

# Affine Classification of $n$ -Curves

Mehdi Nadjafikhah      Ali Mahdipour Sh.

## Abstract

Classification of curves up to affine transformation in a finite dimensional space was studied by some different methods. In this paper, we achieve the exact formulas of affine invariants via the equivalence problem and in the view of Cartan's theorem and then, state a necessary and sufficient condition for classification of  $n$ -Curves.

**A.M.S. 2000 Subject Classification:** 53A15, 53A04, 53A55.

**Key words:** Affine differential geometry, curves in Euclidean space, differential invariants.

## 1 Introduction

This paper devoted to the study of curve invariants, in an arbitrary finite dimensional space, under the group of special affine transformations. This work was done before in some different methods. Furthermore, these invariants were just pointed by Spivak [6], in the method of Cartan's theorem, but they were not determined explicitly. Now, we will exactly determine these invariants in the view of Cartan's theorem and equivalence problem.

An *affine transformation*, in a  $n$ -dimensional space, is generated by the action of the general linear group  $GL(n, \mathbb{R})$  and then, the translation group  $\mathbb{R}^n$ . If we restrict  $GL(n, \mathbb{R})$  to special linear group  $SL(n, \mathbb{R})$  of matrix with determinant equal to 1, we have a *special affine transformation*. The group of special affine transformations has  $n^2 + n - 1$  parameters. This number is also, the dimension of Lie algebra of special affine transformations Lie group. The natural condition of differentiability is  $\mathcal{C}^{n+2}$ .

In next section, we state some preliminaries about Maurer–Cartan forms, Cartan's theorem for the equivalence problem, and a theorem about number of invariants in a space. In section three, we obtain the invariants and then with them, we classify the  $n$ -curves of the space.

## 2 Preliminaries

Let  $G \subset GL(n, \mathbb{R})$  be a matrix Lie group with Lie algebra  $\mathfrak{g}$  and  $P : G \rightarrow Mat(n \times n)$  be a matrix-valued function which embeds  $G$  into  $Mat(n \times n)$  the vector space of  $n \times n$  matrices with real entries. Its differential is  $dP_B : T_B G \rightarrow T_{P(B)} Mat(n \times n) \simeq Mat(n \times n)$ .

**Definition 2.1** The following form of  $G$  is called *Maurer-Cartan form*:

$$\omega_B = \{P(B)\}^{-1} \cdot dP_B$$

that it is often written  $\omega_B = P^{-1} \cdot dP$ . The Maurer-Cartan form is the key to classifying maps into homogeneous spaces of  $G$ , and this process needs to this theorem (for proof refer to [2]):

**Theorem 2.2 (Cartan)** *Let  $G$  be a matrix Lie group with Lie algebra  $\mathfrak{g}$  and Maurer-Cartan form  $\omega$ . Let  $M$  be a manifold on which there exists a  $\mathfrak{g}$ -valued 1-form  $\phi$  satisfying  $d\phi = -\phi \wedge \phi$ . Then for any point  $x \in M$  there exist a neighborhood  $U$  of  $x$  and a map  $f : U \rightarrow G$  such that  $f^* \omega = \phi$ . Moreover, any two such maps  $f_1, f_2$  must satisfy  $f_1 = L_B \circ f_2$  for some fixed  $B \in G$  ( $L_B$  is the left action of  $B$  on  $G$ ).*

**Corollary 2.3** *Given maps  $f_1, f_2 : M \rightarrow G$ , then  $f_1^* \omega = f_2^* \omega$ , that is, this pull-back is invariant, if and only if  $f_1 = L_B \circ f_2$  for some fixed  $B \in G$ .*

The next section, is devoted to the study of the properties of  $n$ -curves invariants, under the special affine transformations group. The number of essential parameters (dimension of the Lie algebra) is  $n^2 + n - 1$ . The natural assumption of differentiability is  $C^{n+2}$ .

We achieve the invariants of the  $n$ -curve in respect to special affine transformations, and with theorem 2.2, two  $n$ -curves in  $\mathbb{R}^n$  will be equivalent under special affine transformations, if they differ with a left action introduced with an element of  $SL(n, \mathbb{R})$  and then a translation.

### 3 Classification of $n$ -curves

Let  $C : [a, b] \rightarrow \mathbb{R}^n$  be a curve of class  $C^{n+2}$  in finite dimensional space  $\mathbb{R}^n$ ,  $n$ -space, which satisfies in the condition

$$(3.1) \quad \det(C', C'', \dots, C^{(n)}) \neq 0,$$

that, we call this curve  $n$ -curve. The condition (3.1) Guarantees that  $C', C'', \dots$ , and  $C^{(n)}$  are independent, and therefore, the curve does not turn into the lower dimensional cases. Also, we may assume that

$$(3.2) \quad \det(C', C'', \dots, C^{(n)}) > 0,$$

to being avoid writing the absolute value in computations.

For the  $n$ -curve  $C$ , we define a new curve  $\alpha_C(t) : [a, b] \rightarrow SL(n, \mathbb{R})$  that is in the following form

$$(3.3) \quad \alpha_C(t) := \frac{(C', C'', \dots, C^{(n)})}{\sqrt[n]{\det(C', C'', \dots, C^{(n)})}}.$$

Obviously, it is well-defined on  $[a, b]$ . We can study this new curve in respect to special affine transformations, that is the action of affine transformations on first, second, ..., and  $n^{\text{th}}$  differentiation of  $C$ . For  $A$ , the special affine transformation, there is a unique representation  $A = \tau \circ B$  which  $B$  is an element of  $\text{SL}(n, \mathbb{R})$  and  $\tau$  is a translation in  $\mathbb{R}^n$ . If two  $n$ -curves  $C$  and  $\bar{C}$  be same under special affine transformations, that is,  $\bar{C} = A \circ C$ , then from [4], we have

$$(3.4) \quad \bar{C}' = B \circ C', \quad \bar{C}'' = B \circ C'', \quad \dots, \quad \bar{C}^{(n)} = B \circ C^{(n)}.$$

We can relate the determinants of these curves as below

$$(3.5) \quad \begin{aligned} \det(\bar{C}', \bar{C}'', \dots, \bar{C}^{(n)}) &= \det(B \circ C', B \circ C'', \dots, B \circ \bar{C}^{(n)}) \\ &= \det(B \circ (C', C'', \dots, C^{(n)})) \\ &= \det(C', C'', \dots, C^{(n)}). \end{aligned}$$

So we can conclude that  $\alpha_{\bar{C}}(t) = B \circ \alpha_C(t)$  and thus  $\alpha_{\bar{C}} = L_B \circ \alpha_C$  that  $L_B$  is a left translation by  $B \in \text{SL}(n, \mathbb{R})$ .

This condition is also necessary because when  $C$  and  $\bar{C}$  are two curves in  $\mathbb{R}^n$  such that for an element  $B \in \text{SL}(n, \mathbb{R})$ , we have  $\alpha_{\bar{C}} = L_B \circ \alpha_C$ , thus we can write

$$(3.6) \quad \begin{aligned} \alpha_{\bar{C}}(t) &= \det(\bar{C}', \bar{C}'', \dots, \bar{C}^{(n)})^{-1/n} (\bar{C}', \bar{C}'', \dots, \bar{C}^{(n)}) \\ &= \det(B \circ (C', C'', \dots, C^{(n)}))^{-1/n} B \circ (C', C'', \dots, C^{(n)}) \\ &= \det(C', C'', \dots, C^{(n)})^{-1/n} B \circ (C', C'', \dots, C^{(n)}). \end{aligned}$$

Therefore, we have  $\bar{C}' = B \circ C'$ , and so there is a translation  $\tau$  such that  $A = \tau \circ B$ , and so, we have  $\bar{C} = A \circ C$  when,  $A$  is a  $n$ -dimensional affine transformation. Therefore, we have

**Theorem 3.1** *Two  $n$ -curves  $C$  and  $\bar{C}$  in  $\mathbb{R}^n$  are same under the special affine transformations that is,  $\bar{C} = A \circ C$ , which  $A = \tau \circ B$  for translation  $\tau$  in  $\mathbb{R}^n$  and  $B \in \text{SL}(n, \mathbb{R})$ ; if and only if,  $\alpha_{\bar{C}} = L_B \circ \alpha_C$ , where  $L_B$  is left translation by  $B$ .*

From Cartan's theorem, a necessary and sufficient condition for  $\alpha_{\bar{C}} = L_B \circ \alpha_C$  by  $B \in \text{SL}(n, \mathbb{R})$ , is that for any left invariant 1-form  $\omega^i$  on  $\text{SL}(n, \mathbb{R})$  we have  $\alpha_{\bar{C}}^*(\omega^i) = \alpha_C^*(\omega^i)$ , that is equivalent with  $\alpha_{\bar{C}}^*(\omega) = \alpha_C^*(\omega)$ , for natural  $\mathfrak{sl}(n, \mathbb{R})$ -valued 1-form  $\omega = P^{-1} \cdot dP$ , where  $P$  is the Maurer–Cartan form.

Thereby, we must compute the  $\alpha_C^*(P^{-1} \cdot dP)$ , which is invariant under special affine transformations, that is, its entries are invariant functions of  $n$ -curves. This  $n \times n$  matrix form, consists of arrays that are coefficients of  $dt$ .

Since  $\alpha_C^*(P^{-1} \cdot dP) = \alpha_C^{-1} \cdot d\alpha_C$ , so for finding the invariants, it is sufficient that we calculate the matrix  $\alpha_C(t)^{-1} \cdot d\alpha_C(t)$ . Thus, we compute  $\alpha_C^*(P^{-1} \cdot dP)$ . We have

$$(3.7) \quad \alpha_C^{-1} = \sqrt[n]{\det(C', C'', \dots, C^{(n)})} \cdot (C', C'', \dots, C^{(n)})^{-1}.$$

We assume that  $C$  is in the form  $(C_1 \ C_2 \ \cdots \ C_n)^T$ . By differentiating of determinant, we have

$$\begin{aligned}
[\det(C', C'', \dots, C^{(n)})]' &= \det(C'', C'', \dots, C^{(n)}) \\
&\quad + \det(C', C''', \dots, C^{(n)}) \\
(3.8) \qquad \qquad \qquad &\vdots \\
&\quad + \det(C', C'', \dots, C^{(n-1)}, C^{(n+1)}) \\
&= \det(C', C'', \dots, C^{(n-1)}, C^{(n+1)}).
\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}
\alpha'_C &= \{\det(C', C'', \dots, C^{(n)})\}^{-1/n} \cdot \begin{pmatrix} C''_1 & C'''_1 & \cdots & C^{(n)}_1 \\ C''_2 & C'''_2 & \cdots & C^{(n)}_2 \\ \vdots & \vdots & & \vdots \\ C''_3 & C'''_3 & \cdots & C^{(n)}_3 \end{pmatrix} \\
(3.9) \qquad \qquad \qquad & \\
& - \frac{1}{n} \det(C', C'', C''')\}^{-(n+1)/n} \cdot \begin{pmatrix} C'_1 & C''_1 & \cdots & C^{(n)}_1 \\ C'_2 & C''_2 & \cdots & C^{(n)}_2 \\ \vdots & \vdots & & \vdots \\ C'_3 & C''_3 & \cdots & C^{(n)}_3 \end{pmatrix}.
\end{aligned}$$

Therefore, we have the  $\alpha_C^{-1} \cdot d\alpha_C$  as the following matrix multiplying with  $dt$ :

$$(3.10) \quad \left( \begin{array}{ccccc} a & 0 & \cdots & 0 & 0 \\ 1 & a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a \\ 0 & 0 & \cdots & 0 & 1 \end{array} \middle| M \cdot C^{(n+1)} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a \end{pmatrix} \right)$$

where, the latest column,  $M \cdot C^{(n+1)} + (0, 0, \dots, a)^T$  is multiple of  $M$  by  $C^{(n+1)}$  added by transpose of  $(0, 0, \dots, a)$ ; which  $M$  is the inverse of matrix  $(C', C'', \dots, C^{(n)})$  and also, we assumed that

$$(3.11) \quad a = -\frac{\det(C', C'', \dots, C^{(n-1)}, C^{(n+1)})}{n \det(C', C'', \dots, C^{(n)})}.$$

But with use of Crammer's law, we compute  $M \cdot C^{(n+1)}$ . If  $M \cdot C^{(n+1)} = X = (X_1, X_2, \dots, X_n)^T$ , then  $M^{-1} \cdot X = C^{(n+1)}$ . So for each  $i = 1, 2, \dots, n$  we conclude that

$$(3.12) \quad X_i = \frac{\det(C', C'', \dots, C^{(i-1)}, C^{(n+1)}, C^{(i+1)}, \dots, C^{(n)})}{\det(C', C'', \dots, C^{(n)})}$$

Finally, the  $\alpha_C^{-1} \cdot d\alpha_C$  is the following multiple of  $dt$ :

$$(3.13) \quad \begin{pmatrix} a & 0 & \cdots & 0 & 0 & (-1)^{n-1} \frac{\det(C'', \dots, C^{(n+1)})}{\det(C', C'', \dots, C^{(n)})} \\ 1 & a & \cdots & 0 & 0 & (-1)^{n-2} \frac{\det(C', C''', \dots, C^{(n)})}{\det(C', C'', \dots, C^{(n)})} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ o & 0 & \cdots & 1 & a & -\frac{\det(C', \dots, C^{(n-2)}, C^{(n)}, C^{(n+1)})}{\det(C', C'', \dots, C^{(n)})} \\ 0 & 0 & \cdots & 0 & 1 & \frac{n-1}{n} \frac{\det(C', \dots, C^{(n-1)}, C^{(n+1)})}{\det(C', C'', \dots, C^{(n)})} \end{pmatrix},$$

which the coefficient  $(-1)^{i-1}$  for  $i^{th}$  entry of the latest column, comes from the translation of  $C^{(n+1)}$  to  $n^{th}$  column of matrix

$$(3.14) \quad (C', C'', \dots, C^{(i-1)}, C^{(n+1)}, C^{(i+1)}, \dots, C^{(n)}).$$

Clearly, the trace of matrix (3.13) is zero. The entries of  $\alpha_C^*(P^{-1} \cdot dP)$  and therefore, arrays of matrix (3.13), are invariants of the group action.

Two  $n$ -curves  $C, \bar{C} : [a, b] \rightarrow \mathbb{R}^n$  are same in respect to special affine transformations, if we have

$$(3.15) \quad \begin{aligned} \frac{\det(C''(t), \dots, C^{(n+1)}(t))}{\det(C'(t), C''(t), \dots, C^{(n)}(t))} &= \frac{\det(\bar{C}''(t), \dots, \bar{C}^{(n+1)}(t))}{\det(\bar{C}'(t), \bar{C}''(t), \dots, \bar{C}^{(n)}(t))} \\ \frac{\det(C'(t), C'''(t), \dots, C^{(n+1)}(t))}{\det(C'(t), C''(t), \dots, C^{(n)}(t))} &= \frac{\det(\bar{C}'(t), \bar{C}'''(t), \dots, \bar{C}^{(n+1)}(t))}{\det(\bar{C}'(t), \bar{C}''(t), \dots, \bar{C}^{(n)}(t))} \\ &\vdots \\ \frac{\det(C'(t), \dots, C^{(n-1)}(t), C^{(n+1)})}{\det(C'(t), C''(t), \dots, C^{(n)}(t))} &= \frac{\det(\bar{C}'(t), \dots, \bar{C}^{(n-1)}(t), \bar{C}^{(n+1)})}{\det(\bar{C}'(t), \bar{C}''(t), \dots, \bar{C}^{(n)}(t))}. \end{aligned}$$

We may use of a proper parametrization  $\gamma : [a, b] \rightarrow [0, l]$ , such that the parametrized curve,  $\gamma = C \circ \sigma^{-1}$ , satisfies in condition

$$(3.16) \quad \det(\gamma'(s), \gamma''(s), \dots, \gamma^{(n-1)}(s), \gamma^{(n+1)}(s)) = 0,$$

then, the arrays on main diagonal of  $\alpha_\gamma^*(dP \cdot P^{-1})$  will be zero. But the latest determinant is differentiation of  $\det(\gamma'(s), \gamma''(s), \dots, \gamma^{(n)}(s))$  thus, it is sufficient that we suppose

$$(3.17) \quad \det(\gamma'(s), \gamma''(s), \dots, \gamma^{(n)}(s)) = 1.$$

On the other hand, we have

$$(3.18) \quad \begin{aligned} C' &= (\gamma \circ \sigma)' = \sigma' \cdot (\gamma' \circ \sigma) \\ C'' &= (\sigma')^2 \cdot (\gamma'' \circ \sigma) + \sigma'' \cdot (\gamma' \circ \sigma) \\ &\vdots \\ C^{(n)} &= (\sigma')^{(n)} \cdot (\gamma^{(n)} \circ \sigma) + n \sigma^{(n-1)} \sigma' \cdot (\gamma^{(n-1)} \circ \sigma) \\ &\quad + \frac{n(n-1)}{2} \sigma^{(n-2)} \sigma'' \cdot (\gamma^{(n-2)} \circ \sigma) + \cdots + \sigma^{(n)} \cdot (\gamma' \circ \sigma) \end{aligned}$$

Therefore,  $C^{(i)}$ s for  $1 \leq i \leq n$ , are some statements in respect to  $\gamma^{(j)} \circ \sigma$ ,  $1 \leq j \leq n$ . We conclude that

$$\begin{aligned}
(3.19) \quad \det(C', C'', \dots, C^{(n)}) &= \det(\sigma'.(\gamma' \circ \sigma), (\sigma')^2.(\gamma'' \circ \sigma) + \sigma''.(\gamma' \circ \sigma), \\
&\quad \dots, (\sigma')^{(n)}.(\gamma^{(n)} \circ \sigma) + n \sigma^{(n-1)} \sigma'.(\gamma^{(n-1)} \circ \sigma) \\
&\quad + \frac{n(n-1)}{2} \sigma^{(n-2)} \sigma''.(\gamma^{(n-2)} \circ \sigma) + \dots + \sigma^{(n)}.(\gamma' \circ \sigma)) \\
&= \det(\sigma'.(\gamma' \circ \sigma), (\sigma')^2.(\gamma'' \circ \sigma), \dots, (\sigma')^n.(\gamma^{(n)} \circ \sigma)) \\
&= \sigma'^{\frac{n(n-1)}{2}}. \det(\gamma' \circ \sigma, \gamma'' \circ \sigma, \dots, \gamma^{(n)} \circ \sigma) \\
&= \sigma'^{\frac{n(n-1)}{2}},
\end{aligned}$$

The latest expression signifies  $\sigma$  therefore, we define the *special affine arc length* as follows

$$(3.20) \quad \sigma(t) := \int_a^t \left\{ \det(C'(u), C''(u), \dots, C^{(n)}(u)) \right\}^{\frac{2}{n(n-1)}} du.$$

So,  $\sigma$  is the natural parameter for  $n$ -curves under the action of special affine transformations, that is, when  $C$  be parameterized with  $\sigma$ , then for each special affine transformation  $A$ ,  $A \circ C$  will also be parameterized with the same  $\sigma$ . Furthermore, every  $n$ -curve parameterized with  $\sigma$  in respect to special affine transformations, will be introduced with the following invariants

$$\begin{aligned}
(3.21) \quad \chi_1 &= (-1)^{n-1} \det(C'', \dots, C^{(n+1)}) \\
\chi_2 &= (-1)^{n-2} \det(C', C''', \dots, C^{(n)}) \\
&\quad \vdots \\
\chi_{n-1} &= \det(C', \dots, C^{(n-2)}, C^{(n)}, C^{(n+1)}).
\end{aligned}$$

We call  $\chi_1, \chi_2, \dots$ , and  $\chi_{n-1}$  as (respectively) the first, second, ..., and  $n-1$ <sup>th</sup> *special affine curvatures*. In fact, we proved the following theorem

**Theorem 3.2** *A curve of class  $C^{n+2}$  in  $\mathbb{R}^n$  with condition (3.1), up to special affine transformations has  $n-1$  invariants  $\chi_1, \chi_2, \dots$ , and  $\chi_{n-1}$ , the first, second, ..., and  $n-1$ <sup>th</sup> affine curvatures that are defined as formulas (3.3).*

**Theorem 3.3** *Two  $n$ -curves  $C, \bar{C} : [a, b] \rightarrow \mathbb{R}^n$  of class  $C^{n+2}$ , that satisfy in the condition (3.1), are special affine equivalent, if and only if,  $\chi_1^C = \chi_1^{\bar{C}}, \dots$ , and  $\chi_{n-1}^C = \chi_{n-1}^{\bar{C}}$ .*

*Proof:* Proof is completely similar to the three dimensional case [5]. The first side of the theorem was proved in above descriptions. For the other side, we assume that  $C$  and  $\bar{C}$  are  $n$ -curves of class  $C^{n+2}$  with conditions (resp.):

$$(3.22) \quad \det(C', C'', \dots, C^{(n)}) > 0, \quad \det(\bar{C}', \bar{C}'', \dots, \bar{C}^{(n)}) > 0,$$

with this mean that they are not  $(n - 1)$ -curves. Also, we suppose that they have same  $\chi_1, \dots$ , and  $\chi_{n-1}$ .

By changing the parameter to the natural parameter  $(\sigma)$ , discussed above, we obtain new curves  $\gamma$  and  $\bar{\gamma}$  resp. that the determinants (3.22) will be equal to 1. We prove that  $\gamma$  and  $\bar{\gamma}$  are special affine equivalent, so there is a special affine transformation  $A$  such that  $\bar{\gamma} = A \circ \gamma$  and then we have  $\bar{C} = A \circ C$  and proof will be completed.

At first, we replace the curve  $\gamma$  with  $\delta := \tau(\gamma)$  properly, in which case that  $\delta$  intersects  $\bar{\gamma}$ , that  $\tau$  is a translation defined by translating one point of  $\gamma$  to one point of  $\bar{\gamma}$ . We correspond  $t_0 \in [a, b]$ , to the intersection point of  $\delta$  and  $\bar{\gamma}$  thus,  $\delta(t_0) = \bar{\gamma}(t_0)$ . One can find a unique element  $B$  of the general linear group  $GL(n, \mathbb{R})$ , such that maps the base  $\{\delta'(t_0), \delta''(t_0), \dots, \delta^{(n)}(t_0)\}$  of tangent space  $T_{\delta(t_0)}\mathbb{R}^3$  to the base  $\{\bar{\gamma}'(t_0), \bar{\gamma}''(t_0), \dots, \bar{\gamma}^{(n)}(t_0)\}$  of it. So, we have  $B \circ \delta'(t_0) = \bar{\gamma}'(t_0)$ ,  $B \circ \delta''(t_0) = \bar{\gamma}''(t_0)$ ,  $\dots$ , and  $B \circ \delta^{(n)}(t_0) = \bar{\gamma}^{(n)}(t_0)$ .  $B$  also is an element of the special linear group,  $SL(n, \mathbb{R})$ , since we have

$$(3.23) \quad \begin{aligned} \det(\gamma'(t_0), \gamma''(t_0), \dots, \gamma^{(n)}(t_0)) &= \\ &= \det(\delta'(t_0), \delta''(t_0), \dots, \delta^{(n)}(t_0)), \end{aligned}$$

and

$$(3.24) \quad \begin{aligned} \det(\delta'(t_0), \delta''(t_0), \dots, \delta^{(n)}(t_0)) &= \\ \det\left(B \circ (\bar{\gamma}'(t_0), \bar{\gamma}''(t_0), \dots, \bar{\gamma}^{(n)}(t_0))\right), \end{aligned}$$

so,  $\det(B) = 1$ . If we denote that  $\eta := B \circ \delta$  is equal to  $\bar{\gamma}$  on  $[a, b]$ , then by choosing  $A = \tau \circ B$ , there will remind nothing for proof.

For the curves  $\eta$  and  $\bar{\gamma}$  we have (resp.)

$$(3.25) \quad \begin{aligned} (\eta', \eta'', \dots, \eta^{(n)})' &= \\ &= (\eta', \eta'', \dots, \eta^{(n)}) \cdot \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \chi_1^\eta \\ 1 & 0 & \dots & 0 & 0 & -\chi_2^\eta \\ 0 & 1 & \dots & 0 & 0 & \chi_3^\eta \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & (-1)^{(n-2)}\chi_{n-1}^\eta \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}, \end{aligned}$$

and

$$(3.26) \quad \begin{aligned} (\bar{\gamma}', \bar{\gamma}'', \dots, \bar{\gamma}^{(n)})' &= \\ &= (\bar{\gamma}', \bar{\gamma}'', \dots, \bar{\gamma}^{(n)}) \cdot \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & \chi_1^{\bar{\gamma}} \\ 1 & 0 & \dots & 0 & 0 & -\chi_2^{\bar{\gamma}} \\ 0 & 1 & \dots & 0 & 0 & \chi_3^{\bar{\gamma}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & (-1)^{(n-2)}\chi_{n-1}^{\bar{\gamma}} \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}, \end{aligned}$$

Since,  $\chi_1, \dots$ , and  $\chi_{n-1}$ , are invariants under special affine transformations so, we have

$$(3.27) \quad \chi_i^\eta = \chi_i^\gamma = \chi_i^{\bar{\gamma}}, \quad (i = 1, \dots, n-1).$$

Therefore, we conclude that  $\eta$  and  $\bar{\gamma}$  are solutions of ordinary differential equation of degree  $n+1$ :

$$Y^{n+1} + (-1)^{n-1} \chi_{n-1} Y^{(n)} + \dots + \chi_2 Y'' - \chi_1 Y' = 0,$$

where,  $Y$  depends to parameter  $t$ . Because of same initial conditions

$$(3.28) \quad \eta^{(i)}(t_0) = B \circ \delta(t_0) = \bar{\gamma}^{(i)}(t_0),$$

for  $i = 0, \dots, n$ , and the generalization of the existence and uniqueness theorem of solutions, we have  $\eta = \bar{\gamma}$  in a neighborhood of  $t_0$ , that can be extended to all  $[a, b]$ .  $\diamond$

**Corollary 3.4** *The number of invariants of special affine transformations group acting on  $\mathbb{R}^n$  is  $n-1$ , that is same with results provided with other methods such as [1].*

## References

- [1] H. Guggenheimer, *Differential Geometry*, Dover Publ., New York (1977).
- [2] T.A. Ivey and J.M. Landsberg, *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential System*, A.M.S. (2003).
- [3] P.J. Olver, *Equivalence, invariants, and symmetry*, Cambridge Univ. Press, Cambridge (1995).
- [4] B. O'Neill, *Elementary Differential Geometry*, Academic Press, London–New York (1966).
- [5] M. Nadjafikhah and A. Mahdipour Sh., *Geometry of Space Curves up to Affine Transformations*, preprint, <http://aps.arxiv.org/abs/0710.2661>.
- [6] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. II and III, Publish or Perish, Wilmington, Delaware (1979).

Mehdi Nadjafikhah  
*Department of Mathematics,*  
*Iran University of Science and Technology, Narmak-16, Tehran, Iran.*  
*E-mail: m\\_nadjafikhah@iust.ac.ir*

Ali Mahdipour Sh.  
*Department of Mathematics,*  
*Iran University of Science and Technology, Narmak-16, Tehran, Iran.*  
*E-mail: mahdi\\_psh@mathdep.iust.ac.ir*