Branes on Poisson varieties

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Dedicated to Nigel Hitchin on the occasion of his sixtieth birthday.

Abstract

We first extend the notion of connection in the context of Courant algebroids to obtain a new characterization of generalized Kähler geometry. We then establish a new notion of isomorphism between holomorphic Poisson manifolds, which is non-holomorphic in nature. Finally we show an equivalence between certain configurations of branes on Poisson varieties and generalized Kähler structures, and use this to construct explicitly new families of generalized Kähler structures on compact holomorphic Poisson manifolds equipped with positive Poisson line bundles (e.g. Fano manifolds). We end with some speculations concerning the connection to non-commutative algebraic geometry.

1 Introduction

In this paper we shall take a second look at a classical structure in differential and algebraic geometry, that of a holomorphic Poisson structure, which is a complex manifold with a holomorphic Poisson bracket on its sheaf of regular functions. The structure is determined, on a real smooth manifold M, by the choice of a pair (I, σ_I), where I is an integrable complex structure tensor and σ_I is a holomorphic Poisson tensor. We shall view (I, σ_I) not as we normally do but instead as a *generalized complex structure*, in the sense of [1]. In so doing, we shall obtain a new notion of equivalence between the pairs (I, σ_I) which does not imply the holomorphic equivalence of the underlying complex structures.

In studying this equivalence relation we are naturally led to an unexpected connection to *generalized Kähler geometry*, as defined in [2], and to a method for constructing certain examples of these structures which extends the recent work of Hitchin constructing bi-Hermitian metrics on Del Pezzo surfaces [3]; in particular we obtain similar families of bi-Hermitian metrics on all smooth Poisson Fano varieties, and in fact on any smooth Poisson variety admitting a positive Poisson line bundle. We therefore give an explicit construction of a subclass of the generalized Kähler structures proven to exist by the generalized Kähler stability theorem of Goto [4].

In both these efforts we shall find it useful to introduce an extension of the notion of connection on a vector bundle, to allow differentiation not only in the tangent but also the cotangent directions; we call such a structure a *generalized connection*. We also show that in the presence of a generalized metric, there is a canonical connection D which plays the role of the Levi-Civita connection in Kähler geometry: namely, we show that $(\mathcal{J}, G)$ is generalized Kähler if and only if $D\mathcal{J} = 0$.

In the final section we make some speculative comments concerning the relationship between generalized Kähler geometry and non-commutative geometry, a topic we hope to clarify in the future.

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2 Gerbe trivializations

Let M be a manifold equipped with a $U(1)$ gerbe with connection (specifically, a gerbe with connective structure in the sense of [5]). This determines canonically a *Courant algebroid* E over M, in the same way that a $U(1)$ principal bundle P determines an Atiyah Lie algebroid $E = TP/U(1)$ over M. See [6, 7] for details of this construction, and see [8, 9, 10] for details concerning Courant algebroids; we review their main properties presently.

The Courant algebroid E is an extension of real vector bundles

$$0 \longrightarrow T^* \xrightarrow{\pi^*} E \xrightarrow{\pi} T \longrightarrow 0\,, \tag{1}$$

where T and T^* denote the tangent and cotangent bundles of M. Further, E is equipped with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ of split signature, such that $\langle \pi^*\xi, a\rangle = \xi(\pi(a))$. Finally, there is a bilinear *Courant bracket* $[\cdot, \cdot]$ on $C^\infty(E)$ such that

- $[[a, b], c] = [a, [b, c]] - [b, [a, c]]$ (Jacobi identity),
- $[a, fb] = f[a, b] + (\pi(a)f)b$ (Leibniz rule),
- $\pi(a)\langle b, c\rangle = \langle [a, b], c\rangle + \langle b, [a, c]\rangle$ (Invariance of bilinear form),
- $[a, a] = \pi^* d\langle a, a\rangle$ (Skew-symmetry anomaly).

The choice of an isotropic complement to T^* in E is a contractible one, and so an isotropic splitting $s : T \longrightarrow E$ of the sequence (1) always exists. Each such splitting determines a closed 3-form $H \in \Omega^3(M)$, given by

$$(i_Y i_X H)(Z) = \langle [s(X), s(Y)], s(Z)\rangle. \tag{2}$$

The cohomology class $[H]/2\pi \in H^3(M, \mathbb{R})$ is independent of the choice of splitting, and coincides with the image of the Dixmier-Douady class of the gerbe in real cohomology. Furthermore, $[H]$ classifies the Courant algebroid up to isomorphism, as shown by [6].

Courant algebroids may be naturally pulled back by the inclusion $S \subset M$ of a submanifold; as a bundle over S, the result is simply given by

$$E_S = \frac{\pi^{-1}(TS)}{\mathrm{Ann}(TS)}, \tag{3}$$

and its bracket and inner product are inherited in a straightforward manner.

A trivialization of the gerbe along S induces a Courant trivialization in the following sense:

Definition 1. *A* Courant trivialization *along S is an integrable isotropic splitting $s : TS \longrightarrow E_S$ of the pullback Courant algebroid. Such trivializations exist if and only if $\iota^*[H] = 0$ for $\iota : S \hookrightarrow M$ the inclusion.*

Integrability is the requirement that the subbundle $s(TS) \subset E_S$ be closed under the Courant bracket. Integrable maximal isotropic subbundles of a Courant algebroid are called Dirac structures; therefore $s(TS)$ is simply a Dirac structure transverse to T^*S. As a result of a Courant trivialization, E_S is canonically isomorphic to $TS \oplus T^*S$ with its natural pairing and the bracket

$$[X + \xi, Y + \eta] = L_X(Y + \eta) - i_Y d\xi.$$

Now suppose that $S_0, S_1 \subset M$ are submanifolds with smooth intersection, and suppose we have gerbe trivializations on each of them. Then on $X = S_0 \cap S_1$ we obtain a pair of gerbe trivializations, which must differ by a line bundle L_{01} with $U(1)$ connection ∇_{01}. Let s_0, s_1 be the splittings of E_X determined by the two gerbe trivializations. Then $s_1 - s_0 : TX \longrightarrow T^*X$ is given by $X \mapsto i_X F_{01}$ where $F_{01} \in \Omega^2(M)$ is the curvature of ∇_{01}.

The notion of Courant trivialization provides a convenient way of characterizing isomorphisms of Courant algebroids, as in the following example. The notation $\overline{E}$ denotes the Courant algebroid E, equipped with the opposite bilinear form $-\langle \cdot, \cdot \rangle$.

Example 2.1. *Let E_M, E_N be Courant algebroids over the manifolds M, N respectively. They are isomorphic precisely if there is a Courant trivialization of the product Courant algebroid $\overline{E}_M \times E_N$ along the graph of a diffeomorphism $\varphi : M \longrightarrow N$ in the product $M \times N$.*

3 Generalized connections

Let E be a Courant algebroid as in the previous section. In keeping with the notion that the Courant algebroid is an analogue of the tangent bundle, we have the following generalization of the usual notion of connection.

Definition 2. *A* generalized connection *on a vector bundle V is a first-order linear differential operator*

$$D : C^\infty(V) \longrightarrow C^\infty(E \otimes V)$$

such that $D(fs) = fDs + (\pi^* df) \otimes s$. Furthermore, if V has a Hermitian metric h, then D is unitary *when*

$$d(h(s,t)) = h(Ds,t) + h(s,Dt).$$

If $s : T \longrightarrow E$ is any splitting (not necessarily isotropic) of the Courant algebroid, then using the decomposition $E = s(T) \oplus T^* \cong T \oplus T^*$, we have

$$D = \nabla + \chi, \tag{4}$$

where ∇ is a usual unitary connection and χ is a vector field with values in the bundle of skew-adjoint endomorphisms of V, i.e. $\chi \in C^\infty(T \otimes \mathfrak{u}(V))$. The tensor χ is independent of the choice of splitting, and we note that if V is of rank 1, χ is simply a vector field on the manifold.

With respect to a different splitting s', such that

$$s' - s = \theta : T \longrightarrow T^*,$$

we obtain a different decomposition $D = \nabla' + \chi$, where $\nabla' = \nabla + \theta(\chi)$.

A generalized connection has a natural curvature operator: for $a, b \in C^\infty(E)$, we define

$$R(a,b) = [D_a, D_b] - D_{[a,b]} \in C^\infty(\mathfrak{u}(V)).$$

This becomes tensorial in a, b when restricted to a Dirac structure $L \subset E$:

$$R|_L \in C^\infty(\wedge^2 L^* \otimes \mathfrak{u}(V)).$$

If $L = T^*$, for example, we obtain a bivector with values in the skew-adjoint endomorphisms, $R|_{T^*} = [\chi, \chi]$.

The tensorial curvatures $R_{s'}, R_s$ associated to integrable splittings s, s' of E with $s' - s = F \in \Omega^2_{cl}(M)$ may be compared by projection to T:

$$R_{s'} = R_s + d^\nabla(a) + a \wedge a,$$

where $a = F(\chi)$. Therefore if V has rank 1, we have that $\chi = iX$ for a real vector field X, and

$$R_{s'} - R_s = d i_X F.$$

In the particular case that we have a generalized connection D on E itself, it is natural to compare the connection derivative with the Courant bracket; we therefore introduce the *torsion* of D, and leave it as an exercise to verify it is well-defined.

Definition 3. *The torsion $T_D \in C^\infty(\wedge^2 E \otimes E)$ of a generalized connection D on E itself is defined by*

$$T(a,b,c) = \langle D_a b - D_b a - [a,b]_{sk}, c\rangle + \tfrac{1}{2}(\langle D_c a, b\rangle - \langle D_c b, a\rangle),$$

where $[a,b]_{sk} = \frac{1}{2}([a,b] - [b,a])$. If D preserves the canonical bilinear form $\langle\cdot,\cdot\rangle$ on E, then T_D is totally skew, i.e. $T_D \in C^\infty(\wedge^3 E)$.

A generalized Riemannian metric on the Courant algebroid E is the choice of a maximal positive-definite subbundle $C_+ \subset E$; this reduces the $O(n,n)$ structure of E to $O(n)\times O(n)$, and defines a positive-definite metric on E:

$$G(\cdot,\cdot) = \langle\cdot,\cdot\rangle|_{C_+} - \langle\cdot,\cdot\rangle|_{C_-},$$

where $C_- = C_+^\perp$ is the orthogonal complement with respect to $\langle\cdot,\cdot\rangle$. We now describe a construction of a canonical connection associated to the choice of such a metric, inspired by calculations in [2, 7] relating metric connections with skew torsion to the Courant bracket.

The G-orthogonal complement to T^* is an isotropic splitting $C_0 \subset E$ and we identify it with T, so that G induces a splitting $E = T \oplus T^*$. The Courant bracket in this splitting is

$$[X+\xi, Y+\eta]_H = [X,Y] + L_X\eta - i_Y d\xi + i_Y i_X H,$$

where $H \in \Omega^3_{cl}(M)$ is defined by (2). The splitting also defines an anti-orthogonal automorphism $C : E \longrightarrow E$ defined by $C(X+\xi) = X - \xi$, which satisfies $C(C_\pm) = C_\mp$. It also has the property, for $Z, W \in C^\infty(E)$:

$$[CZ, CW]_H = C([Z,W]_{-H}).$$

Theorem 3.1. *Let $C_+ \subset E$ be a maximal positive-definite subbundle, i.e. a generalized metric, as above. Let $C : E \longrightarrow E$ be the map defined above. Write $Z = Z_+ + Z_-$ for the orthogonal projections of $Z \in C^\infty(E)$ to $C_\pm$. Then the operator*

$$D_Z(W) = [Z_-, W_+]_+ + [Z_+, W_-]_- + [CZ_-, W_-]_- + [CZ_+, W_+]_+ \qquad (5)$$

defines a generalized connection on E, preserving both $\langle\cdot,\cdot\rangle$ and the positive-definite metric G. We call this the generalized Bismut connection.

Proof. Using the properties of the Courant bracket and the orthogonality $C_+ = C_-^\perp$, we have immediately the property $D_{fZ}W = fD_Z W$, for $f \in C^\infty(M)$. We also have

$$\begin{aligned} D_Z(fW) &= fD_Z(W) + (Z_- f)W_+ + (Z_+ f)W_- + (Z_- f)W_- + (Z_+ f)W_+ \\ &= fD_Z(v) + (Zf)W, \end{aligned}$$

proving that D is a generalized connection.

It is clear from (5) that $C_\pm$ are preserved by the connection, since $D_Z W$ has nonzero component in $C_\pm$ if and only if W does. Hence we obtain a decomposition

$$D = D^+ \oplus D^-, \tag{6}$$

where $D^\pm$ are generalized connections on $C_\pm$ respectively.

To prove that D preserves the canonical metric $\langle \cdot, \cdot \rangle$ as well as the metric G, we show that $D^\pm$ preserve the induced metrics on $C_\pm$. Let $V, W \in C^\infty(C_+)$, and $Z \in C^\infty(E)$. Then

$$\begin{aligned} Z_+\langle V, W\rangle &= (CZ_+)\langle V, W\rangle \\ &= \langle [CZ_+, V], W] \rangle + \langle V, [CZ_+, W] \rangle \\ &= \langle D_{Z_+} V, W\rangle + \langle V, D_{Z_+} W\rangle. \end{aligned}$$

Also, we have

$$\begin{aligned} Z_-\langle V, W\rangle &= \langle [Z_-, V], W\rangle + \langle V, [Z_-, W] \rangle \\ &= \langle D_{Z_-} V, W\rangle + \langle V, D_{Z_-} W\rangle. \end{aligned}$$

Summing these two results, we see that D^+ preserves the metric on C_+; the same argument holds for C_-, completing the proof. $\square$

The generalized connections $D^\pm$ in the decomposition (6) define tensors $\chi^\pm \in C^\infty(T \otimes \mathfrak{so}(C_\pm))$, via the decomposition (4). We see now that these vanish, since for $Z \in C^\infty(T^*)$ and $W \in C^\infty(C_\pm)$, we have

$$\begin{aligned} \chi_Z^\pm W = D_Z W &= [Z_\mp, W_\pm]_\pm + [CZ_\pm, W_\pm]_\pm \\ &= [Z_\mp + (CZ)_\mp, W_\pm]_\pm \\ &= 0, \end{aligned}$$

where we use the fact that $Z \in T^*$ if and only if $CZ = -Z$.

As a result of this, we conclude that $D^\pm$ may be viewed as usual metric connections $\nabla^\pm$ on T, via the projection isomorphisms $\pi_\pm : C_\pm \longrightarrow T$, i.e.

$$D^\pm = \pi_\pm^{-1} \nabla^\pm \pi_\pm.$$

The connections $\nabla^\pm$ may be described as follows:

$$\begin{aligned} \nabla_X^\pm Y &= 2\pi_\pm D_X^\pm Y_\pm \\ &= 4\pi_\pm D_{X_\mp}^\pm Y_\pm \\ &= 4\pi_\pm [X_\mp, Y_\pm]_\pm. \end{aligned}$$

We may easily compute the torsion $T^{\pm}$ of the connections $\nabla^{\pm}$, for vector fields X, Y, Z:

$$\begin{aligned}
2g(T^{+}(X,Y),Z) &= \langle T^{+}(X,Y), Z_{+}\rangle \\
&= \langle 4\pi_{+}[X_{-},Y_{+}]_{+} - 4\pi_{+}[Y_{-},X_{+}]_{+} - \pi[X,Y], Z_{+}\rangle \\
&= \langle 2[X_{-},Y_{+}]_{+} + 2[X_{+},Y_{-}]_{+} - [X,Y] + i_Y i_X H, Z_{+}\rangle \\
&= 2H(X,Y,Z) - 2\langle [X_{+}-X_{-}, Y_{+}-Y_{-}], Z_{+}\rangle \\
&= 2H(X,Y,Z),
\end{aligned}$$

by the fact that $[X_{+}-X_{-}, Y_{+}-Y_{-}] = 0$ since the Courant bracket vanishes on 1-forms. A similar calculation gives $g(T^{-}(X,Y),Z) = -H(X,Y,Z)$.

The above calculation shows that $\nabla^{\pm}$ coincide with the Bismut connections with totally skew torsion $\pm H$. In this way, we have essentially repeated the observation of [7] that $\nabla^{\pm}$ may be conveniently expressed in terms of the Courant bracket. To summarize, the generalized Bismut connection is essentially a usual connection on E which restricts to $C_{\pm}$ to give the Bismut connections with torsion $\pm H$.

Proposition 1. *The torsion T_D of the generalized Bismut connection lies in $\wedge^3 C_{+} \oplus \wedge^3 C_{-} \subset \wedge^3 E$, and is given by*

$$T_D = \pi_{+}^{*}H + \pi_{-}^{*}H.$$

Proof. First we show that $T(C_{+}, C_{-}, \cdot) = 0$, so that $T \in C^{\infty}(\wedge^3 C_{+} \oplus \wedge^3 C_{-})$. Let $x \in C^{\infty}(C_{+})$, $y \in C^{\infty}(C_{-})$ and $z \in C^{\infty}(E)$. Then

$$\begin{aligned}
T_D(x,y,z) &= \langle D_x y - D_y x - [x,y], z\rangle \\
&= \langle [x,y]_{-} - [y,x]_{+} - [x,y], z\rangle = 0,
\end{aligned}$$

as required.

Now take $x, y, z \in C^{\infty}(C_{+})$. Since $\chi_D = 0$, we have the identity

$$\begin{aligned}
\langle D_z x, y\rangle - \langle D_z y, x\rangle &= \langle D_{Cz} x, y\rangle - \langle D_{Cz} y, x\rangle \\
&= \langle [Cz, x], y\rangle - \langle [Cz, y], x\rangle \\
&= \langle [x,y] - [y,x], Cz\rangle.
\end{aligned}$$

Therefore the torsion is given by

$$\begin{aligned}
T(x,y,z) &= \langle D_x y - D_y x - [x,y]_{sk}, z\rangle + \tfrac{1}{2}\langle [x,y] - [y,x], Cz\rangle \\
&= \langle D_x y - D_y x, z\rangle + \langle [x,y], Cz - z\rangle \\
&= g(\nabla^{+}_X Y - \nabla^{+}_Y X - [X,Y], Z) \\
&= H(X,Y,Z).
\end{aligned}$$

A similar calculation for $x, y, z \in C^{\infty}(C_{-})$ gives $T(x,y,z) = H(x,y,z)$ as well, yielding the result. □

As we have explained, the generalized Bismut connection D is completely determined by a usual connection on $T \oplus T^*$. Using the decomposition (6), and the fact that the Bismut connections satisfy $\nabla^{\pm} = \nabla \pm \frac{1}{2} g^{-1} H$ for ∇ the Levi-Civita connection, we may write D explicitly with respect to the splitting $E = T \oplus T^*$, and for $X \in C^\infty(T)$, as follows:

$$D_X = \begin{pmatrix} \nabla_X & \frac{1}{2} \wedge^2 g^{-1}(i_X H) \\ \frac{1}{2} i_X H & \nabla^*_X \end{pmatrix}$$

The significance of this connection in the context of generalized geometry was first understood and investigated by Ellwood in [11]. Here we simply view it as a generalized connection[1] mainly for the purpose of highlighting its properties and defining its torsion tensor.

4 Generalized holomorphic bundles and branes

Suppose now that we have a generalized complex structure $\mathcal{J}$ on (M, E), which is an orthogonal almost complex structure $\mathcal{J} : E \longrightarrow E$ whose $+i$ eigenbundle $L \subset E \otimes \mathbb{C}$ is closed under the Courant bracket [1]. We now describe how the structures in the previous two sections may be made *compatible* with $\mathcal{J}$.

4.1 Generalized holomorphic bundles

The integrability of $\mathcal{J}$ guarantees that $L = \ker(\mathcal{J} - i1)$ is a complex Lie algebroid, with associated de Rham complex

$$C^\infty(\wedge^k L^*) \xrightarrow{d_L} C^\infty(\wedge^{k+1} L^*) \tag{7}$$

A complex vector bundle equipped with a flat L-connection is called a generalized holomorphic bundle [12]. Therefore, generalized holomorphic bundles form a category of Lie algebroid representations in the sense of [13].

In the case that $\mathcal{J}$ is a usual complex structure, for instance, a generalized holomorphic bundle consists of a holomorphic bundle V, together with a holomorphic section $\Phi \in H^0(M, T_{1,0}M \otimes \mathrm{End}(V))$ satisfying

$$\Phi \wedge \Phi = 0 \in H^0(M, \wedge^2 T_{1,0}M \otimes \mathrm{End}(V)).$$

Note that if M is holomorphic symplectic, then $T_{1,0}$ is isomorphic to $T^*_{1,0}$, and Φ may be viewed as a Higgs bundle, in the sense of [14].

In the case that $\mathcal{J}$ is a symplectic structure, a generalized holomorphic bundle is simply a flat bundle.

[1] The author thanks Yicao Wang for pointing this out and correcting an error in the previous version.

Definition 4. *A unitary generalized connection D on a complex vector bundle V is compatible with $\mathcal{J}$ when its curvature along $L = \ker(\mathcal{J} - i1)$ is zero.*

It follows immediately that the restriction of D to L defines a flat L-module structure on V, making V a generalized holomorphic bundle. Conversely, suppose V is a $\mathcal{J}$-holomorphic bundle, i.e. it is equipped with an L-connection as follows:

$$\overline{\partial} : C^\infty(V) \longrightarrow C^\infty(L^* \otimes V), \quad \overline{\partial}^2 = 0. \tag{8}$$

This operator has symbol sequence given by wedging with $\sigma(\xi) = \frac{1}{2}(1 + i\mathcal{J})\xi \in \overline{L}$, where we identify $L^* = \overline{L}$ using the metric on E.

Choosing a Hermitian metric h on the bundle V, so that $\overline{V} \simeq V^*$, we may view the complex conjugate of (8),

$$\partial : C^\infty(\overline{V}) \longrightarrow C^\infty(L \otimes \overline{V}),$$

as a L-connection on V^*; we then form the dual ∂^* of this partial connection. Finally we form the sum

$$D = \overline{\partial} + \partial^* : C^\infty(V) \longrightarrow C^\infty((\overline{L} \oplus L) \otimes V) = C^\infty(E \otimes V),$$

which has symbol $\sigma + \bar{\sigma} = \pi^*$. Hence it defines a generalized connection on V. We summarize the above in the following

Proposition 2. *Let V be a complex vector bundle with $\mathcal{J}$-holomorphic structure given by $\overline{\partial}$, and choose a Hermitian metric on V. Then the operator*

$$D = \overline{\partial} + \partial^* : C^\infty(V) \longrightarrow C^\infty(E \otimes V)$$

is the unique unitary generalized connection extending $\overline{\partial}$.

When V is a line bundle, there is a useful formula for the generalized connection 1-form in terms of a holomorphic trivialization, analogous to the Poincaré-Lelong formula for the Chern connection on a Hermitian holomorphic line bundle.

Proposition 3 (Generalized Poincaré-Lelong formula)**.** *Let V be a generalized holomorphic hermitian line bundle, and let $s \in C^\infty(V)$ be a holomorphic section. Where it is nonzero, it defines a trivialization of the unitary generalized connection, $D = d + i\mathcal{A}$, where*

$$\mathcal{A} = \mathcal{J} d \log |s| \in C^\infty(E).$$

Proof. Whenever s is nonzero, we have

$$D \frac{s}{|s|} = i\mathcal{A} \frac{s}{|s|}.$$

Taking the projection to $\overline{L}$, we obtain

$$d_L \log |s| = i\mathcal{A}^{0,1},$$

so that $\mathcal{A} = -i(d_L - d_{\overline{L}}) \log |s| = \mathcal{J} d \log |s|$, as required. □

In particular, if s is nonzero on an open dense set, then the vector field $\pi\mathcal{J}d\log|s| = X$ must extend to a smooth vector field on the whole of M, since $\pi(i\mathcal{A})$ coincides with $\chi \in C^\infty(T\otimes\mathfrak{u}(V))$, which is globally defined for any generalized connection. But the map $\pi\mathcal{J}|_{T^*} : T^* \longrightarrow T$ is actually a Poisson structure $Q \in C^\infty(\wedge^2 T)$ (see [12] for a discussion of this fact), and hence s vanishes only along the zero locus of the Poisson structure Q, which is a strong constraint on any generalized holomorphic section.

The above proposition may be used, by invoking the local existence of nonvanishing holomorphic sections near points for which $\mathcal{J}$ is regular (i.e. Q has locally constant rank), to show that the vector field χ of any Hermitian $\mathcal{J}$-holomorphic line bundle must be a Poisson vector field. It therefore defines a characteristic class in the Poisson cohomology of [15], which is the cohomology of the complex $(C^\infty(\wedge^\bullet T), d_Q)$, where $d_Q\Pi = [Q,\Pi]$ is the Schouten bracket with Q.

Corollary 4.1. *The real vector field $X = -i\chi$ associated to any Hermitian generalized holomorphic line bundle preserves the Poisson structure Q, i.e. it is a Poisson vector field. Furthermore its Poisson cohomology class $[X] \in H^1_Q(M)$ is independent of the Hermitian metric.*

Proof. X is Poisson since, by the proposition, it is locally Hamiltonian on an open dense subset of M. Hence $L_X Q = 0$ everywhere. Rescaling the Hermitian metric by a positive function e^f, we obtain a new vector field $X' = X + \frac{1}{2}Qdf$, which differs from X by a global Hamiltonian vector field. Hence $[X] \in H^1_Q(M)$ is independent of the choice of Hermitian structure. □

We may also deduce this result from the more general fact that the tensor product of a L-module with a $\overline{L}$ module is a Poisson module for Q (This is a direct consequence of the fact that the tensor product of the Dirac structures $L, \overline{L}$ is the Dirac structure associated to Q, shown in [12]). For any generalized holomorphic line bundle V, therefore, the trivial bundle $V\otimes\overline{V}$ acquires a Q-module structure, and therefore, as described in [13], a characteristic class in $H^1_Q(M)$.

There are always two natural $\mathcal{J}$-holomorphic line bundles on any generalized complex manifold: the trivial bundle, for which $\chi = 0$ (for the standard Hermitian structure), and the canonical line bundle $K_\mathcal{J}$ of pure spinors associated to the maximal isotropic subbundle $L \subset E\otimes\mathbb{C}$. Since $K_\mathcal{J}\otimes\overline{K_\mathcal{J}}$ is naturally the determinant line $\det T^*$, it follows that $[X] = [-i\chi]$ is actually the *modular class* of the Poisson structure Q, in the sense of [16].

4.2 Generalized complex branes

Suppose we have a submanifold $\iota : S \hookrightarrow M$ equipped with a Courant trivialization $s : TS \longrightarrow E_S$. The Dirac structure $s(TS) \subset E_S$ may be canonically lifted to a maximal isotropic subbundle of ι^*E; this operation is called the push-forward of Dirac structures [17]:

$$\tau_S := \iota_* s(TS) = \{e \in E \;:\; \pi(e) \in TS \text{ and } e + \mathrm{Ann}(TS) \in s(TS)\}.$$

Note that τ_S is an extension of the tangent bundle of S by its conormal bundle:

$$0 \longrightarrow N^*S \longrightarrow \tau_S \longrightarrow TS \longrightarrow 0$$

In the presence of the generalized complex structure, there is a natural compatibility condition, as follows.

Definition 5. *A* generalized complex submanifold *is a trivialization of the Courant algebroid along a submanifold $\iota : S \hookrightarrow M$ which is compatible with the generalized complex structure $\mathcal{J}$, in the sense that*

$$\mathcal{J}\tau_S = \tau_S, \tag{9}$$

where $\tau_S = \iota_* s(TS) \subset \iota^* E$.

As shown in [12], in the complex case (and for the trivial gerbe), generalized complex submanifolds correspond to holomorphic submanifolds equipped with unitary holomorphic line bundles, whereas in the symplectic case they correspond to Lagrangian submanifolds equipped with flat line bundles *or* the co-isotropic A-branes of [18]. A useful general example of a generalized complex submanifold is the graph of an isomorphism of generalized complex manifolds, as follows. The notation $\overline{\mathcal{J}}$ denotes the same endomorphism as $\mathcal{J}$ but in the opposite Courant algebroid $\overline{E}$.

Example 4.2. *Let $(M, \mathcal{J}_M)$, $(N, \mathcal{J}_N)$ be generalized complex manifolds. They are isomorphic when there is a Courant algebroid isomorphism in the sense of Example 2.1 which is a generalized complex submanifold of the product $(M \times N, \overline{\mathcal{J}}_M \times \mathcal{J}_N)$.*

In [12], it is shown that in the eigenspace decomposition with respect to $\mathcal{J}$

$$\tau_S \otimes \mathbb{C} = \ell + \bar{\ell},$$

the $+i$ eigenbundle ℓ inherits a Lie bracket, by extending sections randomly to sections over M which remain $+i$ eigensections of $\mathcal{J}$, taking their Courant bracket and restricting to S. Thus ℓ becomes an elliptic complex Lie algebroid over S. Therefore there is a notion of flat ℓ-module. These ℓ-modules are called branes in analogy to the physics literature.

Definition 6. *A* generalized complex brane *is a vector bundle with flat ℓ-connection, supported over a generalized complex submanifold.*

Remark 1. *Just as for generalized holomorphic bundles, we may choose to represent branes using unitary connections with values in τ_S^*, i.e. operators*

$$D : C^\infty(V) \longrightarrow C^\infty(\tau_S^* \otimes V)$$

with symbol given by the inclusion $T^*S \subset \tau_S^*$, and with vanishing curvature along $\ell \subset \tau_S \otimes \mathbb{C}$.

For a usual complex structure, a brane consists of a holomorphic bundle V supported on a complex submanifold $S \subset M$ together with a choice of holomorphic section $\phi \in H^0(S, N_{1,0}S \otimes \mathrm{End}(V))$ satisfying

$$\phi \wedge \phi = 0 \in H^0(S, \wedge^2 N_{1,0}S \otimes \mathrm{End}(V)),$$

where $N_{1,0}S$ denotes the holomorphic normal bundle of S.

On the other hand, for a symplectic structure, branes are complex flat bundles if they are supported on Lagrangian submanifolds; they may also be supported on coisotropic submanifolds with holomorphic structure transverse to the characteristic foliation [18, 12], in which case they are transversally holomorphic bundles, flat along the leaves.

Example 4.3 (Higgs bundles)**.** *Let $C \subset X$ be a curve in a complex surface equipped with holomorphic symplectic form $\Omega \in H^0(X, \Omega^{2,0})$, e.g. a K3 surface. Also, let $V \longrightarrow C$ be a Higgs bundle in the sense of [19], i.e. a holomorphic bundle together with a Higgs field $\theta \in H^0(C, \Omega^1 \otimes \mathrm{End}(V))$. Since C is Lagrangian with respect to Ω, we have an isomorphism $T_{1,0}C \longrightarrow N^*_{1,0}C$, so that we may form $\phi = \Omega^{-1}\theta \in H^0(C, N_{1,0} \otimes \mathrm{End}(V))$, making (V, ϕ) into a brane for the complex structure.*

On the other hand, if (V, θ) is a stable Higgs bundle, then by the existence theorem [19] for solutions to Hitchin's equations we obtain a complex flat connection ∇ on V, rendering (C, V, ∇) into a symplectic brane with respect to either the real or imaginary parts of Ω.

Example 4.4. *Let V be a generalized holomorphic bundle, i.e. a complex vector bundle equipped with a flat L-connection, where $L = \ker(\mathcal{J} - i1)$ for $\mathcal{J}$ a generalized complex structure. Then the pullback of V to any generalized complex submanifold $S \subset M$ defines a generalized complex brane, as can be seen easily from the inclusion $\ell \subset L$.*

Another simple example of a generalized complex brane occurs when it is supported on an isomorphism of generalized complex manifolds, as in Example 4.2.

Proposition 4. *Let $S \subset M_0 \times M_1$ define an isomorphism of the generalized complex manifolds $(M_0, \mathcal{J}_0)$, $(M_1, \mathcal{J}_1)$. Then the Lie algebroid ℓ is isomorphic to both $L_i = \ker(\mathcal{J}_i - i1)$, so that branes on S may be identified with generalized holomorphic bundles on either manifold.*

Proof. Let $\pi_i : M_0 \times M_1 \longrightarrow M_i$ be the usual projection maps. The subbundle $\ell \subset (\pi_0^* L_0 \oplus \pi_1^* L_1)|_S$ is transverse to both $\pi_i^* L_i$, as we now show. If $x \in \ell \cap \pi_0^* L_0$, for instance, then $(\pi_1)_* x = \varphi_*((\pi_0)_* x)$, where φ is the diffeomorphism defining S. This implies $(\pi_i)_* x = 0$ and $x \in N^*S \otimes \mathbb{C}$, which clearly is transverse to $\pi_0^* L_0$. Hence $x = 0$, and similarly for L_1.

This transversality means that we have isomorphic bundle maps onto each factor:

$$\begin{array}{ccccc} L_0 & \xleftarrow{p_0} & \ell & \xrightarrow{p_1} & L_1 \\ \downarrow & & \downarrow & & \downarrow \\ M_0 & \xleftarrow{\pi_0} & S & \xrightarrow{\pi_1} & M_1 \end{array} \qquad (10)$$

We now show that the projections p_0, p_1 are isomorphisms of Lie algebroids.

Given $Z \in C^\infty(S, \ell)$, let $X = p_0(Z)$ and $Y = p_1(Z)$. Then Z may be expressed as $Z = (\pi_0^* X + \pi_1^* Y)|_S$. Computing the bracket of Z, Z', we may use the given extensions to $M_0 \times M_1$ and compute their Courant bracket:

$$\begin{aligned} [Z, Z'] &= [\pi_0^* X + \pi_1^* Y, \pi_0^* X' + \pi_1^* Y']|_S \\ &= [\pi_0^* X, \pi_0^* X']|_S + [\pi_1^* Y, \pi_1^* Y']|_S \\ &= (\pi_0^*[X, X'] + \pi_1^*[Y, Y'])|_S, \end{aligned} \tag{11}$$

where we use the fact that sections pulled back from opposite factors M_0, M_1 Courant commute in the product. Applying the projections to the final formula, we obtain

$$p_0([Z, Z']) = [p_0(Z), p_0(Z')], \quad p_1([Z, Z']) = [p_1(Z), p_1(Z')],$$

as required. □

We now describe the general form of a generalized complex brane when it is supported on the whole manifold M; these are usually called "space-filling branes". We first observe that the requirement that M be a generalized complex submanifold of itself places a very strong constraint on $\mathcal{J}$.

Proposition 5. *$(M, \mathcal{J})$ is a generalized complex submanifold of itself if and only if there exists an integrable isotropic splitting $E = T \oplus T^*$ of the Courant algebroid with respect to which $\mathcal{J}$ has the form:*

$$\mathcal{J} = \begin{pmatrix} I & Q \\ & -I^* \end{pmatrix}, \tag{12}$$

where I is a usual complex structure on the manifold and $\sigma = P + iQ$, for $P = IQ$, is a holomorphic Poisson structure, i.e. it satisfies $[\sigma, \sigma] = 0$.

Proof. Compatibility of the splitting with $\mathcal{J}$ forces $\mathcal{J}T = T$, which holds iff $\mathcal{J}$ is upper triangular, and the orthogonality of $\mathcal{J}$ together with the fact $\mathcal{J}^2 = -1$ guarantees that I is an almost complex structure and that Q is a bivector of type $(2,0) + (0,2)$. The $-i$-eigenbundle of $\mathcal{J}$ is then the direct sum of $T_{0,1}$ with the graph of $\sigma : T_{1,0}^* \longrightarrow T_{1,0}$. This is closed (involutive) for the Courant bracket if and only if $T_{0,1}$ is integrable and $[\sigma, \sigma] = 0$, as required. □

In the splitting $E = T \oplus T^*$ for which $\mathcal{J}$ has the form (12), we see that $\tau_S = TM$, and further that $\ell = T_{1,0}$, so that ℓ-modules are precisely holomorphic bundles with respect to the complex structure I.

5 Multiple branes and holomorphic Poisson varieties

Suppose that we have a Courant trivialization s making $(M, \mathcal{J})$ a generalized complex submanifold of itself, so that $E = T \oplus T^*$ and $\mathcal{J}$ has the form (12). Now we investigate the consequences of having a second trivialization s' which is also compatible with $\mathcal{J}$. Let $F \in \Omega^2_{cl}(M, \mathbb{R})$ be the 2-form taking s to s'. By Proposition 5, and the fact that the Poisson structure Q is independent of splitting, we have

$$\begin{pmatrix} 1 & \\ -F & 1 \end{pmatrix} \begin{pmatrix} I & Q \\ & -I^* \end{pmatrix} \begin{pmatrix} 1 & \\ F & 1 \end{pmatrix} = \begin{pmatrix} J & Q \\ & -J^* \end{pmatrix}, \tag{13}$$

for a second complex structure J such that $\sigma' = JQ + iQ$ is holomorphic Poisson. In particular we note the important fact that a generalized complex structure may be expressed as a holomorphic Poisson structure *in several different ways*, and with respect to different underlying complex structures, depending on the choice of splitting. Equation (13) is equivalent to the conditions

$$\begin{cases} J - I = QF, \\ FJ + I^*F = 0. \end{cases} \tag{14}$$

Phrased as a single condition on F, we obtain the nonlinear equation

$$FI + I^*F + FQF = 0, \tag{15}$$

which may be viewed as a deformation of the usual condition $FI + I^*F = 0$ that F be of type $(1,1)$ with respect to the complex structure. Equation (15) has been studied in [20], where it was shown that it corresponds to a noncommutative version of the $(1,1)$ condition via the Seiberg-Witten transform on tori. We take a different approach here, focusing rather on a groupoid interpretation of the equivalent system (14).

The set of compatible global Courant trivializations forms a groupoid; we may label each trivialization by the complex structure it induces on the base, and we see from (13) or (14) that if F_{IJ} takes I to J and F_{JK} takes J to another trivialization K, then $F_{IJ} + F_{JK}$ takes I to K.

Definition 7. *Fix a real manifold M with real Poisson structure Q. Let $\mathcal{G}$ be the groupoid whose objects are holomorphic Poisson structures (I_i, σ_i) on M with fixed imaginary part given by* $\mathrm{Im}(\sigma_i) = Q$*, and whose morphisms* $\mathrm{Hom}(i, j)$ *consist of real closed 2-forms $F_{ij} \in \Omega^2_{cl}(M, \mathbb{R})$ such that*

$$\begin{cases} I_j - I_i = QF_{ij}, \\ F_{ij}I_j + I_i^*F_{ij} = 0. \end{cases} \tag{16}$$

The composition of morphisms is then simply addition of 2-forms $F_{ij} + F_{jk}$. In keeping with the interpretation of F_{ij} as differences between gerbe trivializations, we could define $\mathrm{Hom}(i, j)$ *to consist of unitary line bundles L_{ij} with curvature F_{ij}, such that composition of morphisms would coincide with tensor product.*

Automorphisms of the Courant algebroid which fix $\mathcal{J}$ give rise to automorphisms of the groupoid of trivializations defined above; we describe these now. Orthogonal automorphisms of the standard Courant bracket on $T \oplus T^*$ consist of pairs $(\varphi, B) \in \mathrm{Diff}(M) \times \Omega^2_{cl}(M, \mathbb{R})$, which act on $T \oplus T^*$ via $X + \xi \mapsto \varphi_* X + (\varphi^{-1})^* \xi + i_{\varphi_* X} B$. Since our generalized complex structure has the form (12), we may easily determine its automorphism group.

Proposition 6. *The automorphism group* $\mathrm{Aut}(\mathcal{J})$ *of the generalized complex structure* (12) *is the set of pairs* $(\varphi, B) \in \mathrm{Diff}(M) \times \Omega^2_{cl}(M, \mathbb{R})$ *such that*

$$\begin{aligned} Q^\varphi &= Q \\ I^\varphi - I &= QB \\ BI^\varphi + I^* B &= 0, \end{aligned}$$

where $Q^\varphi = \varphi_* Q$ and $I^\varphi = \varphi_* I \varphi_*^{-1}$.

These automorphisms therefore act on the groupoid of global generalized complex submanifolds (16), sending $(I_i, \sigma_i) \mapsto (I_i^\varphi, (\varphi^{-1})^* \sigma_i)$ and $F_{ij} \mapsto (\varphi^{-1})^* F_{ij} + B$. Of course, we may wish to interpret B as the curvature of a unitary line bundle U, in which case it would act on the groupoid line bundles L_{ij} by tensor product $L_{ij} \mapsto (\varphi^{-1})^* L_{ij} \otimes U$.

Instead of viewing F_{ij} as the difference between two generalized complex submanifolds of $(M, \mathcal{J})$, we may interpret Equation (13) as giving an isomorphism between two different generalized complex structures on $T \oplus T^*$. This rephrasing leads immediately to the following.

Proposition 7. *Let (I_i, σ_i) and (I_j, σ_j) be holomorphic Poisson structures on M with associated generalized complex structures $\mathcal{J}_i, \mathcal{J}_j$ on $T \oplus T^*$ via* (12)*, let* $\mathrm{Im}(\sigma_i) = \mathrm{Im}(\sigma_j) = Q$*, and let $F_{ij} \in \Omega^2_{cl}(M, \mathbb{R})$ satisfy equation* (16)*. Then the graph of F_{ij} over the diagonal $\Delta \subset M \times M$ defines a generalized complex submanifold of $(M \times M, \overline{\mathcal{J}}_i \times \mathcal{J}_j)$, yielding an isomorphism of generalized complex manifolds*

$$(M, \mathcal{J}_i) \xrightarrow{\cong} (M, \mathcal{J}_j). \tag{17}$$

In view of Proposition 4, this result implies that a morphism F_{ij} from (I_i, σ_i) to (I_j, σ_j) induces an equivalence between the categories of generalized holomorphic bundles associated to $\mathcal{J}_i, \mathcal{J}_j$. We now explain this equivalence explicitly, and its significance for the holomorphic Poisson structures involved.

Proposition 8. *Let $\mathcal{J}$ be of the form (12), for I a complex structure and $\sigma = P + iQ$ a holomorphic Poisson structure. Then a generalized holomorphic bundle is precisely a holomorphic Poisson module [21], i.e. a holomorphic bundle V with an additional action of the structure sheaf on the sheaf of holomorphic sections, denoted $\{f, s\}$, satisfying*

$$\{f, gs\} = \{f, g\}s + g\{f, s\}, \tag{18}$$

$$\{\{f, g\}, s\} = \{f, \{g, s\}\} - \{g, \{f, s\}\}, \tag{19}$$

where $f, g \in \mathcal{O}$, $s \in \mathcal{O}(V)$, and $\{f, g\}$ denotes the Poisson bracket induced by σ.

Proof. Let $L = \ker(\mathcal{J} + i1)$, so that for $\mathcal{J}$ as in (12), we have

$$L = T_{0,1} \oplus \Gamma_\sigma, \tag{20}$$

where $\Gamma_\sigma = \{\xi + \sigma(\xi) \ : \ \xi \in T^*_{1,0}\}$. Let $\overline{\partial}_V : C^\infty(V) \longrightarrow C^\infty(L^* \otimes V)$ be a generalized holomorphic structure. Decomposing using (20) and identifying $\Gamma_\sigma = T^*_{1,0}$, we write $\overline{\partial}_V = \overline{\partial}' + \overline{\partial}''$, where $\overline{\partial}' : C^\infty(V) \longrightarrow C^\infty(T^*_{0,1} \otimes V)$ is a usual holomorphic structure and $\overline{\partial}'' : C^\infty(V) \longrightarrow C^\infty(T_{1,0} \otimes V)$ satisfies, for $f \in C^\infty(M, \mathbb{C})$ and $s \in C^\infty(V)$,

$$\overline{\partial}''(fs) = f\overline{\partial}''s + Z_f \otimes s,$$

where $Z_f = \sigma(df)$ is the Hamiltonian vector field of f. This is equivalent to condition (18). Furthermore $\overline{\partial}_V^2 = 0$ implies that $\overline{\partial}''$ is holomorphic and defines a Poisson module structure via

$$\{f, s\} = \overline{\partial}''_{\partial f} s,$$

since $(\overline{\partial}'')^2 = 0$ implies $\overline{\partial}''_{\partial\{f,g\}} = [\overline{\partial}''_{\partial f}, \overline{\partial}''_{\partial g}]$, which is equivalent to (19), as required. □

Letting $L_k = \ker(\mathcal{J}_k + i1)$, we see from Equation (13) that $\exp(F_{ij})$ takes L_i to L_j. Hence the map on generalized holomorphic bundles induced by the isomorphism (17) may be described as composition with $\exp(F_{ij})$

$$\overline{\partial}_i \mapsto e^{F_{ij}} \circ \overline{\partial}_i.$$

This map may be made more explicit in terms of the associated generalized connections. Choose a Hermitian structure on the $\mathcal{J}_i$-holomorphic bundle (i.e. σ_i-Poisson module), and let $D = \nabla + \chi$ be the extension of $\overline{\partial}_i$ as in Proposition 2. Then F_{ij} acts on D via

$$\begin{cases} D \mapsto D' = \nabla' + \chi, \\ \nabla' = \nabla + F_{ij}(\chi). \end{cases} \tag{21}$$

which then defines a σ_j-Poisson module. It is important to note that the σ_i-Poisson module, which is I_i-holomorphic, inherits via (21) a I_j-holomorphic structure, without the presence of any holomorphic map between (M, I_j) and (M, I_i).

Given this result, it is natural to ask how restrictive the condition of admitting a Poisson module structure actually is. The following is a simple result describing the complete obstruction to the existence of a Poisson module structure on a holomorphic line bundle.

Proposition 9. *Let M be a holomorphic Poisson manifold, and let V be a holomorphic line bundle on M. Then the Atiyah class of V, $\alpha \in H^1(T^*_{1,0})$, combines with the Poisson structure $\sigma \in H^0(\wedge^2 T_{1,0})$ to give the class $\sigma\alpha \in H^1(T_{1,0})$. If $\sigma\alpha = 0$, then there is a well-defined secondary characteristic class $f_\alpha \in H^2_\sigma(M)$ in Poisson cohomology. V admits a Poisson module structure if and only if both classes $\{\sigma\alpha, f_\alpha\}$ vanish. The space of Poisson module structures is affine, modeled on $H^1_\sigma(M)$.*

Proof. A Poisson module structure on V is a holomorphic differential operator $\partial : \mathcal{O}(V) \longrightarrow \mathcal{O}(T_{1,0} \otimes V)$ satisfying $\partial(fs) = f\partial s + Z_f \otimes s$, for $f \in \mathcal{O}$ and $s \in \mathcal{O}(V)$, where Z_f is the σ-Hamiltonian vector field of f, and such that the curvature vanishes. Let $\{U_i\}$ be an open cover of M and let $\{s_i \in \mathcal{O}(U_i, V)\}$ be a local trivialization of V such that $s_i = g_{ij}s_j$ for holomorphic transition functions g_{ij}; then

$$\partial s_i = X_i \otimes s_i, \tag{22}$$

where X_i are holomorphic Poisson vector fields (since $\partial_{d\{f,g\}} = [\partial_{df}, \partial_{dg}]$) such that

$$X_i - X_j = Z_{\log g_{ij}}. \tag{23}$$

The Hamiltonian vector fields $Z_{\log g_{ij}} = \sigma(d \log g_{ij})$ are a Čech representative for the image of the Atiyah class under σ. Therefore, equation (23) holds if and only if $\sigma\alpha = 0 \in H^1(T_{1,0})$. If $\sigma\alpha = 0$, then we may solve (23) for some holomorphic vector fields $\tilde{X}_i$. We can modify these by a global holomorphic vector field so that they are each Poisson if and only if the global bivector field f_σ defined by $f_\sigma|_{U_i} = [\tilde{X}_i, \sigma]$ vanishes in Poisson cohomology, i.e. $f_\sigma = [Y, \sigma]$ for $Y \in H^0(T_{1,0})$, in which case $X_i = \tilde{X}_i - Y|_{U_i}$ defines a Poisson module structure as required.

Given any holomorphic Poisson vector field $Z \in H^1_\sigma(M)$ and Poisson module structure ∂, the sum $\partial + Z$ defines a new Poisson module structure. Conversely, two Poisson module structures ∂', ∂ must satisfy $\partial' - \partial \in H^1_\sigma(M)$, as claimed. □

It is remarked in [21] that the canonical line bundle K always admits a natural Poisson module structure for any holomorphic Poisson structure σ via the action, for $f \in \mathcal{O}$ and $\rho \in \mathcal{O}(K)$,

$$\{f, \rho\} = L_{Z_f}\rho.$$

Based on these considerations, we obtain the following example.

Example 5.1. *Let $M = \mathbb{C}P^2$, equipped with a holomorphic Poisson structure $\sigma \in H^0(\mathcal{O}(3))$. Note that $H^1(T_{1,0}) = 0$. Then $K = \mathcal{O}(-3)$ is canonically a Poisson module, and since $\mathcal{O}(1)^{-3} = K$, we see that the obstruction f_σ from Proposition 9 must vanish for $\mathcal{O}(1)$ as well* *(note that* $\dim H^2_\sigma(\mathbb{C}P^2) = 2$*, so the obstruction space is nonzero). Hence all holomorphic line bundles* $\mathcal{O}(k)$ *admit Poisson module structures. If* σ *is generic, these Poisson module structures are unique, since* $H^1_\sigma(M) = 0$*, due to the fact that only the zero holomorphic vector field on* $\mathbb{C}P^2$ *stabilizes a smooth cubic curve.*

We conclude this section with a simple example of a generalized complex manifold admitting multiple trivializations with *non-biholomorphic* induced complex structures.

Proposition 10. *Let* $E_0 = E \times \mathbb{C}$*, the trivial line bundle over an elliptic curve* E*, and let* E_c*, for* $c \in \mathbb{R}$*, be the alternative holomorphic structure on* $E \times \mathbb{C}$ *obtained by endowing the bundle* $E \times \mathbb{C}$ *with the holomorphic structure associated to the point* $ic \in H^1(E, \mathcal{O}) = \mathbb{C}$*. Then* E_0 *and* E_c *are diffeomorphic, non-biholomorphic complex manifolds. They are equipped with canonical holomorphic Poisson structures* σ_0, σ_c *vanishing to first order on the zero section, and furthermore* (E_0, σ_0) *and* (E_c, σ_c) *are isomorphic as generalized complex manifolds* $\forall c \in \mathbb{R}$ *(and hence have equivalent categories of Poisson modules).*

Proof. Represent E as $\mathbb{C}^*/\{z \mapsto \lambda z\}$ and let w be the linear coordinate on the fiber of $E \times \mathbb{C}$. Then the holomorphic structure E_c is given by the complex volume form

$$\Omega_c = \frac{dz}{z} \wedge \left(dw + icw \frac{d\bar{z}}{\bar{z}} \right),$$

and the holomorphic Poisson structure σ_c is given by

$$\sigma_c = \left(z\frac{\partial}{\partial z} + ic\bar{w}\frac{\partial}{\partial \bar{w}} \right) \wedge w\frac{\partial}{\partial w}$$

The pure spinor corresponding to the generalized complex structure (E_c, σ_c) is

$$\rho_c = e^{\sigma_c}\Omega_c.$$

Now let $F_c = ic\frac{dz\wedge d\bar{z}}{z\bar{z}}$ be a real multiple of the volume form on E (which may be viewed as a curvature when $c \in 2\pi\mathbb{Z}$). Then we verify that

$$\begin{aligned} e^{F_c}e^{\sigma_0}\Omega_0 &= w + \tfrac{dz}{z} \wedge \left(dw + icw\tfrac{d\bar{z}}{\bar{z}} \right) \\ &= e^{\sigma_c}\Omega_c, \end{aligned}$$

showing that (E_0, σ_0) and (E_c, σ_c) are isomorphic as generalized complex manifolds. □

6 Relation to generalized Kähler geometry

A generalized Kähler structure is a pair $(\mathcal{J}_A, \mathcal{J}_B)$ of commuting generalized complex structures such that

$$G(\cdot, \cdot) = \langle \mathcal{J}_A \cdot, \mathcal{J}_B \cdot \rangle$$

is a generalized Riemannian metric.

In [2] it is shown that the integrability of the pair $(\mathcal{J}_A, \mathcal{J}_B)$ is equivalent to the fact that the induced decomposition of the definite subspaces $C_\pm$ given by

$$C_\pm \otimes \mathbb{C} = L_\pm \oplus \overline{L_\pm},$$

where $L_\pm = \ker(\mathcal{J}_A - i1) \cap \ker(\mathcal{J}_B \mp i1)$, satisfies the condition that $L_\pm$ are each involutive. Using the generalized Bismut connection D introduced in Theorem 3.1, we provide the following equivalent description of generalized Kähler geometry.

Theorem 6.1. *Let G be a generalized metric and let $\mathcal{J}$ be a G-orthogonal almost generalized complex structure. Then $(\mathcal{J}, G)$ defines a generalized Kähler structure if and only if $D\mathcal{J} = 0$ and the torsion $T_D \in C^\infty(\wedge^3 E)$ is of type $(2,1) + (1,2)$ with respect to $\mathcal{J}$.*

Proof. We leave the forward direction to the reader. We show that if $D\mathcal{J} = 0$ (where, as usual, $(D_x\mathcal{J})y = D_x(\mathcal{J}y) - \mathcal{J}(D_x y)$) and the torsion is as above, then $\mathcal{J}$ is integrable as a generalized complex structure. Note that under these assumptions, the complementary generalized complex structure $\mathcal{J}' = G\mathcal{J}$ would also be covariant constant, and be compatible with the torsion as well, by Proposition 1. Therefore by the following argument $\mathcal{J}'$ is also integrable, and we obtain the result.

We compute the Nijenhuis tensor of $\mathcal{J}$, for $x, y, z \in C^\infty(E)$ (in the following, $[\cdot, \cdot]$ refers to the skew-symmetrized Courant bracket):

$$\begin{aligned}
\langle N_{\mathcal{J}}(x,y), z\rangle &= \langle [\mathcal{J}x, \mathcal{J}y] - \mathcal{J}[\mathcal{J}x, y] - \mathcal{J}[x, \mathcal{J}y] - [x,y], z\rangle \\
&= \langle D_z(\mathcal{J}x), \mathcal{J}y\rangle - \langle D_z(\mathcal{J}y), \mathcal{J}x\rangle + \langle D_{\mathcal{J}z}(\mathcal{J}x), y\rangle - \langle D_{\mathcal{J}z}y, \mathcal{J}x\rangle \\
&+ \langle D_{\mathcal{J}z}x, \mathcal{J}y\rangle - \langle D_{\mathcal{J}z}(\mathcal{J}y), x\rangle - \langle D_z x, y\rangle + \langle D_z y, x\rangle \\
&- T_D(\mathcal{J}x, \mathcal{J}y, z) - T_D(\mathcal{J}x, y, \mathcal{J}z) - T_D(x, \mathcal{J}y, \mathcal{J}z) - T(x,y,z).
\end{aligned}$$

The first eight terms cancel since $D_x(\mathcal{J}y) = \mathcal{J}D_x y$, and the last four terms cancel since T_D is of type $(2,1) + (1,2)$. Therefore $\mathcal{J}$ is integrable, as claimed. ☐

We now explain that a solution to the system (14), if positive in a certain sense, gives rise to a generalized Kähler structure. When the Poisson structure Q vanishes, this result specializes to the fact that a positive holomorphic line bundle with Hermitian structure defines a Kähler structure.

Definition 8. *Let (I, J, Q, F) be a solution to the system (14), i.e. it defines two global Courant trivializations compatible with a generalized complex structure, separated by the 2-form F. Then*

$$g = -\tfrac{1}{2}F(I + J)$$

is a symmetric tensor, and if it is positive-definite, we say that F is positive.

If F is positive, then (g, I), (g, J) are both Hermitian structures. Let $\omega_I = gI$, $\omega_J = gJ$ be their associated 2-forms. Then we have the following.

Theorem 6.2. *Let (I, J, Q, F) be as above, and let F be positive. Then the pair*

$$\begin{aligned} \mathcal{J}_B &= \frac{1}{2}\begin{pmatrix} 1 & \\ b & 1 \end{pmatrix}\begin{pmatrix} J+I & -(\omega_J^{-1}-\omega_I^{-1}) \\ \omega_J-\omega_I & -(J^*+I^*) \end{pmatrix}\begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix}, \\ \mathcal{J}_A &= \frac{1}{2}\begin{pmatrix} 1 & \\ b & 1 \end{pmatrix}\begin{pmatrix} J-I & -(\omega_J^{-1}+\omega_I^{-1}) \\ \omega_J+\omega_I & -(J^*-I^*) \end{pmatrix}\begin{pmatrix} 1 & \\ -b & 1 \end{pmatrix}, \end{aligned} \tag{24}$$

defines a generalized Kähler structure on the standard Courant algebroid $(T \oplus T^*, [\cdot,\cdot]_0)$, for $b \in \Omega^2(M, \mathbb{R})$ given by

$$b = -\tfrac{1}{2}F(J-I).$$

Proof. It is easily verified that $\mathcal{J}_A^2 = \mathcal{J}_B^2 = -1$ and that $[\mathcal{J}_A, \mathcal{J}_B] = 0$. To show integrability, we first observe that $\mathcal{J}_A$ has the form of a pure symplectic structure; indeed, with the definitions above,

$$\mathcal{J}_A = \begin{pmatrix} & -F^{-1} \\ F & \end{pmatrix}.$$

We see therefore that $\mathcal{J}_A$ is integrable since $dF = 0$.

The structure $\mathcal{J}_B$ is also integrable, as follows. Let $L_B = \ker(\mathcal{J}_B - i)$ and let $L_B = L_+ \oplus L_-$ be its decomposition into $\pm i$ eigenspaces for $\mathcal{J}_A$. Then

$$\begin{aligned} L_+ &= \{X + (b+g)X \ : \ X \in T_J^{1,0}\} \\ L_- &= \{X + (b-g)X \ : \ X \in T_I^{1,0}\} \end{aligned}$$

It follows from the definitions of b, g that $b + g = -FJ$ whereas $b - g = FI$. As a result we have

$$\begin{aligned} L_+ &= \{X - iFX \ : \ X \in T_J^{1,0}\} \\ L_- &= \{X + iFX \ : \ X \in T_I^{1,0}\} \end{aligned}$$

which are integrable precisely when $i_X i_Y dF = 0$ for all X, Y in $T_I^{1,0}$ or $T_J^{1,0}$. Of course this holds since F is closed. □

We note that the converse of this argument also holds; using the result from [2] that any generalized Kähler structure has the form (24), we may show that any generalized Kähler structure $(\mathcal{J}_A, \mathcal{J}_B)$ with the property that $\mathcal{J}_A$ is symplectic gives rise to a solution to the system (14). More explicitly, given the bi-Hermitian data (g, I, J) we determine F via

$$F = -2g(I+J)^{-1},$$

where $(I + J)$ is invertible by the assumption on $\mathcal{J}_A$, and the Poisson structure Q is given by

$$Q = (J-I)F^{-1} = \tfrac{1}{2}[I, J]g^{-1}.$$

This is consistent with Hitchin's general observation [22] that $[I, J]g^{-1}$ defines a holomorphic Poisson structure for both I and J, for any generalized Kähler structure.

In fact, the interpretation of F_{ij} in Proposition 7 as defining a morphism between holomorphic Poisson structures allows us to view the generalized Kähler structure as a morphism between the holomorphic Poisson structures (I, σ_I), (J, σ_J). This point of view is related to the approach in [23] to defining a generalized Kähler potential, and may help to resolve the problems encountered there at non-regular points.

Given the equivalence between certain generalized Kähler structures and configurations of generalized complex submanifolds shown in this section, we may apply it to produce new examples of generalized Kähler structures, or indeed of configurations of branes. We do this in the following section.

7 Construction of generalized Kähler metrics

Given a generalized complex submanifold, it is natural to construct more by deformation; this is a familiar construction in symplectic geometry, where new Lagrangian submanifolds may be produced by applying Hamiltonian or symplectic diffeomorphisms. Therefore we would like to deform a given generalized complex submanifold by an automorphism of the underlying geometry, as described in Proposition 6. If the automorphism used is positive in the sense of Definition 8, then we will have constructed a generalized Kähler structure, by Theorem 6.2. This construction is inspired by a construction of Joyce contained in [24], and its generalization by [22] to the construction of generalized Kähler structures on Del Pezzo surfaces.

To reiterate, the goal of the construction is as follows: given a holomorphic Poisson structure (I, σ_I) on M, with real and imaginary parts $\sigma_I = P + iQ$, find a second complex structure J and a 2-form F solving the system (14), i.e.

$$\begin{cases} J - I = QF, \\ FJ + I^*F = 0. \end{cases} \tag{25}$$

We are particularly interested in the case where $g = -\frac{1}{2}F(I + J)$ is positive-definite, as this then defines a generalized Kähler structure, however the construction does not depend on it.

In this construction, the complex structure J will be obtained from I by flowing along a vector field; as a result, J will be biholomorphic to I. Also, we shall describe the construction in the case that F is the curvature of a unitary connection, although it will be clear that integrality of the form F is not required.

1. We begin with a Hermitian complex line bundle L over a compact complex manifold M; the 2-form F solving (25) will be chosen from the cohomology class $c_1(L)$. We first assume that L admits a holomorphic structure $\overline{\partial}_0$ with respect to the "initial" complex structure $I = I_0$. The associated Chern connection will be

called ∇_0, and its curvature denoted F_0. Recall that ∇_0 is the unique Hermitian connection on L such that $\nabla_0^{0,1} = \overline{\partial}_0$.

2. We then assume that L admits the structure of a holomorphic Poisson module with respect to a holomorphic Poisson structure σ_I on M, which by Proposition 9 occurs if and only if $[\sigma_I F_0] \in H^1_I(T_{1,0})$ vanishes and the secondary characteristic class in $H^2_{\sigma_I}(M)$ also vanishes. By Proposition 2, we construct the Hermitian generalized connection D associated to this generalized holomorphic structure, and decompose it according to the splitting $T \oplus T^*$:
$$D = \nabla_0 + iX,$$
where X is a real Q-Poisson vector field such that $\overline{\partial} X^{1,0} = \sigma_I F_0$, giving rise to the real equations
$$\begin{cases} L_X Q &= 0, \\ L_X I_0 &= QF_0. \end{cases} \tag{26}$$

3. Let φ_t be the time-t flow of the vector field X. Then we may transport F_0 by the flow, yielding the cohomologous family of 2-forms $F_t = \varphi^*_{-t} F_0$, which satisfies
$$\dot{F}_t = L_X F_t = d i_X F_t.$$
We may also transport I_0 by the flow, obtaining a family $I_t = I_0^{\varphi_t}$ satisfying
$$\dot{I}_t = L_X I_t = QF_t,$$
by Equation (26). Note that F_t is type $(1,1)$ with respect to I_t. Also note that F_t is the curvature of the family of connections
$$\nabla_t = \nabla_0 + \int_0^t i_X F_s ds,$$
which are therefore the Chern connections associated to a family of holomorphic structures $\overline{\partial}_t$ on L, each holomorphic with respect to I_t.

4. We then compute the difference
$$\begin{aligned} I_t - I_0 &= \int_0^t QF_s ds \\ &= tQ\tfrac{1}{t}\int_0^t F_s ds \\ &= tQ\overline{F}_t, \end{aligned} \tag{27}$$
where $\overline{F}_t$ is the curvature of the average Chern connection on L:
$$\overline{\nabla}_t = \tfrac{1}{t}\int_0^t \nabla_s ds.$$
Setting $t = 1$ we obtain a solution to the first part of (25):
$$I_1 - I_0 = Q\overline{F}_1.$$

5. Observe that the second part of (25) is automatically satisfied: from (27) we have $I_t - I_0 = QG_t$, where
$$G_t = \int_0^t F_s ds.$$
For $t = 0$, the expression
$$G_t I_t + I_0^* G_t \tag{28}$$
vanishes, since $G_0 = 0$. Taking the time derivative, we obtain
$$\begin{aligned} \dot{G}_t I_t + G_t \dot{I}_t + I_0^* \dot{G}_t &= F_t I_t + G_t Q F_t + I_0^* F_t \\ &= -(I_t^* - I_0^*) F_t + G_t Q F_t \\ &= -G_t Q F_t + G_t Q F_t = 0. \end{aligned}$$
Therefore (28) vanishes for all t; since $\overline{F_t} = t^{-1} G_t$, we obtain the result.

6. *Positivity:* If F_0 is positive, i.e. if the original line bundle L is positive, then $\overline{F}_t$ is positive for sufficiently small t. By Equation (27), this gives a solution to the system (25) for the Poisson structure $t\sigma_I$ replacing σ_I.

We summarize the main result of this construction in the following.

Theorem 7.1. *Let L be a positive holomorphic line bundle with Poisson module structure over a compact complex manifold with holomorphic Poisson structure σ. Let (g_0, I_0) be the original Kähler structure it determines. Then the choice of Hermitian structure on L determines a canonical family of generalized Kähler structures $\{(g_t, I_t, I_0) : -\epsilon < t < \epsilon\}$ such that the complex structure I_t coincides with I_0 only along the vanishing locus of σ for $t \neq 0$.*

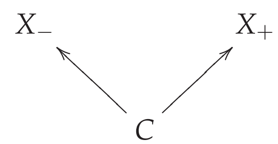
Example 7.2. *One case where the existence of a positive Poisson module is guaranteed is in the case of a Fano Poisson manifold, since the anticanonical bundle, which always admits a Poisson module structure, is positive. This extends the result of [3], who showed that all smooth Fano surfaces (the Del Pezzo surfaces) admit the families of generalized Kähler structures described here.*

We remark finally upon the relation of our construction to Hitchin's result for Del Pezzo surfaces [3]. To obtain the family of generalized Kähler structures, he used a flow generated by a Poisson vector field X which he expressed as the Hamiltonian vector field of $\log |s|^2$, for s a holomorphic section of the anticanonical bundle vanishing at the zero locus of the Poisson structure. From our point of view, he was making use of the generalized Poincaré-Lelong formula of Proposition (3), since in the 2-dimensional case there is always a non-trivial generalized holomorphic section of the anti-canonical bundle of a Poisson surface, namely the Poisson structure itself. However, in higher dimension, there is a dearth of global generalized holomorphic sections; indeed by Proposition (3), such a section (if generically nonzero) must vanish only along the zero locus of σ, which has codimension greater than one in general.

8 Relation to non-commutative algebraic geometry

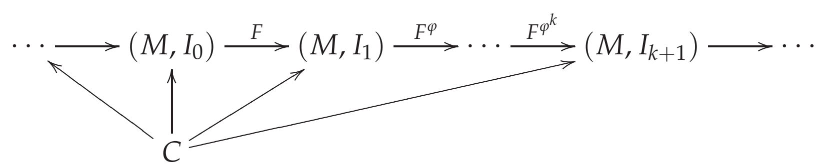
Since the observation in [2] that the deformation space of a complex manifold as a generalized complex manifold includes the "noncommutative" directions in $H^0(\wedge^2 T_{1,0})$, it was hoped that there might be a more precise relationship between generalized complex structures and noncommutativity. The presence of an underlying Poisson structure, for example, lends credence to this idea. In the realm of generalized Kähler 4-manifolds, we have even more evidence in this direction, since, as observed originally in [24], the locus where the bi-Hermitian complex structures (I_+, I_-) coincide is an anti-canonical divisor for both structures.

If smooth, each connected component of this coincidence locus is an elliptic curve C, and we may view it as embedded in two different complex manifolds $X_\pm = (M, I_\pm)$.



In the examples constructed in Section 7 and by [3], $X_\pm$ have natural holomorphic line bundles sitting over them, which we called L_0, L_1. Pulling them back to C, we obtain holomorphic line bundles $\mathcal{L}_0, \mathcal{L}_1$ over C. Furthermore, since the flow satisfied $L_X I_t = QF_t$, we know that the flow restricts to a holomorphic flow on C, the vanishing locus of Q. As a result, $\mathcal{L}_0, \mathcal{L}_1$ are related by an automorphism of C. This data $(C, \mathcal{L}_0, \mathcal{L}_1)$ is precisely what is used in the approach of [25] to the classification of $\mathbb{Z}$-algebras describing noncommutative projective surfaces.

In fact, in our construction we produce an example of an automorphism $(\varphi, F) = (\varphi_1, \overline{F}_1)$ in the sense of Proposition 6. Therefore we may apply it successively, producing an infinite family of generalized complex submanifolds with induced complex structures $\{I_k = I_0^{\varphi_1^k}\}$, each I_k separated from I_0 by the line bundle L^k with connection $\overline{\nabla}_k$, and all coinciding on the vanishing locus C of Q. As a result we obtain an infinite family of embeddings



Where the arrows on the top row indicate morphisms in the sense of the groupoid of Definition 7. This may provide an alternative interpretation of Van den Bergh's construction of the twisted homogeneous coordinate ring (see [26]): let $\mathcal{L} = L_0|_C$, and let $\mathcal{L}^\varphi = (\varphi^{-1})^* \mathcal{L}$. Then define the vector spaces

$$\mathrm{Hom}(i, j) = H^0(C, \mathcal{L}^{\varphi^i} \otimes \mathcal{L}^{\varphi^{i+1}} \otimes \cdots \otimes \mathcal{L}^{\varphi^{j-1}})$$

and define a $\mathbb{Z}^{>0}$-graded algebra structure on

$$A^\bullet = \bigoplus_{k\geq 0} \mathrm{Hom}(0,k), \tag{29}$$

via the multiplication, for $a \in A^p$ and $b \in A^q$:

$$a \cdot b = a \otimes b^{\varphi^p},$$

where we use the natural map $b \mapsto b^{\varphi^p}$ taking $\mathrm{Hom}(0,q) \longrightarrow \mathrm{Hom}(p, p+q)$, and the tensor product is viewed as a composition of morphisms.

Of course this is nothing but a recasting of the Van den Bergh construction; there is a sense in which it captures only certain morphisms between the generalized complex submanifolds, namely those which are visible upon restriction to C. Though rare, there are sometimes generalized holomorphic sections of the bundles L^k supported over all of M. In some sense, these sections must be included in the morphism spaces as well.

For instance, performing our construction for $L = \mathcal{O}(1)$ over $\mathbb{C}P^2$, equipped with a holomorphic Poisson structure $\sigma \in H^0(\mathbb{C}P^2, \mathcal{O}(3))$ with smooth zero locus $\iota : C \hookrightarrow \mathbb{C}P^2$, the graded algebra (29) has linear growth instead of the quadratic growth needed to capture a full non-commutative deformation of the coordinate ring of $\mathbb{C}P^2$ (these are the Sklyanin algebras, classified by [27]). It fails to include an additional generator in degree 3, as can be seen from the fact that the restriction map $H^0(\mathbb{C}P^2, \mathcal{O}(3)) \longrightarrow H^0(C, \iota^*\mathcal{O}(3))$ has 1-dimensional kernel. However it is important to note that neither $\mathcal{O}(1)$ nor $\mathcal{O}(2)$ has generalized holomorphic sections over $\mathbb{C}P^2$, while $\mathcal{O}(3)$ has a 1-dimensional space of them. We end with this vague indication that the morphisms supported on C should be combined with those supported on the whole holomorphic Poisson manifold.

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