Another introduction to the geometry of metric spaces

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Abstract

Here Lipschitz conditions are used as a primary tool, for studying curves in metric spaces in particular.

Contents

1 Basic notions

A *metric space* is a nonempty set M equipped with a distance function $d(x, y)$ defined for $x, y \in M$ such that $d(x, y)$ is a nonnegative real number which is

The present notes are somewhat complementary to [9], and the two can be read in either order.

equal to 0 exactly when $x = y$,

$$d(y, x) = d(x, y) \tag{1.1}$$

for every $x, y \in M$, and

$$d(x, z) \le d(x, y) + d(y, z) \tag{1.2}$$

for every $x, y, z \in M$. This last condition is known as the *triangle inequality*.

As usual, $\mathbf{R}$ denotes the real line, and the *absolute value* $|r|$ of $r \in \mathbf{R}$ is defined to be r when $r \ge 0$ and $-r$ when $r \le 0$. It is easy to check that

$$|r + t| \le |r| + |t| \tag{1.3}$$

for every $r, t \in \mathbf{R}$, which implies that $|r - t|$ is a metric on $\mathbf{R}$.

Let $(M, d(x, y))$ be a metric space. For each $x \in M$ and $r > 0$, the *open ball* with center x and radius r is defined by

$$B(x, r) = \{y \in M : d(x, y) < r\}, \tag{1.4}$$

and the corresponding *closed ball* is defined to be

$$\overline{B}(x, r) = \{y \in M : d(x, y) \le r\}. \tag{1.5}$$

A set $E \subseteq M$ is said to be *bounded* if E is contained in a ball in M. For any $p, q \in M$ and $r > 0$,

$$B(p, r) \subseteq B(q, r + d(p, q)) \tag{1.6}$$

and

$$\overline{B}(p, r) \subseteq \overline{B}(q, r + d(p, q)), \tag{1.7}$$

by the triangle inequality. It follows that a bounded set $E \subseteq M$ is contained in a ball centered at any point in M.

2 Norms on $\mathbf{R}^n$

For each positive integer n, $\mathbf{R}^n$ is the space of n-tuples $x = (x_1, \ldots, x_n)$ of real numbers, i.e., $x_1, \ldots, x_n \in \mathbf{R}$. This is a vector space with respect to coordinatewise addition and scalar multiplication by real numbers.

Suppose that $N(x)$ is a function defined on $\mathbf{R}^n$ with values in the nonnegative real numbers. We say that $N(x)$ is a *norm* on $\mathbf{R}^n$ if $N(x) = 0$ exactly when $x = 0$,

$$N(r\, x) = |r|\, N(x) \tag{2.1}$$

for every $r \in \mathbf{R}$ and $x \in \mathbf{R}^n$, and

$$N(x + y) \le N(x) + N(y) \tag{2.2}$$

for every $x, y \in \mathbf{R}^n$. If $N(x)$ is a norm on $\mathbf{R}^n$, then

$$d_N(x, y) = N(x - y) \tag{2.3}$$

is a metric on $\mathbf{R}^n$.

The absolute value function is a norm on the real line, and any norm on $\mathbf{R}$ can be expressed as $a\,|x|$ for some $a > 0$. The standard Euclidean norm on $\mathbf{R}^n$ is defined by

$$|x| = \Big(\sum_{j=1}^n x_j^2\Big)^{1/2}, \tag{2.4}$$

and determines the standard metric on $\mathbf{R}^n$. It is not completely obvious that this satisfies the triangle inequality, and one way to show this will be mentioned in the next section. It is much easier to check directly that

$$\|x\|_1 = \sum_{j=1}^n |x_j| \tag{2.5}$$

and

$$\|x\|_\infty = \max(|x_1|, \ldots, |x_n|) \tag{2.6}$$

are norms on $\mathbf{R}^n$. Note that the standard norm may also be denoted $\|x\|_2$.

If $N(x)$ is any norm on $\mathbf{R}^n$, then

$$N(x) \le \max(N(e_1), \ldots, N(e_n))\,\|x\|_1, \tag{2.7}$$

where $e_1, \ldots, e_n$ are the standard basis vectors in $\mathbf{R}^n$, which is to say that the jth coordinate of e_ℓ is equal to 1 when $j = \ell$ and to 0 otherwise. Indeed,

$$x = \sum_{j=1}^n x_j\, e_j, \tag{2.8}$$

and therefore

$$N(x) \le \sum_{j=1}^n N(e_j)\,|x_j|. \tag{2.9}$$

In particular,

$$\|x\|_\infty \le \|x\|_2 \le \|x\|_1 \tag{2.10}$$

for every $x \in \mathbf{R}^n$, since the first inequality holds by inspection. One can also get the second inequality by observing that

$$\sum_{j=1}^n x_j^2 \le \|x\|_1\,\|x\|_\infty, \tag{2.11}$$

since $\|x\|_\infty \le \|x\|_1$ also holds by inspection. In the other direction, it is easy to see that

$$\|x\|_1 \le n\,\|x\|_\infty \tag{2.12}$$

and

$$\|x\|_2 \le \sqrt{n}\,\|x\|_\infty, \tag{2.13}$$

and one can use the convexity of $\phi(t) = t^2$ on the real line to show that

$$\|x\|_1 \le \sqrt{n}\,\|x\|_2. \tag{2.14}$$

3 Convex sets in $\mathbf{R}^n$

A set $E \subseteq \mathbf{R}^n$ is said to be *convex* if for every $x, y \in E$ and every real number t, $0 \le t \le 1$,

$$t\,x + (1-t)\,y \in E. \tag{3.1}$$

For example, open and closed balls with respect to the metric associated to a norm on $\mathbf{R}^n$ are convex.

Conversely, suppose that $N(x)$ is a nonnegative real-valued function on $\mathbf{R}^n$ that satisfies $N(x) > 0$ when $x \ne 0$ and the homogeneity condition (2.1). If

$$B_N = \{x \in \mathbf{R}^n : N(x) \le 1\} \tag{3.2}$$

is convex, then N satisfies the triangle inequality (2.2) and hence is a norm. To see this, let $x, y \in \mathbf{R}^n$ be given, and let us check (2.2). The inequality is trivial when $x = 0$ or $y = 0$, and so we may suppose that $x, y \ne 0$. Put

$$x' = \frac{x}{N(x)}, \quad y' = \frac{y}{N(y)}, \tag{3.3}$$

which automatically satisfy

$$N(x') = N(y') = 1. \tag{3.4}$$

For $0 \le t \le 1$, convexity of B_N implies that

$$N(t\,x' + (1-t)\,y') \le 1. \tag{3.5}$$

If

$$t = \frac{N(x)}{N(x) + N(y)}, \tag{3.6}$$

then

$$1 - t = \frac{N(y)}{N(x) + N(y)} \tag{3.7}$$

and

$$t\,x' + (1-t)\,y' = \frac{x+y}{N(x) + N(y)}, \tag{3.8}$$

which means that (2.2) follows from (3.5).

One can use the convexity of the function $\phi(r) = r^2$ on the real line to show directly that the closed unit ball with respect to the standard Euclidean norm is a convex set in $\mathbf{R}^n$, and hence that the Euclidean norm satisfies the triangle inequality and is therefore a norm. For each real number p, $1 \le p < \infty$, put

$$\|x\|_p = \Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p}. \tag{3.9}$$

One can use the convexity of the function $\phi_p(r) = |r|^p$ on the real line to show that the closed unit ball associated to $\|x\|_p$ is a convex set in $\mathbf{R}^n$, and therefore that $\|x\|_p$ is a norm on $\mathbf{R}^n$.

By inspection,

$$\|x\|_\infty \le \|x\|_p \tag{3.10}$$

for every $x \in \mathbf{R}^n$ and $1 \le p < \infty$. If $1 \le p < q < \infty$, then

$$\sum_{j=1}^n |x_j|^q \le \Big(\sum_{j=1}^n |x_j|^p\Big) \, \|x\|_\infty^{q-p}, \tag{3.11}$$

which implies that

$$\|x\|_q \le \|x\|_p^{p/q} \, \|x\|_\infty^{1-(p/q)} \tag{3.12}$$

and thus

$$\|x\|_q \le \|x\|_p \tag{3.13}$$

for every $x \in \mathbf{R}^n$.

4 Lipschitz conditions, 1

Let $(M_1, d_1(x, y))$, $(M_2, d_2(u, v))$ be metric spaces. A mapping $f : M_1 \to M_2$ is said to be *Lipschitz* with constant $C \ge 0$ or C-Lipschitz if

$$d_2(f(x), f(y)) \le C \, d_1(x, y) \tag{4.1}$$

for every $x, y \in M_1$. More precisely, f is Lipschitz of order 1 if this holds for some $C \ge 0$, and we shall discuss Lipschitz conditions of any order $a > 0$ a bit later. Note that f is Lipschitz with $C = 0$ if and only if f is constant, and that Lipschitz mappings are automatically uniformly continuous.

If M_2 is the real line with the standard metric, then the preceding Lipschitz condition is equivalent to

$$f(x) \le f(y) + C \, d_1(x, y). \tag{4.2}$$

This follows by interchanging the order of x and y. In particular,

$$f_p(x) = d_1(x, p) \tag{4.3}$$

is Lipschitz with $C = 1$ for every $p \in M_1$, by the triangle inequality. If f, $\widetilde{f}$ are real-valued Lipschitz functions on M_1 with constants C, $\widetilde{C}$, respectively, then $f + \widetilde{f}$ is Lipschitz with constant $C + \widetilde{C}$. Moreover, $a\, f$ is Lipschitz with constant $|a| \, C$ for every $a \in \mathbf{R}$. The product of bounded real-valued Lipschitz functions is also Lipschitz. If $(M_3, d_3(w, z))$ is another metric space, and $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$ are Lipschitz mappings with constants C_1, C_2, respectively, then the composition $f_2 \circ f_1 : M_1 \to M_3$ defined by

$$(f_2 \circ f_1)(x) = f_2(f_1(x)) \tag{4.4}$$

is Lipschitz with constant $C_1 \, C_2$.

For any mapping $f : M_1 \to M_2$ and set $A \subseteq M_1$,

$$f(A) = \{f(x) : x \in A\} \subseteq M_2. \tag{4.5}$$

Let $B_1(x, r)$ and $B_2(p, t)$ be the open balls in M_1, M_2 with centers $x \in M_1$, $p \in M_2$ and radii $r, t > 0$, respectively. It is easy to see that $f : M_1 \to M_2$ is Lipschitz with constant $C > 0$ if and only if

$$f(B_1(x, r)) \subseteq B_2(f(x), C\, r) \tag{4.6}$$

for every $x \in M_1$ and $r > 0$. This is also equivalent to the analogous condition

$$f(\overline{B}_1(x, r)) \subseteq \overline{B}_2(f(x), C\, r) \tag{4.7}$$

for closed balls. In particular, if A is a bounded set in M_1, then $f(A)$ is bounded in M_2.

Suppose that f is a real-valued function on the real line, equipped with the standard metric. If f is differentiable at a point $x \in \mathbf{R}$, and f is C-Lipschitz for some $C \geq 0$, then the derivative $f'(x)$ of f at x satisfies

$$|f'(x)| \leq C. \tag{4.8}$$

This follows from the definition of the derivative. Conversely, if f is differentiable and satisfies (4.8) at every point in $\mathbf{R}$, then f is C-Lipschitz, by the mean value theorem. Note that $f(x) = |x|$ is 1-Lipschitz on $\mathbf{R}$ and not differentiable at $x = 0$.

5 Lipschitz curves

Let $(M, d(x, y))$ be a metric space, and suppose that a, b are real numbers with $a \leq b$. As usual, the closed interval $[a, b]$ in the real line consists of the $r \in \mathbf{R}$ such that $a \leq r \leq b$. Suppose also that $p : [a, b] \to M$ is Lipschitz with constant k for some $k \geq 0$. If $\{t_j\}_{j=0}^n$ is a finite sequence of real numbers such that

$$a = t_0 < t_1 < \cdots < t_n = b, \tag{5.1}$$

then

$$\sum_{j=1}^n d(p(t_j), p(t_{j-1})) \leq k \sum_{j=1}^n (t_j - t_{j-1}) = k\,(b - a). \tag{5.2}$$

This is often described by saying that the length of the curve determined by $p(t)$, $a \leq t \leq b$, has length $\leq k\,(b - a)$.

Of course, one can use translations on the real line to shift the interval on which a path is defined without changing the Lipschitz constant. One can use affine mappings on $\mathbf{R}$ to change the length of the interval on which a path is defined, with a corresponding change in the Lipschitz constant. The product of the Lipschitz constant and the length of the interval would remain the same.

If $c \in \mathbf{R}$, $c \geq b$, $q : [b, c] \to M$ is k-Lipschitz, and $p(b) = q(b)$, then the mapping from $[a, c]$ into M defined by combining p and q is k-Lipschitz too. This is easy to verify, directly from the definitions. If the Lipschitz constants for p and q are different, then it may be preferable to rescale the intervals so that the Lipschitz constants are the same. If p is constant on an interval $[a_1, b_1] \subseteq [a, b]$, then one can remove (a_1, b_1) from $[a, b]$ and combine the remaining pieces to get a curve with the same Lipschitz constant on a smaller interval.

6 Minimality

Let $(M, d(x, y))$ be a metric space in which closed and bounded sets are compact. Suppose that $x, y \in M$ can be connected by a Lipschitz curve in M. This means that there is a Lipschitz mapping $p : [0, 1] \to M$ such that $p(0) = x$ and $p(1) = y$. Using the Arzela-Ascoli theorem, one can show that there is such a path whose Lipschitz constant is as small as possible. For suppose that $p_1, p_2, \ldots$ is a sequence of Lipschitz mappings from $[0, 1]$ into M whose Lipschitz constants $k_1, k_2, \ldots$, respectively, converge to the infimum k of the possible Lipschitz constants. By passing to a subsequence, we may suppose that the sequence of mappings converges uniformly on $[0, 1]$. The limiting mapping sends 0 to x and 1 to y, and it is easy to check that it is Lipschitz with constant k. Note that $k \geq d(x, y)$.

7 Affine paths in $\mathbf{R}^n$

Fix a positive integer n, and consider an affine mapping $p : \mathbf{R} \to \mathbf{R}^n$, given by $p(r) = u + r\,v$ for some $u, v \in \mathbf{R}^n$. If N is any norm on $\mathbf{R}^n$, then p is Lipschitz with constant $N(v)$ with respect to the standard metric on $\mathbf{R}$ and the metric associated to N on $\mathbf{R}^n$, since

$$N(p(r) - p(t)) = |r - t|\, N(v) \tag{7.1}$$

for every $r, t \in \mathbf{R}^n$. For any $a, b \in \mathbf{R}$ with $a \leq b$, the restriction of p to $[a, b]$ is a Lipschitz curve connecting $p(a)$ to $p(b)$ with constant $N(v)$, and $N(v)$ is the smallest possible Lipschitz constant for such a curve on $[a, b]$.

A norm N on $\mathbf{R}^n$ is said to be *strictly convex* if the corresponding closed unit ball B_N as in (3.2) is strictly convex. This means that for every $x, y \in \mathbf{R}^n$ with $N(x) = N(y) = 1$ and $x \neq y$, we have that

$$N(t\,x + (1 - t)\,y) < 1 \tag{7.2}$$

when $0 < t < 1$. Equivalently, if $w, z \in \mathbf{R}^n$ and

$$N(w + z) = N(w) + N(z), \tag{7.3}$$

then either $w = 0$, $z = 0$, or $z = r\,w$ for some $r > 0$. This follows from an argument like the one used in Section 3 to show that convexity of B_N implies the triangle inequality for N when N is homogeneous.

One can show that the standard Euclidean norm on $\mathbf{R}^n$ is strictly convex, using strict convexity of the function $\phi(r) = r^2$ on the real line. Similarly, $\|x\|_p$ is strictly convex on $\mathbf{R}^n$ when $1 < p < \infty$, as a consequence of the strict convexity of $\phi_p(r) = |r|^p$. In particular, the absolute value is strictly convex as a norm on $\mathbf{R}$, if not in the ordinary sense for arbitrary functions, because equality holds in (1.3) only when $r, t \ge 0$ or $r, t \le 0$. However, $\|x\|_1$ and $\|x\|_\infty$ are not strictly convex norms on $\mathbf{R}^n$ when $n \ge 2$.

Suppose that N is a strictly convex norm on $\mathbf{R}^n$. If $x, y, z \in \mathbf{R}^n$ satisfy

$$d_N(x, z) = d_N(x, y) + d_N(y, z), \tag{7.4}$$

where d_N is as defined in (2.3), then

$$y = r\,x + (1 - r)\,z \tag{7.5}$$

for some $r \in [0, 1]$. If $q : [a, b] \to \mathbf{R}^n$ is k-Lipschitz and

$$N(q(b) - q(a)) = k\,(b - a), \tag{7.6}$$

then

$$d_N(q(a), q(b)) = d_N(q(a), q(t)) + d_N(q(t), q(b)) \tag{7.7}$$

for every $t \in [a, b]$. One can use this to show that $q(t)$ is affine. This does not work for the norms $\|x\|_1$, $\|x\|_\infty$ on $\mathbf{R}^n$ when $n \ge 2$. For example, there is a 1-Lipschitz path from $[0, 2]$ into $\mathbf{R}^2$ equipped with the norm $\|x\|_1$ that connects $(0, 0)$ to $(1, 1)$ by following the horizontal segment to $(1, 0)$ and then the vertical segment to $(1, 1)$. If $\phi : [0, 1] \to \mathbf{R}$ is any 1-Lipschitz function with respect to the standard metric on the real line which satisfies $\phi(0) = \phi(1) = 0$, then $\Phi(t) = (t, \phi(t))$ is a 1-Lipschitz mapping from $[0, 1]$ into $\mathbf{R}^2$ equipped with the norm $\|x\|_\infty$ that connects $(0, 0)$ to $(1, 0)$.

8 C^1 paths in $\mathbf{R}^n$

Let N be a norm on $\mathbf{R}^n$. As in Section 4, the triangle inequality implies that N is 1-Lipschitz with respect to the associated metric d_N. As in Section 2, one can show that N is less than or equal to a constant multiple of the standard Euclidean norm on $\mathbf{R}^n$. It follows that N is also a Lipschitz function with respect to the standard metric on $\mathbf{R}^n$.

Let $p : [a, b] \to \mathbf{R}^n$ be a continuously-differentiable curve with derivative $p'(t)$. This implies that $N(p'(t))$ is a continuous function on $[a, b]$. By the fundamental theorem of calculus,

$$p(t) - p(r) = \int_r^t p'(u)\,du \tag{8.1}$$

when $a \le r \le t \le b$. Hence

$$N(p(t) - p(r)) \le \int_r^t N(p'(u))\,du, \tag{8.2}$$

using an extension of the triangle inequality from sums to integrals. If

$$N(p'(u)) \le k \tag{8.3}$$

for every $u \in [a,b]$, then it follows that p is k-Lipschitz with respect to the metric associated to N on $\mathbf{R}^n$.

Alternatively, let $\epsilon > 0$ be given. For each $r \in [a,b]$,

$$p(t) - p(r) - p'(r)\,(t-r) \tag{8.4}$$

is ϵ-Lipschitz as a function of t on sufficiently small neighborhoods of r in $[a,b]$, since p is continuously-differentiable. Under the hypothesis (8.3), we get that p is $(k+\epsilon)$-Lipschitz with respect to N on sufficiently small neighborhoods of every point in $[a,b]$. One can use this to show that p is $(k+\epsilon)$-Lipschitz on $[a,b]$, and therefore k-Lipschitz because $\epsilon > 0$ is arbitrary. Note that (8.3) holds when p is k-Lipschitz with respect to N on $[a,b]$.

In order for the product of the Lipschitz constant and the length of the parameter interval to be as small as possible, it would be nice to have $N(p')$ constant on $[a,b]$. As in the classical situation, one can try to get this by reparameterizing p. This is easy to do when $p'(t) \ne 0$ for every $t \in [a,b]$. Specifically,

$$\phi(t) = \int_a^t N(p'(u))\,du \tag{8.5}$$

is a continuously-differentiable function on $[a,b]$ with

$$\phi'(t) = N(p'(t)) > 0 \tag{8.6}$$

for each $t \in [a,b]$. If $q = p \circ \phi^{-1}$, then

$$N(q'(r)) = 1 \tag{8.7}$$

when $\phi(a) \le r \le \phi(b)$.

9 Lipschitz conditions, 2

Let $(M_1, d_1(x,y))$ and $(M_2, d_2(u,v))$ be metric spaces. A mapping $f : M_1 \to M_2$ is said to be *Lipschitz of order* $\alpha > 0$ with constant $C \ge 0$ if

$$d_2(f(x), f(y)) \le C\, d_1(x,y)^\alpha \tag{9.1}$$

for every $x, y \in M_1$. As before, this holds with $C = 0$ if and only if f is constant, and Lipschitz mappings of any order are uniformly continuous. If a real-valued function on the real line is Lipschitz of order $\alpha > 1$, then it is constant, because it has derivative 0, although one could also show this more directly. It follows that a Lipschitz mapping of order $\alpha > 1$ from an interval in the real line into any metric space is constant as well, by composing with real-valued Lipschitz functions of order 1 on the range, such as the distance to a fixed point.

Suppose that $0 < \beta < 1$. If $r, t \geq 0$, then

$$\max(r, t) \leq (r^\beta + t^\beta)^{1/\beta}. \tag{9.2}$$

Therefore

$$r + t \leq \max(r, t)^{1-\beta} \, (r^\beta + t^\beta) \leq (r^\beta + t^\beta)^{1/\beta}, \tag{9.3}$$

or equivalently

$$(r + t)^\beta \leq r^\beta + t^\beta. \tag{9.4}$$

This is also very easy to check algebraically when $\beta = 1/2$, for instance.

If $(M, d(w, z))$ is a metric space, then it follows that $d(w, z)^\beta$ is a metric on M too when $0 < \beta < 1$. This does not work when $\beta > 1$, even for the real line. Observe that $f : M_1 \to M_2$ is Lipschitz of order α with respect to $d_1(x, y)$ on M_1 if and only if f is Lipschitz of order α/β with respect to $d_1(x, y)^\beta$, keeping $d_2(u, v)$ fixed on M_2. Similarly, f is Lipschitz of order α with respect to $d_2(u, v)$ on M_2 if and only if f is Lipschitz of order $\alpha \, \beta$ with respect to $d_2(u, v)^\beta$ on M_2, keeping $d_1(x, y)$ fixed on M_1.

A curve $p : [a, b] \to M$ in a metric space $(M, d(w, z))$ parameterized by a Lipschitz mapping of order $\alpha < 1$ can be quite different from the case where $\alpha = 1$. The length of p can be infinite, and moreover $p([a, b])$ can be fractal. This includes common examples of snowflake curves in the plane. Instead one can show that the α-dimensional Hausdorff measure of $p([a, b])$ is finite.

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