symmetric functions has the structure of a Hopf algebra, $Symm = (\Lambda, m, \Delta, \iota, \epsilon, \mathbf{S})$, by virtue of the following identification of operators and their action [1, 2]:

- Product m : $m(s_{\lambda} \otimes s_{\mu})(x) = s_{\lambda}(x) \cdot s_{\mu}(x) = s_{\lambda}(x) s_{\mu}(x) = \sum_{\nu} c_{\lambda\mu}^{\nu} s_{\nu}(x)$.
- Unit ι : $\iota(1) = s_0(x)$.
- Coproduct Δ : $s_{\nu}(x, y) = \sum_{\lambda, \mu} c_{\lambda\mu}^{\nu} s_{\lambda}(x) s_{\mu}(y) = \sum_{\lambda} s_{\lambda}(x) \otimes s_{\nu/\lambda}(y) = \sum_{\mu} s_{\nu/\mu}(x) \otimes s_{\mu}(y)$.
- Counit ϵ : $\epsilon(s_{\lambda}(x)) = \delta_{\lambda,0}$.
- Antipode \mathbf{S} : $\mathbf{S}(s_{\lambda}(x)) = (-1)^{|\lambda|} s_{\lambda'}(x)$.

Here $x = (x_1, x_2, \dots)$, $y = (y_1, y_2, \dots)$ and $x, y = (x_1, x_2, \dots, y_1, y_2, \dots)$, which is sometimes written as $x + y$, the addition of two alphabets of indeterminates. In what follows we tend to favour the use of \cdot rather than m to signify a product.

This particular Hopf algebra $Symm$ has the following properties:

Commutativity: $s_{\lambda} \cdot s_{\mu} = s_{\mu} \cdot s_{\lambda}$ since $s_{\lambda}(x) s_{\mu}(x) = s_{\mu}(x) s_{\lambda}(x)$.

Cocommutativity: $\Delta(s_\nu) = \sum_\zeta s_\zeta \otimes s_{\nu/\zeta} = \sum_\zeta s_{\nu/\zeta} \otimes s_\zeta$ since $s_\nu(x, y) = s_\nu(y, x)$.

Associativity: $s_\rho \cdot (s_\sigma \cdot s_\tau) = (s_\rho \cdot s_\sigma) \cdot s_\tau$ since $s_\rho(x) (s_\sigma(x) s_\tau(x)) = (s_\rho(x) s_\sigma(x)) s_\tau(x)$.

Coassociativity: $(I \otimes \Delta)(\Delta(s_\lambda)) = (\Delta \otimes I)(\Delta(s_\lambda))$ since $s_\lambda(x, (y, z)) = s_\lambda((x, y), z)$.

In addition we may verify the following general requirements of any Hopf algebra:

Antipode identity: $\cdot(S \otimes I) \Delta = \iota \epsilon = \cdot(I \otimes S) \Delta$ since, thanks to (6), we have

$$\begin{aligned} \cdot(S \otimes I) \Delta(s_\lambda) &= \sum_\mu \cdot(S \otimes I)(s_\mu \otimes s_{\lambda/\mu}) = \sum_\mu S(s_\mu) \cdot s_{\lambda/\mu} \\ &= \sum_\mu (-1)^{|\mu|} s_{\mu'} \cdot s_{\lambda/\mu} = \delta_{\lambda 0} s_0 = \iota \delta_{\lambda 0} = \iota \epsilon(s_\lambda). \end{aligned}$$

Counitarity: $\cdot(\epsilon \otimes I) \Delta = I = \cdot(I \otimes \epsilon) \Delta$ since

$$\cdot(\epsilon \otimes I) \Delta(s_\lambda) = \sum_\mu \cdot(\epsilon \otimes I)(s_\mu \otimes s_{\lambda/\mu}) = \sum_\mu \cdot(\delta_{\mu 0} \otimes s_{\lambda/\mu}) = 1 \cdot s_\lambda = s_\lambda = I(s_\lambda).$$

Product and coproduct compatibility: $\Delta(\cdot) = (\cdot \otimes \cdot)(\Delta \otimes \Delta)$ since

$$\begin{aligned} \Delta(\cdot)(s_\lambda(z) s_\mu(w)) &= \Delta(s_\lambda(z) s_\mu(z)) = s_\lambda(x, y) s_\mu(x, y) \\ &= (\cdot \otimes \cdot)(s_\lambda(x, y) s_\mu(u, v)) = (\cdot \otimes \cdot)(\Delta \otimes \Delta)(s_\lambda(z) s_\mu(w)). \end{aligned}$$

This last property is the homomorphism property of the coproduct:

$$\Delta(s_\lambda \cdot s_\mu) = \Delta(s_\lambda) \cdot \Delta(s_\mu) \quad \text{or more generally} \quad \Delta(X \cdot Y) = \Delta(X) \cdot \Delta(Y), \quad (7)$$

for any $X, Y \in \Lambda$.

At this stage it is convenient to introduce a bilinear scalar product, $(\cdot | \cdot)$, on $Symm$. This is defined by $(s_\lambda | s_\mu) = \delta_{\lambda\mu}$ [4]. With this definition we have:

$$\begin{aligned} (s_\nu | s_\lambda \cdot s_\mu) &= \sum_\zeta c_{\lambda\mu}^\zeta (s_\nu | s_\zeta) = c_{\lambda\mu}^\nu; & (s_{\nu/\lambda} | s_\mu) &= \sum_\eta c_{\lambda\eta}^\nu (s_\eta | s_\mu) = c_{\lambda\mu}^\nu; \\ (\Delta(s_\nu) | s_\lambda \otimes s_\mu) &= \sum_{\sigma, \tau} c_{\sigma\tau}^\nu (s_\sigma \otimes s_\tau | s_\lambda \otimes s_\mu) = \sum_{\sigma, \tau} c_{\sigma\tau}^\nu \delta_{\sigma\lambda} \delta_{\tau\mu} = c_{\lambda\mu}^\nu, \end{aligned}$$

so that

$$(s_\nu | s_\lambda \cdot s_\mu) = (s_{\nu/\lambda} | s_\mu) \quad \text{and} \quad (s_\nu | s_\lambda \cdot s_\mu) = (\Delta(s_\nu) | s_\lambda \otimes s_\mu). \quad (8)$$

For any $X = \sum_\sigma a_\sigma s_\sigma$ we let $s_{\lambda \cdot X} = s_\lambda \cdot X = \sum_\sigma a_\sigma (s_\lambda \cdot s_\sigma)$ and $s_{\lambda/X} = \sum_\sigma a_\sigma s_{\lambda/\sigma}$. With this notation, we can extend the first part of (8) so that for any $X = \sum_\sigma a_\sigma s_\sigma$ we have

$$(s_\lambda | X \cdot s_\mu) = \sum_\sigma a_\sigma (s_\lambda | s_\sigma \cdot s_\mu) = \sum_\sigma a_\sigma (s_{\lambda/\sigma} | s_\mu) = (s_{\lambda/X} | s_\mu). \quad (9)$$

In addition, it should be noted that for $X = \sum_\sigma a_\sigma s_\sigma$ we have $a_\sigma = (X | s_\sigma)$ so that

$$X = \sum_\sigma (X | s_\sigma) s_\sigma \quad (10)$$

Finally, for any $X = \sum_\sigma a_\sigma s_\sigma$ and $Y = \sum_\sigma b_\sigma s_\sigma$, we have $X = Y$ if and only if $(X | s_\sigma) = a_\sigma = b_\sigma = (Y | s_\sigma)$ for all σ . It follows that

$$s_{(\lambda/\mu)/\nu} = s_{\lambda/(\mu \cdot \nu)} \quad (11)$$

since $(s_{(\lambda/\mu)/\nu} | s_\sigma) = (s_{\lambda/\mu} | s_\nu s_\sigma) = (s_\lambda | s_\mu s_\nu s_\sigma) = (s_\lambda | s_{\mu \cdot \nu} s_\sigma) = (s_{\lambda/(\mu \cdot \nu)} | s_\sigma)$ for all σ , and

$$s_{(\mu \cdot \nu)/\rho} = \sum_{\sigma, \tau} c_{\sigma\tau}^\rho s_{\mu/\sigma} \cdot s_{\nu/\tau}, \quad (12)$$

$$\begin{aligned} \text{since } (s_{(\mu \cdot \nu)/\rho} | s_\lambda) &= (s_\mu \cdot s_\nu | s_\rho \cdot s_\lambda) = (s_\mu \otimes s_\nu | \Delta(s_\rho \cdot s_\lambda)) = \sum_{\sigma, \tau} c_{\sigma\tau}^\rho (s_\mu \otimes s_\nu | s_\sigma \otimes s_\tau \Delta(s_\lambda)) \\ &= \sum_{\sigma, \tau} c_{\sigma\tau}^\rho (s_{\mu/\sigma} \otimes s_{\nu/\tau} | \Delta(s_\lambda)) = \sum_{\sigma, \tau} c_{\sigma\tau}^\rho (s_{\mu/\sigma} \cdot s_{\nu/\tau} | s_\lambda) \text{ for all } \lambda. \end{aligned}$$

3. Characters of the classical groups

Let $M(m, n)$ be the set of all $m \times n$ matrices over \mathbb{C} . Then the classical groups under consideration here are:

$$\begin{aligned} GL(n) &= \{X \in M(n, n) \mid \det X \neq 0\}; \\ O(n) &= \{X \in GL(n) \mid X G_n X^t = G_n\} \quad \text{with } G_n^t = G_n; \\ Sp(n) &= \{X \in GL(n) \mid X J_n X^t = J_n\} \quad \text{with } J_n^t = -J_n. \end{aligned}$$

It might be noted that for $n = 2k + 1$ the matrix J_n is necessarily singular, and may be chosen [5] so that:

$$Sp(2k + 1) = \begin{bmatrix} Sp(2k) & & & M(2k, 1) \\ & \dots & & \\ 0 & & 0 & GL(1) \end{bmatrix}. \quad (13)$$

Thus $Sp(2k + 1)$ is not semisimple. Nor is it reductive.

The eigenvalues of an arbitrary group element X may be parametrised as follows [3]:

$$\begin{aligned} GL(n) &: x_1, x_2, \dots, x_n \quad \text{with } x_1 x_2 \cdots x_n \neq 0. \\ SL(n) &: x_1, x_2, \dots, x_n \quad \text{with } x_1 x_2 \cdots x_n = 1. \\ SO(2k + 1) &: x_1, x_2, \dots, x_k, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_k, 1. \\ O(2k + 1) \setminus SO(2k + 1) &: x_1, x_2, \dots, x_k, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_k, -1. \\ Sp(2k) &: x_1, x_2, \dots, x_k, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_k. \\ SO(2k) &: x_1, x_2, \dots, x_k, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_k. \\ O(2k) \setminus SO(2k) &: x_1, x_2, \dots, x_{k-1}, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_{k-1}, 1, -1. \\ Sp(2k + 1) &: x_1, x_2, \dots, x_k, \bar{x}_1, \bar{x}_2, \dots, \bar{x}_k, x_{2k+1}. \end{aligned}$$

where $\bar{x}_i = x_i^{-1}$ for all i .

Each covariant tensor irreducible representation, $V_{GL(n)}^\lambda$, of $GL(n)$ is specified by a partition λ of length $\ell(\lambda) \leq n$. Let $X \in GL(n)$ have eigenvalues (x_1, x_2, \dots, x_n) and let $\rho = (n - 1, n - 2, \dots, 1, 0)$. Then the character of this irreducible representation is given by [3, 4]:

$$\text{ch } V_{GL(n)}^\lambda = \frac{a_{\lambda+\rho}(x)}{a_\rho(x)} = \frac{|x_i^{\lambda_j+n-j}|}{|x_i^{n-j}|} = s_\lambda(x). \quad (14)$$

Thanks to the stability property of Schur functions with respect to the number n of indeterminates, we may define the corresponding universal character of $GL(n)$ by

$$\text{ch } V_{GL}^\lambda = \{\lambda\}(x) = s_\lambda(x), \quad (15)$$

where $x = (x_1, x_2, \dots)$. For each finite n the characters $\text{ch } V_{GL(n)}^\lambda$ are recovered from the universal characters $\text{ch } V_{GL}^\lambda$ merely by setting $x = (x_1, x_2, \dots, x_n, 0, 0, \dots, 0)$.

In a similar way, there exist covariant tensor irreducible representation, $V_{O(n)}^\lambda$ and $V_{Sp(n)}^\lambda$, of $O(n)$ and $Sp(n)$, respectively. The corresponding characters $\text{ch } V_{O(n)}^\lambda$ and $\text{ch } V_{Sp(n)}^\lambda$ may each be defined in terms of determinants. More important, from our point of view, is that there exist corresponding universal characters [6, 7] denoted by

$$\text{ch } V_O^\lambda = [\lambda](x) \quad \text{and} \quad \text{ch } V_{Sp}^\lambda = \langle \lambda \rangle(x), \quad (16)$$

with $x = (x_1, x_2, \dots)$ arbitrary. These are universal in the sense that for any finite n the characters $\text{ch } V_{O(n)}^\lambda$ and $\text{ch } V_{Sp(n)}^\lambda$ are obtained by specialising x to $(x_1, x_2, \dots, x_n, 0, 0, \dots, 0)$ with x_1, x_2, \dots, x_n restricted to the eigenvalues of the appropriate group elements parametrised as above.

The universal characters (16) are themselves defined by means of the generating functions [3]:

$$\prod_{i,a} (1 - x_i y_a)^{-1} \prod_{a \leq b} (1 - y_a y_b) = \sum_{\lambda} [\lambda](x) \{\lambda\}(y); \quad (17)$$

$$\prod_{i,a} (1 - x_i y_a)^{-1} \prod_{a < b} (1 - y_a y_b) = \sum_{\lambda} \langle \lambda \rangle(x) \{\lambda\}(y). \quad (18)$$

This leads to [3, 8, 6, 7]:

Theorem 3.1 *The universal characters $\text{ch } V_G^\lambda$ of the orthogonal and symplectic groups are given by*

$$[\lambda](x) = \{\lambda/C\}(x) = s_{\lambda/C}(x) \quad \text{where} \quad C(x) = \prod_{i \leq j} (1 - x_i x_j); \quad (19)$$

$$\langle \lambda \rangle(x) = \{\lambda/A\}(x) = s_{\lambda/A}(x) \quad \text{where} \quad A(x) = \prod_{i < j} (1 - x_i x_j), \quad (20)$$

respectively.

Proof: For $O(n)$, it follows from (17) that the character $[\lambda](x)$ is the coefficient of $s_\lambda(y)$ in $J(x, y) C(y)$. Hence, from (4), (9) and (10) we have

$$\begin{aligned} [\lambda](x) &= \left(\prod_{i,a} (1 - x_i y_a)^{-1} \prod_{a \leq b} (1 - y_a y_b) \mid \{\lambda\}(y) \right) \\ &= \left(\sum_{\sigma} \{\sigma\}(x) \{\sigma\}(y) C(y) \mid \{\lambda\}(y) \right) \\ &= \sum_{\sigma} \{\sigma\}(x) (\{\sigma\}(y) \cdot C(y) \mid \{\lambda\}(y)) \\ &= \sum_{\sigma} \{\sigma\}(x) (\{\sigma\}(y) \mid \{\lambda/C\}(y)) = \{\lambda/C\}(x). \end{aligned}$$

For $Sp(n)$ one simply replaces $a \leq b$ by $a < b$, and $C(y)$ by $A(y)$. This gives: $\langle \lambda \rangle(x) = \{\lambda/A\}(x)$, as required. \blacksquare

4. Branching rules

In order to allow the possibility of extending results to a wider class of subgroups of the general linear group, we consider any subgroup $H(n)$ of $GL(n)$, whose group elements $X \in H(n) \subset GL(n)$ have eigenvalues $x = (x_1, x_2, \dots, x_n)$. We assume, just as in the case of $O(n)$ and $Sp(n)$, that there exist irreducible representations $V_{H(n)}^\lambda$ of $H(n)$, specified by partitions

λ , with characters $\text{ch } V_{H(n)}^\lambda$ that may be determined by specialising from $x = (x_1, x_2, \dots)$ to $x = (x_1, x_2, \dots, x_n, 0, \dots, 0)$, with appropriate x_1, x_2, \dots, x_n , the universal characters

$$\text{ch } V_H^\lambda = \llbracket \lambda \rrbracket(x). \quad (21)$$

If the embedding $H(n) \subset GL(n)$ is such that there exists two series of Schur functions $S(x)$ and $T(x)$ with the properties:

$$\llbracket \lambda \rrbracket(x) = \{\lambda/S\}(x) \quad \text{with} \quad S(x) T(x) = 1 \quad (22)$$

then

$$\{\lambda\}(x) = \llbracket \lambda/T \rrbracket(x). \quad (23)$$

As a result we immediately have the branching rule:

$$GL(n) \supset H(n) : \quad \{\lambda\} \rightarrow \llbracket \lambda/T \rrbracket. \quad (24)$$

Applying this to $O(n)$ and $Sp(n)$, we immediately have [3, 8]:

Theorem 4.1 *The branching rules for the decomposition of representations of $GL(n)$ under restriction to the subgroups $O(n)$ and $Sp(n)$ take the form:*

$$GL(n) \supset O(n) : \quad \{\lambda\} \rightarrow [\lambda/D] \quad \text{with} \quad D = C^{-1} = \prod_{i \leq j} (1 - x_i x_j)^{-1}; \quad (25)$$

$$GL(n) \supset Sp(n) : \quad \{\lambda\} \rightarrow \langle \lambda/B \rangle \quad \text{with} \quad B = A^{-1} = \prod_{i < j} (1 - x_i x_j)^{-1}. \quad (26)$$

It is well known that [8, 9]

$$\begin{aligned} B &= \{0\} + \{1^2\} + \{2^2\} + \{1^4\} + \{3^2\} + \{2^2 1^2\} + \{1^6\} + \dots \\ D &= \{0\} + \{2\} + \{4\} + \{2^2\} + \{6\} + \{42\} + \{2^3\} + \dots \end{aligned}$$

where D involves partitions all of whose parts are even, and B the conjugate of such partitions. The branching rules obtained using this identification of D and B are exemplified in Table 1.

Table 1. Branching rule examples.

$GL(n) \supset O(n):$	$\{\lambda\} \rightarrow [\lambda/D]$
	$\{4\} \rightarrow [4] + [2] + [0]$
	$\{1^4\} \rightarrow [1^4]$
	$\{2^2 1^2\} \rightarrow [2^2 1^2] + [21^2] + [1^2]$
$GL(n) \supset Sp(n):$	$\{\lambda\} \rightarrow \langle \lambda/B \rangle$
	$\{4\} \rightarrow \langle 4 \rangle$
	$\{1^4\} \rightarrow \langle 1^4 \rangle + \langle 1^2 \rangle + \langle 0 \rangle$
	$\{2^2 1^2\} \rightarrow \langle 2^2 1^2 \rangle + \langle 2^2 \rangle + \langle 21^2 \rangle + \langle 1^4 \rangle + 2 \langle 1^2 \rangle + \langle 0 \rangle$

5. Tensor products

We return to the general case of the subgroup $H(n)$ of $GL(n)$ with universal characters defined by $[[\lambda]] = \{\lambda/S\}$. Let the coproduct of $T = S^{-1}$ take the form

$$\Delta(T) = (T \otimes T) \cdot \Delta''(T) \quad \text{with} \quad \Delta''(T) = \sum_{\sigma, \tau} b_{\sigma\tau}^T \{\sigma\} \otimes \{\tau\}, \quad (27)$$

for some, as yet undetermined, coefficients $b_{\sigma\tau}^T$. Then, we have

Theorem 5.1 *The decomposition of products of universal characters of $H(n)$ takes the form:*

$$[[\lambda]] \cdot [[\mu]] = \sum_{\sigma, \tau} b_{\sigma\tau}^T [[(\lambda/\sigma) \cdot (\mu/\tau)]. \quad (28)$$

Proof: Note that from (22) and (23)

$$[[\lambda]] \cdot [[\mu]] = \{\lambda/S\} \cdot \{\mu/S\} = [((\lambda/S) \cdot (\mu/S))/T]. \quad (29)$$

This implies that the multiplicity of $[[\nu]]$ in $[[\lambda]] \cdot [[\mu]]$ is the same as the multiplicity of $\{\nu\}$ in $[((\lambda/S) \cdot (\mu/S))/T]$, that is:

$$\begin{aligned} ([((\lambda/S) \cdot (\mu/S))/T] | \{\nu\}) &= ([(\lambda/S) \cdot (\mu/S)] | T \cdot \{\nu\}) \\ &= ([\lambda/S] \otimes [\mu/S] | \Delta(T \cdot \{\nu\})) \\ &= ([\lambda/S] \otimes [\mu/S] | (T \otimes T) \cdot \Delta''(T) \cdot \Delta(\{\nu\})) \\ &= ([\lambda/S] \otimes [\mu/S] | \Delta''(T) \cdot \Delta(\{\nu\})) \\ &= ([\lambda] \otimes [\mu] | \sum_{\sigma, \tau} b_{\sigma\tau}^T \{\sigma\} \otimes \{\tau\} \cdot \Delta(\{\nu\})) \\ &= \sum_{\sigma, \tau} b_{\sigma\tau}^T ([\lambda/\sigma] \otimes [\mu/\tau] | \Delta(\{\nu\})) \\ &= \sum_{\sigma, \tau} b_{\sigma\tau}^T ([\lambda/\sigma] \cdot [\mu/\tau] | \{\nu\}), \end{aligned}$$

from which the required formula (28) follows ■

In order to apply this to any given subgroup $H(n)$ of $GL(n)$ it is necessary to evaluate the coefficients $b_{\sigma\tau}^T$ appearing in the coproduct of T . In the case of $O(n)$ and $Sp(n)$ we have $T = D$ and $T = B$, respectively, for which we have the coproduct expansions

$$\Delta(D) = (D \otimes D) \cdot \Delta''(D) \quad \text{with} \quad \Delta''(D) = \sum_{\sigma} \{\sigma\} \otimes \{\sigma\}; \quad (30)$$

$$\Delta(B) = (B \otimes B) \cdot \Delta''(B) \quad \text{with} \quad \Delta''(B) = \sum_{\sigma} \{\sigma\} \otimes \{\sigma\}. \quad (31)$$

This can be seen by noting that:

$$\begin{aligned} D(x, y) &= \prod_{i < j} (1 - x_i x_j)^{-1} \prod_{i, a} (1 - x_i y_a)^{-1} \prod_{a < b} (1 - y_a y_b)^{-1} \\ &= D(x) \sum_{\sigma} \{\sigma\}(x) \{\sigma\}(y) D(y); \\ B(x, y) &= \prod_{i < j} (1 - x_i x_j)^{-1} \prod_{i, a} (1 - x_i y_a)^{-1} \prod_{a < b} (1 - y_a y_b)^{-1} \\ &= B(x) \sum_{\sigma} \{\sigma\}(x) \{\sigma\}(y) B(y). \end{aligned}$$

Now we are in a position to apply (28) to the case of $O(n)$ and $Sp(n)$. We find [10, 11, 9, 6, 7]:

Theorem 5.2 *The tensor product decomposition rules for universal characters of $O(n)$ and $Sp(n)$ take the form:*

$$[\lambda] \cdot [\mu] = \sum_{\sigma} [(\lambda/\sigma) \cdot (\mu/\sigma)] \quad \text{and} \quad \langle \lambda \rangle \cdot \langle \mu \rangle = \sum_{\sigma} \langle (\lambda/\sigma) \cdot (\mu/\sigma) \rangle. \quad (32)$$

Proof: In the case of $O(n)$ it is merely necessary to note that $T = D$ and $b_{\sigma\tau}^D = \delta_{\sigma,\tau}$ from (30). Using this in (28) gives

$$[\lambda] \cdot [\mu] = \sum_{\sigma,\tau} b_{\sigma\tau}^D [(\lambda/\sigma) \cdot (\mu/\tau)] = \sum_{\sigma} [(\lambda/\sigma) \cdot (\mu/\sigma)]. \quad (33)$$

Similarly, in the case $Sp(n)$ we have $T = B$ and $b_{\sigma\tau}^B = \delta_{\sigma,\tau}$ from (31). Again using this in (28) gives

$$\langle \lambda \rangle \cdot \langle \mu \rangle = \sum_{\sigma,\tau} b_{\sigma\tau}^B \langle (\lambda/\sigma) \cdot (\mu/\tau) \rangle = \sum_{\sigma} \langle (\lambda/\sigma) \cdot (\mu/\sigma) \rangle. \quad (34)$$

■

These rules are exemplified in Table 2, along with an example for $GL(n)$ that is included for comparative purposes. Remarkably, the universal tensor product rules for $O(n)$ and $Sp(n)$ are identical. However, it is important to note that for finite n , when $x = (x_1, x_2, \dots)$ is suitably specialised, modification rules given elsewhere [12, 9] distinguish them.

Table 2. Tensor product decompositions.

$GL(n):$	$\{2^2\} \cdot \{21\}$	$= \{43\} + \{421\} + \{3^21\} + \{32^2\} + \{321^2\} + \{2^31\}$
$O(n):$	$[2^2] \cdot [21]$	$= [43] + [421] + [3^21] + [32^2] + [321^2] + [2^31]$ $+ [41] + 2[32] + 2[31^2] + 2[2^21] + [21^3]$ $+ [3] + 2[21] + [1^3] + [1].$
$Sp(n):$	$\langle 2^2 \rangle \cdot \langle 21 \rangle$	$= \langle 43 \rangle + \langle 421 \rangle + \langle 3^21 \rangle + \langle 32^2 \rangle + \langle 321^2 \rangle + \langle 2^31 \rangle$ $+ \langle 41 \rangle + 2\langle 32 \rangle + 2\langle 31^2 \rangle + 2\langle 2^21 \rangle + \langle 21^3 \rangle$ $+ \langle 3 \rangle + 2\langle 21 \rangle + \langle 1^3 \rangle + \langle 1 \rangle.$

6. Classical group character rings

Following an approach described in [2], the universal characters $\{\lambda\}$, $[\lambda]$ and $\langle \lambda \rangle$, of the general linear, orthogonal and symplectic groups are linked to each other by means of the following identities:

$$\{\lambda\} = [\lambda/D] = \langle \lambda/B \rangle; \quad \{\lambda/C\} = [\lambda] = \langle \lambda/BC \rangle; \quad \{\lambda/A\} = [\lambda/AD] = \langle \lambda \rangle. \quad (35)$$

By virtue of these identities each of these sets of characters $\{\lambda\}$, $[\lambda]$ and $\langle \lambda \rangle$ forms a basis of Λ . However, as we have seen, their product rules within the character rings $CharGL$, $CharO$ and $CharSp$ are different. They are tabulated in the first line of Table 3. Moreover, within these same character rings, the coproduct of the Hopf algebra $Symm$ induces the coproducts given in the second line of Table 3. These coproduct formulae are just the universal form of the branching rules [8] for the group-subgroup restrictions $GL(n+m) \supset GL(n) \times GL(m)$, $O(n+m) \supset O(n) \times O(m)$, and $Sp(n+m) \supset Sp(n) \times Sp(m)$. To complete the specification of the Hopf algebra structure of $CharGL$, $CharO$ and $CharSp$ it is only necessary to specify the unit ι , counit ϵ and antipode S . These are also given in Table 3, where \mathcal{A} and \mathcal{C} signify the sets of partitions α and γ appearing in the expansions of A and C [8, 9].

Table 3. Hopf algebra structure of group character rings [2]

<i>CharGL</i>	<i>CharO</i>	<i>CharSp</i>
$m(\{\mu\} \otimes \{\nu\}) = \{\mu\} \cdot \{\nu\}$	$m([\mu] \otimes [\nu]) = \sum_{\zeta} [(\mu/\zeta) \cdot (\nu/\zeta)]$	$m(\langle\mu\rangle \otimes \langle\nu\rangle) = \sum_{\zeta} \langle(\mu/\zeta) \cdot (\nu/\zeta)\rangle$
$\Delta(\{\lambda\}) = \sum_{\zeta} \{\lambda/\zeta\} \otimes \{\zeta\}$	$\Delta([\lambda]) = \sum_{\zeta} [\lambda/\zeta] \otimes [\zeta/D]$	$\Delta(\langle\lambda\rangle) = \sum_{\zeta} \langle\lambda/\zeta\rangle \otimes \langle\zeta/B\rangle$
$\iota(1) = \{0\}$	$\iota(1) = [0]$	$\iota(1) = \langle 0 \rangle$
$\epsilon(\{\lambda\}) = \delta_{\lambda,0}$	$\epsilon([\lambda]) = \sum_{\gamma \in \mathcal{C}} (-1)^{ \gamma /2} \delta_{\lambda,\gamma}$	$\epsilon(\langle\lambda\rangle) = \sum_{\alpha \in \mathcal{A}} (-1)^{ \alpha /2} \delta_{\lambda,\alpha}$
$S(\{\lambda\}) = (-1)^{ \lambda } \{\lambda'\}$	$S([\lambda]) = (-1)^{ \lambda } [\lambda'/AD]$	$S(\langle\lambda\rangle) = (-1)^{ \lambda } \langle\lambda'/CB\rangle$

7. Conclusions

Universal characters of irreducible representations of the classical groups have been identified. These have been expressed in terms of Schur functions, and Hopf algebra manipulations have allowed us to calculate branching rules and tensor product decompositions. The analysis covers covariant tensor representations of $GL(n)$, $O(n)$ and $Sp(n)$, and may be extended to other subgroups of $GL(n)$ [1]. Such an extension using higher rank invariants, leads in some cases to finite subgroups.

It should be stressed that, for any finite n , modification rules are needed to interpret the results [10, 12, 9]. In addition, for n odd, the subgroup $Sp(n)$ of $GL(n)$ is neither semisimple nor reductive. However, for $n = 2k + 1$ the results remain valid if each character $\langle\lambda\rangle$ is interpreted as the character of a representation $V_{Sp(2k+1)}^{\lambda}$ that is no longer irreducible but is indecomposable [5].

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