LOWER BOUNDS ON THE COEFFICIENTS OF EHRHART POLYNOMIALS

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Abstract. We present lower bounds for the coefficients of Ehrhart polynomials of convex lattice polytopes in terms of their volume. Concerning the coefficients of the Ehrhart series of a lattice polytope we show that Hibi's lower bound is not true for lattice polytopes without interior lattice points. The counterexample is based on a formula of the Ehrhart series of the join of two lattice polytope. We also present a formula for calculating the Ehrhart series of integral dilates of a polytope.

1. Introduction

Let $\mathcal{P}^d$ be the set of all convex d-dimensional lattice polytopes in the d-dimensional Euclidean space $\mathbb{R}^d$ with respect to the standard lattice $\mathbb{Z}^d$, i.e., all vertices of $P \in \mathcal{P}^d$ have integral coordinates and $\dim(P) = d$. The lattice point enumerator of a set $S \subset \mathbb{R}^d$, denoted by $\mathrm{G}(S)$, counts the number of lattice (integral) points in S, i.e., $\mathrm{G}(S) = \#(S \cap \mathbb{Z}^d)$. In 1962, Eugéne Ehrhart (see e.g. [3, Chapter 3], [7]) showed that for $k \in \mathbb{N}$ the lattice point enumerator $\mathrm{G}(k\,P)$, $P \in \mathcal{P}^d$, is a polynomial of degree d in k where the coefficients $\mathrm{g}_i(P)$, $0 \leq i \leq d$, depend only on P:

$$\mathrm{G}(k\,P) = \sum_{i=0}^{d} \mathrm{g}_i(P)\,k^i. \tag{1.1}$$

The polynomial on the right hand side is called the Ehrhart polynomial, and regarded as a formal polynomial in a complex variable $z \in \mathbb{C}$ it is denoted by $\mathrm{G}_P(z)$. Two of the $d+1$ coefficients $\mathrm{g}_i(P)$ are almost obvious, namely, $\mathrm{g}_0(P) = 1$, the Euler characteristic of P, and $\mathrm{g}_d(P) = \mathrm{vol}(P)$, where $\mathrm{vol}()$ denotes the volume, i.e., the d-dimensional Lebesgue measure on $\mathbb{R}^d$. It was shown by Ehrhart (see e.g. [3, Theorem 5.6], [8]) that also the second leading coefficient admits a simple geometric interpretation as lattice surface area of P

$$\mathrm{g}_{d-1}(P) = \frac{1}{2} \sum_{F \text{ facet of P}} \frac{\mathrm{vol}_{d-1}(F)}{\det(\mathrm{aff}\,F \cap \mathbb{Z}^d)}. \tag{1.2}$$

Here $\mathrm{vol}_{d-1}(\cdot)$ denotes the $(d-1)$-dimensional volume and $\det(\mathrm{aff}\,F \cap \mathbb{Z}^d)$ denotes the determinant of the $(d-1)$-dimensional sublattice contained in the affine hull of F. All other coefficients $\mathrm{g}_i(P)$, $1 \leq i \leq d-2$, have no such known explicit

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geometric meaning, except for special classes of polytopes. For this and as a general reference on the theory of lattice polytopes we refer to the recent book of Matthias Beck and Sinai Robins [3] and the references within. For more information regarding lattices and the role of the lattice point enumerator in convexity see [9].

In [4, Theorem 6] Ulrich Betke and Peter McMullen proved the following upper bounds on the coefficients $\mathrm{g}_i(P)$ in terms of the volume:

$$\mathrm{g}_i(P) \leq (-1)^{d-i}\mathrm{stirl}(d,i)\mathrm{vol}(P) + (-1)^{d-i-1}\frac{\mathrm{stirl}(d,i+1)}{(d-1)!}, \quad i = 1,\dots,d-1.$$

Here $\mathrm{stirl}(d,i)$ denote the Stirling numbers of the first kind which can be defined via the identity $\prod_{i=0}^{d-1}(z-i) = \sum_{i=1}^{d}\mathrm{stirl}(d,i)\,z^i$.

In order to present our lower bounds on $\mathrm{g}_i(P)$ in terms of the volume we need some notation. For an integer i and a variable z we consider the polynomial

$$(z+i)(z+i-1)\cdot\ldots\cdot(z+i-(d-1)) = d!\binom{z+i}{d},$$

and we denote its r-th coefficient by $C^d_{r,i}$, $0 \leq r \leq d$. For instance, it is $C^d_{d,i} = 1$, and for $0 \leq i \leq d-1$ we have $C^d_{0,i} = 0$. For $d \geq 3$ we are interested in

$$M_{r,d} = \min\{C^d_{r,i} : 1 \leq i \leq d-2\}. \tag{1.3}$$

Obviously, we have $M_{0,d} = 0$, $M_{d,d} = 1$ and it is also easy to see that (cf. Proposition 2.1 iii))

$$M_{d-1,d} = C^d_{d-1,1} = -\frac{d(d-3)}{2}. \tag{1.4}$$

With the help of these numbers $M_{r,d}$ we obtain the following lower bounds.

Theorem 1.1. *Let $P \in \mathcal{P}^d$, $d \geq 3$. Then for $i = 1,\dots,d-1$ we have*

$$\mathrm{g}_i(P) \geq \frac{1}{d!}\left\{(-1)^{d-i}\mathrm{stirl}(d+1,i+1) + (d!\,\mathrm{vol}(P)-1)M_{i,d}\right\}.$$

We remark that the coefficients $\mathrm{g}_i(P)$, $1 \leq i \leq d-2$, might be negative and thus also the lower bounds given above. In general, the bounds of Theorem 1.1 are not best possible. For instance, in the case $i = d-1$ we get together with (1.4) the bound

$$\mathrm{g}_{d-1}(P) \geq \frac{1}{(d-1)!}\left\{d-1-\frac{d-3}{2}d!\,\mathrm{vol}(P)\right\}.$$

On the other hand, since the lattice surface area of any facet is at least $1/(d-1)!$ we have the trivial inequality (cf. (1.2))

$$\mathrm{g}_{d-1}(P) \geq \frac{1}{2}\frac{d+1}{(d-1)!}. \tag{1.5}$$

Hence the lower bound on $\mathrm{g}_{d-1}(P)$ given in Theorem 1.1 is only best possible if $\mathrm{vol}(P) = 1/d!$. In the cases $i \in \{1,2,d-2\}$, however, Theorem 1.1 gives best possible bounds for any volume.

Corollary 1.2. *Let $P \in \mathcal{P}^d$, $d \geq 3$. Then*

i) $\mathrm{g}_1(P) \geq 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{d-2} + \dfrac{2}{d-1} - (d-2)!\,\mathrm{vol}(P)$,

ii) $$\mathrm{g}_2(P) \geq \frac{(-1)^d}{d!} \times \left\{\mathrm{stirl}(d+1,3) + \Big((-1)^d (d-2)! + \mathrm{stirl}(d-1,2)\Big)\,(d!\,\mathrm{vol}(P) - 1)\right\},$$

iii) $$\mathrm{g}_{d-2}(P) \geq \begin{cases} \frac{1}{d!}\frac{(d-1)d(d+1)}{24}\left\{3(d+1) - d!\,\mathrm{vol}(P)\right\} : & \textit{if } d \textit{ odd}, \\ \frac{1}{d!}\frac{(d-1)d}{24}\left\{3d(d+2) - (d-2)\,d!\,\mathrm{vol}(P)\right\} : & \textit{if } d \textit{ even}. \end{cases}$$

And the bounds are best possible for any volume.

For some recent inequalities involving more coefficients of Ehrhart polynomials we refer to [2]. Next we come to another family of coefficients of a polynomial associated to lattice polytopes.

The generating function of the lattice point enumerator, i.e., the formal power series
$$\mathrm{Ehr}_P(z) = \sum_{k \geq 0} \mathrm{G}_P(k)\, z^k,$$
is called the Ehrhart series of P. It is well known that it can be expressed as a rational function of the form
$$\mathrm{Ehr}_P(z) = \frac{\mathrm{a}_0(P) + \mathrm{a}_1(P)\,z + \cdots + \mathrm{a}_d(P)\,z^d}{(1-z)^{d+1}}.$$
The polynomial in the numerator is called the $h^\star$-polynomial. Its degree is also called the degree of the polytope [1] and it is denoted by $\deg(P)$. Concerning the coefficients $\mathrm{a}_i(P)$ it is known that they are integral and that
$$\mathrm{a}_0(P) = 1, \quad \mathrm{a}_1(P) = \mathrm{G}(P) - (d+1), \quad \mathrm{a}_d(P) = \mathrm{G}(\mathrm{int}(P)),$$
where $\mathrm{int}(\cdot)$ denotes the interior. Moreover, due to Stanley's famous non-negativity theorem (see e.g. [3, Theorem 3.12], [17]) we also know that $\mathrm{a}_i(P)$ is non-negative, i.e., for these coefficients we have the lower bounds $\mathrm{a}_i(P) \geq 0$. In the case $\mathrm{G}(\mathrm{int}(P)) > 0$, i.e., $\deg(P) = d$, these bounds were improved by Takayuki Hibi [13] to
$$\mathrm{a}_i(P) \geq \mathrm{a}_1(P),\ 1 \leq i \leq \deg(P) - 1. \tag{1.6}$$
In this context it was a quite natural question whether the assumption $\deg(P) = d$ can be weaken (see e.g. [15]), i.e., whether these lower bounds (1.6) are also valid for polytopes of degree less than d. As we show in Example 1.4 the answer is already negative for polytopes having degree 3. The problem in order to study such a question is that only very few geometric constructions of polytopes are known for which we can explicitly calculate the Ehrhart series. In [3, Theorem 2.4, Theorem 2.6] the Ehrhart series of special pyramids and double pyramids over a basis Q are determined in terms of the Ehrhart series of Q. In a recent paper Braun [6] gave a very nice product formula for the Ehrhart series of the free sum of two lattice polytopes, where one of the polytopes has to be reflexive. Here we consider a related construction, known as the join of two

polytopes [11]. As we learned by Matthias Beck the Ehrhart series of such a join is already described as Exercise 3.32 in the book [3] and it was personally communicated to the authors of the book by Kevin Woods. For completeness' sake we present its short proof in Section 3.

Lemma 1.3. *For $P \in \mathcal{P}^p$ and $Q \in \mathcal{P}^q$ let $P \star Q$ be the join of P and Q, i.e.,*

$$P \star Q = \operatorname{conv}\{(x, 0_q, 0)^\intercal, (0_p, y, 1)^\intercal : x \in P,\, y \in Q\} \in \mathcal{P}^{p+q+1},$$

where 0_p and 0_q denote the p- and q-dimensional 0-vector, respectively. Then

$$\mathrm{Ehr}_{P\star Q}(z) = \mathrm{Ehr}_P(z) \cdot \mathrm{Ehr}_Q(z).$$

In order to apply this lemma we consider two families of lattice simplices. For an integer $m \in \mathbb{N}$ let

$$\begin{aligned} T_d^{(m)} &= \operatorname{conv}\{o, e_1, e_1 + e_2, e_2 + e_3, \ldots, e_{d-2} + e_{d-1}, e_{d-1} + m\, e_d\}, \\ S_d^{(m)} &= \operatorname{conv}\{o, e_1, e_2, e_3, \ldots, e_{d-1}, m\, e_d\}, \end{aligned}$$

where e_i denotes the i-th unit vector. It was shown in [4] that

$$\mathrm{Ehr}_{T_d^{(m)}}(z) = \frac{1 + (m-1)\, z^{\lceil \frac{d}{2} \rceil}}{(1-z)^{d+1}} \text{ and } \mathrm{Ehr}_{S_d^{(m)}}(z) = \frac{1 + (m-1)\, z}{(1-z)^{d+1}}. \tag{1.7}$$

Actually, in [4] the formula for $T_d^{(m)}$ was only proved for odd dimensions, but the even case can be treated completely analogously.

Example 1.4. *For $q \in \mathbb{N}$ odd and $l, m \in \mathbb{N}$ we have*

$$\mathrm{Ehr}_{T_q^{(l+1)} \star S_p^{(m+1)}}(z) = \frac{1 + m\, z + l\, z^{\frac{q+1}{2}} + m\, l\, z^{\frac{q+3}{2}}}{(1-z)^{p+q+2}}.$$

In particular, for $q \geq 3$ and $l < m$ this shows that (1.6) *is, in general, false for lattice polytopes without interior lattice points.*

Another formula for calculating the Ehrhart Series from a given one concerns dilates. Here we will show

Lemma 1.5. *Let $P \in \mathcal{P}^d$, $k \in \mathbb{N}$ and let ζ be a primitive k-th root of unity. Then*

$$\mathrm{Ehr}_{k\,P}(z) = \frac{1}{k} \sum_{i=0}^{k-1} \mathrm{Ehr}_P(\zeta^i\, z^{\frac{1}{k}}).$$

The lemma can be used, for instance, to calculate the Ehrhart series of the cube $C_d = \{x \in \mathbb{R}^d : |x_i| \leq 1,\, 1 \leq i \leq d\}$.

Example 1.6. *For two integers j, d, $0 \leq j \leq d$, let*

$$A(d, j) = \sum_{k=0}^{j} (-1)^k \binom{d+1}{k} (j-k)^d$$

be the Eulerian numbers (see e.g. [3, pp. 28]*). Furthermore, we set* $A(d,j)=0$ *if* $j\notin\{0,\dots,d\}$*. Then, for* $0\le i\le d$*, we have*

$$\mathrm{a}_i(C_d)=\sum_{j=0}^{d+1}\binom{d+1}{j}A(d,2\,i+1-j).$$

Of course, the cube C_d may be also regarded as a prism over a $(d-1)$-cube, and as a counterpart to the bipyramid construction in [3] we calculate here also the Ehrhart series of some special prism.

Example 1.7. *Let* $Q\in\mathcal{P}^{d-1}$*,* $m\in\mathbb{N}$*, and let* $P=\{(x,x_d)^\intercal : x\in Q,\, x_d\in[0,m]\}$ *be the prism of height* m *over* Q*. Then*

$$\mathrm{a}_i(P)=(m\,i+1)\mathrm{a}_i(Q)+(m(d-i+1)-1)\,\mathrm{a}_{i-1}(Q),\ 0\le i\le d,$$

where we set $\mathrm{a}_d(Q)=\mathrm{a}_{-1}(Q)=0$*.*

It seems to be quite likely that for the class of 0-symmetric lattice polytopes $\mathcal{P}^d_o$ the lower bounds on $\mathrm{a}_i(P)$ can considerably be improved. In [5] it was conjectured that for $P\in\mathcal{P}^d_o$

$$\mathrm{a}_i(P)+\mathrm{a}_{d-i}(P)\ge\binom{d}{i}\left(\mathrm{a}_d(P)+1\right),$$

where equality holds for instance for the cross-polytopes $C^\star_d(2\,l-1)=\mathrm{conv}\{\pm l\,e_1, \pm e_i : 2\le i\le d\}$, $l\in\mathbb{N}$, with $2l-1$ interior lattice points. It is also conjectured that these cross-polytopes have minimal volume among all 0-symmetric lattice polytopes with a given number of interior lattice points. The maximal volume of those polytopes is known by the work of Blichfeldt and van der Corput (cf. [9, p. 51]) and, for instance, the maximum is attained by the boxes $Q_d(2\,l-1)=\{|x_1|\le l,\,|x_i|\le 1,\,2\le i\le d\}$ with $2\,l-1$ interior points. By the Examples 1.6 and 1.7 we can easily calculate the Ehrhart series of these boxes.

Example 1.8. *Let* $l\in\mathbb{N}$*. Then, for* $0\le i\le d$*,*

$$\mathrm{a}_i(Q_d(2\,l-1))=(2\,l\,i+1)\,a_i(C_{d-1})+(2\,l(d-i+1)-1)\ a_{i-1}(C_{d-1}).$$

It is quite tempting to conjecture that the box $Q_d(2\,l-1)$ maximizes $\mathrm{a}_i(P)+\mathrm{a}_{d-i}(P)$ for 0-symmetric polytope with $2\,l-1$ interior lattice points. In the 2-dimensional case this follows easily from a result of Paul Scott [16] which implies that $\mathrm{a}_1(P)\le 6\,l=\mathrm{a}_1(Q_2(2\,l-1))$ for any 0-symmetric convex lattice polygon with $2\,l-1$ interior lattice points. In fact, the result of Scott was recently generalized by Jaron Treutlein [19] to all degree 2 polytopes.

Theorem 1.9 (Treutlein)**.** *Let* $P\in\mathcal{P}^d$ *of degree 2 and let* $\mathrm{a}_i=\mathrm{a}_i(P)$*. Then*

$$\mathrm{a}_1\le\begin{cases}7, & \text{if } \mathrm{a}_2=1,\\ 3\,\mathrm{a}_2+3, & \text{if } \mathrm{a}_2\ge 2.\end{cases}\tag{1.8}$$

In Section 3 we will show that these conditions indeed classify all $h^\star$-polynomials of degree 2.

Proposition 1.10. *Let* $f(z)=\mathrm{a}_2\,z^2+\mathrm{a}_1\,z+1$*,* $\mathrm{a}_i\in\mathbb{N}$*, satisfying the inequalities in* (1.8)*. Then* f *is the* $h^\star$*-polynomial of a lattice polytope.*

Concerning lower bounds on the coefficients $\mathrm{g}_i(P)$ for 0-symmetric polytopes P we only know, except the trivial case $i = d$, a lower bound on $\mathrm{g}_{d-1}(P)$ (cf. (1.5)). Namely

$$\mathrm{g}_{d-1}(P) \geq \mathrm{g}_{d-1}(C_d^\star) = \frac{2^{d-1}}{(d-1)!},$$

where $C_d^\star = \mathrm{conv}\{\pm e_i : 1 \leq i \leq d\}$ denotes the regular cross-polytope. This follows immediately from a result of Richard P. Stanley [18, Theorem 3.1] on the h-vector of "symmetric" Cohen-Macaulay simplicial complex.

Motivated by a problem in [12] we study in the last section also the related question to bound the surface area $\mathrm{F}(P)$ of a lattice polytope P. In contrast to the $\mathrm{g}_i(P)$'s the surface area is not invariant under unimodular transformations. In order to describe our result we denote by T_d the standard simplex $T_d = \mathrm{conv}\{0, e_1, \dots, e_d\}$.

Proposition 1.11. *Let $P \in \mathcal{P}^d$. Then*

$$\mathrm{F}(P) \geq \begin{cases} \mathrm{F}(C_d^\star) = \frac{2^d}{d!}\, d^{\frac{3}{2}}, \text{ if } P = -P, \\ \mathrm{F}(T_d) = \frac{d+\sqrt{d}}{(d-1)!}, \text{ otherwise} . \end{cases}$$

The paper is organized as follows. In the next section we give the proof of our main Theorem 1.1. Then, in Section 3, we prove the Lemmas 1.3 and 1.5 and show how the Ehrhart series in the Examples 1.4 and 1.6 can be deduced. Moreover, we will give the proof of Proposition 1.10. Finally, in the last section we provide a proof of Proposition 1.11 which in the symmetric cases is based on a isoperimetric inequality for cross-polytopes (cf. Lemma 4.1).

2. Lower bounds on $\mathrm{g}_i(P)$

In the following we denote for an integer r and a polynomial $f(x)$ the r-th coefficient of $f(x)$, i.e. the coefficient of x^r, by $f(x)|_r$. Before proving Theorem 1.1 we need some basic properties of the numbers $C_{r,i}^d$ and $M_{r,d}$ defined in the introduction (see (1.3)). We begin with some special cases.

Proposition 2.1. *Let $d \geq 3$. Then $M_{0,d} = 0$, $M_{d,d} = 1$ and*

$$\text{i) } M_{1,d} = C_{1,d-2}^d = -(d-2)!,$$

$$\text{ii) } M_{2,d} = C_{2,d-2}^d = (d-2)! + (-1)^d \mathrm{stirl}(d-1,2),$$

$$\text{iii) } M_{d-1,d} = C_{d-1,1}^d = -\frac{d(d-3)}{2},$$

$$\text{iv) } M_{d-2,d} = \begin{cases} C^d_{d-2,\frac{d-1}{2}} = -\frac{1}{4}\binom{d+1}{3}, & \text{if } d \text{ odd}, \\ C^d_{d-2,\frac{d}{2}} = -\frac{1}{4}\binom{d}{3}, & \text{if } d \text{ even}. \end{cases}$$

Proof. The cases $M_{0,d}$ and $M_{d,d}$ are trivial. Since $C_{r,l}^d$ is the $(d-r)$-th elementary symmetric function of $\{l, l-1, \dots, l-(d-1)\}$ we have $C_{1,i}^d = (-1)^{d-i-1}\, i!\,(d-$

$i-1)!$ and

$$M_{1,d} = \min\{C^d_{1,i} : 1 \le i \le d-2\} = C^d_{1,d-2} = -(d-2)!$$

In the case $r=2$ we obtain by elementary calculations that

$$\begin{aligned} C^d_{2,i} &= i!\,\mathrm{stirl}(d-i,2) + (-1)^d\,(d-i-1)!\,\mathrm{stirl}(i+1,2) \\ &= i!\,(d-i-1)!\,(-1)^{d-i}\left(\sum_{k=1}^{d-i-1}\frac{1}{k} - \sum_{k=1}^{i}\frac{1}{k}\right), \end{aligned}$$

from which we conclude $M_{2,d} = C^d_{2,d-2} = (d-2)! + (-1)^d\mathrm{stirl}(d-1,2)$.

For iii) we note that

$$C^d_{d-1,i} = \sum_{j=i-(d-1)}^{i} j = -\frac{d}{2}(d-1-2\,i),$$

and so $M_{d-1,d} = C^d_{d-1,1}$. Finally, for the value of $M_{d-2,d}$ we first observe that

$$\begin{aligned} C^d_{d-2,i} - C^d_{d-2,i-1} &= (z+i)\,(z+i-1)\cdot\ldots\cdot(z+i-(d-1))\big|_{d-2} \\ &\quad - (z+i-1)\cdot\ldots(z+i-(d-1))\,(z+i-d)\big|_{d-2} \\ &= \sum_{j=-d+i+1}^{i-1} j\,(i-(-d+i)) = d\sum_{j=-d+i+1}^{i-1} j \\ &= d\frac{(d-1)(-d+2\,i)}{2}. \end{aligned}$$

Thus the function $C^d_{d-2,i}$ is decreasing in $0 \le i \le \lfloor d/2 \rfloor$ and increasing in $\lfloor d/2 \rfloor \le i \le d$. So it takes its minimum at $i = \lfloor d/2 \rfloor$. First let us assume that d is odd. Then

$$\begin{aligned} M_{d-2,d} = C^d_{d-2,\frac{d-1}{2}} &= d!\left.\binom{z+(d-1)/2}{d}\right|_{d-2} \\ &= z\,(z^2-1)\,(z^2-4)\cdot\ldots\cdot(z^2-((d-1)/2)^2)\Big|_{d-2} = -\sum_{i=0}^{(d-1)/2} i^2 \\ &= -\frac{1}{4}\binom{d+1}{3}. \end{aligned}$$

The even case can be treated similarly. □

In addition to the previous proposition we also need

Lemma 2.2.

i) $C^d_{r,i} = (-1)^{d-r}C^d_{r,d-1-i}$ *for* $0 \le i \le d-1$.
ii) *Let* $d \ge 3$. *Then* $M_{r,d} \le 0$ *for* $1 \le r \le d-1$, *and* $M_{r,d} = 0$ *only in the case* $d=3$ *and* $r=2$.

Proof. The first statement is just a consequence of the fact that $C^d_{r,l}$ is the $(d-r)$-th elementary symmetric function of $\{l, l-1, \ldots, l-(d-1)\}$. For ii) we first observe that the case $d=3$ follows directly from Proposition 2.1. Hence it remains to show that $M_{r,d} < 0$ for $d \geq 4$ and $1 \leq r \leq d-1$. On account of i) it suffices to prove this when $d-r$ is even and we will proceed by induction on d.

The case $d = 4$ is covered by Proposition 2.1. So let $d \geq 5$. By Proposition 2.1 i) we also may assume $r \geq 2$. It is easy to see that

$$C^d_{r,i} = (i - d + 1)\, C^{d-1}_{r,i} + C^{d-1}_{r-1,i}, \tag{2.1}$$

and by induction we may assume that there exists a $j \in \{1, \ldots, d-3\}$ with $C^{d-1}_{r-1,j} < 0$. Observe that $d-1-(r-1)$ is even. If $C^{d-1}_{r,j} \geq 0$ we obtain by (2.1) that $C^d_{r,j} < 0$ and we are done. So let $C^{d-1}_{r,j} < 0$. By part i) we know that

$$C^{d-1}_{r,j} = (-1)^{d-1-r} C^{d-1}_{r,d-2-j} \text{ and } C^{d-1}_{r-1,j} = (-1)^{d-r}\, C^{d-1}_{r-1,d-2-j}.$$

Since $d-r$ is even we conclude $C^{d-1}_{r,d-2-j} > 0$ and $C^{d-1}_{r-1,d-2-j} < 0$. Hence, on account of (2.1) we get $C^d_{r,d-2-j} < 0$ and so $M_{r,d} < 0$. □

Now we are able to give the proof of our main Theorem.

Proof of Theorem 1.1. We follow the approach of Betke and McMullen used in [4, Theorem 6]. By expanding the Ehrhart series at $z = 0$ one gets (see e.g. [3, Lemma 3.14])

$$\mathrm{G}_P(z) = \sum_{i=0}^{d} \mathrm{a}_i(P) \binom{z+d-i}{d}. \tag{2.2}$$

In particular, we have

$$\frac{1}{d!} \sum_{i=0}^{d} \mathrm{a}_i(P) = \mathrm{g}_d(P) = \mathrm{vol}(P). \tag{2.3}$$

For short, we will write a_i instead of $\mathrm{a}_i(P)$ and g_i instead of $\mathrm{g}_i(P)$. With this notation we have

$$\begin{aligned} d!\, \mathrm{g}_r &= d!\, \mathrm{G}_P(z)|_r = d! \sum_{i=0}^{d} \mathrm{a}_i \binom{z+d-i}{d}\Bigg|_r \\ &= C^d_{r,d} + (\mathrm{a}_1\, C^d_{r,d-1} + \mathrm{a}_d\, C^d_{r,0}) + \sum_{i=2}^{d-1} \mathrm{a}_i\, C^d_{r,d-i}. \end{aligned} \tag{2.4}$$

Since $C^d_{r,d-1} \geq 0$ we get with Lemma 2.2 i) that $C^d_{r,d-1} = |C^d_{r,0}|$. Together with $\mathrm{a}_1 = \mathrm{G}(P) - (d+1) \geq \mathrm{G}(\mathrm{int}(P)) = \mathrm{a}_d$ and $C^d_{r,d} = (-1)^{d-r}\mathrm{stirl}(d+1, r+1)$ we find

$$\begin{aligned} d!\, \mathrm{g}_r &\geq (-1)^{d-r}\mathrm{stirl}(d+1, r+1) + \sum_{i=2}^{d-1} \mathrm{a}_i\, C^d_{r,d-i} \\ &= (-1)^{d-r}\mathrm{stirl}(d+1, r+1) + \sum_{i=2}^{d-1} \mathrm{a}_i \left(C^d_{r,d-i} - M_{r,d}\right) + \sum_{i=1}^{d} \mathrm{a}_i\, M_{r,d} \\ &\qquad - (\mathrm{a}_1 + \mathrm{a}_d) M_{r,d} \\ &\geq (-1)^{d-r}\mathrm{stirl}(d+1, r+1) + (d!\,\mathrm{vol}(P) - 1) M_{r,d}, \end{aligned} \tag{2.5}$$

where the last inequality follows from the definition of $M_{r,d}$ and the non-positivity of $M_{r,d}$ (cf. Proposition 2.1 and Lemma 2.2 ii)). $\square$

We remark that for $d \geq 3$, $r \in \{1, \ldots, d-1\}$ and $(r,d) \neq (2,3)$ we can slightly improve the inequalities in Theorem 1.1, because in these cases we have $M_{r,d} < 0$ (cf. Lemma 2.2 ii)), and since $C^d_{r,d-1}$ is the $(d-r)$-th elementary symmetric function of $\{0, \ldots, d-1\}$ we also know $C^d_{r,d-1} > 0$ for $1 \leq r \leq d-1$. Hence we get (cf. (2.4) and (2.5))

$$\begin{aligned} d!\,\mathrm{g}_r &= C^d_{r,d} + \sum_{i=1}^{d} \mathrm{a}_i\, C^d_{r,d-i} \\ &= C^d_{r,d} + \mathrm{a}_1 \left(C^d_{r,d-1} - M_{r,d} \right) + \sum_{i=2}^{d} \left(C^d_{r,d-i} - M_{r,d} \right) + \sum_{i=1}^{d} \mathrm{a}_i\, M_{r,d} \\ &\geq (-1)^{d-r} \mathrm{stirl}(d+1, r+1) + 2\,\mathrm{a}_1(P) + (d!\,\mathrm{vol}(P) - 1) M_{r,d} \\ &= (-1)^{d-r} \mathrm{stirl}(d+1, r+1) - 2(d+1) + 2\mathrm{G}(P) + (d!\,\mathrm{vol}(P) - 1) M_{r,d}. \end{aligned}$$

Corollary 1.2 is an immediate consequence of Theorem 1.1 and Proposition 2.1.

Proof of Corollary 1.2. The inequalities just follow by inserting the value of $M_{r,d}$ given in Proposition 2.1 in the general inequality of Theorem 1.1. Here we also have used the identities

$$\mathrm{stirl}(d+1, 2) = (-1)^{d+1}\, d! \sum_{i=1}^{d} \frac{1}{i} \text{ and } \mathrm{stirl}(d+1, d-1) = \frac{3\,d+2}{4} \binom{d+1}{3}.$$

It remains to show that the inequalities are best possible for any volume. For $r = d-2$ we consider the simplex $T^{(m)}_d$ (cf. (1.7)) with $\mathrm{a}_0(T^{(m)}_d) = 1$, $\mathrm{a}_{\lceil d/2 \rceil}(T^{(m)}_d) = (m-1)$ and $\mathrm{a}_i(T^{(m)}_d) = 0$ for $i \notin \{0, \lceil d/2 \rceil\}$. Then $\mathrm{vol}(T^{(m)}_d) = m/d!$ and on account of Proposition 2.1 we have equality in (2.4) and (2.5).

For $r = 1, 2$ and $d \geq 4$ we consider the $(d-4)$-fold pyramid $\tilde{T}^{(m)}_d$ over $T^{(m)}_4$ given by $\tilde{T}^{(m)}_d = \mathrm{conv}\{T^{(m)}_4, e_5, \ldots, e_d\}$. Then $\mathrm{vol}(\tilde{T}^{(m)}_d) = m/d!$ and in view of (1.7) and [3, Theorem 2.4] we obtain

$$\mathrm{a}_0(\tilde{T}^{(m)}_d) = 1,\ \mathrm{a}_2(\tilde{T}^{(m)}_d) = m-1 \text{ and } \mathrm{a}_i(\tilde{T}^{(m)}_d) = 0,\ i \notin \{0, 2\}.$$

Again, by Proposition 2.1 we have equality in (2.4) and (2.5). $\square$

3. Ehrhart series of some special polytopes

We start with the short proof of Lemma 1.3.

Proof of Lemma 1.3. Since

$$\mathrm{Ehr}_P(z)\,\mathrm{Ehr}_Q(z) = \sum_{k \geq 0} \left(\sum_{m+l=k} \mathrm{G}_P(m) \mathrm{G}_Q(l) \right) z^k,$$

it suffices to prove that the Ehrhart polynomial $\mathrm{G}_{P\star Q}(k)$ of the lattice polytope $P\star Q\in\mathcal{P}^{p+q+1}$ is given by

$$\mathrm{G}_{P\star Q}(k)=\sum_{m+l=k}\mathrm{G}_P(m)\mathrm{G}_Q(l).$$

This, however, follows immediately from the definition since

$$k\,(P\star Q)=\{\lambda\,(x,o_q,0)^\intercal+(k-\lambda)\,(o_p,y,1)^\intercal : x\in P,\ y\in Q, 0\leq\lambda\leq k\}\,.$$

□

Example 1.4 in the introduction shows an application of this construction. For Example 1.6 we need Lemma 1.5.

Proof of Lemma 1.5. With $w=z^{\frac{1}{k}}$ we may write

$$\frac{1}{k}\sum_{i=0}^{k-1}\mathrm{Ehr}_P(\zeta^i\,w)=\frac{1}{k}\sum_{i=0}^{k-1}\sum_{m\geq 0}\mathrm{G}_P(m)(\zeta^i\,w)^m=\frac{1}{k}\sum_{m\geq 0}\mathrm{G}_P(m)w^m\sum_{i=0}^{k-1}\zeta^{i\,m}.$$

Since ζ is a k-th root of unity the sum $\sum_{i=0}^{k-1}\zeta^{i\,m}$ is equal to k if m is a multiple of k and otherwise it is 0. Thus we obtain

$$\frac{1}{k}\sum_{i=0}^{k-1}\mathrm{Ehr}_P(\zeta^i\,w)=\sum_{m\geq 0}\mathrm{G}_P(m\,k)w^{m\,k}=\sum_{m\geq 0}\mathrm{G}_{k\,P}(m)z^m=\mathrm{Ehr}_{k\,P}(z).$$

□

As an application of Lemma 1.5 we calculate the Ehrhart series of the cube C_d (cf. Example 1.6). Instead of C_d we consider the translated cube $2\,\tilde{C}_d$, where $\tilde{C}_d=\{x\in\mathbb{R}^d : 0\leq x_i\leq 1,\ 1\leq i\leq d\}$. In [3, Theorem 2.1] it was shown that $\mathrm{a}_i(\tilde{C}_d)=A(d,i+1)$ where $A(d,i)$ denotes the Eulerian numbers. Setting $w=\sqrt{z}$ Lemma 1.5 leads to

$$\begin{aligned}
\mathrm{Ehr}_{C_d}(z)&=\frac{1}{2}\left(\mathrm{Ehr}_{\tilde{C}_d}(w)+\mathrm{Ehr}_{\tilde{C}_d}(-w)\right)\\
&=\frac{1}{2}\left(\frac{\sum_{i=1}^d A(d,i)\,w^{i-1}}{(1-w)^{d+1}}+\frac{\sum_{i=1}^d A(d,i)\,(-w)^{i-1}}{(1+w)^{d+1}}\right)\\
&=\frac{1}{2}\frac{1}{(1-z)^{d+1}}\left(\sum_{i=1}^d A(d,i)\,w^{i-1}\,(1+w)^{d+1}\right.\\
&\qquad\qquad\left.+\sum_{i=1}^d A(d,i)\,(-w)^{i-1}\,(1-w)^{d+1}\right)\\
&=\frac{1}{(1-z)^{d+1}}\left(\sum_{i=1}^d A(d,i)\sum_{j=0,\ i+j-1\text{ even}}^{d+1}\binom{d+1}{j}w^{i+j-1}\right)
\end{aligned}$$

Substituting $2\,l = i + j - 1$ gives

$$\begin{aligned}
\mathrm{Ehr}_{C_d}(z) &= \frac{1}{(1-z)^{d+1}} \left(\sum_{l=0}^{d} \sum_{i=2\,l-d}^{2\,l+1} \binom{d+1}{2\,l+1-i} A(d,i)\, w^{2\,l} \right) \\
&= \frac{1}{(1-z)^{d+1}} \left(\sum_{l=0}^{d} z^l \sum_{j=0}^{d+1} \binom{d+1}{j} A(d, 2\,l+1-j) \right),
\end{aligned}$$

which explains the formula in Example 1.6.

In order to calculate in general the Ehrhart series of the prism $P = \{(x, x_d)^\intercal : x \in Q, x_d \in [0, m]\}$ where $Q \in \mathcal{P}^{d-1}$, $m \in \mathbb{N}$ (cf. Example 1.7), we use the differential operator T defined by $z\frac{\mathrm{d}}{\mathrm{dz}}$. Considered as an operator on the ring of formal power series we have (cf. e.g. [3, p. 28])

$$\sum_{k \geq 0} f(k)\, z^k = f(T) \frac{1}{1-z} \tag{3.1}$$

for any polynomial f. Since $\mathrm{G}_P(k) = (m\,k+1)\,\mathrm{G}_Q(k)$ we deduce from (3.1)

$$\mathrm{Ehr}_P(z) = (m\,T+1)\mathrm{Ehr}_Q(z) = mz\frac{\mathrm{d}}{\mathrm{dz}}\mathrm{Ehr}_Q(z) + \mathrm{Ehr}_Q(z).$$

Thus

$$\begin{aligned}
\mathrm{Ehr}_P(z) &= m\,z\,\frac{\sum_{i=0}^{d-1} i\,\mathrm{a}_i(Q) z^{i-1}(1-z) + \sum_{i=0}^{d-1} d\,\mathrm{a}_i(Q)\, z^i}{(1-z)^{d+1}} + \frac{\sum_{i=0}^{d-1} \mathrm{a}_i(Q)\, z^i}{(1-z)^d} \\
&= \frac{\sum_{i=0}^{d-1} (m\,i+1)\mathrm{a}_i(Q) z^i (1-z) + \sum_{i=0}^{d-1} m\,d\,\mathrm{a}_i(Q) z^{i+1}}{(1-z)^{d+1}} \\
&= \frac{1}{(1-z)^{d+1}} \sum_{i=1}^{d} \left((m\,i+1)\mathrm{a}_i(Q) + (m(d-i+1)-1)\,\mathrm{a}_{i-1}(Q)\right) z^i,
\end{aligned}$$

which is the formula in Example 1.7.

Finally, we come to the classification of $h^\star$-polynomials of degree 2.

Proof of Proposition 1.10. We recall that $\mathrm{a}_1(P) = \mathrm{G}(P) - (d+1)$ and $\mathrm{a}_d(P) = \mathrm{G}(\mathrm{int}(P))$ for $P \in \mathcal{P}^d$. In the case $\mathrm{a}_2 = 1$, $\mathrm{a}_1 = 7$ the triangle $\mathrm{conv}\{0, 3\,e_1, 3\,e_2\}$ has the desired $h^\star$-polynomial. Next we distinguish two cases:

i) $\mathrm{a}_2 < \mathrm{a}_1 \leq 3\,\mathrm{a}_2 + 3$. For integers k, l, m with $0 \leq l, k \leq m+1$ let $P \in \mathcal{P}^2$ given by $P = \mathrm{conv}\{0, l\,e_1, e_2 + (m+1)\,e_1, 2\,e_2, 2\,e_2 + k\,e_1\}$. Then it is easy to see that $\mathrm{a}_2(P) = m$ and P has $k + l + 4$ lattice points on the boundary. Thus $\mathrm{a}_1(P) = k + l + m + 1$.

ii) $\mathrm{a}_1 \leq \mathrm{a}_2$. For integers l, m with $0 \leq l \leq m$ let $P \in \mathcal{P}^3$ given by $P = \mathrm{conv}\{0, e_1, e_2, -l\,e_3, e_1 + e_2 + (m+1)\,e_3\}$. The only lattice points contained in P are the vertices and the lattice points on the edge $\mathrm{conv}\{0, -l\,e_3\}$. Thus $\mathrm{a}_3(P) = 0$ and $\mathrm{a}_1(P) = l$. On the other hand, since $(l+m+1)/6 = \mathrm{vol}(P) = (\sum_{i=0}^{3} \mathrm{a}_i(P))/6$ (cf. (2.3)) it is $\mathrm{a}_2(P) = m$.

$\square$

4. 0-symmetric lattice polytopes

In order to study the surface area of 0-symmetric polytopes we first prove an isoperimetric inequality for the class of cross-polytopes.

Lemma 4.1. *Let $v_1, \dots, v_d \in \mathbb{R}^d$ be linearly independent and let $C = \mathrm{conv}\{\pm v_i : 1 \le i \le d\}$. Then*

$$\frac{\mathrm{F}(C)^d}{\mathrm{vol}(C)^{d-1}} \ge \frac{2^d}{d!}\, d^{\frac{3}{2}d},$$

and equality holds if and only if C is a regular cross-polytope, i.e., $v_1, \dots, v_d$ form an orthogonal basis of equal length.

Proof. Without loss of generality let $\mathrm{vol}(C) = 2^d/d!$. Then we have to show

$$\mathrm{F}(C) \ge \frac{2^d}{d!} d^{\frac{3}{2}}. \tag{4.1}$$

By standard arguments from convexity (see e.g. [10, Theorem 6.3]) the set of all 0-symmetric cross-polytopes with volume $2^d/d!$ contains a cross-polytope $C^\star = \mathrm{conv}\{\pm w_1, \dots, \pm w_d\}$, say, of minimal surface area. Suppose that some of the vectors are not pairwise orthogonal, for instance, w_1 and w_2. Then we apply to $C^\star$ a Steiner-Symmetrization (cf. e.g. [10, pp. 169]) with respect to the hyperplane $H = \{x \in \mathbb{R}^d : w_i\, x = 0\}$. It is easy to check that the Steiner-symmetral of $C^\star$ is again a cross-polytope $\tilde{C}^*$, say, with $\mathrm{vol}(\tilde{C}^\star) = \mathrm{vol}(C^\star)$ (cf. [10, Proposition 9.1]). Since $C^\star$ was not symmetric with respect to the hyperplane H we also know that $\mathrm{F}(\tilde{C}^*) < \mathrm{F}(C^\star)$ which contradicts the minimality of $C^\star$ (cf. [10, p. 171]).

So we can assume that the vectors w_i are pairwise orthogonal. Next suppose that $\|w_1\| > \|w_2\|$, where $\|\cdot\|$ denotes the Euclidean norm. Then we apply Steiner-Symmetrization with respect to the hyperplane H which is orthogonal to $w_1 - w_2$ and bisecting the edge $\mathrm{conv}\{w_1, w_2\}$. As before we get a contradiction to the minimality of $C^\star$.

Thus we know that w_i are pairwise orthogonal and of same length. By our assumption on the volume we get $\|w_i\| = 1$, $1 \le i \le d$, and it is easy to calculate that $\mathrm{F}(C^\star) = (2^d/d!)d^{3/2}$. So we have

$$\mathrm{F}(C) \ge \mathrm{F}(C^\star) = \frac{2^d}{d!} d^{\frac{3}{2}},$$

and by the foregoing argumentation via Steiner-Symmetrizations we also see that equality holds if and only C is a regular cross-polytope generated by vectors of unit-length. □

The determination of the minimal surface area of 0-symmetric lattice polytopes is an immediate consequence of the lemma above, whereas the non-symmetric case does not follow from the corresponding isoperimetric inequality for simplices.

Proof of Proposition 1.11. Let $P \in \mathcal{P}^d$ with $P = -P$. Then P contains a 0-symmetric lattice cross-polytope $C = \mathrm{conv}\{\pm v_i : 1 \le i \le d\}$, say, and by the monotonicity of the surface area and Lemma 4.1 we get

$$\mathrm{F}(P) \geq \mathrm{F}(C) \geq \left(\frac{2^d}{d!}\right)^{\frac{1}{d}} d^{\frac{3}{2}} \operatorname{vol}(C)^{\frac{d-1}{d}}. \tag{4.2}$$

Since $v_i \in \mathbb{Z}^d$, $1 \leq i \leq d$, we have $\operatorname{vol}(C) = (2^d/d!)|\det(v_1, \ldots, v_d)| \geq 2^d/d!$, which shows by (4.2) the 0-symmetric case.

In the non-symmetric case we know that P contains a lattice simplex $T = \{x \in \mathbb{R}^d : a_i\, x \leq b_i, 1 \leq i \leq d+1\}$, say. Here we may assume that $a_i \in \mathbb{Z}^n$ are primitive, i.e., $\operatorname{conv}\{0, a_i\} \cap \mathbb{Z}^n = \{0, a_i\}$, and that $b_i \in \mathbb{Z}$. Furthermore, we denote the facet $P \cap \{x \in \mathbb{R}^d : a_i\, x = b_i\}$ by F_i, $1 \leq i \leq d+1$. With these notations we have $\det(\operatorname{aff} F_i \cap \mathbb{Z}^n) = \|a_i\|$ (cf. [14, Proposition 1.2.9]). Hence there exist integers $k_i \geq 1$ with

$$\operatorname{vol}_{d-1}(F_i) = k_i \frac{\|a_i\|}{(d-1)!}, \tag{4.3}$$

and so we may write

$$\mathrm{F}(P) \geq \mathrm{F}(T) = \sum_{i=1}^{d+1} \operatorname{vol}_{d-1}(F_i) \geq \frac{1}{(d-1)!} \sum_{i=1}^{d+1} \|a_i\|.$$

We also have $\sum_{i=1}^{d+1} \operatorname{vol}_{d-1}(F_i) a_i/\|a_i\| = 0$ (cf. e.g. [10, Theorem 18.2]) and in view of (4.3) we obtain $\sum_{i=1}^{d+1} k_i\, a_i = 0$. Thus, since the $d+1$ lattice vectors a_i are affinely independent we can find for each index $j \in \{1, \ldots, d\}$ at least two vectors a_{i_1} and a_{i_2} having a non-trivial j-th coordinate. Hence

$$\sum_{i=1}^{d+1} \|a_i\|^2 \geq 2\, d. \tag{4.4}$$

Together with the restrictions $\|a_i\| \geq 1$, $1 \leq i \leq d+1$, it is easy to argue that $\sum_{i=1}^{d+1} \|a_i\|$ is minimized if and only if d norms $\|a_i\|$ are equal to 1 and one is equal to $\sqrt{d}$. For instance, the intersection of the cone $\{x \in \mathbb{R}^{d+1} : x_i \geq 1, 1 \leq i \leq d+1\}$ with the hyperplane $H_\alpha = \{x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i = \alpha\}$, $\alpha \geq d+1$, is the d-simplex $T(\alpha)$ with vertices given by the permutations of the vector $(1, \ldots, 1, \alpha - d)^\intercal$ of length $\sqrt{d + (\alpha-d)^2}$. Therefore, a vertex of that simplex is contained in $\{x \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} x_i^2 \geq 2d\}$ if $\alpha \geq d + \sqrt{d}$. In other words, we always have

$$\sum_{i=1}^{d+1} \|a_i\| \geq d + \sqrt{d},$$

which gives the desired inequality in the non-symmetric case (cf. (4.3)). □

We remark that the proof also shows that equality in Proposition 1.11 holds if and only if P is the o-symmetric cross-polytope $C_d^\star$ or the simplex T_d (up to lattice translations).

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