GEOMETRY OF MULTIPLICATIVE PREPROJECTIVE ALGEBRA

DAISUKE YAMAKAWA

ABSTRACT. Crawley-Boevey and Shaw recently introduced a certain multiplicative analogue of the deformed preprojective algebra, which they called the multiplicative preprojective algebra. In this paper we study the moduli space of (semi)stable representations of such an algebra (the multiplicative quiver variety), which in fact has many similarities to the quiver variety. We show that there exists a complex analytic isomorphism between the nilpotent subvariety of the quiver variety and that of the multiplicative quiver variety (which can be extended to a symplectomorphism between these tubular neighborhoods). We also show that when the quiver is star-shaped, the multiplicative quiver variety parametrizes Simpson's (poly)stable filtered local systems on a punctured Riemann sphere with prescribed filtration type, weight and associated graded local system around each puncture.

CONTENTS

1. Introduction

In this paper we study the geometry of multiplicative preprojective relation.

First let us recall the notion of (deformed) preprojective relation. Let $Q = (I, \Omega)$ be a finite quiver with vertex set I and arrow set Ω, and let (I, H) be its "double"; that is obtained by adding a reverse arrow $\overline{h}$ to Ω for each $h \in \Omega$. For $h \in H$, we denote by $\mathrm{out}(h), \mathrm{in}(h) \in I$ the outgoing, incoming vertex of h, respectively. A representation of (I, H) is given by a pair (V, x) of an I-graded vector space $V = \bigoplus V_i$ and a family $x = (x_h)_{h \in H}$ of linear maps $x_h \colon V_{\mathrm{out}(h)} \to V_{\mathrm{in}(h)}$. Then for $\zeta = (\zeta_i) \in \mathbb{C}^I$, the equation

$$(\mu_V)_i(x) := \sum_{h \in H; \mathrm{in}(h) = i} \epsilon(h) x_h x_{\overline{h}} = \zeta_i 1_{V_i} \qquad (i \in I)$$

is called the *(deformed) preprojective relation*. Here $\epsilon(h) = 1$ if $h \in \Omega$ and $\epsilon(h) = -1$ otherwise. One of the most important properties of this relation is that for a fixed V, the map

$$\mu_V \colon \mathbf{M}(V) := \bigoplus_{h \in H} \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}) \to \bigoplus_{i \in I} \mathrm{End}(V_i)$$

satisfies the defining property of moment map for the natural action of $G_V := \prod_i \mathrm{GL}(V_i)$. Thus taking a stability condition on $\mathbf{M}(V)$ in the sense of geometric invariant theory, the quotient $\mu_V^{-1}(\zeta)^{\mathrm{s}} / G_V$ of the stable locus carries naturally a symplectic structure. Such a quotient is so-called the *quiver variety*. Strictly speaking, there are various choices of stability condition parametrized by $\theta \in \mathbb{Q}^I$ (θ-stability), and the quiver variety $\mathfrak{M}_{\zeta, \theta}(V) = \mu_V^{-1}(\zeta)^{\theta-\mathrm{ss}} /\!\!/ G_V$ is defined as the quotient of the θ-semistable locus. In general its stable locus $\mathfrak{M}_{\zeta, \theta}^{\mathrm{s}}(V)$ does not coincide with the whole space.

An importance of quiver variety in geometry was firstly found by Kronheimer [22]. He described the minimal resolution $\widetilde{\mathbb{C}^2/\Gamma}$ of the Kleinnian singularity as the quiver variety associated to a quiver of the extended Dynkin type corresponding to $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ via the McKay correspondence. Motivated by this fact and the ADHM description of moduli spaces of instantons on such spaces by Kronheimer and him [23], Nakajima introduced the notion of quiver variety in his celebrated paper [29]. In the same paper, developing Lusztig's idea he constructed geometrically all irreducible highest weight representations of Kac-Moody Lie algebras. For further developments in this direction, see [30, 31].

On the other hand, Crawley-Boevey and Shaw recently introduced a certain "multiplicative" analogue of the preprojective relation, called the *multiplicative preprojective relation* [11]; that is

$$(\Phi_V)_i(x) := \prod_{h \in H; \mathrm{in}(h) = i} (1 + x_h x_{\overline{h}})^{\epsilon(h)} = q_i 1_{V_i} \qquad (i \in I),$$

where we have fixed $q = (q_i) \in (\mathbb{C}^\times)^I$ and an ordering for taking product, and have assumed that $\det(1 + x_h x_{\overline{h}}) \neq 0$ for all $h \in H$. They considered such a relation motivated by the *Deligne-Simpson problem*. Fix a number of conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_n$ in $\mathrm{GL}(r, \mathbb{C})$. Then the problem asks if irreducible solutions of the equation

$$A_1 A_2 \cdots A_n = 1 \qquad (A_i \in \mathcal{C}_i)$$

exist. Here the word "irreducible" means that A_i's have no common invariant non-zero proper subspace. There is also an additive version of it; replace conjugacy classes $\mathcal{C}_i$ by coadjoint orbits $\mathcal{O}_i \subset \mathfrak{gl}(r, \mathbb{C})$, and replace the above equation by

$$A_1 + A_2 + \cdots + A_n = 0 \qquad (A_i \in \mathcal{O}_i).$$

It is well-known that the closure of any coadjoint orbit in $\mathfrak{gl}(r, \mathbb{C})$ can be described as the quiver variety associated to a quiver of type A, where the stability is nothing so that the resulting quiver variety is the affine quotient $\mu_V^{-1}(\zeta)/\!\!/G_V$. Based on this fact, Crawley-Boevey observed that the quiver variety associated to a *star-shaped* quiver:
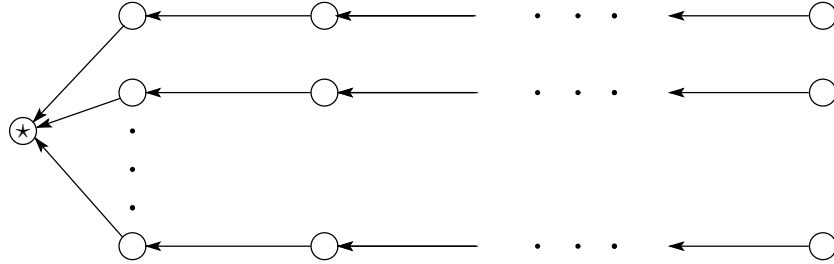
with no stability and an appropriate parameters V and ζ, is isomorphic to the variety

$$\mathcal{Q} := \{ (A_1, A_2, \ldots, A_n) \in \overline{\mathcal{O}}_1 \times \cdots \times \overline{\mathcal{O}}_n \mid A_1 + \cdots + A_n = 0 \}/\!\!/ \mathrm{GL}(r, \mathbb{C}).$$

Here the equation $\sum_i A_i = 0$ arises as the preprojective relation at the vertex $\star$. He solved the additive version [9] using this idea, and in [11], he and Shaw observed that the "multiplicative quiver variety" $\Phi_V^{-1}(q)/\!\!/G_V$ describes the multiplicative analogue of the above variety:

$$\mathcal{R} := \{ (A_1, A_2, \ldots, A_n) \in \overline{\mathcal{C}}_1 \times \cdots \times \overline{\mathcal{C}}_n \mid A_1 \cdots A_n = 1 \}/\!\!/ \mathrm{GL}(r, \mathbb{C}).$$

Note that fixing distinct n points $p_1, \ldots, p_n$ in the Riemann sphere $\mathbb{P}^1$, this variety can be considered as the moduli space of representations of the fundamental group (the *character variety*) of $\mathbb{P}^1 \setminus \{p_i\}$ whose local monodromy around each p_i belongs to $\overline{\mathcal{C}}_i$.

We have mentioned that the preprojective relation can be understood as a moment map. In fact, the multiplicative preprojective relation can be also understood as a "multiplicative analogue" of moment map, called the *group-valued moment map*. The notion of group-valued moment map was introduced by Alekseev-Malkin-Meinrenken [1], and Van den Bergh [36, 37] observed that the map Φ_V together with an appropriate 2-form satisfies the defining properties of group-valued moment map. A general theory of group-valued moment map allows us to take the "quotient" like as moment map; the quotient space $\mathcal{M}_{q,\theta}^s(V) := \Phi_V^{-1}(q)^{\theta-s}/G_V$ of the θ-stable locus has naturally a symplectic structure. We call the quotient $\mathcal{M}_{q,\theta}(V) := \Phi_V^{-1}(q)^{\theta-ss}/\!\!/G_V$ of the semistable locus the *multiplicative quiver variety*, which and its stable locus are the main objects in this paper.

Note that if we consider $\Phi_V(x)$ as a formal series in x_h, then it can be written as

$$\Phi_V(x) = 1 + \mu_V(x) + (\text{higher order terms in } x_h).$$

Thus we may expect a certain direct relation between the quiver variety and the multiplicative quiver variety. In fact, in the case of star-shaped quivers there is a *monodromy map* between them. If each $\mathcal{O}_i$ is semi-simple and eigenvalues are generic, then the variety $\mathcal{Q}$ becomes smooth and there is a map from $\mathcal{Q}$ to the variety $\mathcal{R}$ with $\mathcal{C}_i := \exp \mathcal{O}_i$ given by:

$$(A_1, \cdots, A_n) \longmapsto \text{the monodromy representation of the connection}$$

$$d - \frac{1}{2\pi\sqrt{-1}} \sum_i \frac{A_i}{z - p_i} dz \quad \text{on} \quad \mathbb{P}^1 \setminus \{p_i\}.$$

Such a map was considered by Hitchin [16] and Hausel [14] (Boalch [3, 4] considered its generalization to the case of irregular singularity). Hitchin showed that the monodromy map is a local analytic isomorphism and interchanges the symplectic structures. Hausel conjectured that under this map, the cohomology of $\mathcal{Q}$ is isomorphic to the pure part of one of $\mathcal{R}$. In this direction, he and Rodriguez-Villegas [15] suggested several interesting conjectures for the mixed Hodge polynomial of twisted character varieties of compact Riemann surfaces.

In this paper, using a property of group-valued moment map we show that:

Theorem 1.1 (Corollary 3.11)**.** *There exist an open neighborhood U (resp. U') of $[0] \in \mathcal{M}_{1,0}(V)$ (resp. $[0] \in \mathfrak{M}_{0,0}(V)$) and a commutative diagram*

$$\mathcal{M}_{1,\theta}(V) \supset \pi^{-1}(U) \xrightarrow{\tilde{f}} \pi^{-1}(U') \subset \mathfrak{M}_{0,\theta}(V)$$

$$\pi \downarrow \qquad\qquad \pi \downarrow$$

$$U \xrightarrow{\;f\;} U'$$

such that $f([0]) = [0]$ and both $\tilde{f}$ and f are complex analytic isomorphisms. Moreover $\tilde{f}$ maps the stable locus symplectomorphically onto the stable locus.

Let us consider a star-shaped quiver again. Then the associated $\mathcal{M}_{q,0}(V)$ with appropriate q, V and $\theta = 0$ gives the variety $\mathcal{R}$. Now by definition there is a natural projective morphism $\pi: \mathcal{M}_{q,\theta}(V) \to \mathcal{M}_{q,0}(V) = \Phi_V^{-1}(q)/\!\!/G_V$. We show that:

Theorem 1.2 (Theorem 4.7)**.** *Suppose that $\theta_i > 0$ for any $i \neq \star$. Then the variety $\mathcal{M}_{q,\theta}(V)$ parametrizes Simpson's polystable filtered local systems on $(\mathbb{P}^1, \{p_i\})$ of which the filtration type, weight and the monodromy of the graded local systems are prescribed by V, θ and q, respectively. Moreover π can be understood as the map taking the monodromy representation of the underlying local system.*

For the notion of filtered local system, see [34] or §4 in this paper. This notion naturally arises as an object which should correspond to a *parabolic connection* by the Riemann-Hilbert correspondence. In fact Simpson constructed such a correspondence. On the other hand, the moduli space of parabolic connections on a compact Riemann surface with marked points was constructed by Inaba-Iwasaki-Saito [18] in the case of genus 0 and rank 2, and by Inaba [17] in the case of general genus, rank and

full filtrations. We show that under certain conditions on a stability parameter, Simpson's Riemann-Hilbert correspondence gives a complex analytic symplectomorphism between such a moduli space and a star-shaped multiplicative quiver variety (see Theorem 4.13).

The paper is organized as follows:

- In §2, we give a quick review of some basic facts about quiver variety and group-valued moment map.
- In §3, we define the multiplicative quiver variety, and give some properties of it.
- §4 is devoted to the study in the case of star-shaped quivers.

Results in the rest two sections are little bit modifications of the known results.

- In §5, we show that a functor introduced by Crawley-Boevey and Shaw induces an isomorphism between two multiplicative quiver varieties whose parameters relate by certain reflections. This is a multiplicative version of Maffei's result [27].
- In §6, by the same method as Nakajima [29], we construct all irreducible highest weight representations of a Kac-Moody Lie algebra using the vector spaces of constructible functions on subvarieties of the multiplicative quiver varieties.

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2. Preliminaries

2.1. **Notation and convention.** Throughout this paper we use the following:

- (I, Ω) — a finite quiver whose vertex set is I and arrow set is Ω.
- $(I, \overline{\Omega})$ — the quiver obtained by reversing all arrows in Ω. We set $H := \Omega \sqcup \overline{\Omega}$.
- $\overline{h} \in H$ $(h \in H)$ — the reverse arrow of h.
- $\epsilon \colon H \to \{-1, 1\}$ — the map defined by $\epsilon(h) = 1 = -\epsilon(\overline{h})$ for $h \in \Omega$.
- $\mathrm{in}(h), \mathrm{out}(h) \in I$ — the incoming, outgoing vertex of $h \in H$, respectively.
- H_i $(i \in I)$ — the subset of H consisting of all h with $\mathrm{in}(h) = i$.
- $\alpha \cdot \beta$ — the standard inner product on $\mathbb{Z}^I$; $\alpha \cdot \beta = \sum_{i \in I} \alpha_i \beta_i$.
- $(\alpha, \beta) := 2\alpha \cdot \beta - \sum_{h \in H} \alpha_{\mathrm{out}(h)} \beta_{\mathrm{in}(h)}$.
- $\mathbf{e}_i$ $(i \in I)$ — the i-th coordinate vector in $\mathbb{Z}^I$.

A variety is a complex algebraic variety, not required to be irreducible or reduced. We always work over $\mathbb{C}$, and use the Zariski topology unless otherwise specified.

On a smooth variety, we will treat symplectic structures both in the algebraic sense and in the complex analytic sense. We call the former "algebraic symplectic structures", and the latter "holomorphic symplectic structures". We use the word "algebraic symplectic manifold" as a smooth variety endowed with an algebraic symplectic structure. A morphism or a holomorphic map $f \colon X \to Y$ between algebraic symplectic manifolds is *symplectic* if the pull-back $f^*\omega_Y$ of the symplectic form ω_Y coincides with ω_X.

I-graded vector spaces are always finite dimensional, and whose subspaces are always I-graded.

2.2. **Preliminaries to quiver variety.** Take a non-zero I-graded vector space $V = \bigoplus_{i \in I} V_i$. We denote by $\dim V \in \mathbb{Z}_{\geq 0}^I$ its dimension vector, i.e., $\dim V := (\dim V_i)_{i \in I}$.

Set

$$\mathbf{M}(V) := \bigoplus_{h \in H} \mathrm{Hom}(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}).$$

The reductive group $G_V := \prod_{i \in I} \mathrm{GL}(V_i)$ acts on $\mathbf{M}(V)$ by

$$g \cdot x := \left(g_{\mathrm{in}(h)} x_h g_{\mathrm{out}(h)}^{-1} \right) \quad \text{for} \quad g = (g_i) \in G_V, \ x = (x_h) \in \mathbf{M}(V).$$

We consider $\mathbb{C}^\times$ as a subgroup of G_V by

$$\mathbb{C}^\times \ni \lambda \longmapsto (\lambda 1_{V_i})_{i \in I} \in G_V.$$

Clearly this subgroup acts trivially on $\mathbf{M}(V)$.

Here we recall the notion of θ-stability introduced by King [20].

Take and fix $\theta = (\theta_i) \in \mathbb{Q}^I$ such that $\theta \cdot \dim V = 0$. For $x \in \mathbf{M}(V)$, we say a subspace $S \subset V$ is *x-invariant* if $x_h(S_{\mathrm{out}(h)}) \subset S_{\mathrm{in}(h)}$ for all $h \in H$.

Definition 2.1. We say that a point $x \in \mathbf{M}(V)$ is θ-*semistable* if any subspace $S \subset V$ satisfies $\theta \cdot \dim S \leq 0$. A point x is θ-*stable* if the strict inequality holds unless $S = 0$ or $S = V$.

Here we have changed the sign convention from [20] (this agrees with [32]). King showed that the above stability condition is equivalent to Mumford's stability condition with respect to the linearization given by the trivial bundle with the G_V-action determined by the character $\chi(g) := \prod_i \det(g_i)^{-m\theta_i}$ (see below), where m is any positive integer such that $m\theta \in \mathbb{Z}^I$ (note that the condition for θ-(semi)stability and the one for $m\theta$-(semi)stability are identical).

Set

$$\mathbf{M}_\theta^{\mathrm{ss}}(V) := \{\, x \in \mathbf{M}(V) \mid x \text{ is } \theta\text{-semistable} \,\},$$

$$\mathbf{M}_\theta^{\mathrm{s}}(V) := \{\, x \in \mathbf{M}(V) \mid x \text{ is } \theta\text{-stable} \,\}.$$

Both subsets are G_V-invariant and open.

Let $A_\theta(V)$ be the set consisting of regular functions $f \in \mathbb{C}[\mathbf{M}(V)]$ on $\mathbf{M}(V)$ such that

$$f(g \cdot x) = \chi(g) f(x) \quad \text{for any } (g, x) \in G_V \times \mathbf{M}(V),$$

and set $R_\theta(V) := \bigoplus_{n \in \mathbb{Z}_{\geq 0}} A_{n\theta}(V)$. Then the variety $\mathrm{Proj}\, R_\theta(V)$ gives a *good quotient* of $\mathbf{M}_\theta^{\mathrm{ss}}(V)$; namely, there is a surjective affine G-invariant morphism $\varphi \colon \mathbf{M}_\theta^{\mathrm{ss}}(V) \to \mathrm{Proj}\, R_\theta(V)$ such that the induced map $\varphi^* \colon \mathbb{C}[U] \to \mathbb{C}[\varphi^{-1}(U)]^G$ is an isomorphism for any affine open subset $U \subset \mathrm{Proj}\, R_\theta(V)$. Moreover, a point $x \in \mathbf{M}_\theta^{\mathrm{ss}}(V)$ is θ-stable if and only if the fiber $\varphi^{-1}(\varphi(x))$ consists of a single G_V-orbit and its dimension is equal to $\dim G_V/\mathbb{C}^\times$. In particular $\varphi(\mathbf{M}_\theta^{\mathrm{s}}(V))$ can be identified with the set-theoretic orbit space $\mathbf{M}_\theta^{\mathrm{s}}(V)/G_V$. By the last statement of the proposition below, $\varpi(\mathbf{M}_\theta^{\mathrm{s}}(V))$ is an open subset of $\mathrm{Proj}\, R_\theta(V)$.

Remark 2.2. Both θ-stability and θ-semistability are purely topological conditions. Indeed, let G_V act on $\mathbf{M}(V) \times \mathbb{C}$ by $g \cdot (x, z) := (g \cdot x, \chi(g)^{-1} z)$. Then fixing a non-zero $z \in \mathbb{C}$, a point $x \in \mathbf{M}(V)$ is θ-semistable if and only if

$$\overline{G_V \cdot (x, z)} \cap \mathbf{M}(V) \times \{0\} = \emptyset,$$

and x is θ-stable if and only if $G_V \cdot (x, z)$ is closed and its dimension is equal to $\dim G_V / \mathbb{C}^\times$ (see [20]).

Thus if $f \colon \mathbf{M}(V) \to \mathbf{M}(V)$ is a G_V-equivariant homeomorphism in the sense of usual topology, f preserves both θ-stability and θ-semistability.

We use a standard notation $/\!/$ for good quotient spaces, e.g.,

$$\mathbf{M}^{ss}_\theta(V)/\!/G_V = \operatorname{Proj} R_\theta(V).$$

A good quotient $\varphi \colon X \to Y$ of a G-variety X is called a *geometric quotient* if the induced map $X/G \to Y$ is bijective. $\mathbf{M}^s_\theta(V)$ has a geometric quotient $\mathbf{M}^s_\theta(V)/G_V$ by restricting φ.

Here we introduce several properties of good quotients (see e.g. [33]).

Proposition 2.3. *Let X be a variety acted on by a reductive group G. Suppose that a good quotient $\varphi \colon X \to Y = X/\!/G$ exists.*

(i) A good quotient (Y, φ) is a categorical *quotient; namely, (Y, φ) has the following universal property: if Z is a G-variety and $f \colon X \to Z$ is a G-invariant morphism, then there exists a unique morphism $\psi \colon Y \to Z$ such that $f = \psi \circ \varphi$. In particular Y is unique up to isomorphism.*

(ii) Two points $x, x' \in X$ have the same image $\varphi(x) = \varphi(x')$ if and only if the closures of the two orbits intersect; $\overline{G \cdot x} \cap \overline{G \cdot x'} \neq \emptyset$.

(iii) If $Z \subset X$ is a closed G-invariant subset, then $\varphi(Z) \subset Y$ is closed and the restriction $\varphi \colon Z \to \varphi(Z)$ is a good quotient.

(iv) If $U \subset X$ is open and φ-saturated (namely, $\varphi^{-1}(\varphi(U)) = U$), then $\varphi(U) \subset Y$ is open and the restriction $\varphi \colon U \to \varphi(U)$ is a good quotient.

By the above proposition, two points $x, x' \in \mathbf{M}^{ss}_\theta(V)$ have the same image under φ if and only if

$$\overline{G_V \cdot x} \cap \overline{G_V \cdot x'} \cap \mathbf{M}^{ss}_\theta(V) \neq \emptyset.$$

Since any orbit has a unique closed orbit in its closure (see e.g. [6]), the space $\mathbf{M}^{ss}_\theta(V)/\!/G_V$ parameterizes all closed G_V-orbits in $\mathbf{M}^{ss}_\theta(V)$, where "closed" means "closed in $\mathbf{M}^{ss}_\theta(V)$". A θ-semistable point $x \in \mathbf{M}^{ss}_\theta(V)$ whose orbit is closed in $\mathbf{M}^{ss}_\theta(V)$ is said to be θ-*polystable*.

Proposition 2.4 ([20, Proposition 3.2]). *(i) A point $x \in \mathbf{M}^{ss}_\theta(V)$ is θ-polystable if and only if there is a direct sum decomposition*

$$V = V^1 \oplus V^2 \oplus \cdots \oplus V^n \quad \text{where} \quad \theta \cdot \dim V^i = 0,$$

and a θ-stable point $x^i \in \mathbf{M}^s_\theta(V^i)$ for each i such that $x = x^1 \oplus x^2 \oplus \cdots \oplus x^n$, i.e., x_h is the direct sum of x^i_h's as a linear map for any $h \in H$.

(ii) Every point $x \in \mathbf{M}^{ss}_\theta(V)$ has a filtration

$$V = V^0 \supset V^1 \supset \cdots \supset V^N = 0$$

such that each V^i is x-invariant, $\theta \cdot \dim V^i = 0$ and each point $\operatorname{gr}^i x \in \mathbf{M}(V^i/V^{i+1})$ induced from x is θ-stable. Let us set $\operatorname{gr} V := \bigoplus_i V^i/V^{i+1}$ and $\operatorname{gr} x := \bigoplus_i \operatorname{gr}^i x \in \mathbf{M}(\operatorname{gr} V)$. Then under an identification $V \simeq \operatorname{gr} V$, the orbit $G_V \cdot \operatorname{gr} x$ is a unique closed orbit contained in $\overline{G_V \cdot x} \cap \mathbf{M}^{ss}_\theta(V)$. Here "closed" means "closed in $\mathbf{M}^{ss}_\theta(V)$".

We will often write a point in $\mathbf{M}_\theta^{ss}(V)$ like as $[x]$, where $x \in \mathbf{M}_\theta^{ss}(V)$ is its representative.

Note that clearly $\mathbf{M}_0^{ss}(V) = \mathbf{M}(V)$ and hence the quotient $\mathbf{M}_0^{ss}(V)/\!\!/ G_V$ must be equal to the affine quotient of $\mathbf{M}(V)$;

$$\mathbf{M}(V)/\!\!/ G_V = \operatorname{Spec} \mathbb{C}[\mathbf{M}(V)]^{G_V}.$$

This space parameterizes all closed G_V-orbits in $\mathbf{M}(V)$. By $\mathbb{C}[\mathbf{M}(V)]^{G_V} = A_0(V)$, there is a natural projective morphism

$$\pi \colon \mathbf{M}_\theta^{ss}(V)/\!\!/ G_V \to \mathbf{M}(V)/\!\!/ G_V.$$

Set-theoretically, π sends a point $[x]$ to a unique closed orbit in the closure of $G_V \cdot x$.

Proposition 2.5. *The restriction $\pi \colon \pi^{-1}(\mathbf{M}_0^s(V)/G_V) \to \mathbf{M}_0^s(V)/G_V$ is an isomorphism.*

Proof. By definition we have $\mathbf{M}_0^s(V) \subset \mathbf{M}_\theta^s(V)$. Thus the following diagram is commutative:

$$
\begin{array}{ccc}
\mathbf{M}_\theta^s(V) & \xleftarrow{\text{inclusion}} & \mathbf{M}_0^s(V) \\
\downarrow & & \downarrow \\
\mathbf{M}_\theta^{ss}(V)/\!\!/ G_V & \xrightarrow{\ \pi\ } & \mathbf{M}(V)/\!\!/ G_V
\end{array}
$$

Both of the vertical arrows are the geometric quotients. Hence the assertion follows. □

The following fact is also well-known.

Proposition 2.6. *The stabilizer of any θ-stable point $x \in \mathbf{M}_\theta^s(V)$ is equal to $\mathbb{C}^\times$.*

Proof. Suppose that $g = (g_i) \in G_V$ stabilizes $x \in \mathbf{M}_\theta^s(V)$. Then both $\bigoplus \operatorname{Im}(g_i - \lambda 1_{V_i})$ and $\bigoplus \operatorname{Ker}(g_i - \lambda 1_{V_i})$ are x-invariant subspaces of V for any $\lambda \in \mathbb{C}^\times$. Take λ to be an eigenvalue of g_i for some $i \in I$. By the stability condition, we have

(1) $$\sum_{i \in I} \theta_i \dim \operatorname{Im}(g_i - \lambda 1_{V_i}) \le 0, \qquad \sum_{i \in I} \theta_i \dim \operatorname{Ker}(g_i - \lambda 1_{V_i}) \le 0.$$

On the other hand, we have

$$\sum_{i \in I} \theta_i (\dim \operatorname{Im}(g_i - \lambda 1_{V_i}) + \dim \operatorname{Ker}(g_i - \lambda 1_{V_i})) = \theta \cdot \dim V = 0.$$

Thus the previous two inequalities must be equalities. By the stability condition and the choice of λ, we must have $g = \lambda$. □

For each subspace $S \subset V$, the natural inclusion $\mathbf{M}(S) \hookrightarrow \mathbf{M}(V)$ induces a morphism

$$\mathbf{M}(S)/\!\!/ G_S \to \mathbf{M}(V)/\!\!/ G_V.$$

It is a closed immersion. This follows immediately from the following fact.

Proposition 2.7 ([25, Theorem 1.3])**.** *The invariant subring $\mathbb{C}[\mathbf{M}(V)]^{G_V}$ is generated by functions of the form $\operatorname{Tr}(x_{h_1} x_{h_2} \cdots x_{h_n})$, where $(h_1, h_2, \ldots, h_n)$ is a cycle in H; namely, a sequence in H such that*

$$\operatorname{out}(h_1) = \operatorname{in}(h_2), \ \operatorname{out}(h_2) = \operatorname{in}(h_3), \ \ldots, \ \operatorname{out}(h_{n-1}) = \operatorname{in}(h_n), \ \operatorname{out}(h_n) = \operatorname{in}(h_1).$$

2.3. **Quiver variety.** Let us define the quiver variety. First we define a map $\mu_V \colon \mathbf{M}(V) \to \operatorname{Lie} G_V :=$ $\bigoplus \mathfrak{gl}(V_i)$ by

$$\mu_V(x) := \left(\sum_{h \in H_i} \epsilon(h) x_h x_{\overline{h}} \right)_{i \in I}.$$

It is equivariant with respect to the action of G_V. Thus for a central element $\zeta \in \mathbb{C}^I$ of $\operatorname{Lie} G_V$, the subset $\mu(\zeta)$ is a G_V-invariant closed subvariety of $\mathbf{M}(V)$.

Definition 2.8. For a given $(\zeta, \theta) \in \mathbb{C}^I \times \mathbb{Q}^I$ with $\theta \cdot \dim V = 0$, the *quiver variety* is the good quotient

$$\mathfrak{M}_{\zeta,\theta}(V) := \mathbf{M}_\theta^{\mathrm{ss}}(V) \cap \mu_V^{-1}(\zeta) /\!\!/ G_V.$$

It is well-defined by Proposition 2.3. The equation $\mu_V(x) = \zeta$ is called the *(deformed) preprojective relation*.

The map μ_V has a remarkable property which we explain from now.

Let M be a smooth variety acted on by a reductive algebraic group G. For $\xi \in \operatorname{Lie} G$, we denote by ξ^* the vector field induced from the infinitesimal action of ξ; namely,

$$\xi_x^* := \frac{d}{dt} \exp(t\xi) \cdot x \bigg|_{t=0} \quad \text{for } x \in M.$$

Definition 2.9. A *Hamiltonian G-structure* on M is a pair consisting of a G-invariant 2-form ω on M and a morphism $\mu \colon M \to (\operatorname{Lie} G)^*$, which is equivariant with respect to the coadjoint action on $(\operatorname{Lie} G)^*$, such that:

(H1) $d\omega = 0$;

(H2) $\iota(\xi^*)\omega = d\langle \mu, \xi \rangle$ for any $\xi \in \operatorname{Lie} G$;

(H3) $\operatorname{Ker} \omega_x = 0$ for each $x \in M$.

Here $\operatorname{Ker} \omega_x := \{ v \in T_x M \mid \iota(v)\omega_x = 0 \}$. The triple (M, ω, μ) is called a *Hamiltonian G-space* and μ is called the *moment map*.

We define an algebraic symplectic form ω on $\mathbf{M}(V)$ by

$$\omega := \sum_{h \in \Omega} \operatorname{Tr} dx_h \wedge dx_{\overline{h}} + \sum_{i \in I} \operatorname{Tr} da_i \wedge db_i.$$

Note that $\operatorname{Lie} G_V$ can be identified with its dual by the trace. It is easy to see that, under this identification, the triple $(\mathbf{M}(V), \omega, \mu)$ is a Hamiltonian G_V-space.

Thus the open subvariety

$$\mathfrak{M}_{\zeta,\theta}^{\mathrm{s}}(V) := \mathbf{M}_\theta^{\mathrm{s}}(V) \cap \mu_V^{-1}(\zeta) / G_V$$

of $\mathfrak{M}_{\zeta,\theta}(V)$ is an algebraic symplectic manifold by the following well-known fact.

Theorem 2.10. *Let (M, ω, μ) be a Hamiltonian G-space and $\zeta \in (\operatorname{Lie} G)^*$ be a fixed point with respect to the coadjoint action. Suppose that the stabilizer of each point in $\mu^{-1}(\zeta)$ is trivial. Then $\mu^{-1}(\zeta)$ is smooth. Moreover if a geometric quotient $\mu^{-1}(\zeta)/G$ exists, then it becomes an algebraic symplectic manifold, and for each point $x \in \mu^{-1}(\zeta)$, the tangent space of $\mu^{-1}(\zeta)/G$ at the point represented by x can be naturally identified with the quotient space $\operatorname{Ker} d_x\mu / T_x(G \cdot x)$.*

In the case $\zeta = 0$, we will denote $\mathfrak{M}_\theta(V) = \mathfrak{M}_{0,\theta}(V)$, $\mathfrak{M}_\theta^{\mathrm{s}}(V) = \mathfrak{M}_{0,\theta}^{\mathrm{s}}(V)$.

2.4. **Quasi-Hamiltonian structure.** A notion of *quasi-Hamiltonian structure*, which was intro-duced by Alekseev-Malkin-Meinrenken [1] for C^∞-manifolds with a compact Lie group action, is a "multiplicative" analogue of Hamiltonian structure. This subsection is a quick review of its complex algebraic version which was already treated by Boalch [5] and Van den Bergh [36, 37].

Let G be a reductive algebraic group and $\mathrm{Lie}\,G$ be its Lie algebra. For simplicity, we assume that G is a closed subgroup of $\mathrm{GL}(N, \mathbb{C})$ for some N, and that the symmetric form $\mathrm{Tr}\colon \mathrm{Lie}\,G \otimes \mathrm{Lie}\,G \to \mathbb{C}$ induced from the trace is non-degenerate.

We define

$$\chi := \frac{1}{6}\,\mathrm{Tr}\,(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg) = \frac{1}{6}\,\mathrm{Tr}\,(dg\,g^{-1} \wedge dg\,g^{-1} \wedge dg\,g^{-1}),$$

where $g^{-1}dg$ (resp. $dg\,g^{-1}$) is the left-invariant (resp. right-invariant) Maurer-Cartan form on G.

Definition 2.11. A *quasi-Hamiltonian G-space* is a smooth G-variety M together with a G-invariant 2-form ϖ on M and a G-equivariant morphism $\Phi\colon M \to G$ (where we have let G act on itself by the conjugation) such that:

(QH1) $d\varpi = -\Phi^*\chi$;

(QH2) $\iota(\xi^*)\varpi = \frac{1}{2}\,\mathrm{Tr}\,\xi(\Phi^{-1}d\Phi + d\Phi\,\Phi^{-1})$ for any $\xi \in \mathrm{Lie}\,G$;

(QH3) $\mathrm{Ker}\,\varpi_x = \{\,\xi_x^* \mid \xi \in \mathrm{Ker}(\mathrm{Ad}_{\Phi(x)} +1)\,\}$ for each $x \in M$.

Φ is called the *group-valued moment map*.

There are two typical examples of quasi-Hamiltonian G-space.

Example 2.12 ([1, Proposition 3.1])**.** Let $\mathcal{C} \subset G$ be a conjugacy class with the conjugation action of G. Then there is a quasi-Hamiltonian G-structure on $\mathcal{C}$ whose group-valued moment map is just the inclusion $\mathcal{C} \to G$. Indeed the 2-form is uniquely determined by the condition (QH2) since the action is transitive, and one can easily check that it actually exists.

Example 2.13 ([1, Proposition 3.2])**.** Consider the direct product $G \times G$. Let $G \times G$ act on itself by $(g, h) \cdot (a, b) := (gah^{-1}, hbg^{-1})$. Define a 2-form ϖ on $G \times G$ by

$$\varpi = \frac{1}{2}\,\mathrm{Tr}\,\left(a^{-1}da \wedge db\,b^{-1}\right) - \frac{1}{2}\,\mathrm{Tr}\,\left(b^{-1}db \wedge da\,a^{-1}\right).$$

Then ϖ together with the map

$$G \times G \to G \times G; \qquad (a, b) \mapsto (ab, a^{-1}b^{-1})$$

gives a quasi-Hamiltonian $G \times G$-structure on $G \times G$.

Recall that for a Hamiltonian G-space and a closed subgroup $K \subset G$, the induced K-action is also Hamiltonian in a natural way. Unfortunately, this is not true for a quasi-Hamiltonian G-space in general. However, if $G = K \times K$ and $K \subset G$ is the diagonal subgroup, then an analogous statement holds.

Theorem 2.14 ([1, §6])**.** *Let $(M, \varpi, \Phi = (\Phi_1, \Phi_2, \Psi))$ be a quasi-Hamiltonian $G \times G \times K$-space. Let $G \times K$ act by the diagonal embedding $(g, k) \mapsto (g, g, k)$.*

(1) M with a 2-form

$$\varpi_{12} := \varpi + \frac{1}{2}\operatorname{Tr}\left(\Phi_1^{-1}d\Phi_1 \wedge d\Phi_2\,\Phi_2^{-1}\right)$$

and a morphism

$$(\Phi_{12}, \Psi) := (\Phi_1 \cdot \Phi_2, \Psi)\colon M \to G \times K$$

is a quasi-Hamiltonian $G \times K$-space (This space is called the (internal) fusion).

(2) If we define

$$\varpi_{21} := \varpi + \frac{1}{2}\operatorname{Tr}\left(\Phi_2^{-1}d\Phi_2 \wedge d\Phi_1\,\Phi_1^{-1}\right),$$

$$\Phi_{21} := \Phi_2 \cdot \Phi_1\colon M \to G,$$

then (ϖ_{21}, Φ_{21}) also defines a quasi-Hamiltonian $G \times K$-structure on M. Moreover it is isomorphic to $(M, \varpi_{12}, (\Phi_{12}, \Psi))$.

(3) Let (M, ϖ, Φ) be a quasi-Hamiltonian $G \times G \times G \times K$-space. Let $(\varpi_{(12)3}, \Phi_{(12)3})$ be the quasi-Hamiltonian $G \times K$-structure obtained by first fusioning the first two G-factors, and let $(\varpi_{1(23)}, \Phi_{1(23)})$ be that obtained by first fusioning the last two G-factors. Then the two structures coincide.

The following theorem is a quasi-Hamiltonian version of Theorem 2.10, which provides a new method to construct algebraic symplectic manifolds.

Theorem 2.15 (cf. [1, Theorem 5.1]). *Let $(M, \varpi, (\Phi_1, \Phi_2))$ be a quasi-Hamiltonian $G_1 \times G_2$-space and f be a central element of G_1. Suppose that the stabilizer of each point in $\Phi_1^{-1}(f)$ is trivial. Then $\Phi_1^{-1}(f)$ is a smooth subvariety of M. Moreover if a geometric quotient $\Phi_1^{-1}(f)/G_1$ exists, then $\Phi_1^{-1}(f)/G_1$ becomes a quasi-Hamiltonian G_2-space, and for each point $x \in \Phi_1^{-1}(f)$, the tangent space of $\Phi_1^{-1}(f)$ at the point represented by x can be naturally identified with the quotient space $\operatorname{Ker} d_x\Phi_1/T_x(G_1 \cdot x)$.*

Note that if G_2 is trivial, then the resulting quotient space carries a quasi-Hamiltonian $\{1\}$-structure, which is nothing but an algebraic symplectic structure.

We will use the following example in the next section.

Example 2.16 ([36, 37]). Let V, W be two $\mathbb{C}$-vector spaces. Set

$$M = \operatorname{Hom}(W, V) \oplus \operatorname{Hom}(V, W),$$

$$M^\circ = \{\,(a, b) \in M \mid \det(1 + ab) \neq 0\,\}.$$

We define a 2-form ϖ on M° by

$$\varpi = \frac{1}{2}\operatorname{Tr}(1 + ab)^{-1}da \wedge db - \frac{1}{2}\operatorname{Tr}(1 + ba)^{-1}db \wedge da,$$

and we define a map $(\phi, \psi)\colon M^\circ \to \operatorname{GL}(V) \times \operatorname{GL}(W)$ by

$$\phi(a, b) = 1 + ab, \qquad \psi(a, b) = (1 + ba)^{-1}.$$

Then $(M^\circ, \varpi, \Phi = (\phi, \psi))$ is a quasi-Hamiltonian $\operatorname{GL}(V) \times \operatorname{GL}(W)$-space. The proof needs a long calculation (see [36]). We remark that this quasi-Hamiltonian structure is invertible; the map $\iota\colon M^\circ \to M^\circ$ defined by $\iota(a, b) := (-(1+ab)^{-1}a, b)$ satisfies $\iota^*\varpi = -\varpi$ and $\iota^*(\phi, \psi) = (\phi^{-1}, \psi^{-1})$. It was given by Crawley-Boevey and Shaw [11].

3. Multiplicative quiver variety

3.1. **Definition.** Let us define the main objects in this paper. It is motivated by the paper [11] of Crawley-Boevey and Shaw, who considered a "multiplicative" analogue of the preprojective relation. We require the stability condition for solutions of this equation to obtain a new variety.

Set

$$\mathbf{M}^\circ(V) := \{\, x \in \mathbf{M}(V) \mid \det(1 + x_h x_{\overline{h}}) \neq 0 \text{ for all } h \in H \,\}.$$

Since the function $x \mapsto \prod_{h \in H} \det(1 + x_h x_{\overline{h}})$ is constant along each G_V-orbit, it is a G_V-invariant open subset of $\mathbf{M}(V)$, and the intersection $\mathbf{M}^\circ(V) \cap \mathbf{M}^{\mathrm{ss}}_\theta(V)$ is φ-saturated.

Fix a total order $<$ on H. We define a map $\Phi = (\Phi_i)_{i \in I} \colon \mathbf{M}^\circ(V) \to G_V$ by

$$\Phi_i(x) := \prod_{h \in H_i}^{<} (1 + x_h x_{\overline{h}})^{\epsilon(h)}.$$

We sometimes write $\Phi = \Phi_V$ to emphasize the vector space V. Φ_V is G_V-equivariant with respect to the conjugation. Hence, for any $q \in (\mathbb{C}^\times)^I \subset G_V$, $\Phi_V^{-1}(q)$ is a G_V-invariant closed subvariety of $\mathbf{M}^\circ(V)$. Thus by Proposition 2.3, the subvariety $\mathbf{M}^{\mathrm{ss}}_\theta(V) \cap \Phi_V^{-1}(q)$ has a good quotient, and the subvariety $\mathbf{M}^{\mathrm{s}}_\theta(V) \cap \Phi_V^{-1}(q)$ has a geometric quotient.

Definition 3.1. We define

$$\mathcal{M}_{q,\theta}(V) := \left(\mathbf{M}^{\mathrm{ss}}_\theta(V) \cap \Phi_V^{-1}(q) \right) /\!\!/ G_V,$$

which we call the *multiplicative quiver variety*.

We also define

$$\mathcal{M}^{\mathrm{s}}_{q,\theta}(V) := \left(\mathbf{M}^{\mathrm{s}}_\theta(V) \cap \Phi_V^{-1}(q) \right) / G_V.$$

The equation $\Phi(x) = q$ is called the *multiplicative preprojective relation*. We also use the following notation:

$$\mathcal{M}_\theta(V) = \mathcal{M}_{1,\theta}(V), \qquad \mathcal{M}^{\mathrm{s}}_\theta(V) = \mathcal{M}^{\mathrm{s}}_{1,\theta}(V).$$

$\mathbf{M}^\circ(V)$ has a quasi-Hamiltonian G_V-structure.

Proposition 3.2 ([36, 37])**.** *We define a 2-form ϖ on $\mathbf{M}^\circ(V)$ by*

$$\varpi := \frac{1}{2} \sum_{h \in H} \epsilon(h) \operatorname{Tr} (1 + x_h x_{\overline{h}})^{-1} dx_h \wedge dx_{\overline{h}}$$

$$+ \frac{1}{2} \sum_{h \in H} \operatorname{Tr} \Phi_h^{-1} d\Phi_h \wedge d(1 + x_h x_{\overline{h}})^{\epsilon(h)} (1 + x_h x_{\overline{h}})^{-\epsilon(h)},$$

where

$$\Phi_h := \prod_{h' \in H_i; h' < h}^{<} (1 + x_{h'} x_{\overline{h'}})^{\epsilon(h')} \quad for \ h \in H_i.$$

Then $(\mathbf{M}^\circ(V), \varpi, \Phi)$ is a quasi-Hamiltonian G_V-space.

This proposition was proved by Van den Bergh as the following.

Proof. Set

$$M_h := \operatorname{Hom}(V_{\operatorname{out}(h)}, V_{\operatorname{in}(h)}) \oplus \operatorname{Hom}(V_{\operatorname{in}(h)}, V_{\operatorname{out}(h)}) \quad \text{for } h \in \Omega,$$

and define M_h° as in Example 2.16. Then M_h° has a quasi-Hamiltonian $\operatorname{GL}(V_{\operatorname{in}(h)}) \times \operatorname{GL}(V_{\operatorname{out}(h)})$-structure whose group-valued moment map is

$$(x_h, x_{\overline{h}}) \mapsto \left(1 + x_h x_{\overline{h}}, (1 + x_{\overline{h}} x_h)^{-1}\right).$$

Taking a direct product, we obtain a quasi-Hamiltonian G-structure on $\mathbf{M}^\circ(V)$, where G is given by

$$G = G_V \times \prod_{h \in \Omega} \operatorname{GL}(V_{\operatorname{in}(h)}) \times \prod_{h \in \Omega} \operatorname{GL}(V_{\operatorname{out}(h)})$$

$$= G_V \times \prod_{h \in H} \operatorname{GL}(V_{\operatorname{in}(h)}).$$

Take the internal fusion among the $\operatorname{GL}(V_{\operatorname{in}(h)})$-factors inductively on the total order $<$. Then we get a quasi-Hamiltonian $G_V \times G_V$-structure on $\mathbf{M}^\circ(V)$. Fusioning further the G_V-factors, we obtain finally the desired structure. $\qquad\square$

Note that the above construction of a quasi-Hamiltonian structure depends both on the total order $<$ on H and on the orientation ϵ. (Here an *orientation* is a function $\epsilon' \colon H \to \{-1, 1\}$ satisfying $\epsilon'(\overline{h}) = -\epsilon'(h)$ for all $h \in H$.) However the following holds.

Proposition 3.3. *A quasi-Hamiltonian structure obtained by the method in Proposition 3.2 does not depend on the total order or the orientation up to isomorphism.*

Proof. The assertion follows immediately from Theorem 2.14 and the invertible property of M° mentioned in Example 2.16. $\qquad\square$

It is easy to see that any quasi-Hamiltonian G_V-structure (ϖ, Φ) on $\mathbf{M}^\circ(V)$ naturally descends to a quasi-Hamiltonian $G_V/\mathbb{C}^\times$-structure whose group-valued moment map $\overline{\Phi}$ is obtained by the composition of Φ with the projection $G_V \to G_V/\mathbb{C}^\times$. Notice that for any $x \in \mathbf{M}^\circ(V)$, we have

$$\prod_i \det \Phi_i(x) = \prod_{h \in H_i \cap \Omega} \det(1 + x_h x_{\overline{h}}) \prod_{h \in H_i \cap \overline{\Omega}} \det(1 + x_h x_{\overline{h}})^{-1} = 1.$$

Hence if $\Phi_V^{-1}(q) \neq \emptyset$, q must satisfies the equality

$$q^{\dim V} := \prod_{i \in I} q_i^{\dim V_i} = 1.$$

Moreover if $q^{\dim V} = 1$ then the level set $\Phi_V^{-1}(q)$ coincides with $\overline{\Phi}_V^{-1}(q \mod \mathbb{C}^\times))$. Thus Theorem 2.15 and dimension count implies the following theorem.

Theorem 3.4. $\mathcal{M}_{q,\theta}^s(V)$ *is a pure-dimensional algebraic symplectic manifold, and its dimension is* $2 - (\dim V, \dim V)$.

Proof. Since a quasi-Hamiltonian $\{1\}$-structure is nothing but an algebraic symplectic structure, the first assertion follows from Proposition 2.6. To compute the dimension of $\mathcal{M}_{q,\theta}^s(V)$, note that

$$(\dim V, \dim V) = 2 \sum_i (\dim V_i)^2 - \sum_{h \in H} \left(\dim V_{\operatorname{out}(h)}\right) \left(\dim V_{\operatorname{in}(h)}\right).$$

Since $\dim \mathcal{M}_{q,\theta}^s(V) = \dim \mathbf{M}(V) - 2 \dim G_V/\mathbb{C}^\times$, the assertion follows immediately. $\qquad\square$

Finally we introduce a criterion for the smoothness of $\mathcal{M}_{q,\theta}(V)$.

Set $\mathbf{v} := \dim V$ and

$$R_+ := \{\, \alpha \in \mathbb{Z}_{\geq 0}^I \mid (\alpha, \alpha) \leq 2 \,\} \setminus \{0\},$$

$$R_+(\mathbf{v}) := \{\, \alpha \in R_+ \mid \mathbf{v} - \alpha \in \mathbb{Z}_{\geq 0}^I \,\},$$

$$D_\alpha := \{\, \theta \in \mathbb{Q}^I \mid \theta \cdot \alpha = 0 \,\},$$

$$E_\alpha := \{\, z \in (\mathbb{C}^\times)^I \mid z^\alpha = 1 \,\} \quad \text{for } \alpha \in R_+.$$

Proposition 3.5. (1) $\mathcal{M}_{q,\theta}^{\mathrm{s}}(V)$ *is empty unless* $\mathbf{v} \in R_+$ *and* $(q, \theta) \in E_{\mathbf{v}} \times D_{\mathbf{v}}$.
(2) *If*

$$(q, \theta) \in E_{\mathbf{v}} \times D_{\mathbf{v}} \setminus \bigcup_{\alpha \in R_+(\mathbf{v}) \setminus \{\mathbf{v}\}} E_\alpha \times D_\alpha,$$

then $\mathcal{M}_{q,\theta}^{\mathrm{s}}(V) = \mathcal{M}_{q,\theta}(V)$.

Proof. We have already proved the first assertion. Suppose that there exists a point $x \in \Phi_V^{-1}(q)$ which is θ-semistable but not θ-stable. Then we can find a non-zero proper x-invariant subspace $S \subset V$ such that $\theta \cdot \dim S = 0$. We may assume that S is minimal amongst all non-zero subspaces satisfying such conditions. Set $\alpha := (\dim S_i)$. Then $\theta \in D_\alpha$. Let $x' \in \mathbf{M}(S)$ be the element obtained by the restriction of x to S. Then $\Phi_S(x) = q$ and hence $q \in E_\alpha$. If $T \subset S$ is x'-invariant, then $\theta \cdot \dim T \leq 0$ by the θ-semistability of x, and moreover if $\theta \cdot \dim T = 0$, then $T = 0$ or $T = S$ by the choice of S. Thus x' is θ-stable. In particular $\mathcal{M}_{q,\theta}^{\mathrm{s}}(S)$ is non-empty, so

$$0 \leq \dim \mathcal{M}_{q,\theta}^{\mathrm{s}}(S) = 2 - (\alpha, \alpha).$$

Thus $\alpha \in R_+(\mathbf{v})$. □

3.2. **Some properties.** By Proposition 3.3, we may assume the following:

The total order $<$ satisfies $h < h'$ for any $h \in \Omega$ and $h' \in \overline{\Omega}$.

Then we can decompose $\Phi_i = \Phi_i^+ (\Phi_i^-)^{-1}$, where

$$\Phi_i^+(x) := \prod_{h \in H_i \cap \Omega}^< (1 + x_h x_{\overline{h}}),$$

$$\Phi_i^-(x) := \prod_{h \in H_i \cap \overline{\Omega}}^> (1 + x_h x_{\overline{h}}).$$

Thus the multiplicative preprojective relation $\Phi_i(x) = q_i$ at $i \in I$ is equivalent to

(2) $$\Phi_i^+(x) - q_i \Phi_i^-(x) = 0.$$

Here $\Phi_i^\pm$ is expanded as

$$\Phi_i^+ = 1 + \sum_{h \in H_i \cap \Omega} \Phi_h^+ x_h x_{\overline{h}},$$

$$\Phi_i^- = 1 + \sum_{h \in H_i \cap \overline{\Omega}} x_h x_{\overline{h}} \Phi_h^-,$$

where

$$\Phi_h^+(x) := \prod_{\substack{h' \in H_i \cap \Omega; \\ h' < h}}^{<} (1 + x_{h'} x_{\overline{h'}}),$$

$$\Phi_h^-(x) := \prod_{\substack{h' \in H_i \cap \overline{\Omega}; \\ h' < h}}^{>} (1 + x_{h'} x_{\overline{h'}}) \quad \text{for } h \in H_i.$$

Thus (2) is equivalent to

$$(3) \qquad \sum_{h \in H_i \cap \Omega} \Phi_h^+ x_h x_{\overline{h}} - q_i \sum_{h \in H_i \cap \overline{\Omega}} x_h x_{\overline{h}} \Phi_h^- = q_i - 1.$$

Set $\widehat{V}_i := \bigoplus_{h \in H_i} V_{\mathrm{out}(h)}$. For $h \in H_i$, let $\iota_h \colon V_{\mathrm{out}(h)} \to \widehat{V}_i$ be the natural inclusion and $\pi_h \colon \widehat{V}_i \to V_{\mathrm{out}(h)}$ be the projection. We define

$$\sigma_i(x) := \sum_{h \in H_i \cap \Omega} \iota_h x_{\overline{h}} + \sum_{h \in H_i \cap \overline{\Omega}} \iota_h x_{\overline{h}} \Phi_h^- \colon V_i \to \widehat{V}_i,$$

$$\tau_i(x) := \sum_{h \in H_i \cap \Omega} \Phi_h^+ x_h \pi_h - q_i \sum_{h \in H_i \cap \overline{\Omega}} x_h \pi_h \colon \widehat{V}_i \to V_i.$$

Then by (3), the multiplicative preprojective relation at $i \in I$ is equivalent to $\tau_i \sigma_i = q_i - 1$. In particular, the sequence

$$V_i \xrightarrow{\ \sigma_i\ } \widehat{V}_i \xrightarrow{\ \tau_i\ } V_i$$

is a complex if $\Phi_i(x) = 1$.

Lemma 3.6. *Take $x \in \mathbf{M}^\circ(V)$ and $i \in I$. Suppose that a subspace $S \subset V$ satisfies:*

   (i) $\sigma_i(S_i) \subset \widehat{S}_i$; and
   (ii) $\tau_i(\widehat{S}_i) \subset S_i$.

Then $x_h(S_{\mathrm{out}(h)}) \subset S_{\mathrm{in}(h)}$ for $h \in H_i \cup \overline{H}_i$.

Proof. Suppose that $S \subset V$ satisfies the conditions (i) and (ii). The condition (i) means

$$(4) \qquad\qquad x_{\overline{h}}(S_i) \subset S_{\mathrm{out}(h)} \quad \text{for } h \in H_i \cap \Omega,$$

$$(5) \qquad\qquad x_{\overline{h}} \Phi_h^-(S_i) \subset S_{\mathrm{out}(h)} \quad \text{for } h \in H_i \cap \overline{\Omega},$$

and the condition (ii) means

$$(6) \qquad\qquad \Phi_h^+ x_h(S_{\mathrm{out}(h)}) \subset S_i \quad \text{for } h \in H_i \cap \Omega,$$

$$(7) \qquad\qquad x_h(S_{\mathrm{out}(h)}) \subset S_i \quad \text{for } h \in H_i \cap \overline{\Omega}.$$

Let $h \in H_i \cap \Omega$ be the minimum element in $H_i \cap \Omega$ with respect to $<$. Then $\Phi_h^+ = 1$ and hence $x_h(S_{\mathrm{out}(h)}) \subset S_i$ by (6). Thus $(1 + x_h x_{\overline{h}})(S_i) \subset S_i$ by (4), and hence one can use induction on $<$ to obtain $x_h(S_{\mathrm{out}(h)}) \subset S_{\mathrm{in}(h)}$ for all $h \in H_i \cap \Omega$.

Similarly, if we denote by $h' \in H_i \cap \overline{\Omega}$ the minimum element in $H_i \cap \overline{\Omega}$, then $\Phi_{h'}^- = 1$ and hence $x_{\overline{h'}}(S_i) \subset S_{\mathrm{out}(h')}$ by (5). Thus $(1 + x_{h'} x_{\overline{h'}})(S_i) \subset S_i$ by (7), and hence one can use induction again to obtain $x_{\overline{h}}(S_i) \subset S_{\mathrm{out}(h)}$ for all $h \in H_i \cap \overline{\Omega}$. $\qquad\square$

Proposition 3.7. *Take $x \in \mathbf{M}^\circ(V) \cap \mathbf{M}_\theta^{\mathrm{s}}(V)$ and $i \in I$. Suppose that $\dim V \neq \mathbf{e}_i$.*

(i) *If $\theta_i \geq 0$, then σ_i is injective.*

(ii) *If $\theta_i \leq 0$, then τ_i is surjective.*

Proof. Suppose that $\theta_i \geq 0$. Set

$$S_j = \begin{cases} 0 & \text{if } j \neq i, \\ \operatorname{Ker} \sigma_i & \text{if } j = i. \end{cases}$$

By Lemma 3.6, S is x-invariant. Hence we have $\theta \cdot \dim S \leq 0$ by the stability condition. However

$$\theta \cdot \dim S = \theta_i \dim \operatorname{Ker} \sigma_i \geq 0,$$

so we must have $\theta \cdot \dim S = 0$. Thus $S = 0$ or $S = V$ by the stability condition again. If $S = 0$ we are done. So assume that $S = V$. Then $V_j = 0$ for all $j \neq i$, and hence $x = 0$. Thus any subspace of V is x-invariant, and hence $V_i = \mathbb{C}$ by the stability condition again. This contradicts.

Next suppose that $\theta_i \leq 0$. Set

$$T_j = \begin{cases} V_j & \text{if } j \neq i, \\ \operatorname{Im} \tau_i & \text{if } j = i. \end{cases}$$

By Lemma 3.6, T is x-invariant. Hence we have $\theta \cdot \dim T \leq 0$ by the stability condition. However we have also

$$\theta \cdot \dim T = \theta \cdot \dim V - \theta_i \dim \operatorname{Coker} \tau_i \geq \theta \cdot \dim V = 0.$$

Thus $\theta \cdot \dim T = 0$, which implies that $T = 0$ or $T = V$. If $T = V$ we are done. If $T = 0$ one can deduce a contradiction as above. $\square$

3.3. Singularity at the origin; relation to the quiver variety.

In this subsection we work in the complex analytic category. The following proposition implies that the singularity at the origin of $\Phi_V^{-1}(1)$ and that of $\mu_V^{-1}(0)$ are the same. Recall that we let $\varphi \colon \mathbf{M}(V) \to \mathbf{M}(V) /\!/ G_V$ be the quotient morphism.

Proposition 3.8. *There are φ-saturated open neighborhoods $\mathcal{U}$, $\mathcal{U}'$ of $0 \in \mathbf{M}(V)$, and a G_V-equivariant biholomorphic map $f \colon \mathcal{U} \to \mathcal{U}'$ such that*

$$f(0) = 0, \qquad f(\Phi_V^{-1}(1) \cap \mathcal{U}) = \mu_V^{-1}(0) \cap \mathcal{U}', \qquad (f^*\omega - \varpi)|_{\operatorname{Ker} d\Phi_V} = 0.$$

Proof. We use the following result of Alekseev-Malkin-Meinrenken (they proved it in the case that G is a compact Lie group, but the proof can be extended immediately to the case of complex reductive group).

Lemma 3.9 ([1, Lemma 3.3]). *Let $G \subset \mathrm{GL}(N, \mathbb{C})$ be a complex reductive Lie group. For $s \in [0, 1]$, let $\exp_s \colon \operatorname{Lie} G \to G$ denote a map given by $\exp_s(\xi) := \exp(s\xi)$. Define a holomorphic 2-form ρ on G by*

$$\rho := \frac{1}{2} \int_0^1 ds \, \operatorname{Tr} \left[\exp_s^*(dg \, g^{-1}) \wedge \frac{\partial}{\partial s} \exp_s^*(dg \, g^{-1}) \right].$$

Then ρ is G-invariant and satisfies

$$d\rho = -\exp^* \chi, \qquad \iota(\xi^*)\rho = -d \operatorname{Tr}(\xi \cdot) + \frac{1}{2} \exp^* \operatorname{Tr} \xi (g^{-1} dg + dg \, g^{-1}).$$

Using this, they observed that if (M, ϖ, Φ) is a quasi-Hamiltonian G-space and $\Phi(M)$ is contained in an open subset of G on which an G-equivariant right-inverse log of exp exists, then the triple $(M, \varpi - \Phi^* \log^* \rho, \log \circ \Phi)$ satisfies the conditions (H1) and (H2) [1, Remark 3.2] as follows:

$$d(\varpi - \Phi^* \log^* \rho) = -\Phi^* \chi + \Phi^* \log^* \exp^* \chi = 0,$$

$$\iota(\xi^*)(\varpi - \Phi^* \log^* \rho) = \frac{1}{2} \Phi^* \operatorname{Tr} \xi(g^{-1}dg + dg\, g^{-1}) + \Phi^* \log^* d\operatorname{Tr}(\xi \cdot)$$

$$- \frac{1}{2} \Phi^* \log^* \exp^* \operatorname{Tr} \xi(g^{-1}dg + dg\, g^{-1})$$

$$= \operatorname{Tr}(d(\log \Phi)(\cdot)\, \xi).$$

In fact, we can always find a G-invariant open neighborhood O of 0 in $\operatorname{Lie} G$ such that the restriction $\exp \colon O \to \exp(O)$ has the inverse $\log := (\exp)^{-1}$. Clearly we can take O to be saturated with respect to the quotient map $\operatorname{Lie} G \to (\operatorname{Lie} G)//G$. Then we can apply the above fact to $(\Phi^{-1}(O), \varpi, \Phi)$.

Let us back to our situation. First take a φ-saturated open subset $\mathcal{U}$ to be such that for any $x \in \mathcal{U}$ and $h \in H$, $1 + x_h x_{\overline{h}} \in \exp(O)$, where $O \subset \operatorname{Lie} \operatorname{GL}(V_{\operatorname{in}(h)})$ is the subset taken as above. Then $(\mathcal{U}, \varpi - \Phi_V^* \log^* \rho, \log \circ \Phi_V)$ satisfies (H1) and (H2). Since $x_h x_{\overline{h}} = 0$ and $d(1 + x_h x_{\overline{h}}) = 0$ at the origin, we have $(\Phi_V^* \log^* \rho)_0 = 0$ and

$$\varpi_0 = \frac{1}{2} \sum_{h \in H} \epsilon(h) \operatorname{Tr} dx_h \wedge dx_{\overline{h}} + \frac{1}{2} \sum_{h \in H} \operatorname{Tr} d\Phi_h \wedge d(1 + x_h x_{\overline{h}})^{\epsilon(h)}$$

$$= \frac{1}{2} \sum_{h \in H} \epsilon(h) \operatorname{Tr} dx_h \wedge dx_{\overline{h}} = \omega_0.$$

Thus the 2-form $\varpi - \Phi_V^* \log^* \rho$ coincides with the symplectic form ω at the origin. By the equivariant Darboux theorem (see Remark 3.10 below), taking $\mathcal{U}$ to be small enough if necessary, there is a G_V-equivariant biholomorphic map $f \colon \mathcal{U} \to \mathcal{U}'$ to some φ-saturated open neighborhood $\mathcal{U}'$ such that

$$f(0) = 0, \qquad f^* \omega = \varpi - \Phi_V^* \log^* \rho, \qquad \mu_V \circ f = \log \circ \Phi_V.$$

This gives the desired map since the form $\Phi_V^* \log^* \rho$ vanishes on $\operatorname{Ker} d\Phi_V$. $\qquad\square$

Remark 3.10. The equivariant Darboux theorem asserts for C^∞-manifolds with a compact Lie group action. However we can generalize this theorem to our case as the following (This is due to Nakajima. See [31]).

It is easy to see that the equivariant Darboux theorem can be generalized for complex manifolds with a compact Lie group action. Thus, in order to show our claim, we first apply the theorem for the maximal compact subgroup $U_V := \prod \operatorname{U}(V_i) \subset G_V$. Then there is an open ball $B \in \mathbf{M}(V)$ centered at the origin and a U_V-equivariant open embedding $f \colon B \to \mathbf{M}(V)$ such that $f(0) = 0$. By [35, Proposition 1.4, Lemma 1.14], we can extend uniquely this map to a G_V-equivariant open embedding $f \colon G_V \cdot B \to \mathbf{M}(V)$. We claim that $G_V \cdot B$ is φ-saturated. This can be proved using the map $F_\infty \colon \mathbf{M}(V) \to \mathbf{M}(V)$ introduced in [35], which is U_V-equivariant and has the following property: for any point $x \in \mathbf{M}(V)$, F_∞ maps $G_V \cdot x$ onto $U_V \cdot y$, where y is a point whose G_V-orbit is a unique closed orbit in $\overline{G_V \cdot x}$. Moreover $F_\infty(G_V \cdot B) \subset B$ and there is a continuous family $\{F_t\}$ of diffeomorphisms whose limit is F_∞ and the differential $dF_t(x)/dt|_{t=0}$ at any x is tangent to the orbit $G_V \cdot x$. Thus if $x, x' \in \mathbf{M}(V)$ have the same image under φ and $x \in G_V \cdot B$, then $F_\infty(x') \in U_V \cdot F_\infty(x) \subset B$, and hence $F_t(x') \in B$ for sufficiently large t. Thus $x' \in G_V \cdot B$. Hence $G_V \cdot B$ is φ-saturated.

We take some open ball B' in $f(G_V \cdot B)$ centered at the origin, and set $\mathcal{U} := f^{-1}(G_V \cdot B')$. Then $\mathcal{U}$ is also φ-saturated. To see this, suppose $x' \in \mathbf{M}(V)$ is in the orbit closure $\overline{G_V \cdot x}$ of some $x \in \mathcal{U}$. Then $x' \in G_V \cdot B$ by the above argument, and $f(x') \in \overline{G_V \cdot f(x)}$ since f is continuous. Since $G_V \cdot B'$ is φ-saturated and $f(x) \in G_V \cdot B'$, we see that $f(x') \in G_V \cdot B'$. Thus $x' \in \mathcal{U}$.

Setting $\mathcal{U}' := G_V \cdot B'$, we obtain a desired map $f \colon \mathcal{U} \to \mathcal{U}'$.

Recall the projective morphisms $\pi \colon \mathcal{M}_\theta(V) \to \mathcal{M}_0(V)$ and $\pi \colon \mathfrak{M}_\theta(V) \to \mathfrak{M}_0(V)$.

Corollary 3.11. *There exist an open neighborhood U (resp. U') of $[0] \in \mathcal{M}_0(V)$ (resp. $[0] \in \mathfrak{M}_0(V)$) and a commutative diagram*

$$
\begin{array}{ccc}
\mathcal{M}_\theta(V) \supset \pi^{-1}(U) & \xrightarrow{\ \tilde{f}\ } & \pi^{-1}(U') \subset \mathfrak{M}_\theta(V) \\
\pi \downarrow & & \pi \downarrow \\
U & \xrightarrow{\ f\ } & U'
\end{array}
$$

such that:

(i) $f([0]) = [0]$;

(ii) *both $\tilde{f}$ and f are complex analytic isomorphisms;*

(iii) *$\tilde{f}$ maps $\pi^{-1}(U) \cap \mathcal{M}_\theta^{\mathrm{s}}(V)$ onto $\pi^{-1}(U') \cap \mathfrak{M}_\theta^{\mathrm{s}}(V)$ as a symplectic biholomorphic map; and*

(iv) *if $x \in \varphi^{-1}(U)$ and $y \in \varphi^{-1}(U')$ have closed orbits and $f([x]) = [y]$, then the stabilizers of the two are conjugate. Thus f preserves the orbit-type.*

Proof. Since both $\mathcal{U}$ and $\mathcal{U}'$ are φ-saturated and $f \colon \mathcal{U} \to \mathcal{U}'$ is a biholomorphic map, f sends a closed orbit to a closed orbit and a stable/semistable point to a stable/semistable point (see Remark 2.2). So the result follows. $\qquad\square$

The fiber $\pi^{-1}([0]) \subset \mathcal{M}_\theta(V)$ is called the *nilpotent subvariety*. The above corollary implies that the nilpotent subvarieties of the quiver variety and the multiplicative quiver variety are complex analytically isomorphic.

4. Moduli of filtered local systems and star-shaped quiver

This section is devoted to the study in the case of star-shaped quivers. In particular, we prove Theorem 1.2.

4.1. Star-shaped quiver.
Suppose that conjugacy classes $\mathcal{C}_1, \ldots \mathcal{C}_n$ in $\mathfrak{gl}(r, \mathbb{C})$ for a fixed $r > 0$ are given. Choose $A_i \in \mathcal{C}_i$ and take $\xi_{i,j} \in \mathbb{C}^\times$ which satisfies
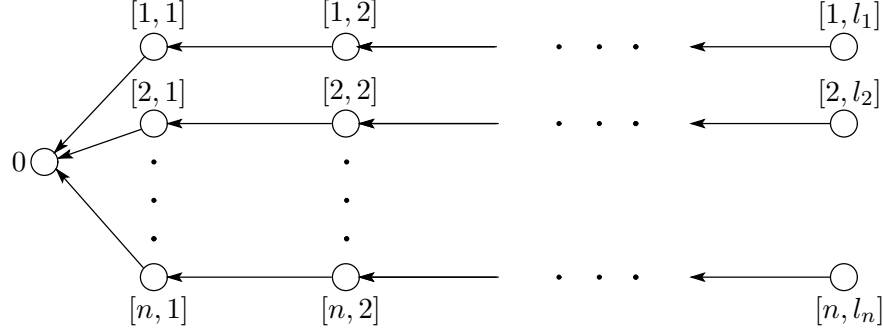
$$
(A_i - \xi_{i,0})(A_i - \xi_{i,1}) \cdots (A_i - \xi_{i,r}) = 0.
$$

Set

$$
v_{i,j} = \mathrm{rank}(A_i - \xi_{i,0}) \cdots (A_i - \xi_{i,j-1}), \qquad l_i = \min\{j; v_{i,j} > 0\}.
$$

Note that each $v_{i,j}$ does not depend on a choice of A_i.

Following Crawley-Boevey, we associate to $\mathcal{C}_1, \ldots \mathcal{C}_n$ the following quiver (I, Ω):

Such a quiver is called a *star-shaped* quiver. We denote the vertex set by $I = \{0\} \cup \{[i,j]\}$ as in the picture, and set $I_0 := I \setminus \{0\}$. We define an I-graded vector space V by

$$V_0 := \mathbb{C}^r, \qquad V_{i,j} := \mathbb{C}^{v_{i,j}} \quad \text{for } [i,j] \in I_0,$$

and use the convention $V_{i,0} = V_0$ and $V_{i,l_i+1} = 0$. For an element $x \in \mathbf{M}(V)$ we will denote its components by $a_{i,j} \in \mathrm{Hom}(V_{i,j+1}, V_{i,j})$, $b_{i,j} \in \mathrm{Hom}(V_{i,j}, V_{i,j+1})$ and write simply as $x = (a,b)$.

The following proposition was proved by Crawley-Boevey (and Shaw).

Proposition 4.1. (i) *Define*

$$\zeta_0 := -\sum_{i=1}^{n} \xi_{i,0}, \qquad \zeta_{i,j} := \xi_{i,j-1} - \xi_{i,j} \quad \text{for } [i,j] \in I_0.$$

Then the morphism

$$\mathfrak{M}_{\zeta,0}(V) \to \{\, (B_1, \ldots, B_n) \in \overline{\mathcal{C}_1} \times \cdots \times \overline{\mathcal{C}_n} \mid B_1 + \cdots + B_n = 0 \,\} /\!\!/ \mathrm{GL}(r, \mathbb{C}),$$

$$x = (a,b) \mapsto B_i = \xi_{i,0} + a_{i,0}b_{i,0}$$

is an isomorphism. Moreover, the variety of the right hand side includes

$$\mathcal{Q} := \{\, (B_1, \ldots, B_n) \in (\mathcal{C}_1 \times \cdots \times \mathcal{C}_n)^{\mathrm{irr}} \mid B_1 + \cdots + B_n = 0 \,\} / \mathrm{GL}(r, \mathbb{C})$$

as the image of $\mathfrak{M}_{\zeta,0}^{\mathrm{s}}(V)$ *under the above map. Here,* $(\mathcal{C}_1 \times \cdots \times \mathcal{C}_n)^{\mathrm{irr}}$ *denotes the set consisting of all* $(B_1, \ldots, B_n) \in \mathcal{C}_1 \times \cdots \times \mathcal{C}_n$ *such that there is no non-zero proper subspace* $S \subset \mathbb{C}^r$ *which is preserved by* B_i *for any* i.

(ii) *Suppose that each* $\xi_{i,j}$ *is non-zero. Define* $q \in (\mathbb{C}^\times)^I$ *by*

$$q_0 := \prod_i \xi_{i,0}^{-1}, \qquad q_{i,j} := \frac{\xi_{i,j-1}}{\xi_{i,j}} \quad \text{for } [i,j] \in I_0.$$

Then the morphism

$$\mathcal{M}_{q,0}(V) \to \{\, (B_1, \ldots, B_n) \in \overline{\mathcal{C}_1} \times \cdots \times \overline{\mathcal{C}_n} \mid B_1 \cdots B_n = 1 \,\} /\!\!/ \mathrm{GL}(r, \mathbb{C}),$$

$$x = (a,b) \mapsto B_i = \xi_{i,0}(1 + a_{i,0}b_{i,0})$$

is an isomorphism. Moreover, the variety of the right hand side includes

$$\mathcal{R} := \{\, (B_1, \ldots, B_n) \in (\mathcal{C}_1 \times \cdots \times \mathcal{C}_n)^{\mathrm{irr}} \mid B_1 \cdots B_n = 1 \,\} / \mathrm{GL}(r, \mathbb{C})$$

as the image of $\mathcal{M}_{q,0}^{\mathrm{s}}(V)$ *under the above map.*

Proof. For a proof of (i), see [8, 9]. (ii) also can be proved similarly, using [10, Theorem 2.1] and the method of Kraft-Procesi [21]. $\square$

Remark 4.2. Recall that every coadjoint orbit has a canonical symplectic structure. Thus identifying $\mathfrak{gl}(r, \mathbb{C})$ with its dual via the trace, each $\mathcal{C}_i$ carries naturally an algebraic symplectic structure. The product of these symplectic forms defines an algebraic symplectic structure on $\prod_{i=1}^n \mathcal{C}_i$, and it has a moment map for the $\mathrm{GL}(r, \mathbb{C})$-action given by $(B_1, \dots, B_n) \mapsto \sum B_i$. Thus $\mathcal{Q}$ also carries naturally an algebraic symplectic structure by Theorem 2.10. Then one can prove that the restriction of the map defined in (i) is a symplectic isomorphism between $\mathfrak{M}_{\zeta,0}^{\mathrm{s}}(V)$ and $\mathcal{Q}$.

Recall further that every conjugacy class $\mathcal{C} \subset G$ of a complex reductive group has a canonical quasi-Hamiltonian G-structure (see [1]). So under the same assumption as in (ii), the product $\prod_i \mathcal{C}_i$ carries a quasi-Hamiltonian $\mathrm{GL}(r, \mathbb{C})$-structure by Theorem 2.14. Its group-valued moment map is $(B_1, \dots, B_n) \mapsto \prod B_i$, and hence the variety $\mathcal{R}$ carries an algebraic symplectic structure by Theorem 2.15. One can also prove that the map defined in (ii) induces a symplectic isomorphism between $\mathcal{M}_{q,0}^{\mathrm{s}}(V)$ and $\mathcal{R}$.

From now on, we assume that $\xi_{i,j} \neq 0$ for all i, j. Take n distinct points $p_1, \dots, p_n$ in the Riemann sphere $\mathbb{P}^1$, and set $D = \{p_1, \dots, p_n\}$. Choose a base point $* \in \mathbb{P}^1 \setminus D$ and consider the fundamental group $\pi_1(\mathbb{P}^1 \setminus D, *)$. It has a presentation $\langle \gamma_1, \gamma_2, \dots, \gamma_n \mid \gamma_1 \cdots \gamma_n = 1 \rangle$, where γ_i represents a loop going from $*$ toward near p_i, once around counterclockwise and back to $*$. Hence the map

$$\mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus D, *), \mathrm{GL}(r, \mathbb{C})) \to \{\, (A_1, \dots, A_n) \in \mathrm{GL}(r, \mathbb{C})^n \mid A_1 \cdots A_n = 1 \,\},$$

$$\rho \mapsto A_i = \rho(\gamma_i)$$

is bijective. If we consider $\mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus D, *), \mathrm{GL}(r, \mathbb{C}))$ as an affine algebraic variety via this bijection, then the space $\mathcal{M}_{q,0}(V)$ can be described as

$$\{\, \rho \in \mathrm{Hom}(\pi_1(\mathbb{P}^1 \setminus D, *), \mathrm{GL}(r, \mathbb{C})) \mid \rho(\gamma_i) \in \overline{\mathcal{C}_i} \,\} /\!/ \mathrm{GL}(r, \mathbb{C}).$$

In fact, the multiplicative quiver variety $\mathcal{M}_{q,\theta}(V)$ with $\theta_{i,j} > 0$ can be also described as a moduli space of local systems on $\mathbb{P}^1 \setminus D$ equipped with a certain additional structure, called a *filtered structure*.

4.2. Filtered local system.
Suppose that the stability parameter $\theta \in \mathbb{Q}^I$ satisfies

$$\theta_{i,j} > 0, \quad \text{and} \quad \theta \cdot \dim V = 0, \text{ i.e., } \theta_0 = -\frac{\sum_{[i,j] \in I_0} \theta_{i,j} \dim V_{i,j}}{\dim V_0}.$$

Let $x = (a, b) \in \Phi_V^{-1}(q)$ be a θ-semistable point. For $i = 1, \dots, n$, define a filtration $F_i = (F_i^j)$ of V_0 by

$$F_i^0(V_0) := V_0, \qquad F_i^j(V_0) := \mathrm{Im}\, a_{i,0} \cdots a_{i,j-1} \quad \text{for } j = 1, 2, \dots, l_i + 1,$$

and set $A_i := \xi_{i,0}(1 + a_{i,0} b_{i,0}) \in \mathrm{GL}(V_0)$.

Lemma 4.3. *For each $[i, j] \in I_0$, we have:*

(i) $(A_i - \xi_{i,j})(F_i^j) \subset F_i^{j+1}$; *and*

(ii) $\dim F_i^j = v_{i,j}$.

Proof. Using induction on j, we first prove the following formula which implies (i):

$$(A_i - \xi_{i,j}) a_{i,0} \cdots a_{i,j-1} = \xi_{i,j} a_{i,0} \cdots a_{i,j-1} a_{i,j} b_{i,j}.$$

If $j = 0$, by definition we have $A_i - \xi_{i,0} = \xi_{i,0} a_{i,0} b_{i,0}$. If $j > 0$, using the hypothesis of induction we have

$$(A_i - \xi_{i,j}) a_{i,0} \cdots a_{i,j-1} = (A_i - \xi_{i,j-1}) a_{i,0} \cdots a_{i,j-2} a_{i,j-1} + (\xi_{i,j-1} - \xi_{i,j}) a_{i,0} \cdots a_{i,j-1}$$

$$= \xi_{i,j-1} a_{i,0} \cdots a_{i,j-1} b_{i,j-1} a_{i,j-1} + (\xi_{i,j-1} - \xi_{i,j}) a_{i,0} \cdots a_{i,j-1}$$

$$= \xi_{i,j-1} a_{i,0} \cdots a_{i,j-1} (1 + b_{i,j-1} a_{i,j-1}) - \xi_{i,j} a_{i,0} \cdots a_{i,j-1}.$$

By the multiplicative preprojective relation at $[i, j]$, we have $1 + b_{i,j-1} a_{i,j-1} = q_{i,j}^{-1}(1 + a_{i,j} b_{i,j})$. Thus we obtain the desired formula as the following:

$$(A_i - \xi_{i,j}) a_{i,0} \cdots a_{i,j-1} = q_{i,j}^{-1} \xi_{i,j-1} a_{i,0} \cdots a_{i,j-1} (1 + a_{i,j} b_{i,j}) - \xi_{i,j} a_{i,0} \cdots a_{i,j-1}$$

$$= \xi_{i,j} a_{i,0} \cdots a_{i,j-1} (1 + a_{i,j} b_{i,j}) - \xi_{i,j} a_{i,0} \cdots a_{i,j-1}$$

$$= \xi_{i,j} a_{i,0} \cdots a_{i,j-1} a_{i,j} b_{i,j}.$$

To prove (ii), it is enough to show that each $a_{i,j-1}$ is injective. Fix $[i, j] \in I_0$ and define a subspace $S \subset V$ by

$$S_0 := 0, \qquad S_{k,m} := \begin{cases} 0 & \text{if } k \neq i \text{ or } m < j, \\ \operatorname{Ker} a_{i,j-1} & \text{if } [k, m] = [i, j], \\ b_{i,m-1} b_{i,m-2} \cdots b_{i,j} (\operatorname{Ker} a_{i,j-1}) & \text{if } k = i \text{ and } m > j. \end{cases}$$

As above one can easily prove the following formula:

$$\xi_{i,j} a_{i,m-1} b_{i,m-1} b_{i,m-2} \cdots b_{i,j} = \xi_{i,m-2} b_{i,m-2} \cdots b_{i,j} b_{i,j-1} a_{i,j-1} + (\xi_{i,m-2} - \xi_{i,j}) b_{i,m-2} \cdots b_{i,j}.$$

This implies that S is (a, b)-invariant. By the stability condition we have

$$\sum_{[k,m] \in I_0} \theta_{k,m} \dim S_{k,m} = \theta \cdot \dim S \leq 0.$$

Since $\theta_{k,m} > 0$ we must have $S_{k,m} = 0$ for each $[k, m]$. Thus $a_{i,j-1}$ is injective. $\square$

The multiplicative preprojective relation at 0 implies $\prod A_i = 1$. Thus setting $\rho(\gamma_i) = A_i$ ($i = 1, \ldots n$), we get a representation ρ of $\pi_1(\mathbb{P}^1 \setminus D, *)$ on V_0. Let L be the corresponding local system on $\mathbb{P}^1 \setminus D$. For $i = 1, \ldots, n$, let U_i be a simply connected open neighborhood of p_i which contains γ_i, and we set $U_i^* = U_i \setminus \{p_i\}$. Note that $\pi_1(U_i^*, *)$ is a free group generated by γ_i. Thus $\rho(\gamma_i)$ determines a representation of $\pi_1(U_i^*, *)$ on V_0 which corresponds to the restriction of L on U_i^*. Since each $F_i^j \subset V_0$ is preserved by $\rho(\gamma_i)$, it induces a local subsystem $\mathbb{F}_i^j(L)$ of $L|_{U_i^*}$. So we get a filtration

$$\mathbb{F}_i \colon L|_{U_i^*} = \mathbb{F}_i^0(L) \supset \mathbb{F}_i^1(L) \supset \cdots \supset \mathbb{F}_i^{l_i+1}(L) = 0$$

by local subsystems of $L|_{U_i^*}$. Note that the local monodromy of $\mathbb{F}_i^j(L)/\mathbb{F}_i^{j+1}(L)$ ($j = 0, 1, \ldots, l_i$) around p_i is given by the scalar multiplication by $\xi_{i,j}$.

Lemma 4.4. $(L, \mathbb{F})$ *satisfies the following property:*

(†) *For any non-zero proper local subsystem $M \subset L$, the following inequality holds:*

$$\frac{\sum_{[i,j] \in I_0} \theta_{i,j} \operatorname{rank} \left(M \cap \mathbb{F}_i^j(L) \right)}{\operatorname{rank} M} \leq \frac{\sum_{[i,j] \in I_0} \theta_{i,j} \operatorname{rank} \mathbb{F}_i^j(L)}{\operatorname{rank} L}.$$

If (a, b) is θ-stable, then the strict inequality holds.

Proof. For a non-zero proper local subsystem $M \subset L$, define a subspace $S \subset V$ by

$$S_0 = M_* \subset V_0, \qquad S_{i,j} = (a_{i,0} \cdots a_{i,j-1})^{-1}(M_* \cap \mathbb{F}_i^j(L)_*),$$

where $M_*, \mathbb{F}_i^j(L)_*$ mean the stalks at $*$. Then S is (a,b)-invariant and non-zero proper by the assumption. On the other hand, $\theta \cdot \dim V = 0$ implies

$$\theta \cdot \dim S = \theta_0 \operatorname{rank} M + \sum_{i,j} \theta_{i,j} \operatorname{rank}(M \cap \mathbb{F}_i^j(L))$$

$$= -\frac{\sum_{i,j} \theta_{i,j} \operatorname{rank} \mathbb{F}_i^j(L)}{\operatorname{rank} L} \operatorname{rank} M + \sum_{i,j} \theta_{i,j} \operatorname{rank}(M \cap \mathbb{F}_i^j(L)).$$

Thus $\theta \cdot \dim S \le 0$ (resp. < 0) if and only if the inequality (resp. the strict inequality) in (†) holds. So the assertion follows. $\qquad\square$

Motivated on the above argument, we introduce the following notion.

Definition 4.5. Let X be a compact Riemann surface and let $D \subset X$ be a finite subset. Let L be a local system on $X \setminus D$. For a tuple of non-negative integers $l = (l_p)_{p \in D}$, a *filtered structure* on L of *filtration type* l is a tuple $(U_p, \mathbb{F}_p)_{p \in D}$, where for each $p \in D$:

(i) U_p is a neighborhood of p in X (we set $U_p^* := U_p \setminus \{p\}$); and

(ii) $\mathbb{F}_p$ is a filtration

$$L|_{U_p^*} = \mathbb{F}_p^0(L) \supset \mathbb{F}_p^1(L) \supset \cdots \supset \mathbb{F}_p^{l_p}(L) \supset \mathbb{F}_p^{l_p+1}(L) = 0$$

by local subsystems of $L|_{U_p^*}$.

Two filtered structures $(U_p, \mathbb{F}_p)_{p \in D}, (U_p', \mathbb{F}_p')_{p \in D}$ of the same filtration type are *equivalent* if for each $p \in D$, there exists a neighborhood $V_p \subset U_p \cap U_p'$ of p such that $\mathbb{F}_p$ and $\mathbb{F}_p'$ coincide on V_p^*. A local system L together with an equivalence class of filtered structures $\mathbb{F} = [(U_p, \mathbb{F}_p)_{p \in D}]$ is called a *filtered local system* on (X, D) of filtration type l.

Definition 4.6. Let $(L, \mathbb{F})$ be a filtered local system on (X, D) of filtration type l. Let $\beta = (\beta_p^j \mid p \in D, \ j = 0, \ldots, l_p)$ be a tuple of rational numbers satisfying $\beta_p^i < \beta_p^j$ for any p and $i < j$ (Such a tuple is called a *weight*).

$(L, \mathbb{F})$ on (X, D) is said to be β-*semistable* if for any non-zero proper local subsystem $M \subset L$ the following inequality holds:

$$\sum_{p \in D} \sum_j \beta_p^j \frac{\operatorname{rank}\left(M \cap \mathbb{F}_p^j(L)\right) / \left(M \cap \mathbb{F}_p^{j+1}(L)\right)}{\operatorname{rank} M} \le \sum_{p \in D} \sum_j \beta_p^j \frac{\operatorname{rank}\left(\mathbb{F}_p^j(L)/\mathbb{F}_p^{j+1}(L)\right)}{\operatorname{rank} L}.$$

$(L, \mathbb{F})$ is β-*stable* if the strict inequality always holds.

Clearly, $(L, \mathbb{F})$ constructed from a θ-semistable point $x = (a, b) \in \Phi_V^{-1}(q) \cap \mathbf{M}_\theta^{ss}(V)$ defines a filtered local system on $(\mathbb{P}^1, \{p_i\})$, where the filtration type l is given by $l_{p_i} := l_i$. Moreover this filtered local system satisfies the stability condition. Fix arbitrary $\beta_i^0 \in \mathbb{Q}$ for each i and set $\beta_{p_i}^j := \beta_i^0 + \sum_{s=1}^j \theta_{i,s}$. Then we have

$$\sum_{i=1}^n \sum_{j \ge 0} \beta_{p_i}^j \frac{\operatorname{rank} \mathbb{F}_{p_i}^j(L)/\mathbb{F}_{p_i}^{j+1}(L)}{\operatorname{rank} L} = \sum_{i=1}^n \beta_{i,0} + \sum_{[i,j] \in I_0} \theta_{i,j} \frac{\operatorname{rank} \mathbb{F}_{p_i}^j(L)}{\operatorname{rank} L},$$

so the β-semistability condition for filtered local systems on $(\mathbb{P}^1, \{p_i\})$ is equivalent to the property (†), and the β-stability condition is equivalent to that the strict inequality always holds in (†). In particular our $(L, \mathbb{F})$ is β-semistable, and if x is θ-stable then $(L, \mathbb{F})$ is β-stable.

It is easy to see that the above construction sends a G_V-orbit in $\Phi_V^{-1}(q) \cap \mathbf{M}_\theta^{ss}(V)$ to an isomorphism class of filtered local systems, and preserves the direct sum operation, where the direct sum of two filtered local systems of the same filtration type means the direct sum of local systems with filtrations induced from those of the two. In particular, this map sends a θ-polystable point $x = x_1 \oplus x_2 \oplus \cdots \oplus x_N$ to a direct sum of β-stable filtered local systems $(L, \mathbb{F}) = (L_1, \mathbb{F}_1) \oplus (L_2, \mathbb{F}_2) \oplus \cdots \oplus (L_N, \mathbb{F}_N)$. Note that each $(L_i, \mathbb{F}_i)$ satisfies

$$\sum_{p \in D} \sum_j \beta_p^j \frac{\mathrm{rank}\left((\mathbb{F}_i)_p^j(L_i)/(\mathbb{F}_i)_p^{j+1}(L_i)\right)}{\mathrm{rank}\, L_i} = \sum_{p \in D} \sum_j \beta_p^j \frac{\mathrm{rank}\left(\mathbb{F}_p^j(L)/\mathbb{F}_p^{j+1}(L)\right)}{\mathrm{rank}\, L},$$

since $\theta \cdot \dim V^i = 0$ (see Proposition 2.4). Such a filtered local system is said to be β-*polystable*.

Conversely, suppose that a β-semistable filtered local system $(L, \mathbb{F})$ of filtration type l with $\mathrm{rank}\, L = r$, $\mathrm{rank}\, L = v_{i,j}$ is given. Suppose that the local monodromy of $\mathbb{F}_{p_i}^j(L)/\mathbb{F}_{p_i}^{j+1}$ around p_i is given by the scalar multiplication $\xi_{i,j}$ for all i, j. We define an I-graded vector space V by $V_0 := L_*$, $V_{i,j} := \mathbb{F}_{p_i}^j(L)_*$, and define a point $(a, b) \in \mathbf{M}(V)$ by

$$b_{i,j} := (\xi_{i,j}^{-1}\rho(\gamma_i) - 1)|_{V_{i,j}} : V_{i,j} \to V_{i,j+1}, \qquad a_{i,j} : V_{i,j+1} \hookrightarrow V_{i,j} \quad \text{the inclusion.}$$

Then $(a, b) \in \mathbf{M}(V)$ satisfies the multiplicative preprojective relation. To check the stability condition, suppose that a non-zero proper (a, b)-invariant subspace $S \subset V$ is given. Then there is a local subsystem $M \subset L$ whose stalk at $*$ is S_0. By the property (†), we have

$$\frac{\sum_{[i,j]\in I_0} \theta_{i,j} \mathrm{rank}\left(M \cap \mathbb{F}_{p_i}^j(L)\right)}{\mathrm{rank}\, M} \leq \frac{\sum_{[i,j]\in I_0} \theta_{i,j} \mathrm{rank}\, \mathbb{F}_{p_i}^j(L)}{\mathrm{rank}\, L}.$$

Since $a_{i,j}$'s are injective, we have $\dim S_{i,j} \leq \mathrm{rank}(M \cap \mathbb{F}_{p_i}^j(L))$, and hence

$$\frac{\sum_{[i,j]\in I_0} \theta_{i,j} \dim S_{i,j}}{\dim S_0} \leq \frac{\sum_{[i,j]\in I_0} \theta_{i,j} \dim V_{i,j}}{\dim V_0},$$

which implies $\theta \cdot \dim S \leq 0$. Thus (a, b) is θ-semistable. Clearly, if $(L, \mathbb{F})$ is β-stable then (a, b) is θ-stable. It is also easy to see that this construction sends an isomorphism class of filtered local systems to a G_V-orbit, and preserves the polystability.

We have obtained maps of both directions between $\mathcal{M}_{q,\theta}(V)$ and the set of isomorphism classes of β-polystable filtered local systems $(L, \mathbb{F})$ of filtration type l satisfying $\mathrm{rank}\, L = r$, $\mathrm{rank}\, \mathbb{F}_{p_i}^j(L) = v_{i,j}$ and that the local monodromy of $\mathbb{F}_{p_i}(L)/\mathbb{F}_{p_i}^{j+1}(L)$ is given by the scalar $\xi_{i,j}$ for all i, j. Clearly each one is the inverse of the other. So we get the following result.

Theorem 4.7. *Let $D = \{p_1, \ldots, p_n\}$ be a finite subset of $\mathbb{P}^1$ with cardinality n. Take an arbitrary $l \in \mathbb{Z}_{\geq 0}^D$, and let $\xi = (\xi_p^j \mid p \in D, \; j = 0, \ldots, l_p)$ be a tuple of non-zero complex numbers, $\beta = (\beta_p^j \mid p \in D, \; j = 0, \ldots, l_p)$ be a tuple of rational numbers satisfying $\beta_p^i < \beta_p^j$ for any p and $i < j$. Take a star-shaped quiver (I, Ω) with n arms such that the length l_i of the i-th arm is equal to l_{p_i}. Then for*

any I-graded vector space V, setting $(q, \theta) \in (\mathbb{C}^\times)^I \times \mathbb{Q}^I$ by

$$\theta_{i,j} := \beta_{p_i}^j - \beta_{p_i}^{j-1}, \qquad\qquad \theta_0 := -\frac{\sum_{[i,j] \in I_0} \theta_{i,j} \dim V_{i,j}}{\dim V_0},$$

$$q_{i,j} := \xi_{p_i}^{j-1}/\xi_{p_i}^j, \qquad\qquad q_0 := \prod_i (\xi_{p_i}^0)^{-1},$$

there is a natural bijection between the multiplicative quiver variety $\mathcal{M}_{q,\theta}(V)$ and the set of isomorphism classes of β-polystable filtered local systems $(L, \mathbb{F})$ on $(\mathbb{P}^1, D)$ satisfying:

- $\operatorname{rank} L = \dim V_0$, $\operatorname{rank} \mathbb{F}_{p_i}^j(L) = \dim V_{i,j}$;
- *the local monodromy of $\mathbb{F}_{p_i}^j(L)/\mathbb{F}_{p_i}^{j+1}(L)$ around p_i is given by the scalar multiplication by $\xi_{p_i}^j$ for all i, j.*

Under this map, a point in $\mathcal{M}_{q,\theta}^s(V)$ corresponds to an isomorphism class of β-stable filtered local systems.

Remark 4.8. The word "filtered local system" is originally due to Simpson [34]. Simpson's filtered local system $(L, \mathbb{F})$ is a pair of a local system L on $X \setminus D$ and a tuple $\mathbb{F} = (\mathbb{F}_p)_{p \in D}$, where for each $p \in D$, $\mathbb{F}_p = (\mathbb{F}_p^\beta)_{\beta \in \mathbb{R}}$ is a filtration of the restriction $L|_{U_p^*}$ of L on some punctured neighborhood U_p^* of p indexed by real number $\beta \in \mathbb{R}$. $\mathbb{F}_p^\beta$ is required to be left continuous, i.e., $\mathbb{F}_p^{\beta-\varepsilon} = \mathbb{F}_p^\beta$ for small $\varepsilon > 0$. Filtered local systems in the sense of Simpson form a category on which direct sum, tensor product, dual, etc. are defined. Moreover the notion of "degree" for a filtered local system in the sense of Simpson is naturally defined and provides a slope stability condition. Our notion of filtered local system $(L, \mathbb{F})$ together with a weight β can be considered as Simpson's filtered local system as follows: for each $\beta \in \mathbb{R}$, define $\mathbb{F}_p^\beta(L) \subset L|_{U_p^*}$ by

$$\mathbb{F}_p^\beta(L) := \begin{cases} L|_{U_p^*} & \text{when } \beta \leq \beta_p^0, \\ \mathbb{F}_p^j(L) & \text{when } \beta \in (\beta_p^{j-1}, \beta_p^j], \\ 0 & \text{when } \beta > \beta_p^{l_p}. \end{cases}$$

Then $(L, \{\mathbb{F}_p^\beta\})$ is a filtered local system in the sense of Simpson. Moreover one can easily check that if our $(L, \mathbb{F})$ is β-stable/semistable/polystable, then $(L, \{\mathbb{F}_p^\beta\})$ is stable/semistable/polystable. Note that in the previous theorem, the star-shaped multiplicative quiver varieties parametrize only polystable filtered local systems $(L, \{\mathbb{F}_p^\beta\})$ such that the local monodromy of $\mathbb{F}_p^\beta/\mathbb{F}_p^{>\beta}$ around p is *scalar* for each p, β. This is because we have considered only the case that all $\theta_{i,j}$'s are positive. In fact, if we allow $\theta_{i,j} = 0$ for some i, j, then a point in the multiplicative quiver variety represents a polystable filtered local system $(L, \{\mathbb{F}_p^\beta\})$ such that the local monodromy of $\mathbb{F}_p^\beta/\mathbb{F}_p^{>\beta}$ is in the closure of some fixed conjugacy class, which may not be a scalar.

4.3. Riemann-Hilbert correspondence.

Definition 4.9. Let X be a compact Riemann surface and let $D \subset X$ be a finite subset. A *logarithmic connection* (E, ∇) on (X, D) is a pair of a holomorphic vector bundle E on X and a morphism of sheaves $\nabla \colon E \to E \otimes \Omega_X^1(\log D)$ satisfying the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla(s) \quad \text{for } f \in \mathcal{O}_X, s \in E,$$

where we have used the same symbol E for the sheaf of holomorphic sections of E, $\Omega^1_X(\log D)$ is the sheaf of meromorphic 1-forms on X with logarithmic poles on D and no poles on $X \setminus D$, and $\mathcal{O}_X$ is the sheaf of holomorphic functions on X.

For each $p \in D$, a logarithmic connection (E, ∇) induces canonically an endomorphism

$$\mathrm{Res}_p \nabla \colon E|_p \to E|_p$$

of the fiber $E|_p$ of E at p. Such an endomorphism is called the *residue* of (E, ∇) at p. Using a trivialization $E|_{U_p} \simeq U_p \times \mathbb{C}^r$ on a neighborhood of p and a local coordinate z centered at p, the logarithmic connection ∇ is written as $\nabla = d + A(z)dz/z$ for some holomorphic function $A(z)$. Then $\mathrm{Res}_p \nabla = A(0)$.

One can also define the notion of logarithmic connection in the algebro-geometric sense. However by GAGA, it is equivalent to the above notion.

By Deligne's Riemann-Hilbert correspondence [12], there is a natural equivalence between the category of local systems L on $X \setminus D$ and the category of logarithmic connections (E, ∇) on (X, D) such that the real parts of eigenvalues of the residue $\mathrm{Res}_p \nabla$ are in $[0, 1)$ for any $p \in D$. A "filtered" version of it was proved by Simpson [34]. To explain it, we introduce a "filtered" structure on a logarithmic connection, so-called a *parabolic structure*.

Definition 4.10. Let X be a compact Riemann surface and let $D \subset X$ be a finite subset. Let (E, ∇) be a logarithmic connection on (X, D).

For $l = (l_p)_{p \in D} \in \mathbb{Z}^D_{\geq 0}$, a *parabolic structure* on (E, ∇) of *filtration type l* is a tuple $\mathcal{F} = (\mathcal{F}_p)_{p \in D}$, where for each $p \in D$, $\mathcal{F}_p$ is a filtration

$$E|_p = \mathcal{F}^0_p(E) \supset \mathcal{F}^1_p(E) \supset \cdots \supset \mathcal{F}^{l_p}_p(E) \supset \mathcal{F}^{l_p+1}_p(E) = 0$$

by vector subspaces of the fiber $E|_p$ at p.

A logarithmic connection (E, ∇) together with a parabolic structure $\mathcal{F} = (\mathcal{F}_p)_{p \in D}$ is called a *parabolic connection* on (X, D).

Definition 4.11. Let $\alpha = (\alpha^j_p \mid p \in D, \ j = 0, \ldots, l_p)$ be a tuple of rational numbers in $[0, 1)$ such that $\alpha^i_p < \alpha^j_p$ for any p and $i < j$. A parabolic connection $(E, \nabla, \mathcal{F})$ is said to be *α-semistable* if for any non-zero proper subbundle $F \subset E$ preserved by ∇, the following inequality holds:

$$\sum_{p \in D} \sum_j \alpha^j_p \frac{\dim\left(F|_p \cap \mathcal{F}^j_p(E)\right) / \left(F|_p \cap \mathcal{F}^{j+1}_p(E)\right)}{\mathrm{rank}\, F} \leq \sum_{p \in D} \sum_j \alpha^j_p \frac{\dim\left(\mathcal{F}^j_p(E)/\mathcal{F}^{j+1}_p(E)\right)}{\mathrm{rank}\, E}.$$

$(E, \nabla, \mathcal{F})$ is *α-stable* if the strict inequality always holds. A direct sum $(E, \nabla, \mathcal{F}) = \bigoplus_i (E_i, \nabla_i, \mathcal{F}_i)$ of α-stable parabolic connections satisfying

$$\sum_{p \in D} \sum_j \alpha^j_p \frac{\dim\left((\mathcal{F}_i)^j_p(E_i)/(\mathcal{F}_i)^{j+1}_p(E_i)\right)}{\mathrm{rank}\, E_i} = \sum_{p \in D} \sum_j \alpha^j_p \frac{\dim\left(\mathcal{F}^j_p(E)/\mathcal{F}^{j+1}_p(E)\right)}{\mathrm{rank}\, E}$$

for all i is said to be *α-polystable*.

We now introduce the filtered version of Deligne's Riemann-Hilbert correspondence.

Theorem 4.12 ([34, Lemma 3.2]). *Let X be a compact Riemann surface and $D \subset X$ be a finite subset. Then there is a natural bijective correspondence between:*

(i) *isomorphism classes of filtered local systems $(L, \mathbb{F})$ on (X, D) together with a weight β; and*

(ii) *isomorphism classes of parabolic connections $(E, \mathcal{F})$ on (X, D) together with a weight α.*

For each $p \in D$, this correspondence induces a bijection between:

$$\left\{ (\lambda, \alpha_p^j) \in \mathbb{C} \times [0,1) \mid \text{the action of } \mathrm{Res}_p \nabla \text{ on } \mathcal{F}_p^j(E)/\mathcal{F}_p^{j+1}(E) \text{ has an eigenvalue } \lambda \right\}; \quad \text{and}$$

$$\left\{ (\xi, \beta_p^k) \in \mathbb{C}^\times \times \mathbb{R} \;\middle|\; \begin{array}{l} \text{the monodromy of } \mathbb{F}_p^k(L)/\mathbb{F}_p^{k+1}(L) \\ \text{along a simple loop around } p \text{ (counterclockwise) has an eigenvalue } \xi \end{array} \right\},$$

which is explicitly given by $(\lambda, \alpha) \mapsto (\xi, \beta)$, where

$$\beta := \alpha - \mathrm{Re}\,\lambda, \qquad \xi := \exp(-2\pi\sqrt{-1}\lambda).$$

Furthermore, if (λ, α_p^j) corresponds to (ξ, β_p^k) under this bijection, then the generalized λ-eigen space of $\mathcal{F}_p^j(E)/\mathcal{F}_p^{j+1}(E)$ and the generalized ξ-eigen space of $\mathbb{F}_p^k(L)/\mathbb{F}_p^{k+1}(L)$ have the same dimension.

Recently, Inaba constructed the moduli space of α-semistable λ-*parabolic connections* $(E, \nabla, \mathcal{F})$ on (X, D) [17] of rank $r > 0$ for a given tuple $\lambda = (\lambda_p^j \mid p \in D, j = 0, \ldots, r-1)$, where λ-parabolic connection means a parabolic connection of full filtration type (i.e., $l_p = r-1$ and $\dim \mathcal{F}_p^j(E) = r-j$) and $(\mathrm{Res}_p \nabla - \lambda_p^j)(\mathcal{F}_p^j(E)) \subset \mathcal{F}_p^{j+1}(E)$ for each p, j. (We will use the word "ξ-filtered local system" by a similar manner.) We denote this moduli space by $\mathcal{M}_{\lambda,\alpha}(X, D; r)$. Its stable locus $\mathcal{M}_{\lambda,\alpha}^{\mathrm{s}}(X, D; r)$ has naturally an algebraic symplectic structure.

Now consider the case of $X = \mathbb{P}^1$. We can take α to be generic so that

$$\mathcal{M}_{\lambda,\alpha}(\mathbb{P}^1, D; r) = \mathcal{M}_{\lambda,\alpha}^{\mathrm{s}}(\mathbb{P}^1, D; r).$$

Inaba showed that if $rn - 2r - 2 > 0$ and $r \geq 2$ (n is the cardinality of D), then $\mathcal{M}_{\lambda,\alpha}(\mathbb{P}^1, D; r)$ is an irreducible variety of dimension $(r-1)(rn - 2r - 2)$ [17, Proposition 4.3]. We assume further that $\alpha_p^i - \mathrm{Re}\,\lambda_p^i \neq \alpha_p^j - \mathrm{Re}\,\lambda_p^j$ for $i \neq j$ so that one can take a permutation $\sigma_p \in \mathfrak{S}_{l_p+1}$ such that

$$i < j \quad \Longrightarrow \quad \alpha_p^{\sigma_p(i)} - \mathrm{Re}\,\lambda_p^{\sigma_p(i)} < \alpha_p^{\sigma_p(j)} - \mathrm{Re}\,\lambda_p^{\sigma_p(j)}.$$

Then under Simpson's Riemann-Hilbert correspondence, an α-semistable λ-parabolic connection correspond to a β-semistable ξ-filtered local system, where β and λ are given by

$$\beta_p^j := \alpha_p^{\sigma_p(j)} - \mathrm{Re}\,\lambda_p^{\sigma_p(j)} \qquad \text{and} \qquad \xi_p^j := \exp(-2\pi\sqrt{-1}\lambda_p^{\sigma_p(j)}).$$

Assume $\mathrm{Re}\,\lambda_p^k \in \mathbb{Q}$ so that $\beta_p^j \in \mathbb{Q}$.

Theorem 4.13. *Under the above notation and assumptions, let (I, Ω) be a star-shaped quiver with n arms such that the length l_i of the i-th arm is equal to $r-1$ for any i. Set q, θ as in Theorem 4.7, and take an I-graded vector space V with $\dim V_0 = r$, $\dim V_{i,j} = r - j$. Then Simpson's Riemann-Hilbert correspondence gives a symplectic biholomorphic map between $\mathcal{M}_{\lambda,\alpha}(\mathbb{P}^1, D; r)$ and $\mathcal{M}_{q,\theta}^{\mathrm{s}}(V) = \mathcal{M}_{q,\theta}(V)$.*

Proof. First of all we recall Simpson's Riemann-Hilbert correspondence. This correspondence can be constructed locally, so we may replace $X = \mathbb{P}^1$ with the unit open disk $\{ z \in \mathbb{C} \mid |z| < 1 \}$ and assume $D = \{0\}$. Also, for simplicity we assume that the permutation σ_p for $p = 0$ is an identity. Let

$(L, \mathbb{F})$ be a ξ-filtered local system on (X, D). L corresponds to a holomorphic bundle with connection (E', ∇) on $X \setminus D$. Take multi-valued flat sections $u_0, u_1, \ldots, u_{r-1}$ of E' such that $u_j \in \mathbb{F}^j(L) \setminus \mathbb{F}^{j+1}(L)$ (we omit the subscript $0 \in D$). Let $M \in \operatorname{End} E'$ be the monodromy operator and let R be a unique operator such that $e^{-2\pi\sqrt{-1}R} = M$ and the eigenvalues of R are $\lambda^0, \ldots, \lambda^{r-1}$. Then

$$v_j(z) := e^{R \log z} u_j(z)$$

becomes a single-valued holomorphic section of E', since when z moves along a simple loop around p once counterclockwise, v_j goes to

$$e^{R \log z} e^{2\pi\sqrt{-1}R} M u_j = v_j.$$

If we denote by $\tilde{E}'$ the sheaf of meromorphic section of the Deligne extension of E' having pole only at D, then v_j can be considered as a section of $\tilde{E}'$. Let E be the subsheaf of $\tilde{E}'$ generated by $v_0, \ldots, v_{r-1}$. Then E is locally free of rank r, and ∇ defines a logarithmic connection on E since $\nabla v_j = R v_j dz/z$ (Note that since $e^{R \log z}$ commutes with R, the representation matrix of R with respect to the framing $(v_0, \ldots, v_{r-1})$ is the same as the one with respect to $(u_0, \ldots, u_{r-1})$, and so it is a constant matrix). Moreover since $u_j \in \mathbb{F}^j \setminus \mathbb{F}^{j+1}$, if we let N be the nilpotent part of R then $R v_j = (\lambda^j + N) v_j$. Thus $v_j(0) \in E|_0$ lies in the generalized eigenspace for $\operatorname{Res}_0 \nabla$ with eigenvalue λ^j. Hence setting

$$\mathcal{F}^j(E) := \bigoplus_{k \geq j} \mathbb{C} v_k(0) \subset E|_0,$$

we get a λ-parabolic connection $(E, \nabla, \mathcal{F})$ on (X, D).

This construction gives a bijection from the set of isomorphism classes of β-stable ξ-filtered local systems on $(\mathbb{P}^1, D)$ to the set of isomorphism classes of α-stable λ-parabolic connections on $(\mathbb{P}^1, D)$ (see [34]). Using this fact, let us consider the inverse map.

Assume again that X is the unit open disk in $\mathbb{C}$ and $D = \{0\}$. Let $(E, \nabla, \mathcal{F})$ be a λ-parabolic connection on (X, D). Let L be the corresponding local system on $X \setminus D$, and let M, R, N be as in the previous paragraph. Take a basis $(e_0, e_1, \ldots, e_{r-1})$ of $E|_0$ compatible with the filtration $\mathcal{F}$. By the above fact, we can take a framing $(v_0, \ldots, v_{r-1})$ of E such that

$$\nabla v_j = R v_j dz/z, \qquad v_j(0) = e_j.$$

Then setting $u_j := e^{-R \log z} v_j$, we get multi-valued flat sections of L. Since $(R v_j)(0) = (\operatorname{Res}_0 \nabla) e_j$, we have $(R - \lambda_j) u_j \in \sum_{k>j} \mathbb{C} u_k$. Thus if we set $\mathbb{F}^j(L) \subset L$ by the subsheaf generated by $u_j, \ldots, u_{r-1}$, then $(L, \mathbb{F})$ is a ξ-filtered local system on (X, D). Note that v_j is uniquely determined by the differential equation $\nabla v_j = R v_j dz/z$ and the condition $v_j(0) = e_j$. Now recall that if a differential equation has complex analytic parameters then a solution of it also depends complex analytically on the parameters. Thus if $(E, \nabla, \mathcal{F})$ varies complex analytically, then the corresponding local system L, the monodromy operator M and the filtration on L which determined by v_j also vary complex analytically. This implies that the map $\operatorname{RH} \colon \mathcal{M}_{\lambda,\alpha}(\mathbb{P}^1, D; r) \to \mathcal{M}_{q,\theta}(V)$ given by Simpson's correspondence is complex analytic.

Next we prove that the map RH is symplectic. First note that since the claim is a closed condition, we may assume that λ is generic so that the morphism $\pi \circ \operatorname{RH} \colon \mathcal{M}_{\lambda,\alpha}(\mathbb{P}^1, D; r) \to \mathcal{M}_{q,0}(V)$ is a complex analytic isomorphism (see [17, Theorem 2.2]), where $\pi \colon \mathcal{M}_{q,\theta}(V) \to \mathcal{M}_{q,0}(V)$ is the canonical projective morphism. This implies that $\mathcal{M}_{q,0}^s(V) = \mathcal{M}_{q,0}(V)$, that π is a symplectic isomorphism, and

that $\mathrm{RH} = \pi^{-1} \circ (\pi \circ \mathrm{RH})$ is biholomorphic. Now take an arbitrary $[\rho] \in \mathcal{M}_{q,0}(V)$ and let $\mathcal{C}_i$ denote the conjugacy class of $\rho(\gamma_i)$. Then we can write $\mathcal{M}_{q,0}(V) = \mathcal{M}_{q,0}^{\mathrm{s}}(V)$ as the variety $\mathcal{R}$ associated to $\mathcal{C}_i$ by Proposition 4.1. We have remarked that $\mathcal{R}$ has naturally an algebraic symplectic structure, and it is isomorphic to $\mathcal{M}_{q,0}(V)$ as an algebraic symplectic manifold via this identification (see Remark 4.2). Thus the remaining task is to compare the symplectic structure on $\mathcal{R}$ and that on $\mathcal{M}_{\lambda,\alpha}(\mathbb{P}^1, D; r)$. To do this, we use the following fact proved by Alekseev-Malkin-Meinrenken. Let Σ be the compact Riemann surface with boundary obtained by cutting out an open disk U_p centered at p for each $p \in D$. Then we have $\pi_1(\mathbb{P}^1 \setminus D, *) \simeq \pi_1(\Sigma, *)$ canonically, and hence we can identify the variety $\mathcal{R}$ with the moduli space of irreducible flat C^∞-connections on Σ with the holonomy along ∂U_{p_i} lying in $\mathcal{C}_i$ for each i. This moduli space is actually smooth, and by the method of Atiyah-Bott we can construct naturally a symplectic structure on it. Alekseev-Malkin-Meinrenken [1] showed that this symplectic structure coincides with the one on $\mathcal{R}$. On the other hand, Biquard [2] constructed a natural isomorphism between the Zariski tangent space of $\mathcal{M}_{\lambda,\alpha}(\mathbb{P}^1, D; r)$ at a point $[(E, \nabla, \mathcal{F})]$ and the degree 1 L^2-cohomology of the complex $\Omega^\bullet(X \setminus D, \mathrm{End}\, E)$ of the spaces of C^∞-forms on $X \setminus D$ with coefficients in $\mathrm{End}\, E$, with the differential given by the flat C^∞-connection $D = \nabla + \overline{\partial}$. One can easily check that Inaba's symplectic form on $\mathcal{M}_{\lambda,\alpha}(\mathbb{P}^1, D; r)$ goes to the form on L^2-cohomology induced from $(u, v) \mapsto \int_X \mathrm{Tr}\,(u \wedge v)$, and this pairing comes from the Atiyah-Bott symplectic structure on $\mathcal{R}$ via the map $\mathcal{M}_{\lambda,\alpha}(\mathbb{P}^1, D; r) \to \mathcal{R}$. Hence RH is symplectic.

Since the determinant of the Jacobian of a symplectic map is everywhere non-vanishing, RH is biholomorphic. $\qquad\square$

4.4. **Higher genus case.** In this subsection let us consider the higher genus case. In this case, we cannot describe the moduli space of filtered local systems as some multiplicative quiver variety, but the quasi-Hamiltonian method still goes through.

First of all let us consider any quiver (I, Ω). Let H^ℓ be the subset of H which consists of all loops in H; $H^\ell := \{\, h \in H \mid \mathrm{in}(h) = \mathrm{out}(h) \,\}$, and set $\Omega^\ell := H^\ell \cap \Omega$. For an I-graded vector space V, we define an open subset $\mathbf{M}^\ell(V) \subset \mathbf{M}(V)$ by

$$\mathbf{M}^\ell(V) := \{\, x \in \mathbf{M}(V) \mid \det x_h \neq 0 \text{ for } h \in H^\ell, \text{ and } \det(1 + x_h x_{\overline{h}}) \neq 0 \text{ for } h \in H \setminus H^\ell \,\}.$$

This is a φ-saturated open subset of $\mathbf{M}(V)$, where $\varphi \colon \mathbf{M}(V) \to \mathbf{M}(V) /\!\!/ G_V$ is the quotient morphism. For any $i \in I$, the variety $\mathrm{GL}(V_i) \times \mathrm{GL}(V_i)$ has a quasi-Hamiltonian $\mathrm{GL}(V_i) \times \mathrm{GL}(V_i)$-structure whose group-valued moment map is $(a, b) \mapsto (ab, a^{-1}b^{-1})$ (see Example 2.13). Thus by fusioning, we can construct a quasi-Hamiltonian G_V-structure on $\mathbf{M}^\ell(V)$ whose group-valued moment map $\Psi_V \colon \mathbf{M}^\ell(V) \to G_V$ is given by

$$(\Psi_V)_i(x) := \prod_{h \in H_i \cap \Omega^\ell}^{<} [x_h, x_{\overline{h}}]^{\mathrm{m}} \prod_{h \in H_i \setminus H^\ell}^{<} (1 + x_h x_{\overline{h}})^{\epsilon(h)},$$

where $[x_h, x_{\overline{h}}]^{\mathrm{m}} := x_h x_{\overline{h}} x_h^{-1} x_{\overline{h}}^{-1}$ and we have fixed a total order $<$ on Ω^ℓ and that on $H \setminus H^\ell$. Thus for each $(q, \theta) \in (\mathbb{C}^\times)^I \times \mathbb{Q}^I$, we get a variety

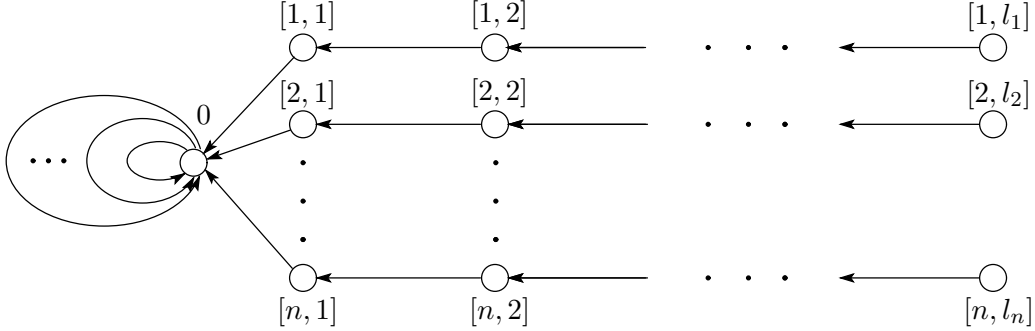$$\mathcal{M}_{q,\theta}^\ell(V) = \left(\Psi_V^{-1}(q) \cap \mathbf{M}^{\mathrm{ss}}(V) \right) /\!\!/ G_V,$$

and its open subset

$$\mathcal{M}_{q,\theta}^{\ell\,\mathrm{s}}(V) = \left(\Psi_V^{-1}(q) \cap \mathbf{M}^{\mathrm{s}}(V) \right) / G_V$$

which carries an algebraic symplectic structure.

Now consider a star-shaped quiver with g loops (I, Ω) as the following picture:

Theorem 4.14. *Let X be a compact Riemann surface with genus $g > 0$, and let $D = \{p_1, \ldots, p_n\}$ be a finite subset of X with cardinality n. Take an arbitrary $l \in \mathbb{Z}_{\geq 0}^D$, and let $\xi = (\xi_p^j \mid p \in D, \ j = 0, \ldots, l_p)$ be a tuple of non-zero complex numbers, $\beta = (\beta_p^j \mid p \in D, \ j = 0, \ldots, l_p)$ be a tuple of rational numbers such that $\beta_p^i < \beta_p^j$ for any p and $i < j$. Take a star-shaped quiver (I, Ω) with g loops as above, such that the number of arms is n and the length of the i-th arm is l_{p_i}. Then for any I-graded vector space V, setting $(q, \theta) \in (\mathbb{C}^\times)^I \times \mathbb{Q}^I$ by*

$$\theta_{i,j} := \beta_{p_i}^j - \beta_{p_i}^{j-1}, \qquad \theta_0 := -\frac{\sum_{[i,j] \in I_0} \theta_{i,j} \dim V_{i,j}}{\dim V_0},$$

$$q_{i,j} := \xi_{p_i}^{j-1} / \xi_{p_i}^j, \qquad q_0 := \prod_i (\xi_{p_i}^0)^{-1},$$

there is a natural bijection between $\mathcal{M}_{q,\theta}^\ell(V)$ and the set of isomorphism classes of β-polystable filtered local systems $(L, \mathbb{F})$ on (X, D) satisfying:

- *$\operatorname{rank} L = \dim V_0$, $\operatorname{rank} \mathbb{F}_{p_i}^j(L) = \dim V_{i,j}$;*
- *the local monodromy of $\mathbb{F}_{p_i}^j(L)/\mathbb{F}_{p_i}^{j+1}(L)$ around p_i is given by the scalar multiplication by $\xi_{p_i}^j$ for all i, j.*

Under this map, a point in $\mathcal{M}_{q,\theta}^{\ell\,\mathrm{s}}(V)$ corresponds to an isomorphism class of β-stable filtered local systems.

Theorem 4.15. *Let X be a compact Riemann surface with genus $g > 0$, and let $D = \{p_1, \ldots, p_n\}$ be a finite subset of X with cardinality n. Under the same notation and assumptions as in Theorem 4.12, assume further that $n > 1$ if $g = 1$, $l_p = r - 1$ for all p and fixed $r > 0$, and that α is generic so that $\mathcal{M}_{\lambda,\alpha}^{\mathrm{s}}(X, D; r) = \mathcal{M}_{\lambda,\alpha}(X, D; r)$. Let (I, Ω) be a star-shaped quiver with g loops such that the number of arms is n and the length of each arm is $r - 1$, and set q, θ as in Theorem 4.14. Then Simpson's Riemann-Hilbert correspondence gives a symplectic biholomorphic map between $\mathcal{M}_{\lambda,\alpha}(X, D; r)$ and $\mathcal{M}_{q,\theta}^{\ell\,\mathrm{s}}(V) = \mathcal{M}_{q,\theta}^\ell(V)$, where V is given by $V_0 = \mathbb{C}^r$, $V_{i,j} = \mathbb{C}^{r-j}$.*

We omit proofs of the above two theorem, since they are almost the same as in the previous two subsections. Notice only that the fundamental group of a punctured Riemann surface $X \setminus D$ of genus $g > 0$ has a presentation

$$\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_n \mid \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\alpha_2\beta_2\alpha_2^{-1}\beta_2^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}\gamma_1 \cdots \gamma_n = 1 \rangle.$$

Let us back to the case of an arbitrary quiver and consider similarities between $\mathcal{M}^\ell$ and $\mathfrak{M}$. Similar to Proposition 3.8 we can show the following.

Proposition 4.16. *Let $0^\ell \in \mathbf{M}(V)$ denote the point whose component x_h is given by $x_h = 1$ for $h \in H^\ell$ and $x_h = 0$ for $h \in H \setminus H^\ell$. Then there are φ-saturated open neighborhoods $\mathcal{U}$, $\mathcal{U}'$ of 0^ℓ in $\mathbf{M}(V)$ and a G_V-equivariant biholomorphic map $f \colon \mathcal{U} \to \mathcal{U}'$ such that*

$$f(0^\ell) = 0^\ell, \qquad f(\Psi_V^{-1}(1) \cap \mathcal{U}) = \mu_V^{-1}(0) \cap \mathcal{U}', \qquad (f^*\omega - \varpi^\ell)|_{\mathrm{Ker}\, d\Psi_V} = 0,$$

where ϖ^ℓ is the 2-form associated to the quasi-Hamiltonian G_V-structure on $\mathbf{M}^\ell(V)$.

Proof. First notice that $0^\ell \in \Psi_V^{-1}(1) \cap \mu_V^{-1}(0)$. The 2-form ϖ^ℓ on $\mathbf{M}^\ell(M)$ is given by

$$\begin{aligned}
\varpi^\ell := {} & \frac{1}{2} \sum_{h \in H \setminus H^\ell} \epsilon(h) \, \mathrm{Tr}\, (1 + x_h x_{\overline{h}})^{-1} dx_h \wedge dx_{\overline{h}} \\
& + \frac{1}{2} \sum_{h \in H^\ell} \epsilon(h) \, \mathrm{Tr}\, x_h^{-1} dx_h \wedge dx_{\overline{h}}\, x_{\overline{h}}^{-1} \\
& + \frac{1}{2} \sum_{h \in \Omega^\ell} \mathrm{Tr}\, [x_{\overline{h}}, x_h]^{\mathrm{m}} d(x_h x_{\overline{h}}) \wedge d(x_h^{-1} x_{\overline{h}}^{-1}) \\
& + \frac{1}{2} \sum_{h \in \Omega^\ell} \mathrm{Tr}\, \Psi_h^{-1} d\Psi_h \wedge d[x_h, x_{\overline{h}}]^{\mathrm{m}}\, [x_{\overline{h}}, x_h]^{\mathrm{m}} \\
& + \frac{1}{2} \sum_{h \in H \setminus H^\ell} \mathrm{Tr}\, \Psi_h^{-1} d\Psi_h \wedge d(1 + x_h x_{\overline{h}})^{\epsilon(h)} (1 + x_h x_{\overline{h}})^{-\epsilon(h)},
\end{aligned}$$

where

$$\Psi_h := \begin{cases} \prod_{h \in H_i \cap \Omega^\ell}^< [x_h, x_{\overline{h}}]^{\mathrm{m}} & \text{if } h \in \Omega^\ell, \\ \prod_{h \in H_i \cap \Omega^\ell}^< [x_h, x_{\overline{h}}]^{\mathrm{m}} \prod_{h' \in H_i; h' < h}^< (1 + x_{h'} x_{\overline{h'}})^{\epsilon(h')} & \text{if } h \in H_i \setminus H^\ell. \end{cases}$$

For a loop $h \in \Omega^\ell$, we have $d(x_h x_{\overline{h}})_{0^\ell} = dx_h + dx_{\overline{h}} = -d(x_h^{-1} x_{\overline{h}}^{-1})_{0^\ell}$. Thus we get

$$\varpi_{0^\ell}^\ell = \frac{1}{2} \sum_{h \in H \setminus H^\ell} \epsilon(h) \, \mathrm{Tr}\, dx_h \wedge dx_{\overline{h}} + \frac{1}{2} \sum_{h \in H^\ell} \epsilon(h) \, \mathrm{Tr}\, dx_h \wedge dx_{\overline{h}} = \omega_{0^\ell}.$$

Moreover we have $(d\Psi_V)_{0^\ell} = 0$. Hence Lemma 3.9 and the equivariant Darboux theorem imply the assertion. $\qquad\square$

Since 0^ℓ is a fixed point for the G_V-action and $\mu_V(0^\ell) = 0$, applying the equivariant Darboux theorem again one can find a φ-saturated open neighborhood $\mathcal{U}''$ of $0 \in \mathbf{M}(V)$ and a G_V-equivariant biholomorphic map $F \colon \mathcal{U}' \to \mathcal{U}''$ such that

$$F(0^\ell) = 0, \qquad F^*\omega = \omega, \qquad \mu_V \circ F = \mu_V.$$

Together with the above proposition, we get:

Corollary 4.17. *Let $\mathcal{M}_\theta^\ell(V) = \mathcal{M}_{1,\theta}^\ell(V)$ and $\pi \colon \mathcal{M}_\theta^\ell(V) \to \mathcal{M}_0^\ell(V)$ denote the natural projective morphism. Then there exist an open neighborhood U (resp. U') of $[0^\ell] \in \mathcal{M}_0^\ell(V)$ (resp. $[0] \in \mathfrak{M}_0(V)$)*

and a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_\theta^\ell(V) \supset \pi^{-1}(U) & \xrightarrow{\tilde{f}} & \pi^{-1}(U') \subset \mathfrak{M}_\theta(V) \\ \pi \downarrow & & \pi \downarrow \\ U & \xrightarrow{f} & U' \end{array}$$

such that:

(i) $f([0^\ell]) = [0]$;

(ii) *both $\tilde{f}$ and f are complex analytic isomorphisms;*

(iii) *$\tilde{f}$ maps $\pi^{-1}(U) \cap \mathcal{M}_\theta^{\ell s}(V)$ onto $\pi^{-1}(U') \cap \mathfrak{M}_\theta^s(V)$ as a symplectic biholomorphic map; and*

(iv) *if $x \in \varphi^{-1}(U)$ and $y \in \varphi^{-1}(U')$ have closed orbits and $f([x]) = [y]$, then the stabilizers of the two are conjugate. Thus f preserves the orbit-type.*

5. Middle convolution

Multiplicative preprojective relation has a certain surprising similarity to preprojective relation. Let $i \in I$ be a *loop-free* vertex, i.e., there is no $h \in H$ such that $\mathrm{out}(h) = \mathrm{in}(h) = i$. In this section we fix such an $i \in I$. Let $s_i \colon \mathbb{C}^I \to \mathbb{C}^I$ be the reflection defined by $s_i(\alpha) := \alpha - (\alpha, \mathbf{e}_i)\mathbf{e}_i$. There is a reflection $r_i \colon \mathbb{C}^I \to \mathbb{C}^I$ which is dual to s_i with respect to the standard inner product:

$$r_i(\zeta) := (\zeta_j'), \qquad \zeta_j' = \zeta_j - (\mathbf{e}_i, \mathbf{e}_j)\zeta_i.$$

Then the i-th *reflection functor* is defined as a certain equivalence between the category of representations (V, x) of (I, H) satisfying the preprojective relation $\mu_V(x) = \zeta$ with a fixed ζ such that $\zeta_i \neq 0$, and the category of those (V', x') satisfying the preprojective relation $\mu_{V'}(x') = r_i(\zeta)$. This functor transforms the dimension vector $\dim V$ to $\dim V' = s_i(\dim V)$. Crawley-Boevey and Shaw [11] constructed its multiplicative analogue by generalizing an algebraic formulation of Katz' middle convolution given by Dettweiler-Reiter [13]. In other words, they constructed an equivalence between the category of the representations (V, x) satisfying the relation $\Phi_V(x) = \zeta$ with a fixed q such that $q_i \neq 1$, and the category of those (V', x') satisfying $\Phi_{V'}(x') = u_i(q)$, where

$$u_i(q) := (q_j'), \qquad q_j' = q_j q_i^{-(\mathbf{e}_i, \mathbf{e}_j)}.$$

This functor is called the *middle convolution functor*.

On the other hand, Maffei [27] showed that the reflection functor sends a θ-stable representation to a $r_i(\theta)$-stable representation if $\dim V \neq \mathbf{e}_i$. Thus the reflection functor induces an isomorphism

$$\mathfrak{M}_{\zeta,\theta}^s(V) \simeq \mathfrak{M}_{r_i(\zeta),r_i(\theta)}^s(V'),$$

where V' is an I-graded vector space with $\dim V' = s_i(\dim V)$. Moreover he proved that the above isomorphism can be defined in the case of $\zeta_i = 0$.

In this section we show a multiplicative version of his result.

Theorem 5.1. *If $\dim V \neq \mathbf{e}_i$ and $s_i(\dim V) \notin \mathbb{Z}_{\geq 0}^I$, then $\mathcal{M}_{q,\theta}^s(V)$ is empty.*

If $s_i(\dim V) \in \mathbb{Z}_{\geq 0}^I$, take an I-graded vector space V' with $\dim V' = s_i(\dim V)$. Then there is an isomorphism of algebraic varieties

$$\mathcal{M}_{q,\theta}^s(V) \simeq \mathcal{M}_{u_i(q),r_i(\theta)}^s(V').$$

The proof of the first statement is easy. Indeed if $\dim V \neq \mathbf{e}_i$ and $\mathcal{M}^{\mathrm{s}}_{q,\theta}(V) \neq \emptyset$, then Proposition 3.7 implies

$$0 \geq \dim \widehat{V}_i - \dim V_i = -(\mathbf{e}_i, \dim V) + \dim V_i.$$

The right hand side is just the coefficient of $s_i(\dim V)$ in $\mathbf{e}_i$, so $s_i(\dim V) \in \mathbb{Z}^I_{\geq 0}$. Moreover if we assume further that $\theta_i = 0$ and $q_i = 1$, then the sequence

$$0 \longrightarrow V_i \xrightarrow{\sigma_i} \widehat{V}_i \xrightarrow{\tau_i} V_i \longrightarrow 0$$

is exact at any point in $\mathcal{M}^{\mathrm{s}}_{q,\theta}(V)$ by Proposition 3.7 again. The exactness implies that the right hand side of the previous inequality is equal to $\dim V_i$, and hence that $s_i(\dim V) = \dim V$. Since we have $r_i(\theta) = \theta$, $u_i(q) = q$ under the assumption, the second statement in the case $\theta_i = 0, q_i = 1$ is clear.

The rest of this section is devoted to the proof of the general case. In fact, all the proofs are very similar to the case of reflection functor.

5.1. Middle convolution functor.
First we rewrite the middle convolution functor in our context. See [11] for the original definition.

From now on, we assume that $s_i(\dim V) \in \mathbb{Z}^I_{\geq 0}$. Note that $\dim V \neq \mathbf{e}_i$ under this assumption. For simplicity, we assume further that $H_i \subset \Omega$.

Let us recall the definitions of $\sigma_i(x)$ and $\tau_i(x)$:

$$\sigma_i(x) = \sum \iota_h x_{\overline{h}} \colon V_i \to \widehat{V}_i,$$

$$\tau_i(x) = \sum_{h \in H_i} \Phi_h x_h \pi_h \colon \widehat{V}_i \to V_i,$$

where

$$\Phi_h = \Phi_h(x) = \prod_{h' \in H_i; h' < h}^{<} (1 + x_{h'} x_{\overline{h'}}).$$

Since i is fixed, we will drop the subscript i; $\sigma = \sigma_i$, $\tau = \tau_i$.

For a point $x \in \Phi^{-1}_V(q)$, we define

$$\phi_h := \sum_{h' \in H_i; h' < h} \iota_{h'} x_{\overline{h'}} x_h + \frac{1}{q_i} \sum_{h' \in H_i; h' \geq h} \iota_{h'} x_{\overline{h'}} x_h + \frac{1 - q_i}{q_i} \iota_h \colon V_{\mathrm{out}(h)} \to \widehat{V}_i \quad (h \in H_i).$$

Then one can show that

$$(8) \qquad\qquad\qquad\qquad \tau \phi_h = 0 \quad \text{for all } h \in H_i,$$

and that

$$(9) \qquad\qquad\qquad\qquad \prod_{h \in H_i} (1 + \phi_h \pi_h) = 1 - \frac{1}{q_i}(q_i - 1 - \sigma\tau).$$

For the proof, see [11].

Now suppose $q_i \neq 1$. The equality $\tau\sigma = q_i - 1$ implies that τ is surjective. Thus if we set $V'_i := \operatorname{Ker} \tau$ and $V'_j := V_j$ for $j \neq i$, then $\dim V'_j = s_i(\dim V)$.

Using (8) we define

$$
(10) \qquad x'_h := \begin{cases} \phi_h \colon V_{\mathrm{out}(h)} \to V'_i & \text{if } h \in H_i, \\[2mm] \pi_{\overline{h}}|_{\mathrm{Ker}\,\tau} \colon V'_i \to V_{\mathrm{in}(h)} & \text{if } h \in \overline{H}_i, \\[2mm] x_h & \text{otherwise,} \end{cases}
$$

Then (9) implies

$$
\Phi_i(x') = \prod_{h \in H_i}(1 + x'_h x'_{\overline{h}}) = \frac{1}{q_i} = q'_i.
$$

Moreover for any $h \in H_i$ we have

$$
x'_{\overline{h}} x'_h = \pi_h \phi_h = \frac{1}{q_i} x_{\overline{h}} x_h + \frac{1 - q_i}{q_i},
$$

and hence

$$
1 + x'_{\overline{h}} x'_h = \frac{1}{q_i}(1 + x_{\overline{h}} x_h).
$$

Thus we get

$$
\Phi_j(x') = q_i^{\mathbf{A}_{ij}} \Phi_j(x) = q'_j
$$

for all $j \neq i$, where $\mathbf{A}_{ij}$ is the number of $h \in H$ satisfying $\mathrm{in}(h) = i$ and $\mathrm{out}(h) = j$.

Thus under the assumption $q_i \neq 1$, we have a map

$$
S_i \colon \Phi_V^{-1}(q)/G_V \to \Phi_{V'}(q)/G_{V'}; \qquad G_V \cdot x \mapsto G_{V'} \cdot x'
$$

between the set-theoretical orbit spaces. This is a set-theoretical definition of the *middle convolution functor*.

Crawley-Boevey and Show observed that $S_i^2 = \mathrm{id}$. We use this map to prove Theorem 5.1 for the case $q_i \neq 1$ or $\theta_i < 0$. Note that even if $q_i = 1$, the above definition of x' for a θ-stable point $x \in \Phi_V^{-1}(q)$ with $\theta_i \leq 0$ makes sense since $\tau(x)$ is still surjective by Proposition 3.7. Thus we have a map $S_i \colon \mathcal{M}_{q,\theta}^{\mathrm{s}}(V) \to \Phi_{V'}(q)/G_{V'}$ in the case $\theta_i \leq 0$.

5.2. **Lusztig's correspondence.** To prove Theorem 5.1, we modify a beautiful formulation of the reflection functor by Lusztig [26] for the middle convolution.

From now on, we assume that $\theta_i \leq 0$ and $\epsilon(h) > 0$ for all $h \in H_i$ as in the previous subsection. Both of the assumption lose no generality by $r_i^2 = \mathrm{id}$ and Proposition 3.3. Moreover we exclude the case $\theta_i = 0, q_i = 1$ as we explained before.

Set $q' := u_i(q)$ and $\theta' := r_i(\theta)$. Take an I-graded vector space V' such that $\dim V' = s_i(\dim V)$ and $V'_j = V_j$ for all $j \neq i$.

In this section we use the following notation:

$$
\mathbf{M} = \mathbf{M}(V), \qquad Z = \mathbf{M}_\theta^{\mathrm{s}}(V) \cap \Phi_V^{-1}(q),
$$
$$
\mathbf{M}' = \mathbf{M}(V'), \qquad Z' = \mathbf{M}_{\theta'}^{\mathrm{s}}(V') \cap \Phi_{V'}^{-1}(q').
$$

Definition 5.2. Let P be the subvariety of $\mathbf{M} \times \mathbf{M}'$ which consists of all pairs $(x, x') \in \mathbf{M} \times \mathbf{M}'$ satisfying the following conditions:

(R1) $x_h = x'_h$ for all $h \notin H_i \cup \overline{H}_i$.

(R2) The sequence

$$0 \longrightarrow V_i' \xrightarrow{\sigma'} \widehat{V}_i \xrightarrow{\tau} V_i \longrightarrow 0$$

   is exact. Here $\sigma' = \sigma(x')$.
(R3) $\sigma\tau = q_i\sigma'\tau' + q_i - 1$. Here $\tau' = \tau(x')$.
(R4) $\det(1 + x_h x_{\overline{h}}) \neq 0$ for all $h \in H$.
(R4') $\det(1 + x_h' x_{\overline{h}}') \neq 0$ for all $h \in H$.
(R5) $\Phi_V(x) = q$.
(R5') $\Phi_{V'}(x') = q'$.
(R6) x is θ-stable.
(R6') x' is θ'-stable.

Let $r\colon P \to Z$ (resp. $r'\colon P \to Z'$) be the map induced from the projection to the first (resp. the second) factor. P is naturally acted on by the reductive group

$$G := \mathrm{GL}(V_i) \times \mathrm{GL}(V_i') \times \prod_{j \neq i} \mathrm{GL}(V_j),$$

and r (resp. r') is equivariant through the projections $G \to G_V$ (resp. $G \to G_{V'}$). Note that P has a geometric quotient, because $Z \times Z'$ has a geometric quotient for the action of $G_V \times G_{V'}$ and hence so for the action of its reductive subgroup G, and P is a G-invariant subvariety of $Z \times Z'$.

The second statement of Theorem 5.1 is deduced from the following fact.

Theorem 5.3. *Suppose $\theta_i \leq 0$, and $q_i \neq 1$ if $\theta_i = 0$. Then r and r' induce isomorphisms*

$$\mathcal{M}_{q,\theta}^{\mathrm{s}}(V) \simeq P/G \simeq \mathcal{M}_{q',\theta'}^{\mathrm{s}}(V').$$

We give a proof of this theorem in §5.4. In the next subsection, we give several properties of P, all of which are needed in §5.4.

5.3. Several lemmas.

Lemma 5.4. *Suppose $\theta_i \leq 0$, and $q_i \neq 1$ if $\theta_i = 0$. If a point $(x, x') \in \mathbf{M} \times \mathbf{M}'$ satisfies the conditions* (R1), (R2) *and* (R3), *then*

$$(x, x') \text{ satisfies (R6)} \quad \Longleftrightarrow \quad (x, x') \text{ satisfies (R6')}.$$

Proof. We adapt a beautiful proof of Nakajima [32] for the reflection functor to our case.

First we prove the direction $\Rightarrow$. Suppose that (R1-3) and (R6). If $\theta_i = 0$, suppose further that $q_i \neq 1$.

Let S' be a x'-invariant subspace of V'. Then

(11) $$\sigma'(S_i') \subset \widehat{S}_i', \qquad \tau'(\widehat{S}_i') \subset S_i'.$$

Set

(12) $$S_j := \begin{cases} S_j' & \text{for } j \neq i, \\ \tau(\widehat{S}_i') & \text{for } j = i. \end{cases}$$

Clearly $x_h(S_{\text{out}(h)}) \subset S_{\text{in}(h)}$ if $\text{in}(h) \neq i \neq \text{out}(h)$. By (11), we have

$$\sigma(S_i) = \sigma\tau(\widehat{S}_i)$$
$$= q_i \sigma'\tau'(\widehat{S}_i) + (q_i - 1)(\widehat{S}_i)$$
$$\subset \sigma'(S_i') + \widehat{S}_i \subset \widehat{S}_i.$$

Thus S is x-invariant by Lemma 3.6.

By the θ-stability of x we have

$$(13) \qquad\qquad 0 \geq \theta \cdot \dim S = \sum_{j \neq i} \theta_j \dim S_j + \theta_i \dim S_i$$

and the strict inequality holds unless $S = 0$ or $S = V$.

Consider the following complex.

$$S_i' \xrightarrow{\ \sigma'\ } \widehat{S}_i \xrightarrow{\ \tau\ } S_i$$

The left arrow is injective by (R2) and the right arrow is surjective by the definition of S_i. Hence we have

$$(14) \qquad\qquad \dim S_i \leq \sum \dim S_{\text{out}(h)} - \dim S_i'.$$

Noticing $\theta_i \leq 0$, we substitute this inequality into (13). Then we get

$$0 \geq \sum_{j \neq i} (\theta_j + \mathbf{A}_{ij}\theta_i)\dim S_j - \theta_i \dim S_i' = \theta' \cdot \dim S'.$$

If we have the equality, we must have the equality in (13) which implies $S = 0$ or $S = V$. If $S = 0$, then $S_j' = 0$ for $j \neq i$. Then (11) and the injectivity of σ' imply $S_i' = 0$. Thus $S' = 0$. We assume $S = V$. When $\theta_i \neq 0$, we must also have the equality in (14). Substituting $S = V$ into it, we obtain $\dim S_i' = \dim V_i'$. Thus $S' = V'$. When $\theta_i = 0$, $q_i \neq 1$ by the assumption. Thus by (R2) and (R3) we have

$$0 = \sigma\tau\sigma' = q_i \sigma'\tau'\sigma' + (q_i - 1)\sigma'.$$

By (R2) σ' is injective, so we have $\tau'\sigma' = q_i^{-1} - 1 \neq 0$. Thus τ' is surjective and hence $S_i' \supset \tau'(\widehat{S}_i') = V_i'$. Thus $S' = V'$. Hence x' is θ'-stable.

The proof of the inverse direction $\Leftarrow$ also can be done similarly. Let S be a x-invariant subspace of V. Set

$$S_j' := \begin{cases} S_j & \text{for } j \neq i, \\ (\sigma')^{-1}(\widehat{S}_i) & \text{for } j = i. \end{cases}$$

Then S' is x'-invariant.

By the θ'-stability of x' we have

$$(15) \qquad\qquad 0 \geq (\theta', \dim S') = \sum_{j \neq i} \theta_j' \dim S_j' + \theta_i' \dim S_i'$$

and we have the strict inequality unless $S' = 0$ or $S' = V'$.

Consider the following complex.

$$S_i' \xrightarrow{\ \sigma'\ } \widehat{S}_i \xrightarrow{\ \tau\ } S_i$$

The left arrow is injective by (R2) and its image is equal to the kernel of the right arrow by the definition of S_i' and (R2). Hence we have

(16) $$\dim S_i' \geq \sum \dim S_{\mathrm{out}(h)} - \dim S_i.$$

Noticing $\theta_i' \geq 0$, we substitute this inequality into (15). Then we get

$$0 \geq \sum_{j \neq i} (\theta_j' + \mathbf{A}_{ij}\theta_i') \dim S_j - \theta_i' \dim S_i = (\theta, \dim S).$$

If we have the equality, we must have the equality in (15) which implies $S' = 0$ or $S' = V'$. If $S' = V'$, then $S_j = S_j$ for $j \neq i$. Thus $S_i \supset \tau(\widehat{S}_i) = V_i$ by the surjectivity of τ. Hence $S = V$. We assume $S' = 0$. When $\theta_i \neq 0$, we must also have the equality in (16). This implies $S_i = 0$, and hence $S = 0$. When $\theta_i = 0$, the conditions (R2), (R3) and the assumption $q_i \neq 1$ implies

$$0 = \tau\sigma'\tau' = q_i^{-1}\tau\sigma\tau + (q_i^{-1} - 1)\tau.$$

By (R2), τ is surjective, so we have $\tau\sigma = q_i - 1 \neq 0$. Thus σ is injective and hence $S_i \subset \sigma^{-1}(\widehat{S}_i) = 0$. Thus $S = 0$. Hence x is θ-stable. $\qquad\square$

Lemma 5.5. *Suppose $\theta_i \geq 0$, and $q_i \neq 1$ if $\theta_i = 0$. If a point $(x, x') \in \mathbf{M} \times \mathbf{M}'$ satisfies (R1), (R3) and*

(R2') *The sequence*
$$0 \longrightarrow V_i \overset{\sigma}{\longrightarrow} \widehat{V}_i \overset{\tau'}{\longrightarrow} V_i' \longrightarrow 0$$
   is exact,

then

$$(x, x') \text{ satisfies (R6)} \quad \Longleftrightarrow \quad (x, x') \text{ satisfies (R6').}$$

Proof. The proof is similar to the previous lemma. $\qquad\square$

Proposition 5.6. *Suppose that $q_i \neq 1$ or $\theta_i < 0$. For a point $x \in Z$, let x' be a representative of $S_i(G_V \cdot x)$ which is defined by (10). Then $(x, x') \in P$.*

Proof. (R1) is satisfied by the definition of x'. Moreover both (R4') and (R5') are satisfied by the argument before.

To check (R2), first note that

$$\sigma' = \sum_{h \in H_i} \iota_h x_{\overline{h}}' = \sum_{h \in H_i} \iota_h \pi_h$$

is equal to the inclusion $V_i' = \mathrm{Ker}\,\tau \hookrightarrow \widehat{V}_i$. Thus σ' is injective and $\tau\sigma' = 0$. Since τ is surjective and the Euler number of the complex in (R2) is zero, (x, x') satisfies (R2).

By (9), we have

$$\begin{aligned}
\tau' &= \sum_{h \in H_i} \prod_{h' \in H_i; h' < h} (1 + x_{h'}'x_{\overline{h'}}')\, x_h' \pi_h \\
&= \sum_{h \in H_i} \prod_{h' \in H_i; h' < h} (1 + \phi_{h'}\pi_{h'})\, \phi_h \pi_h \\
&= \prod_{h \in H_i} (1 + \phi_h \pi_h) - 1 \\
&= q_i^{-1}\sigma\tau + q_i^{-1}(1 - q_i).
\end{aligned}$$

Thus (R3) is satisfied.

Lemma 5.4 shows that x' is θ'-stable when $\theta_i < 0$. So we assume that $\theta_i \geq 0$ and $q_i \neq 1$. By (R3) and the equality $\tau'\sigma' = q_i' - 1$, we have

$$\tau'\sigma\tau = q_i\tau'\sigma'\tau' + (q_i - 1)\tau'$$
$$= q_i(q_i' - 1)\tau' + (q_i - 1)\tau' = 0.$$

Since τ is surjective, the above implies $\tau'\sigma = 0$. Note that τ' is surjective and σ is injective by the equalities $\tau'\sigma' = q_i' - 1$ and $\tau\sigma = q_i - 1$. Hence the sequence

$$0 \longrightarrow V_i \overset{\sigma}{\longrightarrow} \widehat{V}_i \overset{\tau'}{\longrightarrow} V_i' \longrightarrow 0$$

is exact. Thus x' is θ'-stable by Lemma 5.5. $\qquad\square$

Proposition 5.7. *Suppose $q_i \neq 1$ or $\theta_i' < 0$. For a point $x \in Z'$, let x' be a representative of $S_i(G_{V'} \cdot x)$. Then $(x', x) \in P$.*

Proof. The proof is similar. $\qquad\square$

5.4. **Proof of the main theorem.** In this subsection we prove Theorem 5.3. First consider the case $q_i \neq 1$.

Proof of Theorem 5.3 for the case $q_i \neq 1$. $r \colon P \to Z$ is surjective by Proposition 5.6. Let $x_0 \in Z$. We construct a section of r over a neighborhood of x^0.

Take an identification $\widehat{V}_i \simeq V_i \oplus V_i'$ such that the first projection coincides with $\tau(x^0)$. Set

$$Z_0 = \{\, x \in Z \mid \tau(x)|_{V_i} \colon V_i \to V_i \text{ is an isomorphism}\,\}.$$

Then Z_0 is a neighborhood of x^0, and for any $x \in Z$,

$$\alpha := \begin{bmatrix} -(\tau(x)|_{V_i})^{-1}\tau(x)|_{V_i'} \\ 1 \end{bmatrix} \colon V_i \to \operatorname{Ker}\tau(x)$$

is an isomorphism. We choose it for the identification $V_i \simeq \operatorname{Ker}\tau(x)$ to define the point $x' \in Z'$, i.e., we define

$$x_h' := \alpha^{-1}\phi_h \colon V_{\text{out}(h)}' \to V_i', \qquad x_{\overline{h}}' := \pi_h\alpha \colon V_i' \to V_{\text{out}(h)}' \quad \text{for } h \in H_i,$$

and define x_h for $h \notin H_i \cup \overline{H}_i$ by the condition (R1). Then $x \mapsto x'$ defines a section of r over Z_0.

Since r has a local section, the induced morphism $P/G \to Z/G_V$ is an isomorphism. The proof for r' is similar (use Proposition 5.7 instead of Proposition 5.6). $\qquad\square$

In the rest of this subsection we assume that $q_i = 1$ and $\theta_i < 0$.

Lemma 5.8. *If a pair $(x, x') \in \mathbf{M} \times \mathbf{M}'$ satisfies the conditions (R1), (R2) and (R3), then*

$$(x, x') \text{ satisfies (R4)} \quad \Longleftrightarrow \quad (x, x') \text{ satisfies (R4').}$$

And under these assumptions, the equality $x_{\overline{h}}x_h = x_{\overline{h}}'x_h'$ holds for all $h \in H_i$.

Proof. Let $H_i = \{\, h_1 < h_2 < \cdots < h_n \,\}$. By (R3),

$$x_{\overline{h}_1} x_{h_1} = \pi_{h_1} \sigma \tau \iota_{h_1} = \pi_{h_1} \sigma' \tau' \iota_{h_1} = x'_{\overline{h}_1} x'_{h_1},$$

and

$$\det(1 + x'_{h_1} x'_{\overline{h}_1}) = \det(1 + x'_{\overline{h}_1} x'_{h_1}) = \det(1 + x_{\overline{h}_1} x_{h_1}) = \det(1 + x_{h_1} x_{\overline{h}_1}).$$

Set $R_1 = \iota_{h_1} \pi_{h_1} \sigma \tau$. Then

$$x_{h_1} x_{\overline{h}_1} \tau = \tau \iota_{h_1} \pi_{h_1} \sigma \tau = \tau R_1,$$

and also

$$x'_{h_1} x'_{\overline{h}_1} \tau' = \tau' \iota_{h_1} \pi_{h_1} \sigma' \tau' = \tau' R_1.$$

by (R3). Suppose now that (x, x') satisfies (R4). Since $\det(1 + R_1) = \det(1 + \pi_{h_1} \sigma \tau \iota_{h_1}) = \det(1 + x_{\overline{h}_1} x_{h_1}) \neq 0$, $(1 + R_1)$ is invertible and hence

$$\begin{aligned}
x_{\overline{h}_2} x_{h_2} &= \pi_{h_2} \sigma (1 + x_{h_1} x_{\overline{h}_1})^{-1} \tau \iota_{h_2} \\
&= \pi_{h_2} \sigma \tau (1 + R_1)^{-1} \iota_{h_2} \\
&= \pi_{h_2} \sigma' \tau' (1 + R_1)^{-1} \iota_{h_2} \\
&= \pi_{h_2} \sigma' (1 + x'_{h_1} x'_{\overline{h}_1})^{-1} \tau' \iota_{h_2} \\
&= x'_{\overline{h}_2} x'_{h_2}.
\end{aligned}$$

Next we define

$$R_2 = (1 + R_1)^{-1} \iota_{h_2} \pi_{h_2} \sigma \tau.$$

Then

$$\begin{aligned}
\det(1 + R_2) &= \det(1 + \iota_{h_2} \pi_{h_2} \sigma \tau (1 + R_1)^{-1}) \\
&= \det(1 + \iota_{h_2} \pi_{h_2} \sigma (1 + x_{h_1} x_{\overline{h}_1})^{-1} \tau) \\
&= \det(1 + x_{\overline{h}_2} x_{h_2}) \neq 0,
\end{aligned}$$

and

$$\begin{aligned}
x_{h_2} x_{\overline{h}_2} \tau &= (1 + x_{h_1} x_{\overline{h}_1})^{-1} \tau \iota_{h_2} \pi_{h_2} \sigma \tau \\
&= \tau (1 + R_1)^{-1} \iota_{h_2} \pi_{h_2} \sigma \tau = \tau R_2, \\
x'_{h_2} x'_{\overline{h}_2} \tau' &= (1 + x'_{h_1} x'_{\overline{h}_1})^{-1} \tau' \iota_{h_2} \pi_{h_2} \sigma' \tau' \\
&= \tau' (1 + R_1)^{-1} \iota_{h_2} \pi_{h_2} \sigma' \tau' = \tau' R_2.
\end{aligned}$$

By induction, one can easily show that

$$R_k := (1 + R_{k-1})^{-1} \cdots (1 + R_2)^{-1} (1 + R_1)^{-1} \iota_{h_k} \pi_{h_k} \sigma \tau$$

is well-defined and

$$\det(1 + R_k) = \det(1 + x_{\overline{h}_k} x_{h_k}),$$
$$x_{h_k} x_{\overline{h}_k} \tau = \tau R_k,$$
$$x'_{h_k} x'_{\overline{h}_k} \tau' = \tau' R_k$$

for $1 \le k \le n$. Hence for $1 \le k \le n$,

$$
\begin{aligned}
x_{\overline{h}_k} x_{h_k} &= \pi_{h_k} \sigma (1 + x_{h_{k-1}} x_{\overline{h}_{k-1}})^{-1} \cdots (1 + x_{h_1} x_{\overline{h}_1})^{-1} \tau \iota_{h_k} \\
&= \pi_{h_k} \sigma \tau (1 + R_{k-1})^{-1} \cdots (1 + R_1)^{-1} \iota_{h_k} \\
&= \pi_{h_k} \sigma' \tau' (1 + R_{k-1})^{-1} \cdots (1 + R_1)^{-1} \iota_{h_k} \\
&= \pi_{h_k} \sigma' (1 + x'_{h_{k-1}} x'_{\overline{h}_{k-1}})^{-1} \cdots (1 + x'_{h_1} x'_{\overline{h}_1})^{-1} \tau' \iota_{h_k} \\
&= x'_{\overline{h}_k} x'_{h_k}.
\end{aligned}
$$

The proof of the inverse direction (R4') $\Rightarrow$ (R4) can be done similarly, so we omit it. $\qquad\square$

Lemma 5.9. *If a pair (x, x') satisfies* (R1), (R2) *and* (R3), *then*

$$(x, x') \text{ satisfies (R4) and (R5)} \quad \Longleftrightarrow \quad (x, x') \text{ satisfies (R4') and (R5').}$$

Proof. Under the conditions (R1), (R2), (R3) and (R4) (or (R4')), the above lemma implies that $\Phi_j(x) = \Phi_j(x')$ for all $j \ne i$. Since $q_j = q'_j$ for $j \ne i$, the result follows. $\qquad\square$

Proof of Theorem 5.3 for the case $q_i = 1$ and $\theta_i < 0$. The proof for r is the same that in the case $q_i \ne 1$. To prove that r' is a geometric quotient, we will construct locally a section of r', as in the other case.

Let $x' \in Z'$. By Proposition 3.7 and $\theta'_i > 0$, σ' is injective. Thus we can identify V_i with $\widehat{V}_i / \operatorname{Im} \sigma'$. Let p be the projection $\widehat{V}_i \to V_i$. Since $\tau' \sigma' = 0$, τ' descends to a linear map $\overline{\tau}' \colon V_i \to V'_i$. We define

$$x_{\overline{h}} = x'_{\overline{h}} \overline{\tau}' \colon V_i \to V_{\mathrm{out}(h)}, \qquad x_h = \Phi_h^{-1} p \iota_h \colon V_{\mathrm{out}(h)} \to V_i$$

for $h \in H_i$. Here we use induction to define x_h.

We define x_h for $h \notin H_i \cup \overline{H}_i$ by the condition (R1). Then

$$\sigma = \sigma' \overline{\tau}', \qquad \tau = p.$$

Thus $\sigma \tau = \sigma' \overline{\tau}' p = \sigma' \tau'$ and $\tau \sigma' = p \sigma' = 0$. Clearly τ is surjective, so (R2) is satisfied. By Lemma 5.4 and Lemma 5.9, the pair of x and x' is an element of P.

The definition of x depends on the identification $V_i \simeq \widehat{V}_i / \operatorname{Im} \sigma'$, but we can choose it locally to be regular in the variable x', as in the case of r. Thus the induced morphism $P/G \to Z'/G_{V'}$ is an isomorphism. $\qquad\square$

6. Representations of Kac-Moody algebra

In [29], Nakajima constructed all irreducible highest weight representations of a Kac-Moody Lie algebra using the vector spaces of constructible functions on the nilpotent subvarieties of the quiver varieties. In this section we observe that the same method can be applied to the case of the multiplicative quiver varieties.

6.1. **Notation.** Suppose that the following data are given:

- P — a free $\mathbb{Z}$-module, called a *weight lattice*.
- I — an index set of simple roots.
- $\alpha_i \in P$ $(i \in I)$ — *simple root*,
- $h_i \in P^* := \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$ $(i \in I)$ — *simple coroot*.

- $(\ ,\ )$ — a symmetric bilinear form on P.

These are required to satisfy:

(i) $\langle h_i, \lambda \rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $i \in I$ and $\lambda \in P$; where $\langle\ ,\ \rangle \colon P^* \otimes P \to \mathbb{Z}$ is the natural pairing;

(ii) $c_{ij} := \langle h_i, \alpha_j \rangle$ forms a *generalized Cartan matrix*, i.e., $c_{ii} = 2$, $c_{ij} \in \mathbb{Z}_{\leq 0}$ $(i \neq j)$ and $c_{ij} = 0 \Leftrightarrow c_{ji} = 0$;

(iii) $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{>0}$;

(iv) $\{\alpha_i\}_{i \in I}$ is linearly independent; and

(v) there exists $\Lambda_i \in P$ $(i \in I)$, called the *fundamental weight*, such that $\langle h_j, \Lambda_i \rangle = \delta_{ij}$.

Such data are so-called *root data*, to which one associates a Kac-Moody Lie algebra $\mathfrak{g}$ (see e.g. [19]). Let $\mathbf{U}$ be the universal enveloping algebra of $\mathfrak{g}$. Recall the defining relations of it:

$$(17) \qquad\qquad [h, h'] = 0 \quad \text{for } h, h' \in P^*,$$

$$(18) \qquad\qquad [h, e_i] = \langle h, \alpha_i \rangle e_i,$$

$$(19) \qquad\qquad [h, f_i] = -\langle h, \alpha_i \rangle f_i,$$

$$(20) \qquad\qquad [e_i, f_j] = \delta_{ij} h_i,$$

$$(21) \qquad\qquad \sum_{n=0}^{1-c_{ij}} (-1)^n \binom{1 - c_{ij}}{n} e_i^n e_j e_i^{1-c_{ij}-n} = 0 \quad (i \neq j),$$

$$(22) \qquad\qquad \sum_{n=0}^{1-c_{ij}} (-1)^n \binom{1 - c_{ij}}{n} f_i^n f_j f_i^{1-c_{ij}-n} = 0 \quad (i \neq j).$$

We also use the following symbols:

- $P^+ := \{\lambda \in P \mid \langle h_i, \lambda \rangle \geq 0 \text{ for any } i \in I\}$ (the semigroup of *dominant weights*),
- $Q := \bigoplus_i \mathbb{Z}\alpha_i \subset P$ (*root lattice*),
- $Q^+ := \sum_i \mathbb{Z}_{\geq 0}\alpha_i \subset Q$.

Let (I, E) be the graph associated to $\mathbf{C}$, i.e., the graph whose vertex set is I and edge set E is given by $2\mathbf{I} - \mathbf{A} = \mathbf{C}$, where $\mathbf{I}$ is the identity matrix and $\mathbf{A}$ is a matrix whose (i, j) entry is just the number of edges joining i and j. Let (I, Ω) be a quiver whose underlying graph is (I, E).

6.2. **Framed multiplicative quiver variety.** For $\mathbf{v} \in Q^+$ and $\mathbf{w} \in P^+$, we define a variety $\mathcal{M}(\mathbf{v}, \mathbf{w})$ which is a multiplicative analogue of the Nakajima quiver variety $\mathfrak{M}(\mathbf{v}, \mathbf{w})$.

Following Crawley-Boevey (see [7, Introduction]), we associate to (I, Ω) and $\mathbf{w}$ another quiver $(\tilde{I}, \tilde{\Omega})$ by setting $\tilde{I} := I \cup \{\infty\}$ and letting $\tilde{\Omega}$ be the set obtained by adding w_i arrows starting at ∞ toward i for each $i \in I$ to Ω, where $w_i := \langle h_i, \mathbf{w} \rangle$.

Take an I-graded vector space V such that $\sum_i (\dim V_i)\, \alpha_i = \mathbf{v}$. To such V, we associate an $\tilde{I}$-graded vector space $\tilde{V}$ by $\tilde{V}_i := V_i$ and $\tilde{V}_\infty := \mathbb{C}$. To a pair $(q, \theta) \in (\mathbb{C}^\times)^I \times \mathbb{Z}^I$, we associate a pair $(\tilde{q}, \tilde{\theta}) \in (\mathbb{C}^\times)^{\tilde{I}} \times \mathbb{Z}^{\tilde{I}}$ by

$$\tilde{q}_i := q_i, \qquad \tilde{q}_\infty := \prod_i q_i^{-\dim V_i},$$

$$\tilde{\theta}_i := \theta_i, \qquad \tilde{\theta}_\infty := -\sum_i \theta_i \dim V_i.$$

We define an I-graded vector space W by $W_i := \mathbb{C}^{w_i}$. Then the vector space $\mathbf{M}(\widetilde{V})$ can be identified with

$$\mathbf{M}(V, W) := \mathbf{M}(V) \oplus \bigoplus_{i \in I} \operatorname{Hom}(W_i, V_i) \oplus \bigoplus_{i \in I} \operatorname{Hom}(W_i, V_i).$$

For an element x in $\mathbf{M}(V, W)$, we usually denote its three components by $B = (B_h)$, $a = (a_i)$, $b = (b_i)$. The multiplicative preprojective relation $\Phi_i(x) = q_i$ at $i \in I$ becomes

$$(1 + a_{i,1} b_{i,1}) \cdots (1 + a_{i,w_i} b_{i,w_i}) \prod_{i \in I} (1 + B_h B_{\overline{h}})^{\epsilon(h)} = q_i,$$

where

$$a_i = \begin{bmatrix} a_{i,1} & a_{i,2} & \cdots & a_{i,w_i} \end{bmatrix} : \mathbb{C}^{w_i} \to V_i, \qquad b_i = \begin{bmatrix} b_{i,1} \\ b_{i,2} \\ \vdots \\ b_{i,w_i} \end{bmatrix} : V_i \to \mathbb{C}^{w_i}.$$

The following can be checked easily.

Proposition 6.1. *A point $x = (B, a, b) \in \mathbf{M}(V, W)$ is $\tilde{\theta}$-semistable if and only if the following conditions are satisfied:*

   (i) *For any B-invariant subspace $S \subset V$ contained in $\operatorname{Ker} b := \bigoplus \operatorname{Ker} b_i$, the inequality $\theta \cdot \dim S \leq 0$ holds.*

   (ii) *For any B-invariant subspace $T \subset V$ containing $\operatorname{Im} a := \bigoplus \operatorname{Im} a_i$, the inequality $\theta \cdot \dim T \leq \theta \cdot \dim V$ holds.*

x is $\tilde{\theta}$-stable if and only if the strict inequalities hold in (i), (ii) *unless $S = 0$, $T = V$ respectively.*

For a subspace $S \subset V$ we usually identify $\dim S \in \mathbb{Z}_{\geq 0}^I$ with $\sum_i (\dim S_i) \alpha_i \in Q^+$.

We define

$$\mathcal{M}_{q,\theta}(\mathbf{v}, \mathbf{w}) := \mathcal{M}_{\tilde{q}, \tilde{\theta}}(\widetilde{V}), \qquad \mathcal{M}_{q,\theta}^{\mathrm{s}}(\mathbf{v}, \mathbf{w}) := \mathcal{M}_{\tilde{q}, \tilde{\theta}}^{\mathrm{s}}(\widetilde{V}),$$

both of which we call the *framed multiplicative quiver varieties*. One can easily check that the dimension of $\mathcal{M}_{q,\theta}^{\mathrm{s}}(\mathbf{v}, \mathbf{w})$ can be written as

$$\dim \mathcal{M}_{q,\theta}^{\mathrm{s}}(\mathbf{v}, \mathbf{w}) = \langle \mathbf{v}^{\vee}, 2\mathbf{w} - \mathbf{v} \rangle,$$

where $\mathbf{v}^{\vee} := \sum_i (\dim V_i) h_i$.

6.3. **Brill-Noether locus, Steinberg variety and Hecke correspondence.** In this subsection, we assume that:

   (i) $h < h'$ for all $h \in \Omega, h' \in \overline{\Omega}$; and

   (ii) $q_i = 1$ and $\theta_i > 0$ for all i.

Note that the stability condition for $(B, a, b) \in \Phi^{-1}(1)$ then becomes

   • If a subspace $S \subset V$ is B-invariant and contained in $\operatorname{Ker} b$, then $S = 0$,

and the semistability coincides with the stability. We write

$$\mathbf{M}^{\mathrm{s}}(V, W) = \mathbf{M}_\theta^{\mathrm{s}}(\widetilde{V}), \qquad Z^{\mathrm{s}}(V, W) = \Phi^{-1}(1) \cap \mathbf{M}^{\mathrm{s}}(V, W),$$

and

$$\mathcal{M}(\mathbf{v}, \mathbf{w}) = \mathcal{M}_{1,\theta}(\mathbf{v}, \mathbf{w}), \qquad \mathcal{M}_0(\mathbf{v}, \mathbf{w}) = \mathcal{M}_{1,0}(\mathbf{v}, \mathbf{w}).$$

Also we write $\mathcal{M}_0^{\mathrm{s}}(\mathbf{v}, \mathbf{w}) = \mathcal{M}_{1,0}^{\mathrm{s}}(\mathbf{v}, \mathbf{w})$.

The purpose of this section is to show that all the results proved by Nakajima in [30, §4] can be shown analogously in the multiplicative case, and that we can define the multiplicative version of Hecke correspondence of quiver varieties.

Since the projection $Z^{\mathrm{s}}(V, W) \to \mathcal{M}(\mathbf{v}, \mathbf{w})$ is a principal G_V-bundle and each V_i, W_i are representation spaces of G_V, we can define associated vector bundles

$$\mathcal{V}_i = Z^{\mathrm{s}}(V, W) \times_{G_V} V_i, \qquad \mathcal{W}_i = Z^{\mathrm{s}}(V, W) \times_{G_V} W_i.$$

We call these the *tautological bundles.*

Consider the following sequence of vector bundles:

$$C_i^{\bullet}(\mathbf{v}, \mathbf{w}) \colon \mathcal{V}_i \xrightarrow{\ \sigma_i\ } \bigoplus_{h \in H_i} \mathcal{V}_{\mathrm{out}(h)} \oplus \mathcal{W}_i \xrightarrow{\ \tau_i\ } \mathcal{V}_i \,,$$

where we have assigned the degree 0 to the middle term. $C_i^{\bullet}(\mathbf{v}, \mathbf{w})$ is a complex by the multiplicative preprojective relation, and the degree (-1) cohomology vanishes by Proposition 3.7. Let $Q_i(\mathbf{v}, \mathbf{w})$ denote the degree 0 cohomology; $Q_i(\mathbf{v}, \mathbf{w}) := H_0(C_i^{\bullet}(\mathbf{v}, \mathbf{w})) = \operatorname{Ker} \tau_i / \operatorname{Im} \sigma_i$.

We introduce the following subset of $\mathcal{M}(\mathbf{v}, \mathbf{w})$:

$$\mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w}) := \{\, [B, a, b] \in \mathcal{M}(\mathbf{v}, \mathbf{w}) \mid \operatorname{corank} \tau_i(B, a, b) = n \,\},$$

$$\mathcal{M}_{i;\leq n}(\mathbf{v}, \mathbf{w}) := \bigcup_{m \leq n} \mathcal{M}_{i;m}(\mathbf{v}, \mathbf{w}).$$

Since $\mathcal{M}_{i;\leq n}(\mathbf{v}, \mathbf{w})$ is an open subvariety of $\mathcal{M}(\mathbf{v}, \mathbf{w})$, $\mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w})$ is a locally closed subvariety. The restriction $Q_{i;n}(\mathbf{v}, \mathbf{w}) := Q_i(\mathbf{v}, \mathbf{w})|_{\mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w})}$ is a vector bundle of rank $\langle h_i, \mathbf{w} - \mathbf{v} \rangle + n$. $\mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w})$ is a multiplicative analogue of the Brill-Noether locus of the quiver variety.

Replacing V_i to $\operatorname{Im} \tau_i$, we have a natural morphism

$$p \colon \mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}_{i;0}(\mathbf{v} - n\alpha_i, \mathbf{w}).$$

Similar to [30, Proposition 4.5], we have:

Proposition 6.2. *Let* $G(n, Q_{i;0}(\mathbf{v} - n\alpha_i, \mathbf{w}))$ *be the Grassmann bundle of n-planes in* $Q_{i;0}(\mathbf{v} - n\alpha_i, \mathbf{w})$. *Then we have the following commutative diagram:*

$$
\begin{array}{ccc}
G(n, Q_{i;0}(\mathbf{v} - n\alpha_i, \mathbf{w})) & \xrightarrow{\ projection\ } & \mathcal{M}_{i;0}(\mathbf{v} - n\alpha_i, \mathbf{w}) \\
\ \downarrow{\simeq} & & \| \\
\mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w}) & \xrightarrow{\quad p \quad} & \mathcal{M}_{i;0}(\mathbf{v} - n\alpha_i, \mathbf{w})\,.
\end{array}
$$

The kernel of the natural surjective homomorphism $p^* Q_{i;0}(\mathbf{v} - n\alpha_i, \mathbf{w}) \to Q_{i;n}(\mathbf{v}, \mathbf{w})$ *is isomorphic to the tautological vector bundle of* $G(n, Q_{i;0}(\mathbf{v} - n\alpha_i, \mathbf{w}))$ *via the isomorphism of the first row. In particular,*

$$\dim \mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w}) = \dim \mathcal{M}_{i;0}(\mathbf{v} - n\alpha_i, \mathbf{w}) + n\left(\langle h_i, \mathbf{w} - \mathbf{v} \rangle + n\right)$$

$$= \dim \mathcal{M}(\mathbf{v}, \mathbf{w}) - n\left(\langle h_i, \mathbf{w} - \mathbf{v} \rangle + n\right).$$

Proof. The proof is almost the same as [30, Proposition 4.5].

The vector bundle $p^* Q_{i;0}(\mathbf{v} - n\alpha_i, \mathbf{w})$ is given by $\operatorname{Ker} \tau_i / \sigma_i(\operatorname{Im} \tau_i)$. Considering the natural surjection $\operatorname{Ker} \tau_i / \sigma_i(\operatorname{Im} \tau_i) \to \operatorname{Ker} \tau_i / \operatorname{Im} \sigma_i$, we have a surjective homomorphism $p^* Q_{i;0}(\mathbf{v} - n\alpha_i, \mathbf{w}) \to$

$Q_{i;n}(\mathbf{v}, \mathbf{w})$. Its kernel $\operatorname{Im}\sigma_i / \sigma_i(\operatorname{Im}\tau_i) \simeq V_i / \operatorname{Im}\tau_i$ has a constant rank n. Thus we get a morphism from $\mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w})$ to the Grassmann bundle.

Conversely suppose that a point ϕ in the Grassmann bundle is given. Take a subspace $V' \subset V$ such that $\dim V' = \mathbf{v} - n\alpha_i$. Let $(B', a', b') \in Z^s(V', W)$ be a representative of the image of ϕ under the projection, and σ_i', τ_i' denote $\sigma_i(B', a', b'), \tau_i(B', a', b')$ respectively. Take an injective homomorphism $\sigma_i \colon V_i \to \widehat{V}_i \oplus W_i$ such that $\sigma_i|_{V_i'} = \sigma_i'$ and $\operatorname{Im}\sigma_i / \operatorname{Im}\sigma_i' = \phi$. Now we define

$$B_h := B_h' \quad \text{for } h \notin H_i \cup \overline{H_i},$$

$$a_j := a_j', \ b_j := b_j' \quad \text{for } j \neq i,$$

$$B_h := B_h' \colon V_{\mathrm{out}(h)} \to V_i' \hookrightarrow V_i \quad \text{for } h \in H_i,$$

$$a_i := a_i' \colon W_i \to V_i' \hookrightarrow V_i,$$

and define b_i and $B_{\overline{h}}$ for $h \in H_i$ by the condition $\sigma_i(B, a, b) = \sigma_i$. Since $\sigma_i|_{V_i'} = \sigma_i'$, one can prove inductively that $b_i|_{V_i'} = b_i'$ and $B_{\overline{h}}|_{V_i'} = B_{\overline{h}}'$ for $h \in H_i$. Thus $\tau_i = \tau_i'$ and hence

$$\operatorname{Im}\sigma_i / \sigma_i(\operatorname{Im}\tau_i) = \operatorname{Im}\sigma_i / \operatorname{Im}\sigma_i' = \phi.$$

By definition we have $\tau_i \sigma_i = 0$, which implies $\Phi_i(B, a, b) = 1$. Moreover $b_i|_{V_i'} = b_i'$ and $B_{\overline{h}}|_{V_i'} = B_{\overline{h}}'$ implies $b_i a_i = b_i' a_i'$ and $B_{\overline{h}} B_h = B_{\overline{h}}' B_h'$, and hence $\Phi_j(B, a, b) = 1$ for all $j \neq i$. Thus $(B, a, b) \in \Phi^{-1}(1)$.

To check the stability condition, suppose that there is a B-invariant subspace S contained in $\operatorname{Ker} b$. We define a subspace $S' \subset V'$ by

$$S_j' = \begin{cases} S_j & \text{if } j \neq i, \\ S_i \cap \operatorname{Im}\tau_i' = S_i \cap \operatorname{Im}\tau_i & \text{if } j = i. \end{cases}$$

Then one can easily check that S' is B'-invariant and contained in $\operatorname{Ker} b'$ using Lemma 3.6. Thus $S' = 0$ by the stability condition for (B', a', b'). In particular $S_j = 0$ for $j \neq i$, which implies $\sigma_i(S_i) = 0$. Since we have taken σ_i to be injective, S_i must be zero. Thus (B, a, b) is stable. Taking a quotient by the G_V-action we obtain a morphism from the Grassmann bundle to $\mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w})$. It is the inverse of the previous morphism.

To prove the last equality we compute that

$$\dim \mathcal{M}(\mathbf{v}, \mathbf{w}) - \dim \mathcal{M}(\mathbf{v} - n\alpha_i, \mathbf{w}) = \langle \mathbf{v}^{\vee}, 2\mathbf{w} - \mathbf{v} \rangle - \langle \mathbf{v}^{\vee} - nh_i, 2\mathbf{w} - \mathbf{v} + n\alpha_i \rangle$$

$$= 2n \left(\langle h_i, \mathbf{w} - \mathbf{v} \rangle + n \right).$$

$\square$

Let $\mathbf{v}_1, \mathbf{v}_2 \in Q^+$ and $\mathbf{w} \in P^+$. Let $\pi \colon \mathcal{M}(\mathbf{v}^i, \mathbf{w}) \to \mathcal{M}_0(\mathbf{v}^i, \mathbf{w})$ $(i = 1, 2)$ be the projection. Recall that $\mathcal{M}_0(\mathbf{v}^i, \mathbf{w})$ is naturally embedded in $\mathcal{M}_0(\mathbf{v}^1 + \mathbf{v}^2, \mathbf{w})$ by Proposition 2.7. Thus we can regard π's as maps to $\mathcal{M}_0(\mathbf{v}^1 + \mathbf{v}^2, \mathbf{w})$. Following Nakajima [30], we define

$$\mathcal{S}(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}) := \left\{ (x^1, x^2) \in \mathcal{M}(\mathbf{v}^1, \mathbf{w}) \times \mathcal{M}(\mathbf{v}^2, \mathbf{w}) \mid \pi(x^1) = \pi(x^2) \right\}$$

$$= \mathcal{M}(\mathbf{v}^1, \mathbf{w}) \times_{\mathcal{M}_0(\mathbf{v}^1 + \mathbf{v}^2, \mathbf{w})} \mathcal{M}(\mathbf{v}^2, \mathbf{w}),$$

which is an analogue of the Steinberg variety.

Definition 6.3. For $n \in \mathbb{Z}_{>0}$ and $\mathbf{v} \in Q^+$, the *Hecke correspondence* $\mathcal{P}_i^{(n)}(\mathbf{v}, \mathbf{w})$ is the variety defined as

$$\mathcal{P}_i^{(n)}(\mathbf{v}, \mathbf{w}) := \left\{ (B, a, b, S) \,\middle|\, \begin{array}{l} (B, a, b) \in Z^{\mathrm{s}}(V, W),\ S \subset V, \\ S \text{ is } B\text{-invariant, Im}\, a \subset S \text{ and } \dim S = \mathbf{v} - n\alpha_i \end{array} \right\} / G_V.$$

We denote $\mathcal{P}_i(\mathbf{v}, \mathbf{w}) = \mathcal{P}_i^{(1)}(\mathbf{v}, \mathbf{w})$.

We have the following diagram:

$$(23) \qquad\qquad \mathcal{M}(\mathbf{v} - n\alpha_i, \mathbf{w}) \xleftarrow{p_1} \mathcal{P}_i^{(n)}(\mathbf{v}, \mathbf{w}) \xrightarrow{p_2} \mathcal{M}(\mathbf{v}, \mathbf{w}).$$

The first map is given by the restriction of (B, a, b) to S, and the second is given by forgetting S (It is clear that $(B, a, b)|_S \in Z^{\mathrm{s}}(S, W)$).

Note that $p_1 \times p_2 \colon \mathcal{P}_i^{(n)}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}(\mathbf{v} - n\alpha_i, \mathbf{w}) \times \mathcal{M}(\mathbf{v}, \mathbf{w})$ is an embedding whose image consists of all pairs $([B'', a'', b''], [B, a, b])$ such that there exists $\xi \in \bigoplus_{i \in I} \operatorname{Hom}(V_i'', V_i)$ satisfying

$$\xi B'' = B\xi, \qquad \xi a'' = a, \qquad b'' = b\xi.$$

Here we fix an I-graded vector space V'' such that $\sum (\dim V_i'')\, \alpha_i = \mathbf{v} - \mathbf{v}'$. Indeed, if such a ξ exists, then $\operatorname{Ker} \xi$ is zero by the stability condition and $\operatorname{Im} \xi$ is B-invariant and contains $\operatorname{Im} a$. Moreover ξ is unique if we fix representatives $(B'', a'', b''), (B, a, b)$. Thus the point $[(B, a, b), \operatorname{Im} \xi] \in \mathcal{P}_i^{(n)}(\mathbf{v}, \mathbf{w})$ is well-defined.

It is clear that this subvariety is contained in $\mathcal{S}(\mathbf{v} - n\alpha_i, \mathbf{v}; \mathbf{w})$. So we may regard $\mathcal{P}_i^{(n)}(\mathbf{v}, \mathbf{w})$ as a subvariety of $\mathcal{S}(\mathbf{v} - n\alpha_i, \mathbf{v}; \mathbf{w})$.

Similar to [30, Lemma 5.12], we can prove the following.

Proposition 6.4. *Consider the diagram* (23) *with* $n = 1$.

(i) $p_1^{-1}(\mathcal{M}_{i;n}(\mathbf{v} - \alpha_i, \mathbf{w}))$ *can be identified with the projective bundle* $\mathbb{P}(Q_{i;n}(\mathbf{v} - \alpha_i, \mathbf{w}))$.

(ii) $p_2^{-1}(\mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w}))$ *can be identified with the projective bundle* $\mathbb{P}\left(H_1(C_i^\bullet(\mathbf{v}, \mathbf{w}))^*|_{\mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w})}\right)$.

Proof. The proof is similar to Proposition 6.2 $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.4. **Constructible functions.** Let X be an algebraic variety. A $\mathbb{Q}$-valued *constructible function* on X is a function $f \colon X \to \mathbb{Q}$ such that $f(X)$ is finite and $f^{-1}(a)$ is constructible for all $a \in \mathbb{Q}$. Let $\operatorname{CF}(X)$ be the $\mathbb{Q}$-vector space consisting of all $\mathbb{Q}$-valued constructible functions on X. If $Y \subset X$ is a subvariety, we regard $\operatorname{CF}(Y)$ as a subspace of $\operatorname{CF}(X)$ by extending with zero on the complement. A typical example of constructible functions is the characteristic function $[A]$ of a constructible subset $A \subset X$;

$$[A](x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$

and by the definition any constructible functions can be written as a linear combination of characteristic functions.

Any morphism $p \colon X \to Y$ induces the *pull-back* and the *push-forward* between the vector spaces of constructible functions:

$$p^* \colon \operatorname{CF}(Y) \to \operatorname{CF}(X); \qquad (p^* g)(x) := g(p(x)),$$

$$p_! \colon \operatorname{CF}(X) \to \operatorname{CF}(Y); \qquad (p_! f)(y) := \sum_{c \in \mathbb{Q}} c\, \chi(p^{-1}(y) \cap f^{-1}(c)),$$

where χ denotes the Euler characteristic. Regarding χ as a "measure" of constructible subsets, $p_!$ is also written as

$$(p_! f)(y) = \int_{x \in p^{-1}(y)} f(x).$$

Let M_1, M_2, M_3 be three varieties and $p_{ij} \colon M_1 \times M_2 \times M_3 \to M_i \times M_j$ $(i, j = 1, 2, 3)$ be the projection to the i-th and j-th factors. For $f \in \mathrm{CF}(M_1 \times M_2)$ and $f' \in \mathrm{CF}(M_2 \times M_3)$, the *convolution product* $f * f'$ of f and f' is defined by

$$f * f' := (p_{13})_! (p_{12}^*(f) p_{23}^*(f')) \in \mathrm{CF}(M_1 \times M_3).$$

Note that it can be written as

$$(f * f')(x_1, x_3) = \int_{x_2 \in M_2} f(x_1, x_2) f'(x_2, x_3).$$

It is easy to see that the convolution product is associative.

Suppose there are morphisms $p_i \colon M_i \to M_0$ $(i = 1, 2, 3)$ to some variety M_0. Then it is clear that if $f \in \mathrm{CF}(M_1 \times_{M_0} M_2)$ and $f' \in \mathrm{CF}(M_2 \times_{M_0} M_3)$, then $f * f' \in \mathrm{CF}(M_1 \times_{M_0} M_3)$, i.e., the support of $f * f'$ is contained in $M_1 \times_{M_0} M_3$.

6.5. **A geometric construction of the universal enveloping algebra.** Let $\mathcal{A}(\mathbf{w})$ be the subspace of the direct product

$$\prod_{\mathbf{v}^1, \mathbf{v}^2} \mathrm{CF}(\mathcal{S}(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$$

consisting of all elements $(F_{\mathbf{v}^1, \mathbf{v}^2})$ such that the following two conditions are satisfied:

(i) For fixed $\mathbf{v}^1$, $F_{\mathbf{v}^1, \mathbf{v}^2} = 0$ for all but finitely many $\mathbf{v}^2$.
(ii) For fixed $\mathbf{v}^2$, $F_{\mathbf{v}^1, \mathbf{v}^2} = 0$ for all but finitely many $\mathbf{v}^1$.

By the convolution product, it is an associative algebra with $1 = \sum_{\mathbf{v}} [\Delta(\mathbf{v}, \mathbf{w})]$, where $\Delta(\mathbf{v}, \mathbf{w})$ denotes the diagonal subset of $\mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v}, \mathbf{w})$.

Let $(\bullet)^\dagger \colon \mathcal{M}(\mathbf{v} - \alpha_i, \mathbf{w}) \times \mathcal{M}(\mathbf{v}, \mathbf{w}) \to \mathcal{M}(\mathbf{v}, \mathbf{w}) \times \mathcal{M}(\mathbf{v} - \alpha_i, \mathbf{w})$ be the flip of the components. The main theorem in this section is the following:

Theorem 6.5 (cf. [30, Theorem 9.4]). *There is an algebra homomorphism* $\mathbf{U} \to \mathcal{A}(\mathbf{w})$ *such that*

$$h \mapsto \sum_{\mathbf{v}} \langle h, \mathbf{w} - \mathbf{v} \rangle \cdot [\Delta(\mathbf{v}, \mathbf{w})], \qquad e_i \mapsto \sum_{\mathbf{v}^2} [\mathcal{P}_i(\mathbf{v}^2, \mathbf{w})], \qquad f_i \mapsto \sum_{\mathbf{v}^2} [\mathcal{P}_i(\mathbf{v}^2, \mathbf{w})^\dagger].$$

The relations (17), (18), (19) are obviously satisfied. We check the relations (20) and (21), (22) by the same method as in [29, 30].

6.6. **The relation** $[e_i, f_j] = \delta_{ij} h_i$**.** In this subsection we check the relation $[e_i, f_j] = \delta_{ij} h_i$. Fix $\mathbf{v} \in Q^+$ and consider the following diagram:

$$
\begin{array}{ccc}
\mathcal{M}(\mathbf{v} - \alpha_i, \mathbf{w}) & \longleftarrow & \mathcal{S}(\mathbf{v} - \alpha_i, \mathbf{v}; \mathbf{w}) \\
& & \downarrow \\
\mathcal{M}(\mathbf{v}, \mathbf{w}) & \longleftarrow & \mathcal{S}(\mathbf{v}, \mathbf{v} - \alpha_j; \mathbf{w}) \\
& & \downarrow \\
& & \mathcal{M}(\mathbf{v} - \alpha_j, \mathbf{w}),
\end{array}
$$

where the arrows are the natural morphisms. Set $\mathbf{v}^1 := \mathbf{v} - \alpha_i$, $\mathbf{v}^2 := \mathbf{v}$, $\mathbf{v}^3 := \mathbf{v} - \alpha_j$ and let $\mathcal{S}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3; \mathbf{w})$ be the fiber product of $\mathcal{S}(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$ and $\mathcal{S}(\mathbf{v}^2, \mathbf{v}^3; \mathbf{w})$ over $\mathcal{M}(\mathbf{v}^2, \mathbf{w})$. Then the above diagram induces the natural morphisms

$$\mathcal{M}(\mathbf{v}^i, \mathbf{w}) \xleftarrow{p_i} \mathcal{S}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3; \mathbf{w}) \xrightarrow{p_j} \mathcal{M}(\mathbf{v}^j, \mathbf{w})$$

for $i, j = 1, 2, 3$. We set $p_{ij} := p_i \times p_j \colon \mathcal{S}(\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3; \mathbf{w}) \to \mathcal{M}(\mathbf{v}^i, \mathbf{w}) \times \mathcal{M}(\mathbf{v}^j, \mathbf{w})$. Then $e_i f_j$ is described as

$$e_i f_j = \sum_{\mathbf{v}^2} (p_{13})_! \left(p_{12}^*[\mathcal{P}_i(\mathbf{v}^2, \mathbf{w})] p_{23}^*[\mathcal{P}_j(\mathbf{v}^2, \mathbf{w})^\dagger] \right)$$

$$= \sum_{\mathbf{v}^2} (p_{13})_! \left[p_{12}^{-1}(\mathcal{P}_i(\mathbf{v}^2, \mathbf{w})) \cap p_{23}^{-1}(\mathcal{P}_j(\mathbf{v}^2, \mathbf{w})^\dagger) \right].$$

Next consider the following diagram:

$$\mathcal{M}(\mathbf{v} - \alpha_i, \mathbf{w}) \longleftarrow \mathcal{S}(\mathbf{v} - \alpha_i, \mathbf{v} - \alpha_j - \alpha_i; \mathbf{w})$$

$$\downarrow$$

$$\mathcal{M}(\mathbf{v} - \alpha_j - \alpha_i, \mathbf{w}) \longleftarrow \mathcal{S}(\mathbf{v} - \alpha_j - \alpha_i, \mathbf{v} - \alpha_j; \mathbf{w})$$

$$\downarrow$$

$$\mathcal{M}(\mathbf{v} - \alpha_j, \mathbf{w}).$$

Set $\mathbf{v}^4 := \mathbf{v} - \alpha_j - \alpha_i$, and define $\mathcal{S}(\mathbf{v}^1, \mathbf{v}^4, \mathbf{v}^3)$ and $q_{ij} \colon \mathcal{S}(\mathbf{v}^1, \mathbf{v}^4, \mathbf{v}^3) \to \mathcal{M}(\mathbf{v}^i, \mathbf{w}) \times \mathcal{M}(\mathbf{v}^j, \mathbf{w})$ as above. Then $f_j e_i$ is described as

$$f_j e_i = \sum_{\mathbf{v}^4} (q_{13})_! \left(q_{14}^*[\mathcal{P}_j(\mathbf{v}^1, \mathbf{w})^\dagger] q_{43}^*[\mathcal{P}_i(\mathbf{v}^3, \mathbf{w})]) \right)$$

$$= \sum_{\mathbf{v}^4} (q_{13})_! \left[q_{14}^{-1}(\mathcal{P}_j(\mathbf{v}^1, \mathbf{w})^\dagger) \cap q_{43}^{-1}(\mathcal{P}_i(\mathbf{v}^3, \mathbf{w})) \right].$$

The following lemma can be proved by the same way as [30, Lemma 9.10].

Lemma 6.6. *Let $U \subset \mathcal{M}(\mathbf{v}^1, \mathbf{w}) \times \mathcal{M}(\mathbf{v}^3, \mathbf{w})$ denotes the outside of the diagonal when $i = j$, and the whole set otherwise. Then there is an isomorphism*

$$\Pi \colon p_{13}^{-1}(U) \cap p_{12}^{-1}(\mathcal{P}_i(\mathbf{v}^2, \mathbf{w})) \cap p_{23}^{-1}(\mathcal{P}_j(\mathbf{v}^2, \mathbf{w})^\dagger) \longrightarrow q_{13}^{-1}(U) \cap q_{14}^{-1}(\mathcal{P}_j(\mathbf{v}^1, \mathbf{w})^\dagger) \cap q_{43}^{-1}(\mathcal{P}_i(\mathbf{v}^3, \mathbf{w}))$$

such that $q_{13} \circ \Pi = p_{13}$.

Thus $e_i f_j - f_j e_i = 0$ if $i \neq j$, and the support of $e_i f_i - f_i e_i$ is contained in $\sqcup_{\mathbf{v}} \Delta(\mathbf{v}, \mathbf{w})$. So it is sufficient to prove

$$(e_i f_i - f_i e_i)(x, x) = \langle h_i, \mathbf{w} - \mathbf{v} \rangle$$

for all $x \in \mathcal{M}(\mathbf{v}, \mathbf{w})$.

Suppose that $x \in \mathcal{M}_{i;n}(\mathbf{v}, \mathbf{w})$. Then using Proposition 6.4 we compute

$$\chi \left(p_{13}^{-1}(x, x) \cap p_{12}^{-1}(\mathcal{P}_i(\mathbf{v}, \mathbf{w})) \cap p_{23}^{-1}(\mathcal{P}_j(\mathbf{v}, \mathbf{w})^\dagger) \right) = \chi \left(\mathbb{P}\left(Q_{i,n}(\mathbf{v} - \alpha_i, \mathbf{w})|_x \right) \right)$$

$$= \langle h_i, \mathbf{w} - \mathbf{v} \rangle + r + 1.$$

Similarly,

$$\chi\left(q_{13}{}^{-1}(x,x) \cap q_{14}^{-1}(\mathcal{P}_j(\mathbf{v}-\alpha_i,\mathbf{w})^\dagger) \cap q_{43}^{-1}(\mathcal{P}_i(\mathbf{v}-\alpha_i,\mathbf{w}))\right) = \chi\left(\mathbb{P}\left(H_1(C_i^\bullet(\mathbf{v}-\alpha_i,\mathbf{w}))|_x\right)\right)$$

$$= r+1.$$

Thus

$$(e_i f_i - f_i e_i)(x,x) = \langle h_i, \mathbf{w}-\mathbf{v}\rangle + r + 1 - (r+1) = \langle h_i, \mathbf{w}-\mathbf{v}\rangle.$$

6.7. **The Serre relations.** In this subsection we check the relations (21) and (22).

Fix vertices i,j with $i \neq j$, and set $N := -c_{ij}$. For $n = 0, 1, \ldots, N+1$, let P_n be the fiber product

$$\mathcal{P}_j^{(N+1-n)}(\mathbf{v}-n\alpha_j-\alpha_i,\mathbf{w}) \times_{\mathcal{M}(\mathbf{v}-n\alpha_j-\alpha_i,\mathbf{w})} \mathcal{P}_i(\mathbf{v}-n\alpha_j,\mathbf{w}) \times_{\mathcal{M}(\mathbf{v}-n\alpha_j,\mathbf{w})} \mathcal{P}_j^{(n)}(\mathbf{v},\mathbf{w}).$$

Consider the variety consisting of all tuples (B, a, b, S^1, S^2, S^3), where $(B, a, b) \in Z^s(V, W)$ and each S^k is B-invariant subspace of V such that

$$S^1 \supset S^2 \supset S^3 \supset \operatorname{Im} a, \quad \text{and}$$

$$\dim S^1 = \mathbf{v}-n\alpha_i, \ \dim S^2 = \mathbf{v}-n\alpha_j-\alpha_i, \ \dim S^3 = \mathbf{v}-(N+1)\alpha_j-\alpha_i.$$

Then the quotient of it by the G_V-action is naturally isomorphic to P_n.

Set

$$P := \left\{(B, a, b, S) \ \middle| \ \begin{array}{l} (B, a, b) \in Z^s(V, W), \ S \subset V, \\ S \text{ is } B\text{-invariant}, \operatorname{Im} a \subset S \text{ and } \dim S = \mathbf{v}-\alpha_i-(N+1)\alpha_j \end{array}\right\} / G_V.$$

It is a subvariety of $\mathcal{S}(\mathbf{v}-\alpha_i-(N+1)\alpha_j, \mathbf{v}; \mathbf{w})$, and we have a natural morphism $r_n \colon P_n \to P$ which sends $[(B, a, b, S^1, S^2, S^3)]$ to $[(B, a, b, S^3)]$.

Lemma 6.7. *We have*

$$e_j^{N+1-n} e_i e_j^n [\Delta(\mathbf{v},\mathbf{w})] = f_j^n f_i f_j^{N+1-n}[\Delta(\mathbf{v},\mathbf{w})] = (N+1-n)! n! (r_n)_! [P_n][\Delta(\mathbf{v},\mathbf{w})].$$

Proof. Consider the variety consisting of all tuples $(B, a, b, \{S_k\}_{k=1}^n)$, where $(B, a, b) \in Z^s(V, W)$ and each S^k is a B-invariant subspace of V such that

$$S^1 \supset S^2 \supset \cdots \supset S^n \supset \operatorname{Im} a, \quad \text{and}$$

$$\dim S^k = \mathbf{v}-k\alpha_i.$$

Let $\mathcal{P}_i^n(\mathbf{v},\mathbf{w})$ be the quotient of it modulo G_V-action. Then $e_j^{N+1-n} e_i e_j^n[\Delta(\mathbf{v},\mathbf{w})]$ is given by the push-forward of

$$[\mathcal{P}_j^{N+1-n}(\mathbf{v}-n\alpha_j-\alpha_i,\mathbf{w})] * [\mathcal{P}_i(\mathbf{v}-n\alpha_j,\mathbf{w})] * [\mathcal{P}_j^n(\mathbf{v},\mathbf{w})]$$

by the obvious morphism.

Let $\pi_n \colon \mathcal{P}_i^n(\mathbf{v},\mathbf{w}) \to \mathcal{P}_i^{(n)}(\mathbf{v},\mathbf{w})$ be the morphism $[(B, a, b, S^1, \ldots, S^n)] \mapsto [(B, a, b, S^n)]$. Then the fiber of π_n is isomorphic to the full flag variety of n-dimensional vector space, and hence its Euler characteristic is just $n!$. Thus

$$(\pi_n)_! [\mathcal{P}_i^n(\mathbf{v},\mathbf{w})] = n! [\mathcal{P}_i^{(n)}(\mathbf{v},\mathbf{w})].$$

This proves the assertion. □

By the above lemma, it is enough to show that

$$\sum_{n=0}^{N+1}(-1)^n\chi(r_n^{-1}(x)) = 0 \qquad \text{for any } x \in \mathcal{P}(\mathbf{v},\mathbf{w};\alpha_i+(N+1)\alpha_j).$$

Take a representative (B,a,b,S) of x.

Recall the complex

$$V_j \xrightarrow{\ \sigma_j\ } \widehat{V}_j \oplus W_j \xrightarrow{\ \tau_j\ } V_j\,.$$

For $k \in I$ with $k \neq j$, let σ_j^k be the projection of σ_j to $\bigoplus_{h\in H_j;\mathrm{out}(h)=k} V_k$, and τ_j^k be the restriction of τ_j on $\bigoplus_{h\in H_j;\mathrm{out}(h)=k} V_k \subset \widehat{V}_j$. Similarly, let σ_j^W be the projection of σ_j to W_j and τ_j^W be the restriction of τ_j on W_j.

Lemma 6.8. *Define $T_1 := S_j + \mathrm{Im}\,\tau_j^i$ and $T_2 := (\sigma_j^i)^{-1}(S_i^{\oplus N})$. Then the fiber $r_n^{-1}(x)$ is isomorphic to the variety consisting of all codimension n subspaces $T \subset V_j$ such that $T_1 \subset T \subset T_2$.*

Proof. For given $(B,a,b,S^1,S^2,S^3) \in r_n^{-1}(x)$, we set $T := S_j^1 = S_j^2$. Since S^1,S^2 are B-invariant and $V_i = S_i^1$, $T_1 \subset T \subset T_2$ is clearly satisfied.

Conversely suppose that a codimension n subspace $T \subset V_j$ with $T_1 \subset T \subset T_2$ is given. Then we set

$$S^3 := S, \qquad S_k^2 := \begin{cases} T & \text{if } k = j, \\ S_k & \text{if } k \neq j, \end{cases} \qquad S_k^1 := \begin{cases} T & \text{if } k = j, \\ V_k & \text{if } k \neq j. \end{cases}$$

$T_1 \subset T$ implies $S^3 = S \subset S^2$. We prove that S^1,S^2 are B-invariant. Since $S_k^2 = S_k$ for $k \neq j$ and $S_k^1 = S_k^2$ for $k \neq i$, it is enough to show that

$$\sigma_j(S_j^2) \subset \widehat{S}_j \oplus W_j, \qquad \mathrm{Im}\,\tau_j \subset S_j^1$$

by Lemma 3.6. $T \subset T_2$ implies $\sigma_j^i(S_j^2) \subset S_i^{\oplus N}$, and $S_k^2 = V_k$ for $k \neq i,j$ implies $\sigma_j^k(S_j^2) \subset S_k^{\oplus -c_{kj}}$. Thus $\sigma_j(S_j^2) \subset \widehat{S}_j \oplus W_j$. Also, $\mathrm{Im}\,\tau_j^k \subset S_j^1$ for $k \neq i,j$ follows from $V_k = S_k$ and $S_j \subset S_j^1$, and $T_1 \subset T$ implies $\mathrm{Im}\,\tau_j^i \subset S_j^1$. Moreover $\mathrm{Im}\,a \subset S \subset S^1$ implies $\mathrm{Im}\,\tau_j^W \subset S_j^1$. So we get $\tau_j(\widehat{S^1}_j \oplus W_j) \subset S_j^1$.

We complete the proof. $\qquad\square$

Lemma 6.9. $T_1 \neq T_2$.

Proof. Note that $\tau_j^i\sigma_j^i = -\tau_j^W\sigma_j^W - \sum_{k\neq i,j}\tau_j^k\sigma_j^k$, and hence

$$\mathrm{Im}\,\tau_j^i\sigma_j^i = \mathrm{Im}\,\tau_j^W + \sum_{k\neq i,j}\tau_j^k\sigma_j^k(V_j) \subset S_j + \sum_{k\neq i,j}\tau_j^k\left(V_k^{\oplus -c_{kj}}\right) = S_j + \sum_{k\neq i,j}\tau_j^k\left(S_k^{\oplus -c_{kj}}\right) \subset S_j.$$

So by the definitions of T_1,T_2 we have a complex

$$0 \longrightarrow V_j/T_2 \xrightarrow{\ \sigma_j^i\ } (V_i/S_i)^{\oplus N} \xrightarrow{\ \tau_j^i\ } T_1/S_j \longrightarrow 0\,,$$

which is exact except possibly at the middle term. Hence we have $\dim V_j/T_2 \leq N - \dim T_1/S_j^3$, and hence

$$\dim T_2 - \dim T_1 \geq \dim V_j - \dim S_j^3 - N = 1.$$

Thus $T_1 \neq T_2$. $\qquad\square$

Set $d_i := \operatorname{codim} T_i$ $(i = 1, 2)$. Then the fiber $r_n^{-1}(x)$ is empty unless $T_1 \subset T_2$ and $d_1 \leq n \leq d_2$, in which case $r_n^{-1}(x)$ is a Grassmannian manifold of $(d_1 - n)$-dimensional subspaces in a $(d_1 - d_2)$-dimensional space. Thus

$$\sum_{n=0}^{N+1} (-1)^n \chi(r_n^{-1}(x)) = \sum_{n=d_2}^{d_1} (-1)^n \binom{d_1 - d_2}{d_1 - n} = 0.$$

We complete the proof.

6.8. **Construction of irreducible highest weight representations.** Let $\mathcal{L}(\mathbf{v}, \mathbf{w})$ denote the nilpotent subvariety $\pi^{-1}([0]) \subset \mathcal{M}(\mathbf{v}, \mathbf{w})$. The vector space $\bigoplus_{\mathbf{v}} \operatorname{CF}(\mathcal{L}(\mathbf{v}, \mathbf{w}))$ becomes a representation space of the algebra $\mathcal{A}(\mathbf{w})$ by the following way:

$$(F * f)(x^1) := \int_{x^2 \in \mathcal{L}(\mathbf{v}^2, \mathbf{w})} F(x^1, x^2) f(x^2) \quad \text{for} \quad F \in \operatorname{CF}(\mathcal{S}(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})), \ f \in \operatorname{CF}(\mathcal{L}(\mathbf{v}^2, \mathbf{w})).$$

Note that $\mathcal{M}(0, \mathbf{w})$ (and hence $\mathcal{L}(0, \mathbf{w})$) consists of a single point. Set

$$L(\mathbf{w}) := \mathbf{U}^- \cdot [\mathcal{L}(0, \mathbf{w})] \subset \bigoplus_{\mathbf{v}} \operatorname{CF}(\mathcal{L}(\mathbf{v}, \mathbf{w})), \qquad L(\mathbf{v}, \mathbf{w}) := \operatorname{CF}(\mathcal{L}(\mathbf{v}, \mathbf{w})) \cap L(\mathbf{w}).$$

By the same way as [29, Lemma 10.13], one can easily show that $f_i^{w_i+1}[\mathcal{L}(0, \mathbf{w})] = 0$ for all $i \in I$, where $w_i = \langle \mathbf{w}, h_i \rangle$. Thus we get the following corollary:

Corollary 6.10 (cf. [29, Theorem 10.14]). *$L(\mathbf{w})$ is the irreducible highest weight integrable $\mathfrak{g}$-module with the highest weight $\mathbf{w}$, and $L(\mathbf{v}, \mathbf{w})$ is the $(\mathbf{w} - \mathbf{v})$-weight space of $L(\mathbf{w})$.*

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO UNIVERSITY, KYOTO, 606-8502, JAPAN
E-mail address: yamakawa@math.kyoto-u.ac.jp