

Hodge-theoretic aspects of the Decomposition Theorem

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Abstract

Given a projective morphism of compact, complex, algebraic varieties and a relatively ample line bundle on the domain we prove that a suitable choice, dictated by the line bundle, of the decomposition isomorphism of the Decomposition Theorem of Beilinson, Bernstein, Deligne and Gabber, yields isomorphisms of pure Hodge structures. The proof is based on a new cohomological characterization of the decomposition isomorphism associated with the line bundle. We prove some corollaries concerning the intersection form in intersection cohomology, the natural map from cohomology to intersection cohomology, projectors and Hodge cycles, and induced morphisms in intersection cohomology.

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1 Introduction

Let $f : X \rightarrow Y$ be a projective map of proper, complex, algebraic varieties. The Decomposition Theorem predicts that the derived direct image complex Rf_*IC_X of the rational intersection cohomology complex IC_X of X splits into the direct sum of shifted intersection cohomology complexes on Y . This splitting is *not* canonical. When viewed in hypercohomology, it yields decompositions of the rational intersection cohomology groups $IH(X, \mathbb{Q})$ into the direct sum of intersection cohomology groups with twisted coefficients of closed subvarieties of Y .

The Decomposition Theorem is the deepest known fact concerning the homology of complex algebraic varieties and it has far-reaching consequences. The following consideration may give a measure of the importance as well as of the special character of this result. The splitting behavior of Rf_*IC_X over Y is dictated in part by the one over any open subset $U \subseteq Y$. This remarkable fact has no counterpart in other geometries, e.g. complex analytic geometry, real algebraic geometry, etc. More precisely: let $U \subseteq Y$ be a Zariski-dense open subset, $S \subseteq U$ be a closed submanifold, \mathcal{L} be a local system, i.e. a locally constant sheaf, on S ; assume that a shift $\mathcal{L}[l]$ is a direct summand of $(Rf_*IC_X)|_U$ on U ; then a certain shift of the intersection cohomology complex $IC_{\bar{S}}(\mathcal{L})$ on the closure $\bar{S} \subseteq Y$ is a direct summand of Rf_*IC_X on Y .

However, the decomposition isomorphism is *not* canonical and it is not clear, and in fact not true, that the various additional structures present in the various intersection cohomology groups involved should be preserved under an arbitrary splitting. Let us consider the example of resolution of singularities. In this case the Decomposition Theorem predicts the existence of splitting injections $IH(Y, \mathbb{Q}) \rightarrow H(X, \mathbb{Q})$. One may ask if it is possible to realize $IH(Y, \mathbb{Q})$ as a sub-Hodge structure of the pure Hodge structure $H(X, \mathbb{Q})$. More generally, given any projective map f , one may ask the same question for all the summands (see (5) and (7)) of $IH(X, \mathbb{Q})$ arising from the Decomposition Theorem.

In this paper, by building on our previous work [7], we answer this question affirmatively in Theorem 2.8.1 by checking that a certain decomposition isomorphism g_η , that depends on the choice of an f -ample line bundle η on X , turns out to do the job.

Let us summarize the contents of this paper. Given a projective map $f : X \rightarrow Y$ as above and an f -ample line bundle η on X , Deligne, in [10] has constructed a distinguished decomposition isomorphism g_η . Theorem 2.8.1 shows that g_η induces an isomorphism (7) of pure Hodge structures. Let us emphasize that while this isomorphism, being an isomorphism in the derived category, is of a local nature, our result implies that it has global consequences, specifically, concerning the pure Hodge structure in intersection

cohomology. The proof is based on Proposition 2.7.1, i.e. on a property of g_η expressed via its primitive components $f_{i,\eta}$.

We obtain the following Hodge-theoretic consequences: Theorem 3.1.1, on the intersection pairing on intersection cohomology; Theorem 3.2.1, on the natural map $a_Y : H(Y, \mathbb{Q}) \rightarrow IH(Y, \mathbb{Q})$; Theorem 3.3.1, on the homological cycles associated with the Decomposition Theorem; Theorem 3.4.1, on the morphism induced by a surjective f in intersection cohomology.

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1.1 Notation and preliminaries

We fix the following notation. See also [7]. For an introduction to the decomposition theorem with some examples worked out see [8].

– $f : X^n \rightarrow Y^m$: a projective map of compact, complex, algebraic varieties of the indicated dimensions.

– η : the first Chern class of a f -ample line bundle on X .

– $f_\bullet = Rf_*$: the derived direct image functor.

– $H(X) = \mathbb{H}(X, \mathbb{Q}_X) = \mathbb{H}(Y, f_\bullet \mathbb{Q}_X)$: the \mathbb{Q} -singular cohomology of X ; at times we omit seemingly unnecessary cohomological degrees.

– IC_X : the intersection cohomology complex X with \mathbb{Q} -coefficients; if X is smooth, then $IC_X = \mathbb{Q}_X[n]$.

– $IH^{n+l}(X) = \mathbb{H}^l(X, IC_X) = \mathbb{H}^l(Y, f_\bullet IC_X), l \in \mathbb{Z}$: the \mathbb{Q} -intersection cohomology groups of X .

– D_Y : the bounded derived category of constructible sheaves on Y of \mathbb{Q} -vector spaces, endowed with the t -structure associated with middle-perversity.

– P_Y : the abelian category of perverse sheaves on Y ; $P_Y \subseteq D_Y$ is the heart of the middle-perversity t -structure.

– ${}^p H^i : D_Y \rightarrow P_Y$: the associated cohomological functors.

– PHS, MHS, SHS: pure, mixed and Hodge sub-structure.

If $a : K \rightarrow K'$ is a morphism in D_Y , then we often use the same symbol for the induced map in hypercohomology.

The category P_Y is Artinian and the Jordan-Hölder Theorem holds. The simple objects are the intersection cohomology complexes $IC_{\overline{Z}}(\mathcal{L})$ where $Z \subseteq Y$ is a smooth locally closed subvariety and \mathcal{L} is a simple local system on Z . A semisimple object of P_Y is a finite direct sum of such objects.

The following results have been first proved by Beilinson, Bernstein, Deligne and Gabber in [2] using algebraic geometry in positive characteristic. M. Saito has proved them in the more general context of mixed Hodge modules in [15]. We have given a proof in

[7] using classical Hodge theory. The earlier paper [5] had dealt with the special, but revealing case of semismall maps.

Theorem 1.1.1 (Decomposition Theorem (DT)) *There exists an isomorphism in the derived category D_Y*

$$\phi : \bigoplus_{i \in \mathbb{Z}} {}^p H^i(f_\bullet IC_X)[-i] \simeq f_\bullet IC_X, \quad {}^p H^i(f_\bullet IC_X) \text{ semisimple in } P_Y.$$

The Chern class η defines a map $\eta : IC_X \rightarrow IC_X[2]$. Taking push-forwards and cohomology we get maps

$$e := {}^p H^j(f_\bullet \eta) : {}^p H^j(f_\bullet IC_X) \longrightarrow {}^p H^{j+2}(f_\bullet IC_X).$$

Theorem 1.1.2 (Relative Hard Lefschetz Theorem) *For all $i \geq 0$ the map*

$$e^i : {}^p H^{-i}(f_\bullet IC_X) \rightarrow {}^p H^i(f_\bullet IC_X),$$

is an isomorphism.

Let us collect together some well-known facts that we shall use.

Let Y be a proper algebraic variety. Goresky-MacPherson defined the intersection homology using a sub-complex of the complex of geometric chains of Y . This gives rise to a natural map $IH_l(Y) \rightarrow H_l(Y)$. Using the perfect pairing in intersection (co)homology mentioned below, there is the natural dual map $H^l(Y) \rightarrow IH^l(Y) = IH_{2m-l}(Y) = IH_l(Y)^\vee$. This map can be described also as the map in hypercohomology stemming from the natural map $a_Y : \mathbb{Q}_Y[m] \rightarrow IC_Y$ that corresponds to $1 \in \mathbb{Q}$ under the isomorphism

$$\mathbb{Q} = H^0(Y) = \text{Hom}(\mathbb{Q}_Y[m], \mathbb{Q}_Y[m]) \simeq \text{Hom}(\mathbb{Q}_Y[m], IC_Y) = IH^0(Y) = \mathbb{Q} \quad (1)$$

The number $1 \in \mathbb{Q} = H^0(X)$, Id and the adjunction map $adj(f) : \mathbb{Q}_Y \rightarrow f_\bullet \mathbb{Q}_X$ correspond to each other under the isomorphisms

$$\mathbb{Q} = H^0(X) = \text{Hom}(\mathbb{Q}_X, \mathbb{Q}_X) \simeq \text{Hom}(\mathbb{Q}_Y, f_\bullet \mathbb{Q}_X). \quad (2)$$

The map $adj(f)$ induces the familiar pull-back in cohomology $f^* : H(Y) \rightarrow H(X)$.

By adjunction and by (1) applied to X :

$$\text{Hom}(\mathbb{Q}_Y[n], f_\bullet IC_X) = \text{Hom}(\mathbb{Q}_X[n], IC_X) \simeq \mathbb{Q}. \quad (3)$$

Remark 1.1.3 The equalities above hold if we replace Y by a *connected* open subset $U \subseteq Y$ and X by $f^{-1}(U)$.

Given a proper variety Y , there is a non-degenerate intersection pairing on intersection cohomology:

$$IH^{n-l}(Y) \times IH^{n+l}(Y) \longrightarrow \mathbb{Q}.$$

It has been first defined by Goresky-MacPherson in terms of geometric cycles. It affords an alternative description as the map in hypercohomology stemming from the duality isomorphism

$$d_Y : IC_Y \simeq IC_Y^\vee,$$

On the regular part, the isomorphism d_Y coincides with the usual Poincaré duality isomorphism. One way to say this is the following: $IC_{Y_{reg}}$ is canonically isomorphic to $\mathbb{Q}_{Y_{reg}}[m]$ and the duality isomorphism for IC_Y is the unique morphism in $\text{Hom}(IC_Y, IC_Y^\vee) \simeq \mathbb{Q}$ extending the duality isomorphism for $\mathbb{Q}_{Y_{reg}}[m]$.

1.2 Review of [7]

We recall some of the result of our paper [7] in the form we need them here.

For every $l \in \mathbb{Z}$, $IH^l(X)$ carries a canonical PHS of weight l .

The subspaces of the perverse filtration (13) $IH_{\leq i}^l(X) \subseteq IH^l(X)$, $i \in \mathbb{Z}$, are SHS. In fact, the filtration $IH_{\leq i}^l(X)$ can be described up to shift as the monodromy weight filtration of the endomorphism $IH(X) \rightarrow IH(X)$ given by the cup-product with the pull back of *any* ample bundle on Y . The graded pieces (14) $IH_i^l(X) = IH_{\leq i}^l(X)/IH_{\leq i-1}^l(X)$ inherit the quotient PHS.

Let $Y = \coprod_{d \geq 0} S_d = \coprod_{d \geq 0} \coprod S$, be a stratification of Y adapted to f , where S ranges over the connected components of the d -dimensional stratum S_d . There is a canonical decomposition given by strata for the semisimple:

$${}^pH^i(f \bullet IC_X) = \bigoplus_{d \geq 0} \bigoplus_{S \subseteq S_d} IC_{\overline{S}}(\mathcal{L}_{i,S}), \quad (4)$$

where $\mathcal{L}_{i,S}$ are semisimple local systems on S .

The ensuing decomposition in hypercohomology is by SHS:

$$IH_i^{n+l}(X) = \mathbb{H}^l(Y, {}^pH^i(f \bullet IC_X)[-i]) = \bigoplus_{d,S} \mathbb{H}^l(Y, IC_{\overline{S}}(\mathcal{L}_{i,S})[-i]), \quad \forall i, l \in \mathbb{Z}, \quad (5)$$

where the first equality stems from (14).

There are Hard Lefschetz isomorphisms for the action of η on the graded pieces:

$$e^i : IH_{-i}^l(X) \simeq IH_i^{l+2i}(X), \quad \forall l \in \mathbb{Z}, \forall i \geq 0. \quad (6)$$

A natural question, see [2] and [14], is whether the decomposition

$$\bigoplus_i \phi(IH_i^{n+l}(X)) = IH^{n+l}(X), \quad (7)$$

its refinements stemming from (5) and the further refinements stemming from the (η, L) -decomposition we prove in [7], are isomorphisms of PHS for a suitable choice of the isomorphism ϕ .

Our main Theorem 2.8.1 gives a positive answer.

We shall need the following simple

Lemma 1.2.1 *Let A and B be rational Hodge structures and*

$$A \xrightarrow{g} B \xrightarrow{p} A$$

be linear maps with $p \circ g = Id$, and p be a map of rational Hodge structures and $g(A) \subseteq B$ a SHS.

Then g is a map of HS.

Proof. We need to show that, after complexification, $g(A^{pq}) \subseteq B^{pq}$. Let $a_{pq} \in A^{pq}$. We have that $g(a_{pq}) = \sum b_{st}$ for unique $b_{st} \in B^{st}$. Noting that g is necessarily injective and since we are assuming that $g(A) = \oplus (g(A) \cap B^{pq})$, then $b_{st} = g(c_{st})$ for a unique collection $c_{st} \in A$. Since $a_{pq} = \sum p(b_{st})$, we have that $p(b_{st}) = 0$ for $(s, t) \neq (p, q)$ and we also have that $0 = p(g(c_{st})) = c_{st}$ for the same (s, t) . It follows that $a_{pq} = c_{pq}$ and that $g(a_{pq}) = g(c_{pq}) = b_{pq}$. \square

Remark 1.2.2 The example of $A = B$ as vector spaces, but with conjugate Hodge structures, shows that having g defined over \mathbb{Q} and having image a SHS is not sufficient to have a map of HS.

2 Formalism for decompositions

The aim of this paper is to show that the isomorphism g_η , constructed by Deligne in [10], gives rise to an isomorphism of PHS.

The morphism g_η is constructed by assembling together certain morphisms $f_{i,\eta}$ defined on the primitive components $P_\eta^{-i} \subseteq {}^pH^*(f_\bullet IC_X)$ for the action of η on the perverse cohomology complexes ${}^pH(f_\bullet IC_X)$.

In this section we review the constructions of g_η and $f_{i,\eta}$ given in [10]. We then prove Proposition 2.7.1 that is the key to our main result Theorem 2.8.1.

To simplify the notation, we present most of the material of this section in the abstract context of a triangulated category with t -structure. For our purposes, the main example of the formalism discussed below is given by $\mathcal{D} = D_Y$, $K = f_\bullet IC_X$, $F(-) = \mathbb{H}^0(Y, -)$, etc.

A geometric example is discussed in §2.9.

2.1 Decomposition via E_2 -degeneration

Let \mathcal{D} be a triangulated category with t -structure. Its heart $\mathcal{C} \subseteq \mathcal{D}$ is an abelian category. This data comes equipped with the corresponding cohomology functors $H^i : \mathcal{D} \rightarrow \mathcal{C}$.

We consider objects K of \mathcal{D} with bounded cohomological amplitude, i.e. such that $H^i(K) = 0$, for every $|i| \gg 0$. We also assume the t -structure non-degenerate, see, [2], 1.7. This implies that $H^i(K) = 0$ for all i if and only if $K = 0$.

For any object X of \mathcal{D} there is a spectral sequence

$$E_2^{pq} = \text{Hom}(X[-p], H^q(K)) \implies \text{Hom}(X[-p], K[q]). \quad (8)$$

If we assume that (8) is E_2 -degenerate for $X = H^i(K)$, for any i , then there exists an isomorphism in \mathcal{D} :

$$\phi := \sum_i \phi_i : \bigoplus_i H^i(K)[-i] \xrightarrow{\simeq} K \quad (9)$$

inducing the identity in cohomology. This can be seen as follows. The E_2 -degeneration ensures that $\text{Hom}(H^i(K), H^i(K))$ is a quotient of $\text{Hom}(H^i(K)[-i], K)$. This implies that every map $H^i(K) \rightarrow H^i(K)$ admits a, *not necessarily unique*, lift to $\text{Hom}(H^i(K)[-i], K)$. By applying this to $\text{Id} : H^i(K) \simeq H^i(K)$, we get a map $\phi_i : H^i(K)[-i] \rightarrow K$ inducing the identity in cohomological degree i and the zero map in the remaining degrees. By putting together these maps, we get the morphism (9) that, being the identity in cohomology, is an isomorphism in \mathcal{D} .

Any isomorphism ϕ as above can be normalized by an automorphism of the lhs so that it induces the identity in cohomology. We always work with such isomorphisms.

In short, the degeneration of (8) implies the existence of a splitting (9). However, as the construction shows, this decomposition is *not unique*.

2.2 E_2 -degeneration via the Deligne-Lefschetz Criterion

Let $F : \mathcal{D} \rightarrow \text{Ab}$ be a cohomological functor. As usual, set $F^0(X) := F(X)$ and $F^l(X) := F^0(X[l])$. Fix a morphism

$$\eta : K \rightarrow K[2]. \quad (10)$$

For $a \in F^l(K)$, denote $\eta(a)$ by $\eta \wedge a \in F^{l+2}(K)$. Set $e := H^l(\eta) : H^l(K) \rightarrow H^{l+2}(K)$.

Assumption 2.2.1 *Assume that η satisfies the following Hard Lefschetz relation:*

$$e^i : H^{-i}(K) \simeq H^i(K), \quad \forall i \geq 0. \quad (11)$$

The Deligne Lefschetz Criterion (cf. [9] and [10], p.116) is a sufficient condition for degeneration and splitting: the Hard Lefschetz relation (11) implies that the spectral sequence (8) is E_2 -degenerate so that there exist splittings ϕ as in (9).

The main example for us is the following. Let $f : X \rightarrow Y$ be a projective morphism of varieties, $\eta \in \text{Hom}(\mathbb{Q}_X, \mathbb{Q}_X[2])$ be the first Chern class of an f -ample line bundle on X . Setting $K := f_*IC_X$. we have $f_*\eta : K \rightarrow K[2]$ etc. The Relative Hard Lefschetz Theorem 1.1.2 holds and one deduces from it the Decomposition Theorem 1.1.1 (without the semisimplicity assertion).

2.3 Primitive Decomposition

Since the heart \mathcal{C} of the given t -structure on \mathcal{D} is an abelian category, with slight abuse of language, we think of kernels and images in \mathcal{C} as subobjects.

By analogy with the classical primitive decomposition of the cohomology of a projective manifold with respect to an ample line bundle we define:

$$P_\eta^{-i} := \text{Ker } \{e^{i+1} : H^{-i}(K) \rightarrow H^{i+2}(K)\}, \quad i \geq 0,$$

$$e^j P_\eta^{-i} := \text{Im } \{e^j : P_\eta^{-i} \rightarrow H^{2j-i}(K)\}, \quad 0 \leq j \leq i.$$

There is the Lefschetz-type canonical decomposition isomorphism in the heart \mathcal{C} :

$$\bigoplus_{l=2j-i; 0 \leq j \leq i} e^j P_\eta^{-i} \simeq H^l(K). \quad (12)$$

2.4 The t -filtration

Let $F : \mathcal{D} \rightarrow Ab$ be a cohomological functor. The t -structure on \mathcal{D} defines a filtration on the groups $F^l(K)$:

$$F_{\leq i}^l(K) := \text{Im } \{F^l(\tau_{\leq i} K) \rightarrow F^l(K)\}. \quad (13)$$

This filtration is the abutment of the spectral sequence (8) and we call it the t -filtration.

In the geometric case, we get an increasing filtration $IH_{\leq i}(X) \subseteq IH(X)$ and we call it the *perverse filtration*.

For every isomorphism $\phi := \sum_i \phi_i : \bigoplus_i H^i(K)[-i] \simeq K$ we have

$$F_{\leq i}^l(K) = F^l(\phi(\bigoplus_{i' \leq i} H^{i'}(K)[-i']));$$

this means that, while the individual summands on the rhs are not canonically embeddable in the lhs, the images of the direct sums above are the canonical subspaces yielding the t -filtration.

By abuse of notation we often denote with the same symbol a map of, say, complexes and the resulting map in, say, hypercohomology.

Since K decomposes, the associated graded pieces satisfy canonically

$$F_i^l(K) := F_{\leq i}^l(K)/F_{\leq i-1}^l(K) \simeq F^l(H^i(K)[-i]), \quad \forall i, l \in \mathbb{Z}. \quad (14)$$

Since η (10) is a 2-morphism, we have

$$\eta^j : F_{\leq i}^l(K) \longrightarrow F_{\leq i+2j}^{l+2j}(K), \quad \forall i, l \in \mathbb{Z}, \forall j \geq 0.$$

For every $i \geq 0$, the composition

$$F^l(H^{-i}(K)[i]) \xrightarrow{\phi_l} (F^l(K)) \xrightarrow{\eta^i} F^l(K[2i]) \xrightarrow{pr_i \circ \phi^{-1}} F^l(H^i(K)[i])$$

coincides, in view of (14) and (11), with the isomorphism (see (6)):

$$e^i : F_{-i}^l(K) \simeq F_i^{l+2i}(K).$$

and the composition

$$\phi(F^l(P_\eta^{-i-2j}(-j)[i])) \xrightarrow{\eta^{i+1}} F_{\leq i+2}^{l+2(i+1)}(K) \longrightarrow F_{i+2}^{l+2(i+1)}(K) \quad (15)$$

is an isomorphism onto its image for every $j > 0$.

The condition $a \in F^l(P_\eta^{-i}[i])$, does not imply that $\phi(a)$ is primitive in the usual sense, i.e. $\eta^{i+1} \wedge \phi(a) = 0$. What is true is (16) below. An element $a \in F^l(P_\eta^{-i}[i])$, $i \geq 0$, satisfies $e^{i+1}a = 0 \in F^l(H^{i+2}(K)[i]) = F^{l+2(i+1)}(H^{i+2}K[-i-2])$. This means that for every splitting ϕ as in (9) we have that $\phi(a) \in F_{\leq -i}^l(K)$ and

$$\eta^{i+1} \wedge \phi(a) \in F_{\leq i+1}^{l+2(i+1)}(K) \subseteq F_{\leq i+2}^{l+2(i+1)}(K) \quad (16)$$

so that $\eta^{i+1} \wedge \phi(a)$ becomes zero in $F_{i+2}^{l+2(i+1)}(K)$.

2.5 The canonical morphisms $f_{i,\eta} : P_\eta^{-i}[i] \longrightarrow K$

For the reader's convenience, we now recall Deligne's construction of the maps $f_{i,\eta}$ (cfr.[10]).

We assume 2.2.1. Since then K splits, for every cohomological functor $F : \mathcal{D} \rightarrow Ab$ we have short exact sequences

$$0 \longrightarrow F^{i+1}H^{-i-1}K \longrightarrow F^0\tau_{\geq -i-1}K \longrightarrow F^0\tau_{\geq -i}K \longrightarrow 0.$$

Let $i \geq 0$ and $t : \mathcal{T} \rightarrow H^{-i}K$ be a morphism in \mathcal{D} that factors through P_η^{-i} . In particular, we have $0 = \eta^s \circ t : \mathcal{T} \rightarrow H^{-i+2s}K$, for every $s > i$. The morphism t induces a morphism $x : \mathcal{T}[i] \rightarrow \tau_{\geq -i}K$. Let $T := \text{Hom}(\mathcal{T}[i], -) : \mathcal{D} \rightarrow Ab$.

Proposition 2.5.1 *Let $t \in T^0H^{-i}K[i]$ and $x \in T^0\tau_{\geq -i}K$ be as above. There exists a unique lift $F^{-i}K \ni \tau : \mathcal{T}[i] \rightarrow K$ of x such that $0 = \eta^s \circ \tau \in T^{2s}\tau_{\geq s}K$, for every $s > i$.*

Proof. (See [10], Lemme 2.2). There is the commutative diagram of short exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T^{i+1}H^{-i-1}K & \longrightarrow & T^0\tau_{\geq -i-1}K & \longrightarrow & T^0\tau_{\geq -i}K \longrightarrow 0 \\ & & \simeq \downarrow e^{i+1} & & \downarrow \eta^{i+1} & & \downarrow \eta^{i+1} \\ 0 & \longrightarrow & T^{i+1}H^{+i+1}K(i+1) & \longrightarrow & T^{2(i+1)}\tau_{\geq i+1}K & \longrightarrow & T^{2(i+1)}\tau_{\geq i+2}K \longrightarrow 0 \end{array}$$

Since $\eta^{i+1} \circ x = 0$ and η^{i+1} is an isomorphism, the Snake Lemma yields the existence of a unique lift of x , $x_{-i-1} \in T^0\tau_{\geq -i-1}K$, with the property that $\eta^{i+1} \circ x_{-i-1} = 0$. Repeating this procedure, with i replaced by $i+1$ and x by x_{-i-1} , that clearly satisfies $\eta^{i+2} \circ x_{-i-2} = 0$, we obtain, for some $r \gg 0$, $\tau := x_{-r} \in T^0\tau_{\geq -r}K = T^0K = \text{Hom}(\mathcal{T}[i], K)$ with the required property. \square

Definition 2.5.2 Fix $i \geq 0$. Let $t : T := P_\eta^{-i} \rightarrow H^{-i}(K)$ be the inclusion. Proposition 2.5.1 yields morphisms:

$$f_{i,\eta} : P_\eta^{-i} \longrightarrow K.$$

These morphisms are characterized by the two properties that

- (i) $H^{-i}(f_{i,\eta}) : P_\eta^{-i} \rightarrow H^{-i}(K)$ is the natural inclusion and
- (ii) for every $s > i$, the composition below is zero:

$$P_\eta^{-i}[i] \longrightarrow K \xrightarrow{\eta^s} K[2s] \longrightarrow (\tau_{\geq s}K)[2s].$$

The second condition implies that if $F : \mathcal{D} \rightarrow Ab$ is any cohomological functor and ϕ is any decomposition isomorphism (9) coinciding with $f_{i,\eta}$ on the summand $P_\eta^{-i}[i]$, then we have

$$\eta^s \wedge \phi(F^l(P_\eta^{-i}[i])) \subseteq F_{\leq s-1}^{l+2s}(K), \quad \forall s > i. \quad (17)$$

By (16), a priori the lhs is contained in the bigger $F_{\leq s-1+(s-i)}^{l+2s}(K)$. This is an important restriction and is used in our proof of the key Proposition 2.7.1. We shall discuss it further in a geometric example in §2.9.

Remark 2.5.3 The objects P_η^{-i} , depend on η and so do the morphisms $f_{i,\eta}$. It may happen that P_η^{-i} is independent of η . It is important to keep in mind that, even in this case, the morphisms $f_{i,\eta}$ depend on η . See the example of §2.9. This explains why in general one cannot hope for a canonical decomposition isomorphism (9). Of course, in special cases, one may have a distinguished choice of η and consider the resulting g_η canonical.

2.6 The isomorphism $g_\eta : \bigoplus H^l(K)[-l] \simeq K$.

We assume 2.2.1 and therefore we have the morphisms $f_{i,\eta}$ of §2.5.

The isomorphism g_η is constructed by assembling together the $f_{i,\eta}$ using the primitive Lefschetz decomposition (12).

We start by defining $g_{l,\eta} : H^l K[-l] \rightarrow K$ by first defining it on the direct summands $e^j P_\eta^{-i}[i-2j]$, $0 \leq j \leq i$, $l = 2j - i$, as the composition

$$g_{l,\eta} : e^j P_\eta^{-i}[i-2j] \xrightarrow{(e^j)^{-1}} P_\eta^{-i}[i-2j] \xrightarrow{f_{i,\eta}[i-2j]} K[-2j] \xrightarrow{\eta^j} K.$$

Collecting together the maps $g_{l,\eta}$, $l \in \mathbb{Z}$, we obtain a decomposition isomorphism

$$g_\eta : \bigoplus_l H^l(K)[-l] \simeq K. \quad (18)$$

It depends on η : the $g_{l,\eta}$ are obtained via the $f_{i,\eta}$ and through repeated applications of η . It induces the identity in cohomology and, by construction, the restriction of g_η to the direct summand $P_\eta^{-i}[i]$ is $f_{i,\eta}$.

The properties of g which are relevant to this paper are the following.

Let $0 \leq j \leq i$. For every j' s.t. $j' + j \leq i$, we have that $g_\eta^{-1} \circ \eta^{j'} \circ g_\eta = e^{j'}$ when restricted to $e^j P^{-i}[i-2j]$. In particular, the cup product with $\eta^{j'}$ has the simplest possible expression in terms of the direct sum decomposition, i.e. the following diagram is commutative:

$$\begin{array}{ccc} F^l(e^j P^{-i}[i-2j]) & \xrightarrow{g_\eta} & g(F^l(K)) \\ \downarrow e^{j'} & & \downarrow \wedge \eta^{j'} \\ F^{l+2j'}(e^{-j'-j} P^{-i}[i-2j]) & \xrightarrow{g_\eta} & g(F^{l+2j'}(K)), \end{array}$$

or, in words, $\eta^{j'}$ and g commute when applied to elements of the primitive decomposition as long as $\eta^{j'}$ stays injective as predicted by the Hard Lefschetz property 2.2.1.

In the remaining range, we have the key restriction (17).

There is the decomposition

$$F^l(K) = \bigoplus_{0 \leq j \leq i} \eta^j \wedge f_{i,\eta}(F^{l-2j}(e^j P_\eta^{-i}[i])), \quad (19)$$

i.e. the lhs can be built inductively using the images of primitives via the maps $f_{i,\eta}$ in degrees $\leq l$ via cup products with η .

In our geometric situation, $K = f_\bullet IC_X$, $F = \mathbb{H}^0(Y, -)$ etc, we get

$$IH^{n+l}(X) = \bigoplus_{0 \leq j \leq i} \eta^j \wedge f_{i,\eta}(\mathbb{H}^{l-2j}(Y, P_\eta^{-i}[i])), \quad l \in \mathbb{Z}.$$

2.7 Characterization of $f_{i,\eta}(\mathbb{H}^l(Y, P_\eta^{-i}[i]) \subseteq IH^{n+l}(X)$

We revert to our geometric situation: $K := f_\bullet IC_X$, $F(-) := \mathbb{H}^0(Y, -)$, etc.

Fix $i \geq 0$. We shall define maps Ψ_t and express the images in hypercohomology

$$f_{i,\eta}(\mathbb{H}^l(Y, P_\eta^{-i}[i])) \subseteq g_\eta(\mathbb{H}^l(Y, {}^p H^{-i}(f_\bullet IC_X)[i])) \subseteq IH_{\leq -i}^{n+l}(X) \subseteq IH^{n+l}(X)$$

as $\text{Ker } \Psi_{r-i}$, where $r = r(f_\bullet IC_X)$ is the cohomological amplitude of $f_\bullet IC_X$. This will be achieved by means of a repeated application of the key restriction (17).

Let g_η be the isomorphism (18) associated with η .

In what follows, for simplicity, we omit some cohomological degrees .

Consider the composition

$$\Psi_0 : IH_{\leq -i}^\bullet(X) \xrightarrow{\eta^{i+1}} IH_{\leq i+2}^{\bullet+2(i+1)}(X) \longrightarrow IH_{i+2}^{\bullet+2(i+1)}(X),$$

and define inductively, for $t \geq 1$:

$$\Psi_t : \text{Ker } \Psi_{t-1} \xrightarrow{\eta^{i+t}} IH_{i+t}^{\bullet+2(i+t)}(X).$$

Proposition 2.7.1

$$\text{Ker } \Psi_{r-i} = f_{i,\eta}(\mathbb{H}^l(Y, (P_\eta^{-i}[i]))).$$

Proof. We are going to prove by induction on t that

$$\text{Ker } \Psi_t = IH_{\leq -i-t-1}^{n+l}(X) \oplus f_{i,\eta}(\mathbb{H}^l(Y, (P_\eta^{-i}[i]))), \quad \forall t \geq 0. \quad (20)$$

Taking $t = r - i$, where r is the cohomological amplitude of $f_\bullet IC_X$, we can draw the desired conclusion, for $IH_{\leq -r-1}(X) = 0$.

We first prove (20) for $t = 0$. We have

$$IH_{\leq -i}^{n+l}(X) = IH_{\leq -i-1}^{n+l}(X) \oplus f_{i,\eta}(\mathbb{H}^l(Y, (P_\eta^{-i}[i])) \oplus \bigoplus_{j>0} f_{i,\eta}(\mathbb{H}^l(Y, (P_\eta^{-i-2j}(-j)[i]))).$$

The first summand lands first in $IH_{\leq i+1}^{\bullet+2(i+1)}(X)$ and is therefore in the kernel of Ψ_0 . So is the second summand since it lands first in $IH_{\leq i}^{\bullet+2(i+1)}(X)$ by virtue of (17). As to the third summand, it maps isomorphically to its image via Ψ_0 by (15). This proves the case $t = 0$.

Assume we have proved (20) for $t - 1$ and let us prove it for t . We have the composition

$$\Psi_t : \text{Ker } \Psi_{t-1} \xrightarrow{\eta^{i+t}} IH_{\leq i+t}^{\bullet+2(i+t)}(X) \longrightarrow IH_{i+t}^{\bullet+2(i+t)}(X)$$

where, by the inductive hypothesis:

$$\text{Ker } \Psi_{t-1} = IH_{\leq -i-t-1}^{n+l}(X) \oplus g_\eta(\mathbb{H}^l(Y, {}^p H^{-i-t}(f_\bullet IC_X)[i+t])) \oplus f_{i,\eta}(\mathbb{H}^l(Y, (P_\eta^{-i}[i]))).$$

We conclude as in the case $t = 0$. □

Using orthogonality with respect to the intersection pairing in $IH(X)$, we can re-word Proposition 2.7.1 as

Corollary 2.7.2

$$f_{i,\eta}(\mathbb{H}^l(Y, (P_\eta^{-i}[i])) = IH_{\leq -i}^{n+l} \cap \bigcap_{t \geq 1} (\eta^{i+t} IH_{-i-t}^{n+l-2(i+t)}(X))^\perp.$$

This formula shows that the realization of intersection cohomology as a sub-Hodge structure of the cohomology of a resolutions of isolated singularities of threefolds and fourfolds worked out in [8] coincides with the one defined by g_η .

2.8 The isomorphism g_η is Hodge-theoretic

For simplicity let us now assume that X is smooth and let us briefly discuss the PHS on the graded spaces $H_i^l(X)$. In the paper [7] we have identified, up to some trivial shifting procedure, the perverse filtration $H_{\leq i}^l(X)$ arising from a map $f : X \rightarrow Y$ with the filtration associated with the nilpotent action on $H^*(X)$ of the first Chern class of an ample line

bundle on Y . Since this action is of type $(1, 1)$, the filtration is given by SHS. Accordingly, the subspaces of the filtration are PHS and the graded pieces, $H_i^l(X)$ inherit canonical PHS.

The Decomposition Theorem does not ensure that the resulting decomposition $H^l(X) = \bigoplus \phi(H_i^l(X))$ (7) into the sum of the graded pieces can be made into an isomorphism of PHS.

We are about to prove that this is achieved by the isomorphisms g_η .

Theorem 2.8.1 *Let $f : X \rightarrow Y$ be a projective morphism of compact varieties, η be an f -ample line bundle on X . Then g_η induces isomorphisms of weight l PHS*

$$g_\eta : \bigoplus_i IH_i^l(X) \simeq IH^l(X).$$

Proof. By Lemma 1.2.1, it is enough to show that $g_\eta(IH_i^l(X)) \subseteq IH^l(X)$ is a SHS for every $i \in \mathbb{Z}$.

The cup product map $\eta : IH(X) \rightarrow IH^{+2}(X)$ is a map of PHS.

By virtue of the η -decomposition (19) associated with g_η , it is enough to show that each subspace $f_{i,\eta}(\mathbb{H}^{l-n}(Y, P_\eta^{-1}[i]))$ is a SHS. This follows from Proposition 2.7.1 that exhibits those subspaces as iterated kernels of maps of PHS. \square

Corollary 2.8.2 *Assume, in addition, that $f : X \rightarrow Y$ is a resolution of singularities. Then $IC_Y \subseteq P_\eta^0 \subseteq {}^pH^0(f_\bullet IC_X)$ and the induced map*

$$f_{0,\eta} : IH(Y) \longrightarrow H(X)$$

is an injection of PHS.

Proof. The inclusion $IC_Y \subseteq {}^pH^0(f_\bullet IC_X)$ holds over the smooth part of Y and the Decomposition Theorem implies that the inclusion must hold over Y . Since the complexes ${}^pH^{\neq 0}(f_\bullet IC_X)$ are supported on a proper subvariety of Y , the simplicity of IC_Y implies the inclusion $IC_Y \subseteq P_\eta^0$.

The summand $IH(Y) \subseteq IH_0(X)$ corresponds to the dense stratum in the strata-like decomposition (5) and is therefore a SHS. We conclude by Theorem 2.8.1. \square

2.9 An example: the blow up of a quadric cone

Let $f : X \rightarrow Y$ be the blowing up at the vertex $v \in Y$ of the projective cone Y over a quadric surface $\mathbb{P}^1 \times \mathbb{P}^1 \simeq Q \subseteq \mathbb{P}^3$. There is the canonical \mathbb{P}^1 -bundle projection $p : X \rightarrow Q$ with sections $D := f^{-1}(v)$ and $D_\infty := f^{-1}(\Delta_\infty)$, where $\Delta_\infty \subseteq Y$ is the quadric at infinity. There are the two surfaces $D_i := p^{-1}(l_i)$, $i = 1, 2$, where l_i are two lines of the two distinct rulings of Q . Let $\Delta_i := f(D_i)$.

We have the following relations in the 3-dimensional group $H^2(X)$:

$$H^2(X) = \langle D, D_1, D_2 \rangle = \langle D_\infty, D_1, D_2 \rangle, \quad D_\infty = D + D_1 + D_2.$$

As to $IH^2(Y)$, the perversity condition is empty for 4-chains since the singular locus is zero-dimensional. Hence Δ_1, Δ_2 define intersection cohomology classes and in fact

$$IH^2(Y) = \langle \Delta_1, \Delta_2 \rangle, \text{ with the relation } \Delta_\infty = \Delta_1 + \Delta_2.$$

which is easily checked to hold on $Y - \{v\}$, hence on Y .

We have $IC_X = \mathbb{Q}_X[3]$. The perverse cohomology complexes are (cf. [8], [7]): ${}^pH^0(f_\bullet IC_X) = P_\eta^0 = IC_Y$; ${}^pH^{-1}(f_\bullet IC_X) = P_\eta^{-1} = H_4(D)_v$ (skyscraper sheaf; in cohomological degree zero; generated by the fundamental class of D); ${}^pH^1(f_\bullet IC_X) = P_\eta^{-1}(-1) = H^4(D)_v$.

The Decomposition Theorem yields the existence of an isomorphism:

$$\phi : H_4(D)[1] \oplus IC_Y \oplus H^4(D)[-1] \simeq f_\bullet IC_X. \quad (21)$$

The resulting inclusion $\phi(IH^2(Y)) \subseteq H^2(X)$ depends on ϕ .

Even with the choice $\phi = g_\eta$, the subspace $g_\eta(IH^2(Y)) = f_{i,\eta}(IH^2(Y)) \subseteq H^2(X)$ still depends on η in a way we now make explicit.

The map $e : {}^pH^{-1}(f_\bullet IC_X) \simeq {}^pH^1(f_\bullet IC_X)$ is the map

$$[D] \longrightarrow e([D]) = \{[D] \longrightarrow \eta \cdot D \cdot D\}, \quad (22)$$

where the product is in $H(X)$.

By Corollary 2.7.2 we have:

$$H^2(X) \supseteq f_{i,\eta}(IH^2(Y)) = \{a \in H^2(X) \mid \eta \cdot D \cdot a = 0\}$$

so that the dependence on η is now transparent.

For example, set $\eta := mD_1 + D_2$, $m \in \mathbb{Q}^+$. Then

$$g_\eta(IH^2(Y)) = \langle D_\infty, -mD_1 + D_2 \rangle \subseteq H^2(X),$$

$$g_\eta(\Delta_1) = D_1 + \frac{1}{m+1}D, \quad g_\eta(\Delta_2) = D_2 + \frac{m}{m+1}D.$$

The conclusion is that different choices of η produce different embeddings on $IH(Y)$ into $H(X)$.

It is amusing to note the following. For $m = 0$, $\eta = D_2$ is no longer f -ample, but the relative Hard Lefschetz still holds since $D_2 \cdot D \cdot D = -1 \neq 0$ and we have $g_{D_2}(\Delta_1) = D_1 + D$ and $g_{D_2}(\Delta_2) = D_2$. This decomposition can be seen as the one that arises canonically by factoring (in precisely one of the two possible ways!) $f : X \rightarrow Y$ through the small resolution of the quadric cone.

Note also that there is no isomorphism ϕ yielding $\phi(\Delta_i) = D_i$, $i = 1, 2$. This is because $\phi(\Delta_1 + \Delta_2) = \phi(\Delta_\infty) = D_\infty \neq D_1 + D_2$.

All the embeddings of $IH^2(Y)$ into $H^2(X)$ are, in this example where everything is of pure type $(1, 1)$, compatible with the Hodge structures. In general, this is not true. Our main result, Theorem 2.8.1, is that we can arrange for this to be true in complete generality.

We conclude this section by remarking that the mechanism in the proof of Proposition 2.5.1 becomes transparent in this example, where $i = 0$. In fact, we start with *any* lift $y : IC_Y \rightarrow \tau_{\geq -1} f_{\bullet} IC_X = f_{\bullet} IC_X$ of the natural map $x : IC_Y \rightarrow {}^p H^0(f_{\bullet} IC_X) \rightarrow \tau_{\geq 0} f_{\bullet} IC_X$. The Snake Lemma allows to correct uniquely y by adding to it a map $IC_Y \rightarrow {}^p H^{-1}(f_{\bullet} IC_X)[1]$ so that the resulting map $f_{0,\eta} = \tau = x_{-1} : IC_Y \rightarrow f_{\bullet} IC_X$ has the property that the composition

$$IC_Y \xrightarrow{f_{0,\eta}} f_{\bullet} IC_X \xrightarrow{\eta} f_{\bullet} IC_X[2] \longrightarrow \tau_{\geq 1} f_{\bullet} IC_X[2] = {}^p H^1(f_{\bullet} IC_X)[1] = H^4(D)[1]$$

is the zero map. In hypercohomology, i.e. in $H(X)$, this translates into the condition

$$\eta \cdot D \cdot f_{0,\eta}(\Delta_i) = 0, \quad i = 1, 2.$$

3 Applications

We give few applications of Theorem 2.8.1.

3.1 The intersection pairing on $IH(Y)$.

Theorem 3.1.1 *Let Y be a compact algebraic variety of dimension n . For every $l \in \mathbb{Z}$ the intersection pairing*

$$d_Y : IH^{n-l}(Y) \longrightarrow IH^{n+l}(Y)^{\vee}(-n)$$

is an isomorphism of weight $(n-l)$ PHS.

Proof. Let $f : X \rightarrow Y$ be a projective resolution of the singularities of Y and η be an f -ample line bundle on X . There is the diagram

$$\begin{array}{ccc} f_{\bullet} IC_X & \xrightarrow{f_{\bullet} d_X} & (f_{\bullet} IC_X)^{\vee} \\ g_{\eta} \uparrow & & \downarrow g_{\eta}^{\vee} \\ IC_Y & \xrightarrow{d_Y} & IC_Y^{\vee}. \end{array}$$

The composition $g_{\eta}^{\vee} \circ f_{\bullet} d_X \circ g_{\eta} \in \text{Hom}(IC_Y, IC_Y^{\vee}) \simeq \mathbb{Q}$ coincides with d_Y on the smooth locus of f on Y and hence on the whole Y ; see Remark 1.1.3.

It follows that the duality isomorphism d_Y can be exhibited as the composition of maps of PHS by virtue of Theorem 2.8.1:

$$IH^{n-l}(Y) \xrightarrow{g_{\eta}} H^{n-l}(X) \xrightarrow{d_X} H^{n+l}(X)^{\vee}(-n) \xrightarrow{\text{proj} \circ g_{\eta}^{\vee}} IH^{n+l}(Y)^{\vee}(-n).$$

□

3.2 The map $H(Y) \rightarrow IH(Y)$

Given a compact algebraic variety Y of dimension n there is the natural map $a_Y : \mathbb{Q}_Y[n] \rightarrow IC_Y$ and the induced map in hypercohomology $a_Y : H(Y) \rightarrow IH(Y)$.

We freely employ the language and basic results of the theory of MHS in [11]. The MHS on $H^l(Y)$ has weights $\leq l$, i.e. $W_l H^l(Y) = H^l(Y)$. In fact, for every resolution of singularities $f : X \rightarrow Y$, $\text{Ker } f^* = W_{l-1} H^l(Y)$. The quotient $H^l(Y)/W_{l-1} H^l(Y)$ is a PHS of weight l .

Theorem 3.2.1 (The natural map $H(Y) \rightarrow IH(Y)$) *Let Y be compact. The natural map*

$$a_Y : H^l(Y) \longrightarrow IH^l(Y)$$

is a map of MHS (the r.h.s. is a PHS) and

$$\text{Ker } a_Y = \text{Ker } f^* = W_{l-1} H^l(Y).$$

Proof. Let $f : X \rightarrow Y$ be a *projective* resolution of singularities. Let g_η be the isomorphism associated with some ample line bundle η on X . As in the proof of Theorem 3.1.1, the formula (2) yields the commutative diagram

$$\begin{array}{ccc} \mathbb{Q}_Y[n] & \xrightarrow{\text{adj}(f)} & f_* \mathbb{Q}_X[n] \\ & \searrow a_Y & \nearrow g_\eta \\ & IC_Y & \end{array}$$

with g_η admitting a splitting g' . Since g_η is injective, $\text{Ker } a_Y = \text{Ker } f^*$ and the second conclusion follows.

We have $a_Y = g' \circ \text{adj}(f)$. The map induced in hypercohomology $a_Y = g' \circ f^*$ is the composition of the map f^* of MHS (Deligne's theory of MHS) and of the splitting g' of PHS (Theorem 2.8.1).

If f is not projective, then the second assertion follows by considering a Chow envelope $f' : X' \rightarrow X \rightarrow Y$ with f' projective and then by recalling that Deligne's theory of MHS ensures that $\text{Ker } f^* = \text{Ker } f'^*$. \square

3.3 Projectors and Hodge classes

Let $f : X \rightarrow Y$ be a projective morphism of proper varieties and η be an f -ample line bundle on X .

Let $IH_i^l(X) = H \oplus H'$ be a direct sum decomposition into SHS.

Using the decomposition isomorphism g_η and the associated projectors we obtain the composition

$$p_H : IH^l(X) \longrightarrow H \longrightarrow IH^l(X)$$

which is a projector, i.e. $p^2 = p$ in the algebra

$$\text{End}(IH(X)) = IH(X) \otimes IH(X)^\vee \simeq IH(X) \otimes IH(X) = IH(X \times X),$$

where the middle isomorphism stems from the nondegenerate intersection pairing of Theorem 3.1.1.

Theorem 3.3.1

$$p_H \in IH_{\mathbb{Q}}^{n,n}(X \times X).$$

Proof. The proof is identical to the analogous one to be found in [8] for the case when X is smooth. The only missing piece is Theorem 3.1.1. \square

It is natural to ask whether the classes p_H of Theorem 3.3.1 are algebraic, i.e. representable in terms of admissible geometric chains arising from algebraic subvarieties.

If X is smooth, then this amounts to ask whether these classes are in

$$\text{Im} (A_n(X \times X) \rightarrow H^{2n}(X \times X)).$$

This takes one to the realm of the Standard Conjectures for algebraic cycles and we have nothing to say in this direction, except for very special, yet non-trivial cases. In [6], we have shown that for proper semismall maps from a smooth space, for every H , one can find canonical *algebraic* projectors c' of type (n, n) . The key point is that $\dim X \times_Y X = n$. The paper [4], dealt with the case of Hilbert schemes of points on surfaces. In [8], we have shown that the same can be done for the resolution of singularities of a threefold. The key point there is the use of the $(1, 1)$ -Theorem.

We remark that if $H = IH_i^l(X)$ is a summand as in (5), a summand of the (η, L) -decomposition of [7], or any intersection of the two, then it can be shown that the cycles p_H are *absolute Hodge classes* in the sense of [12]. We plan to pursue this aspect and some of its consequences in a forthcoming paper.

3.4 Induced morphisms in intersection cohomology

Intersection cohomology is not functorial in the “space” variable. However, the paper [1] constructs, for every proper map $f : X \rightarrow Y$, non-canonical morphisms $IH^l(Y) \rightarrow IH^l(X)$.

If f is surjective, these morphisms stem from the Decomposition Theorem and are splitting injections. We now show how to choose them so that they are map of PHS.

Theorem 3.4.1 *Let $f : X^n \rightarrow Y^m$ be a projective, surjective map of compact varieties of the indicated dimensions, $\rho := n - m$ and η be an f -ample line bundle on X . Then there are a canonical splitting injection*

$$\gamma : IC_Y \longrightarrow {}^pH^{-\rho}(f_{\bullet}IC_X), \tag{23}$$

and a commutative diagram of MHS

$$\begin{array}{ccc}
 H^l(Y) & \xrightarrow{a_Y} & IH^l(Y) \\
 \downarrow f^* & & \downarrow g_\eta \circ \gamma \\
 H^l(X) & \xrightarrow{a_X} & IH^l(X).
 \end{array} \tag{24}$$

Sketch of proof. Let $Y_m \subseteq Y$ be the dense stratum of a stratification for f . The perverse sheaf ${}^pH^{-\rho}(f_\bullet IC_X)$ restricted to Y_m reduces to the shifted local system $\mathcal{L}[m]$, where \mathcal{L} is the semisimple local system of the ρ -dimensional irreducible components of the typical fiber $f^{-1}(y)$.

Since X is irreducible, the $\pi_1(Y_m, y)$ -invariants $\mathcal{L}^{\pi_1(Y_m)} \simeq \mathbb{Q}_{Y_m} \subseteq \mathcal{L}$, and the inclusion splits.

The Decomposition Theorem implies that

$$IC_Y = IC_Y(\mathcal{L}^{\pi_1(Y_m)}) \subseteq IC_Y(\mathcal{L}) \subseteq {}^pH^{-\rho}(f_\bullet IC_X)$$

and that all the inclusions split canonically. This gives the map γ and proves (23).

The diagram

$$\begin{array}{ccc}
 \mathbb{Q}_Y[m] & \xrightarrow{a_Y} & IC_Y \\
 \downarrow \text{adj}(f) & & \downarrow g_\eta \circ \gamma \\
 f_\bullet \mathbb{Q}_X[m] & \xrightarrow{a_X} & f_\bullet IC_X[-\rho]
 \end{array}$$

commutes in view of the formula (3) and of Remark 1.1.3 applied to $U = Y_m$. The diagram (24) is induced by it by taking hypercohomology and is therefore commutative.

The decomposition by strata (5) and Theorem 2.8.1 imply that $g_\eta(IH^{n+l}(Y, \mathcal{L})) \subseteq IH^{n+l}(X)$ is a SHS.

We are left with checking that $IH(Y) \subseteq IH(Y, \mathcal{L})$ is a SHS. Once this is done, we conclude using Theorem 3.2.1.

Without loss of generality, we may assume that X and Y are normal.

There is the Stein factorization $X \xrightarrow{f'} Y' \xrightarrow{\nu} Y$ where f' has connected fibers, Y' is normal and ν is finite. We have $IC_Y(\mathcal{L}) = \nu_* IC_{Y'}$, so that we may replace $f : X \rightarrow Y$, by $\nu : Y' \rightarrow Y$, i.e. we may assume that f is finite.

We have the commutative diagram

$$\begin{array}{ccc}
 Z & \xrightarrow{h} & X \\
 & \searrow k & \swarrow f \\
 & & Y
 \end{array}$$

arising from the Galois closure of $K(X)/K(Y)$. The maps h and k are finite Galois.

The inclusions $IH(Y) \subseteq IH(X) \subseteq IH(Z)$ imply that if we can prove the wanted conclusion for a Galois map, then it will follow for f . This means that we may assume that f is Galois with finite Galois group G .

The group G acts on the PHS $IH(X)$ by automorphisms of PHS: take a G -equivariant resolution of the singularities $p : X' \rightarrow X$, a G -invariant p -ample line bundle η' on X' and use Theorem 2.8.1.

It follows that the g -invariants $IH(Y) = IH(X)^G \subseteq IH(X)$ form a SHS. □

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