Affine Geometry of Space Curves

Mehdi Nadjafikhah[a,1]**, Ali Mahdipour–Shirayeh**[b]

[a,b]**Department of Mathematics, Iran University of Science and Technology, Narmak, Tehran 16846-13114, IRAN**

Abstract

This paper is devoted to the complete classification of space curves under affine transformations in the view of Cartan's theorem. Spivak has introduced the method but has not found the invariants. Furthermore, for the first time, we propound a necessary and sufficient condition for the invariants. Then, we study the shapes of space curves with constant curvatures in detail and suggest their applications in physics, computer vision and image processing.

Keywords. affine geometry, curves in Euclidean space, differential invariants.

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1 Introduction

Classification of curves has a significant place in geometry, physics, mechanics, computer vision and image processing. In geometrical sense, a plane curve with constant curvature, up to special affine transformations may be either an ellipse, a parabola or a hyperbola [14]. This classification will be obtained by the concept of invariants. Geometry of curves in spaces with dimension ≥ 3 has studied with geometers such as Guggenheimer [5], Spivak [14] and etc. The aim was finding the invariants of curves under transformations. On the other hand, in [14], study of space curves in the view of Cartan's theorem was started but has not completed yet.

This paper can be viewed as a continuation of the work [14], where the authors began the classification of space curves up to special affine transformations. We determine all of differential invariants and our method is different from the method of Guggenheimer

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[1]Corresponding Author. E-mail Address: m_nadjafikhah@iust.ac.ir

and other existing methods. Also, for the first time, we prove a necessary and sufficient condition for the invariants in order that complete the classification. Moreover, we classify the shapes of space curves of constant curvatures which has a wide variety of applications in physics, computer vision and image processing. The general form of these shapes are exist in [5], but here we try to discuss them in more details.

In physics, classification of curves up to affine transformations has a special position in the study of rigid motions. Suppose we have a particle moving in 3D space and that we want to describe the trajectory of this particle. Especially, each curve in a three dimensional space could be imagined as a trajectory of a particle with a specified mass in the view of an observer. By classification of curves we can, in fact, obtain conservation laws.

Computer vision deals with image understanding at various levels. At the low level, it addresses issues such us planar shape recognition and analysis. Some results on differential invariants associated to space curves are relevant to space object recognition under different views and partial occlusion. The evolution of space shapes under curvature controlled diffusion have applications in geometric shape decomposition, smoothing, and analysis, as well as in other image processing applications (see, e.g. [8, 9]) and similar to recent results for planer shapes. For instance, there are some important applications of moving frames method in use of the differential invariant signatures [12]. In [1, 2] there exist some applications to the problem of object recognition and symmetry. Also, joint differential invariants has been proposed as noise-resistant alternatives to differential invariant signatures in computer vision [3]. Practical applications of the derived shapes in the latest section are related to invariant signatures, object recognitions, and symmetry of 3D shapes via the generalization of them from 2D shapes to 3D ones.

In the next section, we state some preliminaries about Maurer-Cartan forms and a way of classification of maps with the notable role of Maurer-Cartan forms and Cartan's theorem. In section three, classification of space curves in $\mathbf{R}^3$ under the action of

affine transformations is discussed. Finally, in the last section, we study the shapes of space curves with constant curvatures and propose some applications of these shapes in physics, computer vision and image processing.

2   Maurer-Cartan form

Let $G \subset \mathrm{GL}(n, \mathbf{R})$ be a matrix Lie group with Lie algebra $\mathcal{L}$ and $P : G \to \mathrm{Mat}(n \times n)$ be a matrix-valued function which embeds G into $\mathrm{Mat}(n \times n)$, the vector space of $n \times n$ matrices with real entries. Its differential is $dP_B : T_B G \to T_{P(B)} \mathrm{Mat}(n \times n) \simeq \mathrm{Mat}(n \times n)$.

Definition 2.1   The 1-form $\omega_B = \{P(B)\}^{-1} \cdot dP_B$ of G is called the *Maurer-Cartan form*. It is often written $\omega = P^{-1} \cdot dP$. The Maurer-Cartan form is in fact the unique left invariant $\mathcal{L}-$valued 1-form on G such that $\omega_{\mathrm{Id}} : T_{\mathrm{Id}} G \to \mathcal{L}$ is the identity map. The Maurer-Cartan form ω satisfies in *Maurer-Cartan equation $d\omega = -\omega \wedge \omega$*. The Maurer-Cartan form is the key to classifying maps into homogeneous spaces of G. This process needs to the following theorem (for a proof we refer the reader to [6]):

Theorem 2.2 (Cartan)   *Let G be a matrix Lie group with Lie algebra $\mathcal{L}$ and Maurer-Cartan form ω. Let M be a manifold on which there exists a $\mathcal{L}-$valued 1-form ϕ satisfying $d\phi = -\phi \wedge \phi$. Then for any point $x \in M$ there exist a neighborhood U of x and a map $f : U \to G$ such that $f^* \omega = \phi$. Moreover, Any two such maps f_1, f_2 must satisfy $f_1 = L_B \circ f_2$ for some fixed $B \in G$ (L_B is the left action of B on G).*

Corollary 2.3   *Given maps $f_1, f_2 : M \to G$, then $f_1^* \omega = f_2^* \omega$, that is, this pull-back is invariant, if and only if $f_1 = L_B \circ f_2$ for some fixed $B \in G$.*

In Theorem 2.2, if M is connected and simply-connected, then the desired map f may be extended to all of M [15]. We suppose that G be the special linear group

$SL(3, \mathbf{R})$ as a Lie group and we denote its Lie algebra with $sl(3, \mathbf{R})$. This Lie group is not simply-connected, so our achievements are local.

Definition 2.4   An *affine transformation* of the Euclidean space $\mathbf{R}^3$ is the composition of a translation in $\mathbf{R}^3$ among with an element of the general linear group $GL(3, \mathbf{R})$. An affine transformation is called *special* or *unimodular*, if its matrix part is an element of $SL(3, \mathbf{R})$. The group of special affine transformations is the connected coefficient, closed subgroup of the Lie group of affine transformations.

The following section is devoted to the study of the properties of space curves' invariants under the action of volume–preserving affine transformations, i.e., the special affine group. The number of essential parameters (dimension of the Lie algebra) is 11. The natural assumption of differentiability is $\mathcal{C}^5$.

In the next section, by defining the new curve α_c instead of a considered regular smooth curve c, we will see that the classification of curves in $\mathbf{R}^3$ and in the viewpoint of Theorem 3.1 is equivalent to the ones in $SL(3, \mathbf{R})$. Thus we find the Maurer-Cartan form of $SL(3, \mathbf{R})$ and then its pull-back via the matrix–valued curve α_c. In fact, α_c s play the role of f_i s in Corollary 2.3. This tends to a complete set of invariants of α_c as 1-forms on $\mathbf{R}$. The derived invariants in a corresponding manner determines curves of $SL(3, \mathbf{R})$. Finally in Theorem 3.3, we find that these invariants also provide a necessary and sufficient condition for specifying curves in $\mathbf{R}^3$ when we supposed the action of special affine group $SL(3, \mathbf{R})$ on $\mathbf{R}^3$.

3   Classification of space curves

In the present section, we achieve the invariants of a space curve up to special affine transformations. From Theorem 2.2, two curves in $\mathbf{R}^3$ are equivalent under special affine transformations, if they differ with a left action introduced by an element of $SL(3, \mathbf{R})$ and then a translation.

Let $c : [a, b] \to \mathbf{R}^3$ be a curve in three dimensional space which we call the *space curve*, be of class $\mathcal{C}^5$ and

$$\det(c', c'', c''') \neq 0, \tag{1}$$

for any point of the domain, that is, we assume that c', c'' and c''' are linear independent. Otherwise, if for example, c''' depends to c' and c'' for some interval $[a, b]$, then we can simply observe that the curve c will sit in $\mathbf{R}^2$, which is not our main topic of investigation. Moreover, we can assume that $\det(c', c'', c''') > 0$ for being avoid writing the absolute value in calculations.

For the curve c, we consider a new curve, namely $\alpha_c(t) : [a, b] \to \mathrm{SL}(3, \mathbf{R})$, defined by

$$\alpha_c(t) := \frac{(c', c'', c''')}{\{\det(c', c'', c''')\}^{1/3}}$$

which is well defined on the domain of c into the special linear group $\mathrm{SL}(3, \mathbf{R})$. We can study the new curve in respect to special affine transformations, i.e. the action of special affine transformations on first, second and third differentiations of c. If we assume that A is a three dimensional special affine transformation, then we have the unique representation $A = \tau \circ B$ which B is an element of $\mathrm{SL}(3, \mathbf{R})$ and τ is a translation in $\mathbf{R}^3$. If two curves c and $\bar{c}$ be the same up to an A, $\bar{c} = A \circ c$, then we have

$$\bar{c}' = B \circ c', \qquad \bar{c}'' = B \circ c'', \qquad \bar{c}''' = B \circ c'''.$$

Also from $\det B = 1$ we obtain

$$\begin{aligned} \det(\bar{c}', \bar{c}'', \bar{c}''') &= \det(B \circ c', B \circ c'', B \circ c''') = \det(B \circ (c', c'', c''')) \\ &= \det(c', c'', c'''). \end{aligned}$$

and so we conclude that $\alpha_{\bar{c}}(t) = B \circ \alpha_c(t)$ and $\alpha_{\bar{c}} = L_B \circ \alpha_c$, where L_B is the left translation for $B \in \mathrm{SL}(3, \mathbf{R})$.

This condition is also necessary because when c and $\bar{c}$ are two space curves in which $\alpha_{\bar{c}} = L_B \circ \alpha_c$ for an element $B \in \mathrm{SL}(3, \mathbf{R})$, then we can write

$$
\begin{aligned}
\alpha_{\bar{c}} &= \{\det(\bar{c}', \bar{c}'', \bar{c}''')\}^{-1/3} (\bar{c}', \bar{c}'', \bar{c}''') \\
&= \{\det(B \circ (c', c'', c'''))\}^{-1/3} B \circ (c', c'', c''') \\
&= \{\det(c', c'', c''')\}^{-1/3} B \circ (c', c'', c''').
\end{aligned}
$$

Thus $\bar{c}' = B \circ c'$ and there is a translation τ such that $A = \tau \circ B$ and $\bar{c} = A \circ c$ where A is a three dimensional affine transformation. Therefore we have

Theorem 3.1. *Let c and $\bar{c}$ are two space curves. c and $\bar{c}$ are the same with respect to special affine transformations, i.e. $\bar{c} = A \circ c$ when $A = \tau \circ B$ for translation τ in $\mathbf{R}^3$ and $B \in \mathrm{SL}(3, \mathbf{R})$ if and only if $\alpha_{\bar{c}} = L_B \circ \alpha_c$ where L_B is a left translation generated by B.*

From Cartan's theorem, a necessary and sufficient condition for $\alpha_{\bar{c}} = L_B \circ \alpha_c$ ($B \in \mathrm{SL}(3, \mathbf{R})$) is that for any left invariant 1-form ω^i on $\mathrm{SL}(3, \mathbf{R})$ we have $\alpha_{\bar{c}}^*(\omega^i) = \alpha_c^*(\omega^i)$. It is equivalent to $\alpha_{\bar{c}}^*(\omega) = \alpha_c^*(\omega)$ for natural $\mathrm{sl}(3, \mathbf{R})$-valued 1-form $\omega = P^{-1}.dP$ for matrix-valued function P which embeds $\mathrm{SL}(3, \mathbf{R})$ into $\mathrm{Mat}3 \times 3$, the vector space of 3×3 matrices with real entries, and ω is the Maurer-Cartan form.

We must compute $\alpha_c^*(P^{-1}.dP)$ which is invariant under special affine transformations. Its entries are in fact invariant functions of space curves. It is a 3×3 matrix form which arrays are multiplications of dt (1-forms on $[a, b]$).

Since $\alpha_c^*(P^{-1} \cdot dP) = \alpha_c^{-1} \cdot d\alpha_c$, so we calculate the matrix $\alpha_c^{-1} \cdot \alpha_c'$ and then multiply it by dt to have $\alpha_c^*(P^{-1} \cdot dP)$. we have

$$
\begin{aligned}
\alpha_c^{-1} &= \det(c', c'', c''')^{1/3} \cdot (c', c'', c''')^{-1} \\
&= \det(c', c'', c''')^{-2/3} \cdot
\begin{pmatrix}
c_2'' c_3''' - c_3'' c_2''' & c_3'' c_1''' - c_1'' c_3''' & c_1'' c_2''' - c_2'' c_1''' \\
c_3' c_2''' - c_2' c_3''' & c_3' c_5''' - c_1' c_3''' & c_2' c_1''' - c_1' c_2''' \\
c_2' c_3'' - c_3' c_2'' & c_3' c_1'' c_1' c_3'' & c_1' c_2'' - c_2' c_1''
\end{pmatrix}
\end{aligned}
$$

which $c = (c_1, c_2, g_3)^T$ as a column matrix be the vector representation of curve c. We also have

$$[\det(c', c'', c''')]' = \det(c'', c'', c''') + \det(c', c''', c''') + \det(c', c'', c'''')$$
$$= \det(c', c'', c'''').$$

Thus we see that

$$\alpha'_c = \det(c', c'', c''')^{-1/3} \cdot \begin{pmatrix} c''_1 & c'''_1 & c''''_1 \\ c''_2 & c'''_2 & c''''_2 \\ c''_3 & c'''_3 & c''''_3 \end{pmatrix} - \frac{1}{3} \det(c', c'', c''')^{-4/3} \cdot \begin{pmatrix} c'_1 & c''_1 & c'''_1 \\ c'_2 & c''_2 & c'''_2 \\ c'_3 & c''_3 & c'''_3 \end{pmatrix}.$$

After some computations, finally we find that $\alpha_c^{-1} \cdot \alpha'_c$ is in the following multiple of dt

$$\begin{pmatrix} -\frac{\det(c', c'', c'''')}{3 \det(c', c'', c''')} & 0 & \frac{\det(c'', c''', c'''')}{\det(c', c'', c''')} \\ 1 & -\frac{\det(c', c'', c'''')}{3 \det(c', c'', c''')} & -\frac{\det 8c', c''', c''''}{\det(c', c'', c''')} \\ 0 & 1 & \frac{2 \det(c', c'', c'''')}{3 \det(c', c'', c''')} \end{pmatrix}.$$

Clearly, the trace of the last matrix is zero and entries of $\alpha_c^*(P^{-1} \cdot dP)$ and therefore entries of the above matrix, are invariants of the group action.

Therefore according to Theorem 3.1, two space curves $c, \bar{c} : [a, b] \to \mathbf{R}^3$ are the same under special affine transformations if we have

$$\frac{\det(c', c'', c'''')}{\det(c', c'', c''')} = \frac{\det(\bar{c}', \bar{c}'', \bar{c}'''')}{\det(\bar{c}', \bar{c}'', \bar{c}''')}$$
$$\frac{\det(c'', c''', c'''')}{\det(c', c'', c''')} = \frac{\det(\bar{c}'', \bar{c}''', \bar{c}'''')}{\det(\bar{c}', \bar{c}'', \bar{c}''')}$$
$$\frac{\det(c', c''', c'''')}{\det(c', c'', c''')} = \frac{\det(\bar{c}', \bar{c}''', \bar{c}'''')}{\det(\bar{c}', \bar{c}'', \bar{c}''')}.$$

We may use of a proper parametrization $\sigma : [a, b] \to [0, l]$ such that the parameterized curve $\gamma = c \circ \sigma^{-1}$ satisfies in condition $\det(\gamma'(s), \gamma''(s), \gamma''''(w)) = 0$ and then entries over the principal diagonal of $\alpha_\gamma^*(P^{-1} \cdot dP)$ be zero. But this determinant is in fact

the differentiation of $\det(\gamma'(s), \gamma''(s), \gamma'''(s))$ and for being zero it is sufficient that we assume $\det(\gamma'(s), \gamma''(s), \gamma'''(s)) = 1$. On the other hand, we have

$$
\begin{aligned}
c' &= (\gamma \circ \sigma)' = \sigma' \cdot (\gamma' \circ \sigma) \\
c'' &= (\sigma')^2 \cdot (\gamma'' \circ \sigma) + \sigma'' \cdot (\gamma' \circ \sigma) \\
c''' &= (\sigma')^3 \cdot (\gamma''' \circ \sigma) + 3\,\sigma'\sigma'' \cdot (\gamma'' \circ \sigma) + \sigma''' \cdot (\gamma' \circ \sigma).
\end{aligned}
$$

Thus we conclude that

$$
\begin{aligned}
\det(c', c'', c''') &= \det(\sigma' \cdot (\gamma' \circ \sigma),\ (\sigma')^2 \cdot (\gamma'' \circ \sigma) + \sigma'' \cdot (\gamma' \circ \sigma),\\
&\qquad (\sigma')^3 \cdot (\gamma''' \circ \sigma) + 3 * \sigma'\sigma'' \cdot (\gamma'' \circ \sigma) + \sigma''' \cdot (\gamma' \circ \sigma)) \\
&\qquad \det(\sigma' \cdot (\gamma' \circ \sigma),\ (\sigma')^2 \cdot (\gamma'' \circ \sigma),\ (\sigma')^3 \cdot (\gamma''' \circ \sigma)) \\
&= (\sigma')^6 \cdot \det(\gamma' \circ \sigma,\ \gamma'' \circ \sigma\ \gamma''' \circ \sigma) \\
&= (\sigma')^6,
\end{aligned}
$$

The last expression specifies σ, namely the *special affine arc length*, is defined as follows

$$
\sigma := \int_a^t \Big[\det(c'(u), c''(u), c'''(u)) \Big]^{1/6} du.
$$

So σ is a natural parameter under the action of special affine transformations, that is, when c is parameterized by σ then for each special affine transformation A, $A \circ c$ is also parameterized by the same σ. Furthermore, every curve parameterized by σ up to special affine transformations is introduced with the following invariants

$$
\chi_1 = \det(c'', c''', c''''), \qquad \chi_2 = \det(c', c''', c''''). \tag{2}
$$

We call χ_1 and χ_2 the first and the second *special affine curvatures* resp. Thus finally we have

$$
\alpha_\gamma^*(P^{-1}.dP) = \begin{pmatrix} 0 & 0 & \chi_1 \\ 1 & 0 & -\chi_2 \\ 0 & 1 & 0 \end{pmatrix} d\sigma.
$$

Theorem 3.2 *Every space curve of class $\mathcal{C}^5$ satisfying in condition (1) under the action of special (unimodular) affine transformations is determined by its natural equations $\chi_1 = \chi_1(\sigma)$ and $\chi_2 = \chi_2(\sigma)$ of the first and second special affine curvatures (2) as functions (invariants) of the special affine arc length.*

Theorem 3.3 *Two space curves $c, \bar{c} : [a, b] \to \mathbf{R}^3$ of class $\mathcal{C}^5$ which satisfy in condition (1) are special affine equivalent if and only if $\chi_1^c = \chi_1^{\bar{c}}$ and $\chi_2^c = \chi_2^{\bar{c}}$.*

Proof: The first side of the theorem is trivial with respect to above descriptions. For the other side, we assume that $c, \bar{c}$ are curves of class $\mathcal{C}^5$ satisfying (resp.) in

$$\det(c', c'', c''') > 0, \qquad \det(\bar{c}', \bar{c}'', \bar{c}''') > 0, \tag{3}$$

that is, they are not plane curves. Also we suppose that functions χ_1 and χ_2 are the same for these curves.

By changing the parameter to the natural parameter σ (mentioned above), we obtain new curves γ and $\bar{\gamma}$ resp., that determinant (3) will be equal to 1. We prove that γ and $\bar{\gamma}$ are special affine equivalent and so there is a special affine transformation A sech that $\bar{\gamma} = A \circ \gamma$. Then we have $\bar{c} = A \circ c$ and the proof is complete.

At first, we replace the curve γ with $\delta := \tau(\gamma)$ properly, in which case that δ intersects $\bar{\gamma}$ where τ is a translation defined by translating one point of γ to one point of $\bar{\gamma}$. We correspond $t_0 \in [a, b]$ to the intersection of δ and $\bar{\gamma}$, thus $\delta(t_0) = \bar{\gamma}(t_0)$. One can find a unique element B of the general linear group $\mathrm{GL}(3, \mathbf{R})$ such that maps the frame $\{\delta'(t_0), \delta''(t_0), \delta'''(t_0)\}$ to the frame $\{\bar{\gamma}'(t_0), \bar{\gamma}''(t_0), \bar{\gamma}'''(t_0)\}$. So we have $B \circ \delta'(t_0) = \bar{\gamma}'(t_0)$, $B \circ \delta''(t_0) = \bar{\gamma}''(t_0)$, and $B \circ \delta'''(t_0) = \bar{\gamma}'''(t_0)$. B is also an element of the special linear group $\mathrm{SL}(3, \mathbf{R})$; since we have

$$\det(\gamma'(t_0), \gamma''(t_0), \gamma'''(t_0)) = \det(\delta'(t_0), \delta''(t_0), \delta'''(t_0)) \quad \text{and}$$
$$\det(\delta'(t_0), \delta''(t_0), \delta'''(t_0)) = \det(B \circ (\bar{\gamma}'(t_0), \bar{\gamma}''(t_0), \bar{\gamma}'''(t_0))),$$

and thus $\det(B) = 1$. If we prove that $\eta := B \circ \delta$ is equal to $\bar{\gamma}$ on $[a, b]$, then by choosing $A = \tau \circ B$, there will remain nothing for proof.

For curves η and $\bar{\gamma}$ we have (resp.)

$$(\eta', \eta'', \eta''')' = (\eta', \eta'', \eta''') \begin{pmatrix} 0 & 0 & \chi_1^\eta \\ 1 & 0 & -\chi_2^\eta \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and}$$

$$(\bar{\gamma}', \bar{\gamma}'', \bar{\gamma}''')' = (\bar{\gamma}', \bar{\gamma}'', \bar{\gamma}''') \begin{pmatrix} 0 & 0 & \chi_1^{\bar{\gamma}} \\ 1 & 0 & -\chi_2^{\bar{\gamma}} \\ 0 & 1 & 0 \end{pmatrix}.$$

Since χ_1 and χ_2 remain unchanged under special affine transformations, so we have $\chi_1^\eta = \chi_1^\gamma = \chi_1^{\bar{\gamma}}$ and $\chi_2^\eta = \chi_2^\gamma = \chi_2^{\bar{\gamma}}$, therefore, we conclude that η and $\bar{\gamma}$ are solutions of ordinary differential equation $Y'''' + \chi_2 Y'' - \chi_1 Y' = 0$ where Y depends to t. Because of the same initial conditions

$$\eta(t_0) = B \circ \delta(t_0) = \bar{\gamma}(t_0), \qquad \eta'(t_0) = B \circ \delta'(t_0) = \bar{\gamma}'(t_0),$$

$$\eta''(t_0) = B \circ \delta''(t_0) = \bar{\gamma}''(t_0), \qquad \eta'''(t_) = B \circ \delta'''(t_0) = \bar{\gamma}'''(t_0),$$

and the generalization of the existence and uniqueness theorem of solutions, we have $\eta = \bar{\gamma}$ in a neighborhood of t_0 that can be extended to the whole $[a, b]$. $\diamondsuit$

Corollary 3.4   *The number of invariants of special affine transformation group acting on $\mathbf{R}^3$ is two which is the same with another results provided by other methods such as [5].*

The generalization of the affine classification of curves in an arbitrary finite dimensional space has been discussed in [10] and a complete set of invariants with a necessary and sufficient condition of them for classifying curves up to affine transformations has been derived.

4 Geometric interpretations applied to physics and computer vision

In the present section, the geometric interpretation of the first and the second special affine curvatures and their applications in physics and computer vision is discussed. Since every curve parameterized with special affine arc length σ and with constant first and second affine curvatures χ_1 and χ_2 fulfilled in relation $\alpha_c'(\sigma) = \alpha_c(\sigma).(b)$, for some $b \in \text{sl}(3, \mathbf{R})$ via the right action of the Lie algebra. In fact, we assumed that the action of the Lie group be the left action [11]. Whereof, Maurer-Cartan matrix of $SL(3, \mathbf{R})$ is

a base for Lie algebra $\text{sl}(3, \mathbf{R})$ and one can write $\alpha_c'(\sigma) = \alpha_c(\sigma). \begin{pmatrix} 0 & 0 & \chi_1 \\ 1 & 0 & -\chi_2 \\ 0 & 1 & 0 \end{pmatrix}$. By

solving this first order equation, we obtain $\alpha_c(\sigma) = \exp\left(\sigma. \begin{pmatrix} 0 & 0 & \chi_1 \\ 1 & 0 & -\chi_2 \\ 0 & 1 & 0 \end{pmatrix}\right)$ that, for

different values of χ_1 and χ_2 it has a different forms which we divide these forms in the following cases:

I. The case $\chi_1 = \chi_2 = 0$.

In this case, the curve is in the form $\alpha_c(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ \sigma & 1 & 0 \\ \frac{1}{2}\sigma^2 & \sigma & 1 \end{pmatrix}$. It is clear that the

first column of this matrix is $c'(\sigma)$ and so we have $c(\sigma) = K + (\sigma, \frac{1}{2}\sigma^2, \frac{1}{6}\sigma^3)$ for some constant $K \in \mathbf{R}^3$, that its image is analogous to the image of *twisted cubic* [4]. Also the image is similar to the *Neil* or *semi-cubical* parabola's graph [7]. The projection of this space curve in the direction of $z-$axis is a parabola in plane. This space curve is the simplest curve in $\mathbf{R}^3$ under special affine transformations. Its figure is a translation of Figure 1-(a) by constant K.

Theorem 4.1 *Space curves with zero special affine curvatures are in the form of twisted cubic probably with some translations.*

Figure 1: (a) $\chi_1 = \chi_2 = 0$. (b) $\chi_1 = 0, \chi_2 > 0$.

II. The case $\chi_1 = 0$ and $\chi_2 > 0$.

In this case, we have

$$
\alpha_c(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{\chi_2}} \sin(\sqrt{\chi_2}\,\sigma) & \cos(\sqrt{\chi_2}\,\sigma) & -\sqrt{\chi_2}\sin(\sqrt{\chi_2}\,\sigma) \\ -\frac{1}{\chi_2}\left(\cos(\sqrt{\chi_2}\,\sigma) - 1\right) & \frac{1}{\sqrt{\chi_2}}\sin(\sqrt{\chi_2}\,\sigma) & \cos(\sqrt{\chi_2}\,\sigma) \end{pmatrix}.
$$

So we obtain $c(\sigma) = K + \left(\sigma,\, -\frac{1}{\chi_2}\cos(\sqrt{\chi_2}\,\sigma),\, -\frac{1}{\chi_2\sqrt{\chi_2}}\sin(\sqrt{\chi_2}\,\sigma) + \frac{\sigma}{\chi_2}\right)$ for $K \in \mathbf{R}^3$.
The image of this curve is a translation of Figure 1-(b) by constant K. Its projection in the direction of $z-$axis is similar to the graph of function $\cos(\sigma)$.

III. The case $\chi_1 = 0$ and $\chi_2 < 0$.

If we use $|\chi_2| = -\chi_2$ to be the absolute value of χ_2, in the same way as the previous cases, we find that

$$
\alpha_c(\sigma) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{|\chi_2|}}\sinh(\sqrt{|\chi_2|}\,\sigma) & \cosh(\sqrt{|\chi_2|}\,\sigma) & \sqrt{|\chi_2|}\sinh(\sqrt{|\chi_2|}\,\sigma) \\ \frac{1}{|\chi_2|}\left\{\cosh(\sqrt{|\chi_2|}\,\sigma) - 1\right\} & \frac{1}{\sqrt{|\chi_2|}}\sinh(\sqrt{|\chi_2|}\,\sigma) & \cosh(\sqrt{|\chi_2|}\,\sigma) \end{pmatrix}.
$$

Thus $c(\sigma) = K + \left(\sigma, \frac{1}{|\chi_2|}\cosh(\sqrt{|\chi_2|}\,\sigma), \frac{1}{|\chi_2|\sqrt{|\chi_2|}}\sinh(\sqrt{|\chi_2|}\,\sigma) - \frac{\sigma}{|\chi_2|}\right)$ where K is an element of $\mathbf{R}^3$. Its image is drown in Figure 2-(a) probably after a translation. Its z−axis projection is similar to the graph of the function $\cosh(\sigma)$.

Figure 2: (a) $\chi_1 = 0, \chi_2 < 0$. (b) $\chi_1 > 0, \chi_2 = 0$.

IV. The case $\chi_1 > 0$ and $\chi_2 = 0$.

Under these conditions, the $\alpha_c(\sigma)$ is

$$
\begin{pmatrix}
\frac{2}{3}M + \frac{1}{3}R & -\frac{1}{3}\chi_1^{1/3}(\sqrt{3}N + M - R) & \frac{1}{3}\chi_1^{2/3}(\sqrt{3}N - M + R) \\
\frac{1}{3\chi_1^{1/3}}(\sqrt{3}N - M + R) & \frac{2}{3}M + \frac{1}{3}R & -\frac{1}{3}\chi_1^{1/3}(\sqrt{3}N + M - R) \\
-\frac{1}{3\chi_1^{2/3}}(\sqrt{3}N + M - R) & \frac{1}{3\chi_1^{1/3}}(\sqrt{3}N - M + R) & \frac{2}{3}M + \frac{1}{3}R
\end{pmatrix},
$$

where

$$
M = \exp(-\tfrac{1}{2}\chi_1^{1/3}\sigma)\,\cos(\tfrac{\sqrt{3}}{2}\chi_1^{1/3}\sigma), \qquad R = \exp(\chi_1^{1/3}\sigma)
$$
$$
N = \exp(-\tfrac{1}{2}\chi_1^{1/3}\sigma)\,\sin(\tfrac{\sqrt{3}}{2}\chi_1^{1/3}\sigma).
$$

Therefore with above conditions, we can write

$$
c(\sigma) = K + \left(\frac{1}{3\chi_1^{1/3}}(\sqrt{3}N - M + R), \frac{1}{3\chi_1^{2/3}}(-\sqrt{3}N - M + R), \frac{1}{3\chi_1}(2M + R)\right)
$$

for some $K \in \mathbf{R}^3$. Its figure is similar to Figure 2-(b).

V. The case $\chi_1 < 0$ and $\chi_2 = 0$.

Such as case III, with use of $|\chi_2| = -\chi_2$, the conditions of this case lead to the form of $\alpha_c(\sigma)$:

$$
\begin{pmatrix}
\frac{2}{3}M+\frac{1}{3}R & \frac{1}{3}|\chi_1|^{1/3}(-\sqrt{3}N+M-R) & \frac{1}{3}|\chi_1|^{2/3}(\sqrt{3}N+M-R) \\
\frac{1}{3|\chi_1|^{1/3}}(\sqrt{3}N+M-R) & \frac{2}{3}M+\frac{1}{3}R & \frac{-1}{3}|\chi_1|^{1/3}(\sqrt{3}N-M+R) \\
\frac{-1}{3|\chi_1|^{2/3}}(\sqrt{3}N-M+R) & \frac{1}{3|\chi_1|^{1/3}}(\sqrt{3}N+M-R) & \frac{2}{3}M+\frac{1}{3}R
\end{pmatrix},
$$

where

$$
M = \exp(\tfrac{1}{2}|\chi_1|^{1/3}\sigma)\,\cos(\tfrac{\sqrt{3}}{2}|\chi_1|^{1/3/}\sigma), \qquad R = \exp(-|\chi_1|^{1/3}\sigma)
$$
$$
N = -\exp(\tfrac{1}{2}|\chi_1|^{1/3}\sigma)\,\sin(\tfrac{\sqrt{3}}{2}|\chi_1|^{1/3/}\sigma).
$$

And so we have the following curve

$$
c(\sigma) = K + \Big(\frac{1}{3|\chi_1|^{1/3}}(\sqrt{3}N+M-R), \frac{-1}{3|\chi_1|^{2/3}}(\sqrt{3}N-M+R), \frac{-1}{3|\chi_1|})2M+R\Big)
$$

for some $K \in \mathbf{R}^3$. Its shape is similar to Figure 3-(a).

Figure 3: $\chi_1 < 0, \chi_2 = 0$.

VI. The case $\chi_1, \chi_2 \neq 0$.

In this case, relations are not as simple as previous cases. $\alpha_c(\sigma)$ in this general case, is in the following form $\alpha_c(\sigma) = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}$ where for $1 \leq i, j \leq 3$ the entries B_{ij} for brevity are given in Appendix at the end of the paper.

Thus the relation $c'(\sigma) = (B11, B21, B31)$ signifies the curve $c(\sigma)$ by integrating of the coefficients with respect to σ. Therefore we obtain $c(\sigma) = K + (T1, T2, T3)$ when K is an element of $\mathbf{R}^3$ and T_i s are in the forms of variables which are indicated in Appendix.

For different values of constants χ_1, $\chi_2 \neq 0$, there exist various curves and $c(\sigma)$ is a translation, contraction or extraction of a curve in the form of these cases. Thus we have different figures that each of which is similar to one of the shapes given in Figure 4, (a)–(d).

Figure 4: (a) $\chi_1, \chi_2 > 0$.   (b) $\chi_1, \chi_2 < 0$.   (c) $\chi_1 < 0 < \chi_2$.   (d) $\chi_1 > 0 > \chi_2$.

Corollary 4.2   *In general, every solution of $\alpha_c : \mathbf{R} \to \mathrm{SL}(3, \mathbf{R})$ is provided by multiplying a special linear transformation and a translation from an acquired curve in above cases. In fact, the geometrical sense of above curves can be explained as follows:*
Each curve has two branches. The values of the first and second special affine curvatures determine "rotation quantities" of the branches that by ascending (descending resp.) the values, each branch's bend will increase (decrease resp.). Accordingly, the definitions of χ_1 and χ_2 have geometric interpretations as the usual terminology of cur-

vatures.

In the case of constant χ_1 and χ_2, by using Theorem 3.3, we can classify space curves in these cases via special affine transformations and as a result we have the following theorem:

Theorem 4.3 *Each curve of class $\mathcal{C}^5$ in $\mathbf{R}^3$ satisfied in condition (1) with constant affine curvatures χ_1 and χ_2, up to special affine transformations, is the trajectory of a one–parameter subgroup of special (unimodular) affine transformations, that is, a curve of cases I-VI.*

Finally we give two applications of the classification of space curves by the action of special affine transformations and Theorem 4.3:

Corollary 4.4 *In the physical sense, we may assume that each space curve $\mathbf{X}$: $[a,b] \to \mathbf{R}^3$ is the trajectory of a particle with a specified mass m" in $\mathbf{R}^3$ and in the view of an observer, that is influenced under the effect of a force $\mathbf{F}$. By the action of special affine transformations, particle's path has two conservation laws: $(\mathbf{X}'' \times \mathbf{X}''') \cdot \mathbf{X}''''$ and $(\mathbf{X}' \times \mathbf{X}''') \cdot \mathbf{X}''''$, that are, the first and the second special affine curvatures. Therefore, by multiplying constant m^3 to theses invariants, we find conservation laws as $(\mathbf{F} \times \mathbf{F}') \cdot \mathbf{F}''$ and $(\mathbf{P} \times \mathbf{F}') \cdot \mathbf{F}''$ where $\mathbf{P} = m \cdot \mathbf{v}$ is the momentum of the particle. If these invariants of the trajectory are constant, then the shape of the motion is similar to one of the six cases mentioned in the above theorem.*

The derived invariants in above, may have important applications in astronomy, fluid mechanics, quantum, general relativity and etc. A reason for this importance is that in these areas we deal with the motion of a space particle and it may be of our interest to investigate for symmetry properties and invariants of the particle under rigid

motions.

Corollary 4.5 *In computer vision and image processing, we may suppose that each space curve is one of the characteristic curves on a 3-dimensional object, that are feasible minimum segment curves that completely signify the object in the viewpoint of an observer. Also, if by an effect provided by (orientation-preserving) rotations and translations in $\mathbf{R}^3$ we change the position of a picture without any change in characteristic lines, then these cur es will be equivalent under special affine transformation. If a characteristic line has constant affine curvatures χ_1 and χ_2, then it will be similar to one of the cases of curves mentioned in Theorem 4.3.*

For instance, these image invariants provide the most prominent application fields in 3D medical imagery, including MRI, ultrasound and CT data, in object recognition, symmetry and differential invariant signatures of 3D shapes [8, 9, 12].

Appendix

In case **VI** of section 4, entries B_{ij} $(1 \leq i, j \leq 3)$ are

$$
\begin{aligned}
B_{11} &= \frac{1}{\Delta}\Big\{ A\Big(72\,c_1^{1/3}\,\chi_1\,\chi_2 - 8\,c_1^{1/3}\,c_2\,\chi_2 - 144\,c_2\chi_1 - 192\,\chi_2^3 - 1296\,\chi_1^2 \\
&\quad + 8\,c_1^{2/3}\,\chi_2^2\Big) + B\Big(13.85640646\,c_1^{2/3}\,\chi_2^2 + 12.470765820\,c_1^{1/3}\,\chi_1\,\chi_2 \\
&\quad + 13.85640646\,c_1^{1-3}\,c_2\,\chi_2\Big) + D\Big(-8\,c_1^{2/3}\,\chi_2^2 - 72\,c_2\,\chi_1 - 648\,\chi_1^2 \\
&\quad - 96\,\chi_2^3 - 72\,b_1^{1/3}\,\chi_1\,\chi_2 - 8\,c_1^{1/3}\,c_2\,\chi_2\Big)\Big\}, \\
B_{21} &= -B_{32} = -\frac{1}{\chi_1}B_{13} = \frac{1}{\Delta}\Big\{ A\Big(-18\,c_1^{2/3}\,\chi_1 - 2\,c_1^{2/3}\,c_2 + 24\,c_1^{1/3}\,\chi_2^2\Big) \\
&\quad + B\Big(41.56921940\,c_1^{1/3}\chi_2^2 + 31.176874540c_1^{2/3}\chi_1 + 3.464101616c_1^{2/3}c_2\Big) \\
&\quad + D\Big(-24\,c_1^{1/3}\,\chi_2^2 + 18\,c^{2/3}\,\chi_1 + 2\,c_1^{2/3}\,c_2\Big)\Big\}, \\
B_{31} &= c_1^{2/3}B_{12} = \frac{1}{\Delta}\Big\{ A\Big(-c_1^{4/3} - 12\,c_1^{2/3}\,\chi_2\Big) + B\Big(-1.732050808\,c_1^{4/3} \\
&\quad + 20.7846097\,c_1^{2/3}\,\chi_2\Big) + D\Big(c_1^{4/3} + 12\,c_1^{2/3}\,\chi_2\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
B_{22} \;=\; B_{33} &= \frac{1}{\Delta}\Big\{ A\Big(1296\,\chi_1^2 + 144\,c_2\,\chi_1 + 4\,c_1^{2/3}\,\chi_2^2 + 36\,c_1^{1/3}\,\chi_1\,\chi_2 \\
&\quad + 4\,c_1^{1/3}\,c_2\,\chi_2 + 192\,\chi_2^3\Big) + B\Big(62.353831080\,c_9^{1/3}\,\chi_1\,\chi_2 \\
&\quad + 69.28203232\,c_1^{2/3}\,c_2\,\chi_2 - 69.28203232\,c_1^{2/3}\,\chi_2^2\Big) + D\Big(72\,c_2\,\chi_1 \\
&\quad + 96\,\chi_2^3 + 648\,\chi_1^2 - 36\,c_1^{1/3}\,\chi_1\,\chi_2 - M4\,c_1^{1/3}\,c_2\,\chi_2 - 4\,c_1^{2/3}\,\chi_2^2\Big)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
F_{23} &= \frac{-1}{\Delta}\Big\{ A\Big(108c_1^{1/3}\chi_1^2 + 24c_1^{1/3}\chi_2^3 - 6c_1^{2/3}\chi_1\chi_2 - 2c_1^{1/3}c_2\chi_2 + 12c_1^{1/3}c_2\chi_1\Big) \\
&\quad + B\Big(10.39230485c_1^{2/3}\chi_1\chi_2 + 3.464101616c_1^{1/3}c_2\chi_2 + 187.0614873 \\
&\quad \times c_1^{1/3}\chi_1^2 + 41.5692194c_1^{1/3}\chi_2^3 + 20.7846097c_1^{1/3}c_2\chi_1\Big) + D\Big(6c_1^{2/3}\chi_1\chi_2 \\
&\quad - 24c_1^{1/3}\chi_2^3 + 2c_1^{2/3}c_2\chi_2 - 12c_1^{2/3}c_2\chi_1 - 108c_1^{1/3}\chi_1^2\Big)\Big\},
\end{aligned}
$$

Which in the above relations we assumed that

$$
c_1 = 108\,\chi_1 + 12\big(\sqrt{12\,\chi_2^3 + 81\,\chi_1^2}\big),
$$

$$
c_2 = \sqrt{12\,\chi_2^3 + 81\,\chi_1^2},
$$

$$
A = \exp\Big(-0.08333333333\frac{c_1^{2/3} - 12\chi_1}{c_1^{1/3}}\,\sigma\Big)\,\cos\Big(0.1443375673\frac{c_1^{2/3} + 12\chi_1}{c_1^{1/3}}\,\sigma\Big),
$$

$$
B = \exp\Big(-0.08333333333\frac{c_1^{2/3} - 12\chi_1}{c_1^{1/3}}\,\sigma\Big)\,\sin\Big(0.1443375673\frac{c_1^{2/3} + 12\chi_1}{c_1^{1/3}}\,\sigma\Big),
$$

$$
D = \exp\Big(0.1666666667\frac{c_1^{2/3} - 12\chi_1}{c_1^{1/3}}\,\sigma\Big),
$$

$$
\Delta = (c_1^{2/3}\,\chi_2 - 9c_1^{1/3}\,\chi_1 - c_1^{1/3}\,c_2 - 12\,\chi_2^2)(c_1^{2/3} + 12\,\chi_2).
$$

Also T_i s are in the following forms

$$
\begin{aligned}
T_1 &= \frac{72}{\Delta'}\Big\{ A\Big(5832\chi_1^3\chi_2 + 864\chi_1\chi_2^4 - 24c_1^{1/3}\chi_2^5 - 81c_1^{2/3}\chi_1^3 + 48c_2\chi_2^4 \\
&\quad - 24c_1^{2/3}\chi_1\chi_2^3 + 648c_2\chi_1^2\chi_2 - 30c_1^{1/3}c_2\chi_2^2 - 9c_1^{2/3}c_2\chi_1^2 - 2c_1^{2/3}c_2\chi_2^3 \\
&\quad - 270c_1^{1/3}\chi_1^2\chi_2^2\Big) + B\Big(-41.56921939c_1^{1/3}\chi_2^5 + 140.2961154c_1^{2/3}\chi_1^3 \\
&\quad + 41.56921939c_1^{2/3}\chi_1\chi_2^3 - 467.6537182c_1^{1/3}\chi_1^2\chi_2^2 + 15.58845727c_1^{2/3}c_2\chi_1^2 \\
&\quad - 51.96152424c_1^{1/3}c_2\chi_2^2 + 3.46101616c_1^{2/3}c_2\chi_2^3\Big) + D\Big(432\chi_1\chi_2^4 + 24c_2\chi_2^4 \\
&\quad + 81c_1^{2/3}\chi_1^3 + 2916\chi_1^3\chi_2 + 24c_1^{1/3}\chi_2^5 + 24c_1^{2/3}\chi_1\chi_2^3 + 270c_1^{1/3}\chi_1^2\chi_2^2
\end{aligned}
$$

$$+324c_2\chi_1^2\chi_2 + 2c_1^{2/3}c_2\chi_2^3 + 9\,c_1^{2/3}\,c_2\,\chi_1^2 + 30c_1^{1/3}c_2\chi_1\chi_2^2\Big)\Big\},$$

$$
T_2 = \frac{-1}{\Delta'}\Big\{A\Big(864c_1^{2/3}\chi_1^2\chi_2 + 216c_1^{2/3}c_2\chi_1\chi_2 + 162c_1^{4/3}\chi_1^2 + 18c_1^{4/3}c_2\chi_1\Big)
$$
$$
-B\Big(3367.1067714c_1^{2/3}\chi_1^2\chi_2 - 374.1229746c_1^{2/3}c_2\chi_1\chi_2
$$
$$
+280.59223086c_1^{4/3}\chi_1^2 + 31.176914544c_1^{4/3}c_2\chi_1\Big) - D\Big(162c_1^{4/3}\chi_1^2
$$
$$
-18c_1^{4/3}c_2\chi_1 - 216c_1^{2/3}c_2\chi_1\chi_2\Big)\Big\},
$$

$$
T_3 = \frac{-144}{\Delta'}\Big\{A\Big(12c_1^{1/3}\chi_2^4 - 324c_2\chi_1^2 - 432\chi_1\chi_2^3 - 24c_2\chi_2^3 - 2916\chi_1^3 + 9c_1^{2/3}
$$
$$
\cdot\chi_1\chi_2^2 + c_1^{2/3}c_2\chi_2^2 + 162c_1^{1/3}\chi_1^2\chi_2 + 18c_1^{1/3}c_2\chi_1\chi_2\Big) + B\Big(20.78460970c_1^{1/3}
$$
$$
\cdot\chi_2^4 - 15.58845727c_1^{2/3}\chi_1\chi_2^2 - 1.732050808c_1^{2/3}c_2\chi_2^2 + 280.5922309c_1^{1/3}
$$
$$
\cdot\chi_1^2\chi_2 + 31.17691454c_1^{1/3}c_2\chi_1\chi_2\Big) - D\Big(12c_1^{1/3}\chi_2^4 - 9c_1^{2/3}\chi_1\chi_2^2 - c_1^{2/3}c_2\chi_2^2
$$
$$
-162c_1^{1/3}\chi_1^2\chi_2 - 162c_2\chi_1^2 - 216\chi_1\chi_2^3 - 12c_2\chi_2^3 - 1458\chi_1^3 - 18c_1^{1/3}c_2\chi_1\chi_2\Big)\Big\}.
$$

In the latter relations, A, B, D, c_1 and c_2 are the same with relations mentioned in above and

$$
\Delta' = [9c_1^{1/3}\chi_1 + c_1^{1/3}c_2 - c_1^{2/3}\chi_2 + 12\chi_2^2][9c_1^{1/3}\chi_1 + c_1^{1/3}c_2 + c_1^{2/3}\chi_1 + 12\chi_2^2]
$$
$$
\times[9c_1^{1/3}\chi_1 + c_1^{1/3}c_2 - 12\chi_2^2].
$$

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