SETS NON-THIN AT ∞ IN $\mathbb{C}^m$, AND THE GROWTH OF SEQUENCES OF ENTIRE FUNCTIONS OF GENUS ZERO

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Abstract. In this paper we define the notion of non-thin at ∞ as follows: Let E be a subset of $\mathbb{C}^m$. For any $R>0$ define $E_R = E\cap\{z\in\mathbb{C}^m : |z|\leq R\}$. We say that E is non-thin at ∞ if

$$\lim_{R\to\infty} V_{E_R}(z)=0$$

for all $z\in\mathbb{C}^m$, where V_E is the pluricomplex Green function of E.

This definition of non-thinness at ∞ has good properties: If $E\subset\mathbb{C}^m$ is non-thin at ∞ and A is pluripolar then $E\backslash A$ is non-thin at ∞, if $E\subset\mathbb{C}^m$ and $F\subset\mathbb{C}^n$ are closed sets non-thin at ∞ then $E\times F\subset\mathbb{C}^m\times\mathbb{C}^n$ is non-thin at ∞ (see Lemma 1).

Then we explore the properties of non-thin at ∞ sets and apply this to extend the results in [10] and [4].

1. Introduction

Fix $m\in\boldsymbol{N}$, let $\mathbb{C}^m$ be the usual complex m-dimensional complex space. Before going into the main points, we recall some facts about the potential theory in $\mathbb{C}^m$. Let U be an open subset of $\mathbb{C}^m$. A function $u: U\to[-\infty,\infty)$ (see [5]) is called PSH in U (written $u\in PSH(U)$) if u is upper-semicontinuous and when restricted to any complex line $L\simeq\mathbb{C}$ then u is subharmonic (see [5]).

A $PSH(\mathbb{C}^m)$ function u is said to be of minimal growth if

$$u(z)-\log|z|\leq O(1),\ \text{as}\ |z|\to\infty,$$

here $|z|$ is the usual Euclide norm of an element $z\in\mathbb{C}^m$. We denote the set of all such functions by $\mathbb{L}$.

Let E be a subset in $\mathbb{C}^m$. Then the pluricomplex Green function of the set E with pole at infinity (see [5], page 184) is

$$V_E(z)=\sup\{u(z):\ u\in\mathbb{L},\ u|_E\leq 0\}\ (z\in\mathbb{C}^m).$$

V_E is also called the Siciak extremal function of the set E. By definition we see that $V_E\geq 0$.

For any function $u:\mathbb{C}^m\to[-\infty,\infty)$ we define its upper-semicontinuos regularization u^* (see [5]) by

$$u^*(z)=\limsup_{\zeta\to z} u(\zeta).$$

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Let $V_E^*(z)$ be the regularization of the Green function $V_E(z)$. It is well-known (see [5]) that there are only three cases:

Case 1: $V_E^*(z) \equiv +\infty$. Then E is pluripolar.

Case 2: $V_E^*(z) \in \mathbb{L}$, and $V_E^* \not\equiv 0$.

Case 3: $V_E^*(z) \equiv 0$.

When $m = 1$ then $V_E^*(z) \equiv 0$ iff the set E is non-thin at ∞, or equivalently the set $E^* = \{z : 1/z \in E\}$ is non-thin at 0 (see [10], [3]). Recall that a subset E of $\mathbb{C}^m$ is pluri-thin (or thin for brevity) at a point $a \in \mathbb{C}^m$ if either a is not a limit point of E or there is a neighborhood U of a and a function $u \in PSH(U)$ such that

$$\limsup_{z \to a,\ z \in E \setminus a} u(z) < u(a).$$

If E is not thin then it is called non-thin.

In [10] the authors proved the following result (see also [3], page 270)

Proposition 1. *Let E be a closed subset of $\mathbb{C}$. Then the following four statements are equivalent*

1. E is non-thin at ∞.

2. For any $z \in \mathbb{C}$ we have

$$\lim_{R \to \infty} V_{E_R}(z) = 0,$$

where we define $E_R = E \cap \{z \in \mathbb{C} : |z| \leq R\}$.

3. If sequences (P_n) of polynomials and $k_n \geq deg(P_n)$ satisfy

$$\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in E$, then

$$\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}$.

4. $V_E(z) \equiv 0$.

When $m > 1$, as to our knowledge, there was no conception of non-thin at ∞. The difficulty here is because we can not reduce the definition of non-thinness of E at ∞ to the non-thinness at 0 of some other sets E^*, as it was when $m = 1$. However, Proposition 1 suggests a way to define non-thinness at ∞ in higher dimensions:

Definition 1. *Let E be a subset of $\mathbb{C}^m$. For any $R > 0$ define $E_R = E \cap \{z \in \mathbb{C}^m : |z| \leq R\}$. We say that E is non-thin at ∞ if*

$$\lim_{R \to \infty} V_{E_R}(z) = 0$$

for all $z \in \mathbb{C}^m$.

If E is not non-thin at ∞ then we say E is thin at ∞.

This definition of non-thinness at ∞ has good properties: If $E \subset \mathbb{C}^m$ is non-thin at ∞ and A is pluripolar then $E \backslash A$ is non-thin at ∞, if $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$ are closed sets non-thin at ∞ then $E \times F \subset \mathbb{C}^m \times \mathbb{C}^n$ is non-thin at ∞ (see Lemma 1).

Our first result in this paper is the following, which is an analog to the case $m = 1$:

Theorem 2. *Let E be a closed subset of $\mathbb{C}^m$. Then the following two statments are equivalent:*

1. E is non-thin at ∞.

2. If sequences (P_n) of polynomials and $k_n \geq deg(P_n)$ satisfy

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in E$, then

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}^m$.

If E is non-thin at ∞ then it is easy to see that $V_E \equiv 0$. The converse is not true for $m \geq 2$, by the following example (which essentially is the same as Example 1.1 in [3]).

Example 1. *Let E be a subset of $\mathbb{C}^2$ defined by*

$$E = \{(z_1, z_2) \in \mathbb{C}^2 : \ |z_2| \leq 1\} \bigcup \{(z_1, z_2) \in \mathbb{C}^2 : \ z_1 = 0\}.$$

Then $V_E \equiv 0$ but V_E is thin at ∞.

Proof. Use arguments in Example 1.1 in [3], it is easy to see that $V_E \equiv 0$.

If E was non-thin at ∞ then since $A = \{(z_1, z_2) \in \mathbb{C}^2 : \ z_1 = 0\}$ is pluripolar, by Lemma 1 we should have $E\backslash A$ is non-thin at ∞. In particular we should have $V_{E\backslash A} \equiv 0$. However as computed in [3] we have $V_{E\backslash A}(z) = \log^+ |z|$, which is a contradiction.

Hence E is thin at ∞. □

The following result gives a characterization of sets E with $V_E \equiv 0$, which also shows that the non-thin at ∞ sets are not rare.

Theorem 3. *Let E be a closed subset of $\mathbb{C}^m$. Then $V_E \equiv 0$ iff any open neighborhood of E is non-thin at ∞.*

In [10], the authors applied the result of Proposition 1 to various problems of convergence of sequences of polynomials in one complex variable. In [4] we extended the results to sequences of entire functions of genus zero of one complex variable. With our definition of non-thinness at ∞ for a set $E \subset \mathbb{C}^m$ here, we can extend Theorems 1 and 3 in [4] to the case of entire functions of genus zero of several complex variables, under appropriate assumptions.

If $P(z)$ is an entire function of genus zero of one complex variable then we have a representation of P by (see [9])

$$P(z) = az^{\alpha} \prod_j (1 - \frac{z}{z_j}). \tag{1.1}$$

Moreover

$$\sum_j \frac{1}{|z_j|} < \infty.$$

In higher dimensions we still have a canonical representation of any entire function of genus zero $P(z_1, \ldots, z_m)$ (see Proposition 3.16 in [7], see also [12]). However in this paper we will use instead the reduction to one variable complex as described below.

Let $P(z_1,\ldots,z_m)$ be an entire function of genus zero in $\mathbb{C}^m$. Then for any $\lambda\in\mathbb{P}^{m-1}$ (here $\mathbb{P}^{m-1}$ means the complex projective space of dimension $m-1$) which represents a pluricomplex direction in $\mathbb{C}^m$, we define a one complex variable function

$$P_\lambda(w)=P(w\lambda)$$

where $w\in\mathbb{C}$. Here we understand the expression $w\lambda$ as follows: $w\lambda := w\widetilde{\lambda}\in\mathbb{C}^m$ where $\widetilde{\lambda}\in S^{2m-1}=\{z\in\mathbb{C}^m : |z|=1\}$ is a fixed representative of λ. Then $P_\lambda(w)$ is also an entire function of genus zero of one complex variable hence we can use (1.1) to write

$$P_\lambda(w)=a_\lambda w^{\alpha_\lambda}\prod_j(1-\frac{w}{w_{\lambda,j}}).$$

We define the counting functions of P_λ and P as follows: If $t>0$ then

$$\begin{aligned}\eta_P(t,\lambda) &= \text{the number of elements of the set}\{w\in C : P_\lambda(w)=0,\ |w|\leq t\},\\ \eta_P(t) &= \int_{\mathbb{P}^{m-1}} n_P(t,\lambda)d\lambda\end{aligned}$$

where we integrate using the standard metric on $\mathbb{P}^{m-1}$.

The following result generalizes Theorem 1 in [4]

Theorem 4. *Let $E\subset\mathbb{C}^m$ be non-thin at ∞.*

Let $P_n:\mathbb{C}^m\to\mathbb{C}$ be a sequence of entire functions of genus zero, and let (k_n) be a sequence of positive numbers greater than 1. For $\lambda\in\mathbb{P}^{m-1}$ let us define the function $P_{n,\lambda}:\mathbb{C}\to\mathbb{C}$ by

$$P_{n,\lambda}(w)=P_n(w\lambda).$$

Let

$$P_{n,\lambda}(w)=a_{n,\lambda}w^{\alpha_{n,\lambda}}\prod_j(1-\frac{w}{w_{n,\lambda,j}}),$$

its representation, where

$$\sum_j\frac{1}{|w_{n,\lambda,j}|}<\infty.$$

Let $\eta_{P_n}(t,\lambda)$ and $\eta_{P_n}(t)$ be counting functions of P_n as defined above.

Assume that for any compact set $K\subset\mathbb{C}^m$ there exists N_0 such that the set

$$\{|P_n(z)|^{1/k_n}:\ n\geq N_0, z\in K\}\tag{1.2}$$

is bounded from above.

For each $\lambda\in\mathbb{P}^{m-1}$ assume that the following three conditions are satisfied:

$$\limsup_{n\to\infty}\frac{[\eta_{P_n}(1,\lambda)+\sum_{|w_{n,\lambda,j}|\geq 1}1/|w_{n,\lambda,j}|]}{k_n}<\infty.\tag{1.3}$$

$$\lim_{R\to\infty}\limsup_{n\to\infty}\frac{|\sum_{|w_{n,\lambda,j}|\geq R}1/w_{n,\lambda,j}|}{k_n}=0.\tag{1.4}$$

There exists a sequence $R_{n,\lambda}\to\infty$ such that

$$\limsup_{n\to\infty}\frac{\eta_{P_n}(R_{n,\lambda},\lambda)}{k_n}\leq\kappa<\infty,\tag{1.5}$$

where κ is independent of λ.

Then the following conclusion is true: If

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in E$, then we have

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}^m$.

We mention here some remarks about the result of Theorem 4:

1. Theorem 4 is not a direct consequence of the one-dimensional result in [4]. This is because although E is non-thin at ∞ in $\mathbb{C}^m$, we may have that $E \cap L$ is thin at ∞ in L (even $E \cap L$ may be the empty set) where L is a complex line in $\mathbb{C}^m$.

2. The way of considering the growth of the sequence (P_n) in $\mathbb{C}^m$ by considering the growth of the restricted sequence to any line L_λ is a natural approach.

3. Condition (1.2) is natural. By Theorem 2 in [4] conditions (1.4) and (1.5) (with κ may depend on λ) in Theorem 4 are satisfied in case we have condition (1.3),

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1,$$

for all $z \in \mathbb{C}^m$, and

$$\liminf_{n\to\infty} |P_n(0)|^{1/k_n} > 0.$$

Similarly, we can extend Theorems 3 in [4] by the same manner as that of Theorem 4. Because the statement of the result is rather long, we defer the statement of this result to Section 3 (see Theorem 5). Here we outline the notations and results we will use in the statement and the proof of Theorem 5.

Let $K \subset \mathbb{C}^m$ be compact. Then the Robin constant of K is defined as (see [8])

$$\gamma(K) = \limsup_{z\to\infty} [V_E^*(z) - \log|z|],$$

and the $\mathbb{C}^m$-capacity of K is

$$C(K) = e^{-\gamma(K)}. \tag{1.6}$$

Let $K \subset \{z \in \mathbb{C}^m : \ |z| \leq s\}$ be compact and non-pluripolar, where $s > 0$. By Theorem 2 in [14] and Poisson integration formula for harmonic functions (see Theorem 2.2.3 in [5]), there exists a constant $C_m > 0$ depending only on the dimension m such that

$$\sup_{|z|\leq s} V_K^*(z) \leq C_m(\log s + \log 2 - \log C(K)). \tag{1.7}$$

The following corollary is an analog of Remark 1 in [10]

Corollary 1. *Let $E \subset \mathbb{C}^m$ be closed. Let $0 < C_m < +\infty$ be the constant in (1.7). Then E is non-thin at ∞ if it satisfies the following condition: There exists $\beta > 0$ such that*

$$\limsup_{R\to\infty} \frac{\log C(E_R)}{\log R} \geq \beta > \frac{C_m - 1}{C_m},$$

where $C(E_R)$ is determined from formula (1.6).

Interesting questions may arise such as whether we can in fact get rid of the condition (1.2) or get rid of the constant κ in (1.5) so we have complete analogs with the results in [4], or whether we have a Wienner criterion for non-thin at ∞ in $\mathbb{C}^m$... We hope to return to these issues in a future paper.

This paper consists of three sections. In Section 2 we prove Theorems 2 and 3, as well as some other properties of sets non-thin at ∞. In Section 3 we prove Theorem 4 as well as the analog of Theorem 3 in [4].

2. Non-thin at ∞ sets in $\mathbb{C}^m$

In this section we explore some properties of sets which are non-thin at ∞ in $\mathbb{C}^m$.

Lemma 1. *a) Let E be a subset of $\mathbb{C}^m$. If E is non-thin at ∞ and A is pluripolar then $E\backslash A$ is non-thin at ∞.*

b) Let $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$. Then $E \times F \subset \mathbb{C}^m \times \mathbb{C}^n$ is non-thin at ∞ iff $E \subset \mathbb{C}^m$ and $F \subset \mathbb{C}^n$ are non-thin at ∞.

Proof. a) Define $F = E\backslash A$. For any $R > 0$ the set $E_R = E \cap \{z \in \mathbb{C}^m : |z| \leq R\}$ is bounded. Hence by Corollary 5.2.5 in [5] we have

$$V^*_{E_R} = V^*_{F_R}.$$

Fix a sequence $R_n \to \infty$, using the same argument as in the proof of Theorem 2 there exists a pluripolar set B such that

$$\lim_{n\to\infty} V_{F_{R_n}}(z) = \lim_{n\to\infty} V^*_{F_{R_n}}(z) = \lim_{n\to\infty} V^*_{E_{R_n}}(z) = \lim_{R\to\infty} V_{E_{R_n}}(z) = 0.$$

for all $z \in \mathbb{C}^m\backslash B$. Then a little more argument completes the proof of this part.

b) Let $R > 0$. Then one can check easily that

$$\max\{V_{E_R}(z), V_{F_R}(w)\} \leq V_{E_R\times F_R}(z, w) \leq V_{E_R}(z) + V_{F_R}(w).$$

where $z \in \mathbb{C}^m,\ w \in \mathbb{C}^n$. This completes the proof of case b). □

Now we proceed to proving Theorem 2.

Proof. In this proof fix a sequence $R_n \nearrow \infty$.

($2 \Rightarrow 1$) Since E is closed, for any $R > 0$ we have E_R is compact.

Fixed $z_0 \in \mathbb{C}^m$. By Siciak's theorem (see Theorem 5.1.7 in [5]) there exists a sequence of polynomials (P_n) of degree (k_n) such that

$$||P_n||^{1/k_n}_{E_{R_n}} \leq 1,$$

for all $n = 1, 2, \ldots$, and

$$\lim_{n\to\infty} \log |P_n(z_0)|^{1/k_n} = \lim_{n\to\infty} V_{E_{R_n}}(z_0). \tag{2.1}$$

Then for any $z \in E$ we have

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1.$$

Hence by assumption that Statement 2 is true, we have

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}^m$. In particular, with $z = z_0$ we get from (2.1) that

$$\lim_{n\to\infty} V_{E_{R_n}}(z_0) \leq 0.$$

Since $z_0 \in \mathbb{C}^m$ is arbitrary and obviously $V_E(z) \geq 0$ for all z, we get Statement 1.

(1 $\Rightarrow$ 2) We use the ideas in [10]. Assume that Statement 1 is true. Consider any sequence (P_n) of polynomials and $k_n \geq deg(P_n)$ such that

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1 \tag{2.2}$$

for all $z \in E$. For each n define

$$v_n(z) = \log |P_n(z)|^{1/k_n}.$$

Note that Statement 1 implies that for R large enough then E_R is non-pluripolar. Fix $R > 0$ such that E_R is non-pluripolar.

For any $h, l \in \boldsymbol{N}$ define

$$E_R^{h,l} = \bigcap_{n=h}^{\infty} \{z \in E_R :\ v_n(z) < \frac{1}{l}\}.$$

Then $E_R^{h,l} \subset E_R^{h+1,l}$ and from (2.2)

$$\bigcup_{h=1}^{\infty} E_R^{h.l} = E_R \tag{2.3}$$

for any $l \in \boldsymbol{N}$.

By definition of the pluricomplex Green function, for any $h, l \in \boldsymbol{N},\ n \geq h$ and $z \in \mathbb{C}^m$

$$v_n(z) \leq V_{E_R^{h,l}}(z) + \frac{1}{l} \leq V^*_{E_R^{h,l}}(z) + \frac{1}{l}.$$

Hence take the limsup of the above inequality as $n \to \infty$ we get

$$v(z) = \limsup_{n\to\infty} v_n(z) \leq V_{E_R^{h,l}}(z) + \frac{1}{l} \leq V^*_{E_R^{h,l}}(z) + \frac{1}{l},$$

for all $h, l \in \boldsymbol{N},\ z \in \mathbb{C}^m$. Take the limit of this inequality as $h \to \infty$, using (2.2), we see from Corollary 5.2.6 in [5] that

$$v(z) \leq \lim_{h\to\infty} V^*_{E_R^{h,l}}(z) + \frac{1}{l} = V^*_{E_R}(z) + \frac{1}{l},$$

for all $l \in \boldsymbol{N},\ z \in \mathbb{C}^m,\ R > 0$. Since l is arbitrary, we get

$$v(z) \leq V^*_{E_R}(z), \tag{2.4}$$

for all $z \in \mathbb{C}^m,\ R > 0$.

For each $n \in \boldsymbol{N}$ define

$$A_n = \{z \in \mathbb{C}^m :\ V_{E_{R_n}}(z) < V^*_{E_{R_n}}(z)\}$$

then A_n is pluripolar thus has Lebesgue measure zero, and $V_{E_{R_n}}(z) = V^*_{E_{R_n}}(z)$ for $z \in \mathbb{C}^m \backslash A_n$. Hence

$$A = \bigcup_{n=1}^{\infty} A_n$$

is also pluripolar and of Lebesgue measure zero, and we have $V_{E_{R_n}}(z) = V^*_{E_{R_n}}(z)$ for $z \in \mathbb{C}^m \backslash A$ and $n \in \boldsymbol{N}$. Hence for $z \in \mathbb{C}^m \backslash A$, apply (2.4) we get

$$v(z) \leq \lim_{n\to\infty} V^*_{E_{R_n}}(z) = \lim_{n\to\infty} V_{E_{R_n}}(z) = 0$$

for all $z \in \mathbb{C}^m \backslash A$. Since A is of Lebesgue measure zero, by definition of $v(z)$, we see that $v(z) \leq 0$ for all $z \in \mathbb{C}^m$. This completes the proof of Theorem 2. □

Now we prove Theorem 3.

Proof. (⇒) Let E be a closed subset of $\mathbb{C}^m$. Assume that $V_E \equiv 0$. We will show that any open neighborhood of E is non-thin at ∞. Let F be any open neighborhood of E. Then for any $R > 0$ since E_R is compact, F is open and contains E_R, we can find $1 > \epsilon = \epsilon_R > 0$ such that

$$E_{R,\epsilon} = \{z \in \mathbb{C}^m : \ dist(z, E_R) \leq \epsilon\} \subset F.$$

By Corollary 5.1.5 in [5] we have $V^*_{E_{R,\epsilon}}(z) = 0$ for $z \in E_{R,\epsilon} \supset E_R$. Now it is obvious that $E_{R,\epsilon} \subset F_{R+1}$ hence for $z \in E_R$

$$V^*_{F_{R+1}}(z) = 0. \tag{2.5}$$

Define

$$v(z) = \lim_{R\to\infty} V^*_{F_R}(z),$$

then v is the limit of a decreasing sequence of PSH functions hence v is itself PSH. By (2.5) for $z \in E$ we have $v(z) \equiv 0$. Thus by definition of the pluricomplex Green function

$$v(z) \leq V_E(z) = 0$$

for all $z \in \mathbb{C}^m$. This shows that F is non-thin at ∞.

(⇐) Assume that E is an arbitrary subset of $\mathbb{C}^m$ with $V_E \not\equiv 0$. Then $V_E(z_0) > 0$ for some $z_0 \in \mathbb{C}^m$, hence by definition of the pluricomplex Green function, there exist a function $u \in L$ such that $u(z) \leq 0$ for $z \in E$, and $u(z_0) > 0$. Define

$$F = \{z \in \mathbb{C}^m : \ u(z) < u(z_0)/2\}.$$

F is open because u is upper-semicontinuous, and $E \subset F$ because $u|_E \leq 0$ and $u(z_0) > 0$. Now $u(z) < u(z_0)/2$ for $z \in F$, hence we have

$$u(z) \leq V_F(z) + u(z_0)/2$$

for all $z \in \mathbb{C}^m$. In particular choose $z = z_0$ we have $V_F(z_0) \geq u(z_0)/2 > 0$, hence F is thin at ∞. □

Theorem 3 applied to Example 1 shows that any nonempty open cone in $\mathbb{C}^m$ is non-thin at ∞. The following result provide other sources of sets which are non-thin at ∞.

Remark 1. *Let Δ be a collection of complex lines L in $\mathbb{C}^m$ such that*

$$\bigcup_{L\in\Delta} L = \mathbb{C}^m.$$

Let E be a closed subset of $\mathbb{C}^m$ such that for each $L \in \Delta$ the set $E \cap L$ considered as a subset in the one-dimensional complex line L is non-thin at ∞. Then E is non-thin at ∞ as a subset in $\mathbb{C}^m$.

Proof. By Theorem 2 we need only to show that: If (P_n) is a sequence of polynomials, and $k_n \geq deg(P_n)$ such that

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1$$

for $z \in E$, then

$$\limsup_{n\to\infty} |P_n(z)|^{1/k_n} \leq 1$$

for all $z \in \mathbb{C}^m$.

Let $w \in \mathbb{C}$ be a coordinate for L such that the coordinates $z_1, \ldots, z_n$ of $\mathbb{C}^m$ are linear functions of w when restricted on L, and denote by $P_{n,L}(w)$ the restriction of P_n to L. Then $P_{n,L}(w)$ is a polynomial in one complex variable w of degree $deg(P_{n,L}(w)) \leq deg(P_n) \leq k_n$.

Then since

$$\limsup_{n\to\infty} |P_{n,L}(w)|^{1/k_n} \leq 1$$

for $w \in E \cap L$, and $E \cap L$ is non-thin at ∞ in the complex line L, hence

$$\limsup_{n\to\infty} |P_n(w)|^{1/k_n} \leq 1$$

for all $w \in L$. Since this is true for any complex line $L \in \Delta$, it is also true for their union, which is equal to $\mathbb{C}^m$. □

3. The growth of sequences of entire functions of genus zero

In this section we apply the results in previous section to sequences of entire functions of genus zero in $\mathbb{C}^m$.

First we give a proof of Theorem 4.

Proof. Define functions from $\mathbb{C}^m$ to $[-\infty, +\infty]$ by

$$u(z) = \limsup_{n\to\infty} u_n(z),$$

where

$$u_n(z) = \sup_{j\geq n} \frac{1}{k_j} \log |P_j(z)|.$$

From the assumption (1.2), by Theorem 4.6.3 in [5], u^* is PSH. Moreover there is a pluripolar set $A \subset \mathbb{C}^m$ such that $u(z) = u^*(z)$ for all $z \in \mathbb{C}^m \backslash A$.

Fix $\lambda \in \mathbb{P}^{m-1}$. Choose $R_n = R_{n,\lambda}$ as the sequence in condition (1.5). Then from the proof of Theorem 1 in [4] we see that for all $w \in \mathbb{C}$

$$\limsup_{n\to\infty} u_n(w\lambda) = \limsup_{n\to\infty} \frac{1}{k_n} \log |Q_{n,\lambda}(w)|$$

where $Q_{n,\lambda} : \mathbb{C} \to \mathbb{C}$ is defined by

$$Q_{n,\lambda}(w) = a_{n,\lambda} w^{\alpha_{n,\lambda}} \prod_{|w_{n,\lambda,j}|\leq R_n} (1 - \frac{w}{w_{n,\lambda,j}}).$$

Condition (1.2) shows that

$$\limsup_{n\to\infty} \frac{1}{k_n} \log |Q_{n,\lambda}(w)| \leq \kappa \log C_0$$

for all $w \in \mathbb{C}$ with $|w| \leq \epsilon$. This, together with condition (1.5) and potential theory in one complex variable (Berstein's lemma for polynomials, see also the proof of Theorem 2), show that there exists $C > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{k_n} \log |Q_{n,\lambda}(w)| \leq \kappa \log^+ |w| + C \tag{3.1}$$

for all $w \in \mathbb{C}$. Moreover C is independent of λ.

Hence u^*/κ is in $\mathbb{L}$. Let $F = E \backslash A$, then by Lemma 1 F is non-thin at ∞. In particular $V_F \equiv 0$. Now for all $z \in F$ we have by assumption

$$u^*(z) \leq 0,$$

thus

$$u^*(z) \leq \kappa V_F(z) = 0$$

for all $z \in \mathbb{C}^m$. □

Now we consider the analog of Theorem 3 in [4]. We have the following result.

Theorem 5. *Let $E \subset \mathbb{C}^m$ be closed and satisfy the following condition: there exists $\beta > 0$ such that*

$$\limsup_{R \to \infty} \frac{\log C(E_R)}{\log R} \geq \beta > \frac{C_m - 1}{C_m}, \tag{3.2}$$

where $C(E_R)$ is determined from formula (1.6), and $0 < C_m < \infty$ is the constant in (1.7).

Let (P_n) and (k_n) be sequences satisfying conditions (1.2), (1.3), (1.4) and (1.5) of Theorem 4. Then for $\lambda \in \mathbb{P}^{m-1}$ we have

$$\exp\{\limsup_{n \to \infty} \frac{1}{2\pi k_n} \int_0^{2\pi} \log |P_n(e^{i\theta}\lambda)| d\theta\} = C_\lambda < \infty.$$

Assume that for all $z \in E$ we have

$$\limsup_{n \to \infty} |P_n(z)|^{1/k_n} \leq h(|z|)$$

where

$$\limsup_{R \to \infty} \frac{\log h(R)}{\log R} \leq \tau < \infty. \tag{3.3}$$

Then for any $w \in \mathbb{C}$ and $\lambda \in \mathbb{P}^{m-1}$ we have

$$\limsup_{n \to \infty} |P_n(w\lambda)|^{1/k_n} \leq C_\lambda (1 + |w|)^{\tau/[1 - C_m(1-\beta)]}.$$

Proof. Define functions u and $Q_{n,\lambda}$ as in the proof of Theorem 4. Without loss of generality we may assume that $u \geq 0$. By part (i) of Lemma 1 in [4] and the proof of Theorem 4 there exists a constant $C > 0$ such that

$$u^*(z) \leq \kappa \log^+ |z| + C, \tag{3.4}$$

for all $z \in \mathbb{C}^m$. Now we define

$$\kappa_0 = \limsup_{z \in \mathbb{C}^m,\ z \to \infty} \frac{u^*(z)}{\log |z|}. \tag{3.5}$$

For any $R > 0$, by the definition of the pluricomplex Green function

$$u^*(z) \leq \kappa_0 V_{E_R}(z) + h(R). \tag{3.6}$$

For any $\lambda \in \mathbb{P}^{m-1}$ define

$$\kappa(\lambda) = \limsup_{w \in \mathbb{C},\ w \to \infty} \frac{u(w\lambda)}{\log |w|} \leq \kappa_0.$$

Fixed $\lambda \in \mathbb{P}^{m-1}$. By definition

$$u(w\lambda) = \limsup_{n\to\infty} \frac{1}{k_n} \log |Q_n(w\lambda)|$$

for all $w \in \mathbb{C}$. Hence using (3.6), for any $s > 0$ we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{k_n}\log|Q_n(se^{i\theta}\lambda)|d\theta &\leq \frac{1}{2\pi}\int_0^{2\pi} u(se^{i\theta}\lambda)d\theta \\ &\leq \kappa_0 \frac{1}{2\pi}\int_0^{2\pi} V_{E_s}(se^{i\theta}\lambda)d\theta + h(s).\end{aligned}$$

By the previous inequality, (1.7) and assumption (3.2) we get

$$\liminf_{s\to\infty}\limsup_{n\to\infty} \frac{1}{\log s}\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{k_n}\log|Q_n(se^{i\theta}\lambda)|d\theta \leq \kappa_0 C_m(1-\beta)+\tau. \tag{3.7}$$

Then by Lemma 2 in [4] we have

$$\kappa(\lambda) \leq \kappa_0 C_m(1-\beta) + \tau. \tag{3.8}$$

Since (3.8) is true for any $\lambda \in \mathbb{P}^{m-1}$, by definition (3.5) and Lemma 2 in [4] we have

$$\kappa_0 \leq \kappa_0 C_m(1-\beta) + \tau,$$

or equivalently

$$\kappa_0 \leq \frac{\tau}{1 - C_m(1-\beta)}.$$

From the above inequality, use Lemma 2 in [4] we get the conclusion of Theorem 5. □

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