

Multiplicative properties of Morin maps

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Abstract

In the first part of the paper we construct a ring structure on the rational cobordism classes of Morin maps (i. e. smooth generic maps of corank 1). We show that associating to a Morin map its Σ^{1r} (or A_r) singular strata defines a ring homomorphism to $\Omega_* \otimes \mathbb{Q}$, the rational oriented cobordism ring. This is proved by analyzing the multiple-point sets of a product immersion. Using these homomorphisms we compute the ring of Morin maps.

In the second part of the paper we find the oriented Thom polynomial of the Σ^2 singularity type with \mathbb{Q} coefficients. Then we provide a product formula for the Σ^2 and the $\Sigma^{1,1}$ singularities.

1 Introduction

The results of this paper are the first steps toward understanding how the direct product operation affects the singularities of maps. There are two main problems. The first one is that the direct product of generic maps will not be generic, so one has to take a small perturbation. This makes it hard to understand the singular strata geometrically. The second problem is that in general the product of two singular maps even after a generic perturbation will have more complicated singularities than the original maps had.

In Section 2 we study products of immersions. Here only the first type of problem arises, namely that the self intersections will not be transverse. This can be overcome by employing a general multiple-point formula from [2] that helps to compute the characteristic numbers of multiple-point manifolds.

In Section 3 we study Morin maps. In this case one has to deal with the second kind of problem. We get around this by increasing the dimension of the target space by one.

In Section 4 we set out to compute the ring $\text{Mor}_{\mathbb{Q}}$ (the ring of rational cobordism classes of Morin maps) defined at the end of Section 3. First, in Section 4.1, combining the results of the previous sections we show that the singular strata behave nicely under the multiplication defined in Section 3.2. Then in Section 4.2 we show that this information is actually enough to compute $\text{Mor}_{\mathbb{Q}}$.

Finally Section 5 deals with general singular maps. We show that a Cartan-type formula relates the Σ^1 points of two maps with the Σ^1 points of their direct product. We compute the oriented Thom polynomial of the Σ^2 singularity with \mathbb{Q} coefficients. Finally we derive a Cartan-type formula for the Σ^2 points as well.

2 Products of immersions

We start this section by recalling some basic notions about multiple points and the relevant results of [2].

First we shall introduce a characteristic class β that assigns to any oriented vector bundle ξ over B an element

$$\beta(\xi) = \prod_{i=1}^{\infty} (1 + p_1(\xi)t_i + p_2(\xi)t_i^2 + \dots) \in H^*(B; \mathbb{Q})[[t_1, t_2, \dots]]$$

in the ring of formal power series of the variables t_i over the ring $H^*(B; \mathbb{Q})$. (Here $p_i(\xi) \in H^{4i}(B; \mathbb{Q})$ is the $4i$ -dimensional Pontrjagin class of ξ). Since the Cartan formula holds for Pontrjagin classes modulo 2-torsion it follows that $\beta(\xi \oplus \eta) = \beta(\xi) \cdot \beta(\eta)$. (We have got rid of all torsions by taking \mathbb{Q} coefficients.) It is also easily seen that β is natural, and always has an inverse element. When B is a manifold we shall abbreviate $\beta(TB)$ by $\beta(B)$.

Now let $f : M^n \rightarrow N^{n+k}$ be a generic (i.e. selftransverse) immersion between oriented manifolds. The manifolds and the maps representing the r -fold points of f in the source and the target respectively will be denoted by

$$\begin{aligned}\phi_r(f) &: \tilde{M}_r(f) \rightarrow M, & \text{and} \\ \psi_r(f) &: \tilde{N}_r(f) \rightarrow N.\end{aligned}$$

When the codimension of the map k is even, these manifolds are equipped with a natural orientation. It is easy to see that the cobordism classes of these manifolds depend only on the cobordism class of f . Our goal is to obtain information about these cobordism classes. To this end we compute their characteristic numbers.

Let us denote

$$\begin{aligned}m_r &= m_r(f) = \phi_r(f)_!(\beta(\tilde{M}_r(f))), \\ n_r &= n_r(f) = \psi_r(f)_!(\beta(\tilde{N}_r(f))).\end{aligned}$$

The reason for considering these elements is the following simple observation. Evaluating each coefficient of m_r on the fundamental class of M we get an element in $\mathbb{Q}[[t_1, t_2, \dots]]$. The coefficients of this power series are exactly the Pontrjagin numbers of $\tilde{M}_r(f)$.

The classes m_r and n_r are related by the equality:

$$m_r \cdot \beta(\nu_f) = f^* n_{r-1} - e(\nu_f) m_{r-1} \quad (1)$$

where ν_f is the normal bundle of f and e is the Euler class. This is a generalization of the well-known Herbert-Ronga formula (see the Main formula of [2]).

We are going to apply this in the case when the target is a Euclidean space. Then $f^* = 0$ so (1) is simplified to $m_r \cdot \beta(\nu_f) = -e(\nu_f) \cdot m_{r-1}$. Applying this recursively one gets that $m_r \cdot \beta(\nu_f)^{r-1} = (-e(\nu_f))^{r-1} \cdot m_1$. But $m_1 = \beta(M)$ and $\beta(M) \cdot \beta(\nu_f) = \beta(\mathbb{R}^n) = 1$, so we end up with

$$m_r = (-e(\nu_f))^{r-1} \cdot \beta(M)^r.$$

Now we can state and prove the main result of this section.

Theorem 1. *Let $g_i : M_i^{n_i} \rightarrow \mathbb{R}^{n_i+k_i}; (i = 1, 2)$ be generic immersions. Then we have*

$$\tilde{M}_r(g_1 \times g_2) \sim (-1)^{r-1} \tilde{M}_r(g_1) \times \tilde{M}_r(g_2) \quad (2)$$

where \sim stands for “unoriented-cobordant”.

If both manifolds M_i are oriented and both codimensions k_i are even, then the two sides of 2 are oriented cobordant.

Proof. We will only consider the oriented case. The unoriented version is proved exactly the same way, except that there is no need to study Pontrjagin classes.

Let $f = g_1 \times g_2$. Then

$$\begin{aligned}m_r(f) &= (-e(\nu_f))^{r-1} \cdot \beta((M_1 \times M_2))^r = \\ &= (-e(\nu_{g_1} \times \nu_{g_2}))^{r-1} \cdot \beta(TM_1 \times TM_2)^r = \\ &= (-1)^{r-1} ((-e(\nu_{g_1}))^{r-1} \cdot \beta(M_1)^r) \times ((-e(\nu_{g_2}))^{r-1} \cdot \beta(M_2)^r) = \\ &= (-1)^{r-1} m_r(g_1) \times m_r(g_2).\end{aligned}$$

The following equations are easily checked.

$$\begin{aligned}\langle \beta(\tilde{M}_r(f)), [\tilde{M}_r(f)] \rangle &= \langle m_r(f), [M_1 \times M_2] \rangle = \langle m_r(g_1 \times g_2), [M_1 \times M_2] \rangle = \\ &= (-1)^{r-1} \langle \beta(\tilde{M}_r(g_1), [\tilde{M}_r(g_1)]) \cdot \beta(\tilde{M}_r(g_2), [\tilde{M}_r(g_2)]) \rangle = \\ &= (-1)^{r-1} \langle \beta((\tilde{M}_r(g_1) \times \tilde{M}_r(g_2))), [\tilde{M}_r(g_1) \times \tilde{M}_r(g_2)] \rangle\end{aligned}$$

We have obtained equality of two formal power series, so the corresponding coefficients must be equal on the two sides. As the coefficients are the Pontrjagin numbers of the manifolds involved, we get that the Pontrjagin numbers of the two manifolds are all equal.

To finish the proof we have to repeat the whole argument using an analogous class instead of β , namely

$$\beta'(\xi) = \prod_{i=1}^{\infty} (1 + w_1(\xi)t_i^1 + w_2(\xi)t_i^2 + \dots) \in H^*(B, \mathbb{Z}_2)[[t_1, t_2, \dots]].$$

It is obvious that all the above hold for β' as well. Thus not only the Pontrjagin numbers, but also the Stiefel-Whitney numbers of the two manifolds are equal. Since the oriented cobordism class is determined by these numbers, the claim of the theorem follows. \square

This result will no longer hold if we consider a general target space N . However the Pontrjagin and Stiefel-Whitney numbers of the multiple-point manifolds of $g_1 \times g_2$ are still expressible in terms of g_1, g_2 and their multiple-point manifolds. This expression is particularly simple for the double-point set.

First we need a simple result about the embedded manifold representing a vector bundle's Euler class. Let $\xi \rightarrow B$ be a vector bundle over a manifold B . Let $s : B \rightarrow \xi$ be a section transverse to the 0-section. Let us denote by Δ_ξ the submanifold in B that is the inverse image of the 0-section by s , and let $\delta_\xi : \Delta_\xi \rightarrow B$ denote the inclusion.

Lemma 1. $\langle \beta(\Delta_\xi), [\Delta_\xi] \rangle = \langle \beta(B) \cdot \frac{e(\xi)}{\beta(\xi)}, [B] \rangle$.

Proof. It suffices to show that

$$\delta_{\xi!}(\beta(\Delta_\xi)) = \beta(B) \cdot \frac{e(\xi)}{\beta(\xi)}.$$

By the construction of Δ_ξ we have the following pull-back diagram:

$$\begin{array}{ccc} \Delta_\xi & \xrightarrow{\delta_\xi} & B \\ \downarrow \delta_\xi & & \downarrow \text{0-section} \\ B & \xrightarrow{s} & \xi \end{array}$$

Hence the normal bundle of δ_ξ is just the pull-back of the normal-bundle of the 0-section. This latter is just ξ . Thus we have

$$T\Delta_\xi \oplus \delta_\xi^* \xi = \delta_\xi^* TB,$$

which in turn implies that

$$\beta(\Delta_\xi) = \delta_\xi^* \left(\frac{\beta(B)}{\beta(\xi)} \right).$$

Applying the push-forward to this equation gives the proof of the lemma, since $f_!(f^*x) = f_!(1) \cdot x$ is well known and obviously $\delta_{\xi!}(1) = e(\xi)$. \square

Theorem 2. Let $g_i : M_i^{n_i} \rightarrow N_i^{n_i+k_i}; (i = 1, 2)$ be generic immersions. Then

$$\tilde{M}_2(g_1 \times g_2) \sim \tilde{M}_2(g_1) \times \tilde{M}_2(g_2) + \tilde{M}_2(g_1) \times \Delta_{\nu_{g_2}} + \Delta_{\nu_{g_1}} \times \tilde{M}_2(g_2)$$

where \sim stands for "unoriented-cobordant". (Recall that ν_{g_i} is the normal bundle of g_i and Δ_{g_i} is the zero set of a generic section of ν_{g_i} .) If the M_i are oriented and the k_i are even, then the same is true up to oriented cobordism.

Proof. We proceed in a similar manner as in the previous theorem. Let us put $f = g_1 \times g_2$ and $M = M_1 \times M_2$ again. Then using (1) we get

$$\begin{aligned} \beta(\nu_f) \cdot m_2(f) &= f^* f_!(\beta(M)) - e(\nu_f) \cdot \beta(M) = \\ &= g_1^* g_{1!}(\beta(M_1)) \times g_2^* g_{2!}(\beta(M_2)) - e(\nu_f) \cdot \beta(M) = \\ &= (\beta(\nu_{g_1}) m_2(g_1) + e(\nu_{g_1}) \cdot \beta(M_1)) \times (\beta(\nu_{g_2}) m_2(g_2) + e(\nu_{g_2}) \cdot \beta(M_2)) - \\ &\quad - e(\nu_f) \cdot \beta(M) = \\ &= \beta(\nu_f) \cdot \left(m_2(g_1) \times m_2(g_2) + m_2(g_1) \times \beta(M_2) \frac{e_{\nu_{g_2}}}{\beta(\nu_{g_2})} + \beta(M_1) \frac{e_{\nu_{g_1}}}{\beta(\nu_{g_1})} \times m_2(g_2) \right) \end{aligned}$$

Now we can divide by $\beta(\nu_f)$ as it is an invertible element. We evaluate both sides on $[M] = [M_1] \times [M_2]$. Finally we have to apply the previous lemma to get that all the corresponding characteristic numbers are equal for the two manifolds in question. As before, we can repeat the argument for Stiefel-Whitney numbers in \mathbb{Z}_2 coefficients and Pontrjagin numbers in \mathbb{Q} coefficients, so we get both parts of the theorem at the same time. \square

Remark 1. 1. It is possible to carry out similar calculations for triple points or points of higher (say r) multiplicity. But the number of terms involved in these formulas grow exponentially with r and the authors did not manage to find a nice way to write them down, not even recursively.

2. It would be possible to obtain similar formulas not only for the cobordism classes of the underlying multiple-point manifolds, but for the cobordism classes of the immersions ϕ_r themselves. To do this one would need to consider the characteristic numbers of these immersions instead of the characteristic numbers of the manifolds. These calculations are more or less the same as the ones described here, but they are harder to keep track of.
3. It seems that the same results could be obtained using techniques of Eccles and Grant from [4].
4. We would like to point out that Theorem 2 is a non-trivial generalisation of the oriented case of Theorem A in [3], which considers the case of $n = k$.

3 Ring structure of Morin maps

Given a smooth map $f : M \rightarrow N$, a point $x \in M$ is said to be a Σ^i point if the corank of $df_x : T_x M \rightarrow T_{f(x)} N$ is at least i . The set of such points is denoted by $\Sigma^i(f)$. If $i_1 \geq i_2$ then we can define $\Sigma^{i_1, i_2} f = \Sigma^{i_2} f|_{\Sigma^{i_1} f}$. This method can be continued recursively to give the definition of $\Sigma^{(i_1, i_2, \dots, i_r)}$ points, where $i_1 \geq i_2 \geq \dots \geq i_r$. This classification of singular points is called the Thom-Boardman type. For details see e. g. [1].

A generic smooth map $f : M \rightarrow N$ is called a *Morin* map if it has no Σ^2 points. The singularities of such maps are classified by their Thom-Boardman type, which can only be $\Sigma^{(\overbrace{1, 1, \dots, 1}^r)} = \Sigma^{1r}$ for some $r \geq 0$. (In the notation of [1] this is A_r .)

Cobordism of Morin maps is defined in the usual way: two Morin maps $f : M_1^n \rightarrow N^{n+k}$ and $g : M_2^n \rightarrow N^{n+k}$ are said to be cobordant if there is a Morin map $H : W^{n+1} \rightarrow N^{n+k} \times [0, 1]$ such that $\delta W = M_1 \cup M_2$ and $H|_{M_1} = f, H|_{M_2} = g$.

Let us consider the set of cobordism classes of all Morin maps to Euclidean spaces (for all nonnegative dimensions and all positive codimensions). This set is a commutative group with addition induced by the disjoint union of maps. We can take tensor product with \mathbb{Q} to obtain the rational cobordism group whose elements will be referred to as rational cobordism classes. In this section we endow this rational cobordism group with a ring structure. Further we will show that the singularities can be used to define ring homomorphisms into Ω_* , the oriented cobordism ring of manifolds.

The main tool in constructing the multiplication will be the so-called ‘‘prim maps’’, while the ring homomorphisms will be derived from the results of the previous section.

3.1 Prim maps

Definition 1. A generic map $f : M \rightarrow N$ is called *prim* (*projected immersion*) if it can be lifted to a generic immersion, $\tilde{f} : M \rightarrow N \times \mathbb{R}$. (We will always denote the lifting by a tilde.)

Cobordism of prim maps can be defined in a natural way (the cobordism itself should be a prim map into $N \times [0, 1]$), and disjoint union induces a group operation on the cobordism classes. The class of a prim map f will be denoted by $[f]$. (For details see e.g. [6].)

Clearly a prim map is necessarily a Morin map. Prim maps provide a good link between immersions and Morin maps. We shall first define multiplication of prim maps (using their liftings to immersions) and then show how to extend it to multiplication of Morin maps (using results from [8]). We will only work with prim maps whose target space is Euclidean.

Let us denote $l_0 : pt \hookrightarrow \mathbb{R}$ the inclusion of a point into the line.

Lemma 2.

- a) Any two generic hyperplane projections of an immersion represent the same prim cobordism class.
- b) Projections of cobordant immersions represent the same prim cobordism class.

Proof. a) Instead of taking two projections of the same immersion we can take the same projection of two immersions which differ only by a rotation. This rotation can be realized by a regular homotopy. We can take a generic projection of this homotopy to a hyperplane that is sufficiently close to the original one. This gives a prim cobordism between slightly perturbed versions of the original prim maps, but since generic projections form an open set this perturbation does not effect the prim cobordism class (not even the prim homotopy class).

b) This can be proved in exactly the same way, by taking a generic projection of the cobordism connecting the two immersions. \square

Definition 2. Given two prim maps $f_i : M_i \rightarrow \mathbb{R}^{n_i}$ ($i = 1, 2$) consider the product map

$$g = f_1 \times f_2 \times l_0 : M_1 \times M_2 \rightarrow \mathbb{R}^{n_1+n_2} \times \mathbb{R}.$$

The map g might not yet be prim, but we can turn it into such by a small perturbation. Take liftings \tilde{f}_1 and \tilde{f}_2 that are sufficiently close to $f_1 \times l_0$ and $f_2 \times l_0$. Now $\tilde{f}_1 \times \tilde{f}_2 : M_1 \times M_2 \rightarrow \mathbb{R}^{n_1+n_2} \times \mathbb{R}^2$ is a non-generic immersion. Let us take a sufficiently small perturbation of this product so that it becomes a generic immersion. Finally take a generic projection this immersion to a hyperplane “close” to $\mathbb{R}^{n_1+n_2} \times \mathbb{R}$, where the last \mathbb{R} factor is the diagonal in \mathbb{R}^2 . We obviously get a prim map g' that can be arbitrarily close to g . Let us denote $g' = f_1 * f_2$ and let us define the multiplication on prim cobordism classes as follows: $[f_1] * [f_2] = [f_1 * f_2]$.

Theorem 3. *The above definition is correct, that is $[f_1 * f_2]$ is independent of the choice of f_1 and f_2 within their cobordism class and of any other choices made in the definition. The multiplication defined in this way gives rise to a ring structure with respect to the disjoint union as additon.*

Proof. The liftings are unique up to regular homotopy. Also the perturbation of $\tilde{f}_1 \times \tilde{f}_2$ is unique up to regular homotopy. Thus Lemma 2 implies that the resulting prim map is independent of these choices.

Now suppose $[f_1] = [g_1]$. Then there is a prim cobordism H joining f_1 and g_1 . We can take its lifting \tilde{H} which is an immersed cobordism between \tilde{f}_1 and \tilde{g}_1 , and so $\tilde{f}_1 \times \tilde{f}_2$ and $\tilde{g}_1 \times \tilde{f}_2$ are regularly homotopic via $\tilde{H} \times \tilde{f}_2$. So their projections are prim cobordant, and this is what we wanted to prove. (The definition is symmetric so the other factor can be handled the same way.)

The last claim only requires the checking of distributivity, which is obvious. \square

3.2 Morin maps

In this section we only consider maps between oriented manifolds. Let us denote the group of cobordism classes of oriented Morin maps $f : M^n \rightarrow \mathbb{R}^{n+k}$ by $\text{Cob}_{\Sigma^1}(n, k)$ and the cobordism classes of prim maps $f : M^n \rightarrow \mathbb{R}^{n+k}$ by $\text{Prim}(n, k)$. As a prim map is automatically Morin and prim cobordant maps are Morin cobordant as well, we have a natural forgetting map $F : \text{Prim}(n, k) \rightarrow \text{Cob}_{\Sigma^1}(n, k)$, that induces a map $F_{\mathbb{Q}} : \text{Prim}(n, k) \otimes \mathbb{Q} \rightarrow \text{Cob}_{\Sigma^1}(n, k) \otimes \mathbb{Q}$. The following key result, which says that every Morin map has a non-zero multiple that is Morin-cobordant to a prim map is proved in [8]:

Lemma 3. *The map $F_{\mathbb{Q}}$ is epimorphic.*

Using this result and the construction in the previous section we can now define a multiplication on $\left(\bigoplus_{n,k} \text{Cob}_{\Sigma^1}(n, k)\right) \otimes \mathbb{Q}$.

Definition 3. Let us take two Morin maps $g_i : M_i^{n_i} \rightarrow \mathbb{R}^{n_i+k_i}$. By Lemma 3 we can find prim maps f_1 and f_2 that are rationally Morin cobordant to g_1 and g_2 . Let us define $[g_1] * [g_2] \stackrel{\text{def}}{=} [F_{\mathbb{Q}}(f_1 * f_2)]$, where $[f]$ denotes the rational Morin cobordism class of the Morin map f .

Theorem 4. *The above definition is correct, that is $[g_1] * [g_2]$ is independent of the choices made. The multiplication defined this way gives rise to a ring structure on $\left(\bigoplus_{n,k} \text{Cob}_{\Sigma^1}(n, k)\right) \otimes \mathbb{Q}$.*

Proof. There is only one thing left that needs to be checked: if f_1 and f'_1 are Morin cobordant prim-representatives of g_1 , then $F(f_1 * f_2)$ is indeed Morin cobordant to $F(f'_1 * f_2)$. Let us take the Morin cobordism H connecting f_1 and f'_1 . Then $H \times (f_2 \times l_0)$ is still a Morin cobordism after a sufficiently small perturbation, since the second factor can be perturbed to an immersion. This Morin cobordism connects exactly the two desired maps. \square

Definition 4. Let $\text{Mor}_{\mathbb{Q}}$ denote the group $\bigoplus_{n,k} \text{Cob}_{\Sigma^1}(n, k) \otimes \mathbb{Q}$ with this ring structure. $\text{Mor}_{\mathbb{Q}}$ is a bigraded ring, the two grades being n and $k + 1$.

4 Computing $\text{Mor}_{\mathbb{Q}}$

4.1 Ring homomorphisms

Let k be odd, and let $f : M^n \rightarrow \mathbb{R}^{n+k}$ be a generic oriented Morin map of odd codimension. To such a map we can associate the subset of M^n of those points where the Thom-Boardman singularity type of f is $\overbrace{\Sigma^{1,1,\dots,1}}^r = \Sigma^{1r}$. This subset is actually a submanifold and will be denoted by $\Sigma^{1r}(f)$. The cobordism class of this submanifold is invariant under a Morin cobordism of f , since the Σ^{1r} points of the cobordism of f give a cobordism between the Σ^{1r} points of f . For even r we actually get an oriented cobordism class. We can tensor with \mathbb{Q} and get a map

$$\Sigma^{1r} : \bigoplus_{k \text{ odd}, n} \text{Cob}_{\Sigma^1}(n, k) \otimes \mathbb{Q} \rightarrow \Omega_* \otimes \mathbb{Q}$$

to the rational oriented cobordism ring.

Theorem 5. *If r is even then the map Σ^{1r} is a ring homomorphism or in other words for Morin maps f, g to Euclidean spaces we have*

$$\Sigma^{1r}(f * g) \sim \Sigma^{1r}(f) \times \Sigma^{1r}(g)$$

where \sim now stands for rationally cobordant (in the oriented sense).

Proof. We will proceed along the lines explained earlier, that is we will use prim maps as a link between Morin maps and immersions. Then the multiplicative properties of multiple points of immersions will provide the result.

Let us first consider prim maps. The same argument as above gives a map

$$\Sigma_{P_r}^{1r} : \left(\bigoplus_{k \text{ odd}, n} \text{Prim}(n, k) \right) \otimes \mathbb{Q} \rightarrow \Omega_* \otimes \mathbb{Q}.$$

It is obvious that $\Sigma_{P_r}^{1r} = \Sigma^{1r} \circ F_{\mathbb{Q}}$.

Let us denote the oriented cobordism groups of $k+1$ codimensional immersions from n -dimensional manifolds to Euclidean spaces by $\text{Imm}^{SO}(n, k+1)$. Given an immersion $f : M^n \rightarrow \mathbb{R}^{n+k+1}$, let us denote by $\pi(f)$ its generic projection to a hyperplane. This map is a prim map whose prim cobordism class is well defined according to Lemma 2. The direct sum $\bigoplus_{k \text{ odd}, n} \text{Imm}^{SO}(n, k+1)$ has a natural ring structure with multiplication being the direct product. It is clear from the definitions that

$$\pi : \bigoplus_{k \text{ odd}, n} \text{Imm}^{SO}(n, k+1) \rightarrow \bigoplus_{k \text{ odd}, n} \text{Prim}(n, k)$$

is a ring homomorphism with respect to the direct product on the left, and $*$ -product on the right. The same remains true after forming the tensor product with \mathbb{Q} .

In Theorem 1 we have shown that

$$\tilde{M}_{r+1} : \bigoplus_{k \text{ odd}, n} \text{Imm}^{SO}(n, k+1) \rightarrow \Omega_*$$

is a ring homomorphism, and obviously the same is true after forming the tensor product with \mathbb{Q} .

To finish the proof we have to recall a result from [7] which in our notations reads as:

Theorem 6 ([7]). $\tilde{M}_{r+1} \otimes \text{id}_{\mathbb{Q}} = (\pi \otimes \text{id}_{\mathbb{Q}}) \circ \Sigma_{P_r}^{1r}$ i. e. the rational cobordism class of the manifold of $r+1$ -tuple points of an immersion $f : M^n \rightarrow \mathbb{R}^{n+k+1}$ coincides with that of the manifold of Σ^{1r} (or A_r) points of its hyperplane projection.

All of the above proves that the following diagram is commutative.

$$\begin{array}{ccc}
\left(\bigoplus_{k \text{ odd}, n} \text{Imm}^{SO}(n, k+1) \right) \otimes \mathbb{Q} & & \\
\downarrow \pi \otimes \text{id}_{\mathbb{Q}} & \searrow \tilde{M}_{r+1} \otimes \text{id}_{\mathbb{Q}} & \\
\left(\bigoplus_{k \text{ odd}, n} \text{Prim}(n, k) \right) \otimes \mathbb{Q} & \xrightarrow{\Sigma_{P_r}^{1r}} & \Omega_* \otimes \mathbb{Q} \\
\downarrow F_{\mathbb{Q}} & \nearrow \Sigma^{1r} & \\
\left(\bigoplus_{k \text{ odd}, n} \text{Cob}_{\Sigma^1}(n, k) \right) \otimes \mathbb{Q} & &
\end{array}$$

The vertical maps are ring epimorphisms and \tilde{M}_{r+1} is a ring homomorphism. This implies that $\Sigma_{P_r}^{1r}$ and Σ^{1r} are ring homomorphisms too. \square

4.2 The structure of $\text{Cob}_{\Sigma^1}(n, k)$

In [8] it is shown that the rational cobordism class of an oriented Morin map is actually determined by those of its singular strata. As we have seen the singular strata are ring homomorphisms from $\text{Mor}_{\mathbb{Q}}$. This provides a complete computation of the ring $\text{Mor}_{\mathbb{Q}}$.

For any stable singularity type η there is a bundle $\tilde{\xi}_{\eta}$ that plays the role of the universal normal bundle for this singularity type. This means the following: Whenever for a map $f : M \rightarrow N$ one of its most complicated singularities is η then the η -points of f form a submanifold of M . The restriction of f to this submanifold is an immersion to N . The normal bundle of this immersion is induced from $\tilde{\xi}_{\eta}$. (See [5] for details.)

Let us write $\tilde{\xi}_r = \tilde{\xi}_{\Sigma^{1r}}$ for short. Let $\text{Imm}^{\tilde{\xi}_r}(n, k)$ denote the cobordism group of oriented immersions $f : M^n \rightarrow \mathbb{R}^{n+k}$ whose normal bundles are induced from $\tilde{\xi}_r$.

We need two results from [8] which we state here in a lemma.

Lemma 4.

1. For odd k we have

$$\text{Cob}_{\Sigma^1}(n, k) \otimes \mathbb{Q} = \bigoplus_{i=0}^{\infty} \text{Imm}^{\tilde{\xi}^{2i}}(n - 2i(k+1), 2i(k+1) + k) \otimes \mathbb{Q}. \quad (3)$$

while for even k we have $\text{Cob}_{\Sigma^1}(n, k) \otimes \mathbb{Q} = \text{Imm}^{SO}(n, k)$.

2. For even r we have $H_{n+k}(T\tilde{\xi}_r; \mathbb{Q}) = H_{n-r(k+1)}(BSO(k); \mathbb{Q})$.

Proof. Part (i) is stated explicitly in [8] as Example 119.

For part (ii) we have to recall that the bundle $\tilde{\xi}_{\eta}$ has a counterpart denoted by ξ_{η} which is the universal normal bundle of the η -points of a map in the source manifold. The two bundles ξ_{η} and $\tilde{\xi}_{\eta}$ have the same base space BG_{η} where G_{η} is the maximal compact subgroup of the symmetry group of the singularity η . This implies that the homologies of $T\tilde{\xi}_{\eta}$ and $T\xi_{\eta}$ are the same up to a dimension shift equal to $\text{rank } \tilde{\xi}_{\eta} - \text{rank } \xi_{\eta} = k$, i. e. $H_{n+k}(T\tilde{\xi}_r; \mathbb{Q}) = H_n(T\xi_r; \mathbb{Q})$.

Lemma 103/b in [8] implies that for even r we have $H_n(T\xi_r; \mathbb{Q}) = H_{n-r(k+1)}(BSO(k); \mathbb{Q})$. The statement follows. \square

It is well known that

$$\text{Imm}^{\tilde{\xi}_r}(n, k) \otimes \mathbb{Q} \cong \pi_{n+k}^S(T\tilde{\xi}_r) \otimes \mathbb{Q} \cong H_{n+k}(T\tilde{\xi}_r; \mathbb{Q}) = H_{n-r(k+1)}(BSO(k); \mathbb{Q}).$$

There is the natural forgetting map that assigns to an immersion the cobordism class of its underlying source manifold. This forgetting map on the level of classifying spaces is just the inclusion of the classifying spaces $BSO(k) \hookrightarrow BSO$. The rational cohomology ring of the classifying space for Ω_* is $\mathbb{Q}[p_1, p_2, \dots]$. Since k is odd $H^*(BSO(k); \mathbb{Q}) = \mathbb{Q}[p_1, p_2, \dots, p_{\frac{k-1}{2}}]$. Thus the inclusion map induces a surjective homomorphism between the rings and this means that the forgetting map is actually injective.

Thus for every even r we have a map $\text{Cob}_{\Sigma^1}(n, k) \otimes \mathbb{Q} \rightarrow \text{Imm}^{\xi_r}(n-r(k+1), r(k+1)+k) \otimes \mathbb{Q} \rightarrow \Omega_{n-r(k+1)} \otimes \mathbb{Q}$. The first arrow is just the projection in the splitting (3) while the second arrow is the forgetting map. The composition of the two is obviously the previously defined Σ^{1r} .

This proves that for odd k an element $[f] \in \text{Cob}_{\Sigma^1}(n, k) \otimes \mathbb{Q}$ is indeed determined by the collection of rational cobordism classes of the $\Sigma^{1r}f$ manifolds. It also follows from the previous argument that exactly those cobordism classes are in the image $\Sigma^{1r}(\text{Cob}_{\Sigma^1}(n, k) \otimes \mathbb{Q})$ which do not have non-zero Pontrjagin numbers involving Pontrjagin classes higher than $p_{\frac{k-1}{2}}$.

For even k the situation is simpler. It follows from Lemma 4 that for an element $[f] \in \text{Cob}_{\Sigma^1}(n, k) \otimes \mathbb{Q}$ we have $\Sigma^{1r}(f) = 0$ for every $r \geq 1$ and thus the class of f is completely determined by the cobordism class of its underlying manifold. In other words any even codimensional Morin map is Morin-cobordant to an immersion. It is then clear from the definitions 2 and 3 that multiplying by an even codimensional map annihilates any singularities.

5 Singular strata of direct products

Our goal in this final section is to show that the cohomology class represented by the submanifold formed by the closure of the set of certain singular points of a direct product $f \times g$ depends only on those f and g and some maps closely related to them.

The arguments are based on the well known fact, that the Thom polynomials of the singularity types in question are simple. Before we formulate the theorems, we have to introduce some notation.

Definition 5. For $j \geq 0$ let $q_j : * \rightarrow S^j$ denote the inclusion of a point into S^j and for $j < 0$ let $q_j : S^{|j|} \rightarrow *$ be the map that takes the sphere to a point. Now for any integer j we define $f'_j = f \times q_j$ and take f_j to be a generic perturbation of f'_j .

Finally let $\text{id}_j = \text{id}_M \times q_j$.

5.1 The Σ^1 stratum

Let $\Sigma^1 f$ denote the closure of the set of all singular points in the source manifold of f . The Thom polynomial of this singularity type is w_{k+1} . That is, given a map $f : M^n \rightarrow N^{n+k}$, the cohomology class Poincaré dual to the homology class represented by $\Sigma^1 f$ is equal to $w_{k+1}(\nu_f)$ where ν_f stands for the virtual normal bundle of f . This dual cohomology class will be denoted by $[\Sigma^1 f]$ for simplicity.

Theorem 7. *Let $f : M_1^{n_1} \rightarrow N^{n_1+k_1}, g : M_2^{n_2} \rightarrow N^{n_2+k_2}$ be two generic maps. Then for a generic perturbation of their product we have*

$$[\Sigma^1 f \times g] = \sum_{j \geq 1} \left([\Sigma^1 f_{j-1}] \times \text{id}_j^* [\Sigma^1 g_{(-j)}] + \text{id}_j^* [\Sigma^1 f_{(-j)}] \times [\Sigma^1 g_{j-1}] \right)$$

Proof. As a first step let us notice that since $\nu_{f \times g} = \nu_f \times \nu_g$ we can write

$$\begin{aligned} w_{k_1+k_2+1}(\nu_{f \times g}) &= \sum_{r=0}^{k_1+k_2+1} w_r(\nu_f) \times w_{k_1+k_2+1-r}(\nu_g) = \\ &= \sum_{j \geq 1} \left(w_{k_1+j}(\nu_f) \times w_{k_2-j+1}(\nu_g) + w_{k_1-j+1}(\nu_f) \times w_{k_2+j}(\nu_g) \right) \end{aligned}$$

Now we have to take a closer look at $w_{k_1+j}(\nu_f)$. If k_1+j-1 would be equal to the codimension of f then this characteristic class would just represent the singular locus of f . When this is not the case, we have to find an appropriate replacement of f that has the right codimension, whose normal bundle however is stably equivalent to that of f . This replacement map is exactly f_{j-1} . Indeed, $\nu_{f_{j-1}} = \nu_f \oplus \varepsilon^{j-1}$ so $w_{k_1+j}(\nu_f) = w_{k_1+j}(\nu_{f_{j-1}})$ which in turn is equal to $[\Sigma^1 f_{j-1}]$ since this map has the right codimension.

The argument is just slightly more complicated in the case of w_{k_2-j+1} . Here first we take the map $g_{(-j)} : M_2^{n_2} \times S^j \rightarrow N^{n_2+k_2}$. This has codimension k_2-j so $[\Sigma^1 g_{(-j)}] = w_{k_2-j+1}(\nu_{g_{(-j)}})$. The only problem is that this class lives in the cohomology of $M_2 \times S^j$. This is why we have to pull it back to M_2 by $\text{id}_{(-j)}$. Since the composition of id_j and $g_{(-j)}$ is just a perturbation of g and $w(\nu_{q_j}) = 1$ it follows that $\text{id}_j^* w_{k_2-j+1}(\nu_{g_{(-j)}}) = w_{k_2-j+1}(\nu_g)$.

Putting all these together gives the result of the theorem. \square

5.2 The Σ^2 stratum

A very similar result can be proved about the Σ^2 stratum of oriented maps. First we need to compute the Thom polynomial of the Σ^2 stratum in the oriented case. We will work with rational coefficients.

Theorem 8. *Let $f : M^n \rightarrow N^{n+k}$ be a generic map where $(k = 2t - 2)$. Then the rational cohomology class dual to the closure of the set of Σ^2 -points of f (for short $[\Sigma^2 f]$) equals $p_t(\nu_f)$, where $p_t \in H^{4t}(M; \mathbb{Q})$ is the t^{th} Pontrjagin class.*

Proof. By definition the Thom polynomial tp_{Σ^2} of the Σ^2 -stratum is a cohomology class in $H^{4t}(BSO; \mathbb{Q}) = \mathbb{Q}[p_1, p_2, p_3, \dots]$. We want to show that $tp_{\Sigma^2} = p_t$. It is enough to show that these two cohomology classes evaluated on any homology class in $H_{4t}(BSO; \mathbb{Q})$ are equal.

Lemma 5. *All homology classes in $H_{4t}(BSO; \mathbb{Q})$ can be represented by a normal map, i. e. by a map $h : L^{4t} \rightarrow BSO$ of an oriented $4t$ -manifold L^{4t} corresponding to the stable normal bundle of L^{4t} .*

Proof. It is enough to consider a sufficiently large finite dimensional approximation $BSO(N)$, $(N \gg 1)$. By the Pontrjagin-Thom construction an embedding $L^{4t} \hookrightarrow S^K$ gives a map $h' : S^K \rightarrow MSO(K - 4t)$ that maps L^{4t} into $BSO(K - 4t)$ and the restriction $h'|_{L^{4t}}$ corresponds to the normal bundle of L^{4t} . The homotopy class $[h'] \in \pi_K(MSO(K - 4t))$ is mapped by the composition of the Hurewicz homomorphism and the Thom isomorphism into a homology class $x = h'_*([L^{4t}]) \in H_{4t}(BSO(K - 4t))$. Hence this class x is represented by a normal map. Since the Hurewicz homomorphism in stable dimensions $(K \geq 8t + 2)$ is a rational isomorphism, we obtain the statement of the lemma. \square

To evaluate a $4t$ dimensional cohomology class on a $4t$ dimensional homology class represented by a manifold, one just pulls back the cohomology class to the manifold and evaluates it on the fundamental class.

Now it is enough to prove, that for every oriented M^{4t} the map $\nu^* : H^{4t}(BSO; \mathbb{Q}) \rightarrow H^{4t}(M; \mathbb{Q})$ induced by the normal mapping $\nu : M^{4t} \rightarrow BSO$ takes p_t and tp_{Σ^2} to the same cohomology class in $H^{4t}(M; \mathbb{Q})$. As $\nu^*(p_t) = p_t(\nu_M)$ and $\nu^*(tp_{\Sigma^2})$ is the dual of the Σ^2 stratum of a generic map $M^{4t} \rightarrow \mathbb{R}^{6t-2}$ we reduced the problem of finding the Thom polynomial to the special case of $M^{4t} \rightarrow \mathbb{R}^{6t-2}$ maps.

If we take an immersion $f : M^{4t} \rightarrow \mathbb{R}^{6t}$, and project it to two non-parallel hyperplanes, then we get a map $f' : M^{4t} \rightarrow \mathbb{R}^{6t-2}$. Let us denote the two hyperplanes H_1, H_2 . The projection of f to H_i shall be called f_i . It is obvious that those and only those points belong to $\Sigma^2 f'$ which belong to $\Sigma^1 f_1$ and $\Sigma^1 f_2$ at the same time. This means that for this f' we have $[\Sigma^2 f'] = [\Sigma^1 f_1] \cup [\Sigma^1 f_2]$. The two cohomology classes on the right are both equal to the Thom polynomial of the Σ^1 singularity, which is the Euler class of the normal bundle of f . As this normal bundle has rank $2t$, the square of its Euler class is equal to $p_t(\nu_f)$, which is the same as $p_t(\nu_M)$. So far we have proved our claim for those maps $M^{4t} \rightarrow \mathbb{R}^{6t-2}$ where the source manifold can be immersed into \mathbb{R}^{6t} .

Let us recall that by $\text{Imm}^{SO}(4t, 2t)$ we denoted the cobordism group of oriented immersions from $4t$ dimensional manifolds to \mathbb{R}^{6t} . There is the natural forgetting map $\psi : \text{Imm}^{SO}(4t, 2t) \rightarrow \Omega_{4t}$ taking an immersion to its underlying manifold. To finish the proof of the theorem it is sufficient to show, that this map is a rational epimorphism. According to the Pontrjagin-Thom construction and the stable Hurewicz homomorphism

$$\text{Imm}^{SO}(4t, 2t) \cong \pi_{6t}^S MSO(2t) \cong_{\mathbb{Q}} H_{6t}(MSO(2t); \mathbb{Q})$$

and

$$\Omega_{4t} \cong \pi_{4t}^S(MSO) \cong_{\mathbb{Q}} H_{4t}(MSO; \mathbb{Q}),$$

where $\cong_{\mathbb{Q}}$ means isomorphic if tensored with \mathbb{Q} . Thus ψ being epimorphic is equivalent to

$$\psi_H : H_{6t}(MSO(2t); \mathbb{Q}) \rightarrow H_{4t}(MSO; \mathbb{Q})$$

being epimorphic, which is further equivalent to (by taking the dual morphism in cohomology)

$$\psi^* : H^{4t}(MSO; \mathbb{Q}) \rightarrow H^{6t}(MSO(2t); \mathbb{Q})$$

being monomorphic. We can apply the Thom-isomorphism to further reduce the problem to showing that

$$\psi_B^* : H^{4t}(BSO; \mathbb{Q}) \rightarrow H^{4t}(BSO(2t); \mathbb{Q})$$

is monomorphic. It is easy to see that ψ_B^* is induced by the natural inclusion map $BSO(2t) \hookrightarrow BSO$. The cohomology ring of $BSO(2t)$ is the polynomial ring $\mathbb{Q}[p_1, p_2, \dots, p_{t-1}, \chi_{2t}]$ generated by the Pontrjagin classes and the Euler class, whose square is p_t . On the other hand $H^*(BSO; \mathbb{Q}) \cong \mathbb{Q}[p_1, p_2, \dots]$. As ψ_B^* takes each Pontrjagin class to the same Pontrjagin class, we get that ψ_B^* is indeed injective in dimension $4t$. This completes the proof of $tp_{\Sigma^2} = p_t$. \square

When we want to consider direct products of maps, we will need the Cartan formula. For Pontrjagin classes the Cartan formula only holds mod 2, so we will need to consider everything in $H^*(M; \mathbb{Q})$ to get rid of the 2-torsion.

The proof of the next theorem copies the proof of the previous section.

Theorem 9. *Let $f : M_1^{n_1} \rightarrow N^{n_1+k_1}, g : M_2^{n_2} \rightarrow N^{n_2+k_2}$ be two generic maps of even codimension. Then for a generic perturbation of their product we have*

$$[\Sigma^2 f \times g] = \sum_{j \geq 1} \left([\Sigma^2 f_{2j-2}] \times \text{id}_{2j}^* [\Sigma^2 g_{(-2j)}] + \text{id}_{2j}^* [\Sigma^2 f_{(-2j)}] \times [\Sigma^2 g_{2j-2}] \right)$$

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