

SCHUBERT POLYNOMIALS FOR THE AFFINE GRASSMANNIAN OF THE SYMPLECTIC GROUP

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ABSTRACT. We study the Schubert calculus of the affine Grassmannian Gr of the symplectic group. The integral homology and cohomology rings of Gr are identified with dual Hopf algebras of symmetric functions, defined in terms of Schur's P and Q functions. An explicit combinatorial description is obtained for the Schubert basis of the cohomology of Gr , and this is extended to a definition of the affine type C Stanley symmetric functions. A homology Pieri rule is also given for the product of a special Schubert class with an arbitrary one.

1. INTRODUCTION

Let G be a simply-connected simple complex algebraic group and let Gr_G denote the affine Grassmannian of G . Following Peterson [22] and Lam [15] we study the homology and cohomology Schubert calculus of $\text{Gr}_{Sp_{2n}(\mathbb{C})}$.

The structure of $H_*(\text{Gr}_G)^1$ and $H^*(\text{Gr}_G)$ is particularly rich because of the interaction of two phenomena. On the one hand, Gr_G inherits free \mathbb{Z} -module *Schubert bases* $\{\xi_x \in H_*(\text{Gr}_G)\}$ and $\{\xi^x \in H^*(\text{Gr}_G)\}$ from its presentation $\text{Gr}_G = \mathcal{G}/\mathcal{P}$ where \mathcal{G} is the affine Kac-Moody group associated to G and $\mathcal{P} \subset \mathcal{G}$ is a maximal parabolic subgroup. On the other hand, it is a classical result of Quillen [24] (see also [7] and [23]) that Gr_G is homotopy equivalent to the based loops ΩK into the maximal compact subgroup $K \subset G$. The group structure on ΩK endows $H_*(\text{Gr}_G)$ and $H^*(\text{Gr}_G)$ with the structure of dual Hopf algebras.

The dual Hopf algebras $H_*(\text{Gr}_G)$ and $H^*(\text{Gr}_G)$ were first studied intensively by Bott [2]. Bott gave an algorithm to compute these Hopf algebras in terms of the Cartan data of G , essentially by transgressing elements of $H^*(K)$ to obtain the primitive elements in $H^*(\text{Gr}_G)$. With \mathbb{Q} -coefficients, $H^*(K, \mathbb{Q})$ is an exterior algebra with odd-dimensional generators so $H^*(\text{Gr}_G, \mathbb{Q})$ is a polynomial algebra on even-dimensional generators. The situation is even more favorable when $G = Sp_{2n}(\mathbb{C})$ since $Sp_{2n}(\mathbb{C})$ is torsion-free and $H_*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ is a polynomial algebra over \mathbb{Z} . Bott comments that his description does not give polynomial generators for $H_*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$. We resolve this by producing n special Schubert classes which are polynomial generators over \mathbb{Z} .

Our main result identifies $H_*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ and $H^*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ with certain dual Hopf algebras $\Gamma_{(n)}$ and $\Gamma^{(n)}$ of symmetric functions, defined in terms of Schur's P - and Q -functions [20]. We explicitly describe symmetric functions $Q_w^{(n)} \in \Gamma^{(n)}$ which represents the cohomology Schubert basis. These symmetric functions are constructed using the combinatorics of a remarkable subset $\mathcal{Z} \subset \tilde{C}_n$ of the affine

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¹Our (co)homologies are with \mathbb{Z} -coefficients unless otherwise specified.

Weyl group \tilde{C}_n of $Sp_{2n}(\mathbb{C})$. In fact the cohomology representatives $Q_w^{(n)}$ extend to a larger family of symmetric functions: the type C affine Stanley symmetric functions.

1.1. Peterson's work on affine Schubert calculus. Our results rely heavily on the work of Peterson [22] who defined a Hopf embedding $j : H_T(\text{Gr}_G) \rightarrow \mathbb{A}$ of the T -equivariant cohomology of Gr_G as a commutative subalgebra of the nil-Hecke ring $\mathbb{A} = \mathbb{A}_G$ of Kostant and Kumar [12]. Here $T \subset K$ is a maximal torus. Peterson characterizes the image $j(\xi_x)$ of the Schubert basis of $H_T(\text{Gr}_G)$ in terms of certain identities inside \mathbb{A} . In the non-equivariant case, Lam [15] showed that Peterson's embedding specializes to a Hopf isomorphism $j_0 : H_*(\text{Gr}_G) \rightarrow \mathbb{B}$ with an algebra which he called the affine Fomin-Stanley subalgebra. We give an explicit combinatorial formula for generators of \mathbb{B} in the case $G = Sp_{2n}(\mathbb{C})$.

1.2. Earlier work for $G = SL_n(\mathbb{C})$. For $G = SL_n(\mathbb{C})$, Lam [15] identified the Schubert basis of $H_*(\text{Gr}_{SL_n(\mathbb{C})})$ with symmetric functions, called k -Schur functions, of Lapointe, Lascoux and Morse [18]; these arose in the study of Macdonald polynomials. The Schubert basis of $H^*(\text{Gr}_{SL_n(\mathbb{C})})$ are the dual k -Schur functions [19] which are generalized by the affine Stanley symmetric functions [14]. In [16] Pieri rules were given for the multiplication of Bott's generators on the Schubert bases of Bott's realization of $H_*(\text{Gr}_{SL_n(\mathbb{C})})$ and $H^*(\text{Gr}_{SL_n(\mathbb{C})})$. Furthermore, a combinatorial interpretation of the pairing between $H_*(\text{Gr}_{SL_n(\mathbb{C})})$ and $H^*(\text{Gr}_{SL_n(\mathbb{C})})$ is given.

1.3. Two Hopf algebras of symmetric functions. Let Λ denote the ring of symmetric functions over \mathbb{Z} . Let P_i and Q_i denote the Schur P - and Q -functions with a single part [20, III.8]. Define the Hopf subalgebras of Λ given by $\Gamma_* = \mathbb{Z}[P_1, P_2, \dots]$ and $\Gamma^* = \mathbb{Z}[Q_1, Q_2, \dots]$. There is a natural pairing (see (2.14)) $[\cdot, \cdot] : \Gamma_* \times \Gamma^* \rightarrow \mathbb{Z}$ making Γ_* and Γ^* into dual Hopf algebras. For $n \geq 1$ the subspace $\Gamma_{(n)} = \mathbb{Z}[P_1, P_2, \dots, P_{2n}] \subset \Gamma_*$ is a Hopf subalgebra and we let $\Gamma^* \twoheadrightarrow \Gamma^{(n)}$ be the dual quotient Hopf algebra.

1.4. Special classes. The affine Weyl group of $Sp_{2n}(\mathbb{C})$, denoted \tilde{C}_n , has simple generators s_0, s_1, \dots, s_n with the relations (3.2). Let \tilde{C}_n^0 denote the minimal length coset representatives of \tilde{C}_n/C_n , also called the *Grassmannian* elements of \tilde{C}_n . Define $\rho_i \in \tilde{C}_n^0$ by

$$(1.1) \quad \rho_i = \begin{cases} s_{i-1}s_{i-2} \cdots s_1s_0 & \text{for } 1 \leq i \leq n \\ s_{2n+1-i}s_{2n+2-i} \cdots s_{n-1}s_n s_{n-1} \cdots s_1s_0 & \text{for } n+1 \leq i \leq 2n. \end{cases}$$

The homology Schubert classes $\xi_{\rho_i} \in H_*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ for $1 \leq i \leq 2n$, are called *special classes*.

1.5. Zee-s. Let \mathcal{Z} be the Bruhat order ideal in \tilde{C}_n generated by the conjugates of the element ρ_{2n} , that is, the set of $w \in \tilde{C}_n$ which have a reduced word that is a subword of a rotation of the unique reduced word of ρ_{2n} . An element of \mathcal{Z} is called a Z . Let $\mathcal{Z}_r = \{w \in \mathcal{Z} \mid \ell(w) = r\}$ denote the set of Z -s of length $\ell(w)$ equal to r .

Example 1.1. Let $n = 2$. Then \mathcal{Z} consists of the elements of \tilde{C}_2 which have a reduced word that is a subword of one of the words 1210, 2101, 1012, 0121.

Given a word u with letters in I_{af} , its support $\text{Supp}(u) \subset I_{\text{af}}$ is by definition the set of letters appearing in u . For $w \in \tilde{C}_n$ define $\text{Supp}(w) = \text{Supp}(u)$ for any reduced word u ; this is independent of the choice of u . A component of a subset of I_{af} is by definition a maximal nonempty subinterval. Let $c(w)$ denote the number of components of $\text{Supp}(w)$.

1.6. Affine type C Stanley symmetric functions. For $w \in \tilde{C}_n$ we define the generating function

$$(1.2) \quad Q_w^{(n)}[Y] = \sum_{(v^1, v^2, \dots)} \prod_i 2^{c(v^i)} y_i^{\ell(v^i)},$$

where the sum runs over the factorizations $v^1 v^2 \dots = w$ of w such that $v^i \in \mathcal{Z}$ and $\ell(v^1) + \ell(v^2) + \dots = \ell(w)$.

Theorem 1.2. *The series $Q_w^{(n)}$ is symmetric and defines an element of $\Gamma^{(n)}$. The subset $\{Q_v^{(n)} \mid v \in \tilde{C}_n^0\}$ forms a basis of $\Gamma^{(n)}$ such that all product and coproduct structure constants are positive and every $Q_w^{(n)}$ for $w \in \tilde{C}_n$ is positive in this basis.*

The symmetric functions $Q_w^{(n)}$ are type C analogues of the affine Stanley symmetric functions studied in [14]. Some examples for the type C affine Stanley symmetric functions are given in Appendix B. They have the following geometric interpretation.

Let $LSp(n)$ and $\Omega Sp(n)$ denote respectively the space of all loops and based loops, into the maximal compact subgroup $Sp(n) \subset Sp_{2n}(\mathbb{C})$ and let $T \subset Sp(n)$ be the maximal torus. Let $p : \Omega Sp(n) \rightarrow LSp(n)/T$ denote the composition $\Omega Sp(n) \hookrightarrow LSp(n) \rightarrow LSp(n)/T$ of the inclusion and natural projection. The type C affine Stanley symmetric functions $Q_w^{(n)}$ can be identified via Theorem 1.3 (see below) with the pullbacks $p^*(\xi^w)$ of the Schubert classes $\xi^w \in H^*(LSp(n)/T)$. This follows from (5.7) and [15, Remark 8.6]. See also [15, Remark 4.6]. For $w \in \tilde{C}_n^0$, $p^*(\xi^w)$ is itself a Schubert class in $H^*(\Omega Sp(n)) \cong H^*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ as detailed below.

1.7. (Co)homology Schubert polynomials. The Hopf algebras $H^*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ and $H_*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ are dual via the cap product. The Schubert bases $\{\xi_x \in H_*(\text{Gr}_G)\}$ and $\{\xi^x \in H^*(\text{Gr}_G)\}$ are dual under the cap product and are both indexed by the Grassmannian elements $x \in \tilde{C}_n^0$.

Theorem 1.3. *There are dual Hopf algebra isomorphisms*

$$\begin{aligned} \Phi : \Gamma_{(n)} &\rightarrow H_*(\text{Gr}_{Sp_{2n}(\mathbb{C})}) \\ \Psi : H^*(\text{Gr}_{Sp_{2n}(\mathbb{C})}) &\rightarrow \Gamma^{(n)} \end{aligned}$$

such that

$$\begin{aligned} \Phi(P_i) &= \xi_{\rho_i} && \text{for } 1 \leq i \leq 2n, \text{ and} \\ \Psi(\xi^w) &= Q_w^{(n)} && \text{for } w \in \tilde{C}_n^0. \end{aligned}$$

Since $\Gamma_{(n)} = \mathbb{Z}[P_1, P_3, \dots, P_{2n-1}]$, we obtain in particular that $H_*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ is a polynomial algebra on $\xi_{\rho_1}, \xi_{\rho_3}, \dots, \xi_{\rho_{2n-1}}$. It also follows that the basis $\{P_w^{(n)} \mid w \in \tilde{C}_n^0\}$ of $\Gamma_{(n)}$ dual to $\{Q_w^{(n)} \mid w \in \tilde{C}_n^0\} \subset \Gamma^{(n)}$ maps to the homology Schubert classes $\xi_w \in H^*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$. The symmetric functions $P_w^{(n)}$ are Schubert polynomials for

$H_*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})})$ and are the type C analogue of k -Schur functions [15, 18]. Examples are given in Appendix C.

1.8. Pieri rule for $H_*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})})$. We also give a positive formula for the multiplication of an arbitrary homology class by a special class.

Theorem 1.4. *Let $w \in \tilde{C}_n^0$. Then in $H_*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})})$ we have*

$$\xi_{\rho_i} \xi_w = \sum_{v \in \mathcal{Z}_i} 2^{c(v)-1} \xi_{vw}$$

where the sum is taken over all $v \in \mathcal{Z}_i$ such that $\ell(vw) = \ell(v) + \ell(w)$ and $vw \in \tilde{C}_n^0$.

1.9. Future work and other directions.

1.9.1. Pieri rule for $H^*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})})$ and explicit description of homology Schubert basis. We hope to describe the symmetric functions $\{P_w^{(n)} \mid w \in \tilde{C}_n^0\} \subset \Gamma_{(n)}$ explicitly in the future, perhaps in a manner similar to the *strong tableaux* in [16]. As is explained in [16], the description of $P_w^{(n)}$ is essentially equivalent to the description of a Pieri rule for $H^*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})})$.

1.9.2. The special orthogonal groups. A generalization of our work to the special orthogonal groups $G = SO_n(\mathbb{C})$, together with the $G = SL_n(\mathbb{C})$ case in [15], would complete the analysis of the classical groups. The symmetric function description of $H_*(\mathrm{Gr}_{SO_n(\mathbb{C})})$ is likely to be more involved as it is not a polynomial algebra over \mathbb{Z} .

1.9.3. Comparison with finite case. We hope to explore the relationship between our symmetric functions and the “type B ” Stanley symmetric functions and classical type Schubert polynomials of Fomin and Kirillov [6], and of Billey and Haiman [1]. In particular, specializing $A_n = 0$ in Theorem 5.1 we obtain an expression nearly the same as the formula [6, (4.1)].

1.9.4. Embedding of groups and branching of Schubert classes. We intend to study the behavior of the affine Schubert classes studied here and in [15] induced by the inclusions of compact groups:

$$SU(n) \subset SU(n+1) \quad Sp(n) \subset Sp(n+1) \quad Sp(n) \subset SU(2n) \quad SU(n) \subset Sp(n).$$

In particular, the symmetric functions $P_w^{(n)}$ and $Q_w^{(n)}$ have positivity properties with respect to expansions involving Schur P -functions, Schur Q -functions, and ordinary Schur functions.

1.9.5. Work of Ginzburg and Bezrukavnikov, Finkelberg and Mirkovic. The rings $H_*(\mathrm{Gr}_G)$ and $H^*(\mathrm{Gr}_G)$ were also studied by Ginzburg [8] and by Bezrukavnikov, Finkelberg and Mirkovic [3] from the point of view of geometric representation theory. The connection with our point of view is unclear since the Schubert basis is as yet unavailable in their descriptions, although part of the Schubert basis is described by Ginzburg.

1.10. Organization. In section 2 we give notation for symmetric functions and describe the dual Hopf algebras $\Gamma_{(n)}$ and $\Gamma^{(n)}$. In section 3 we fix notation concerning affine root systems and Weyl groups. In section 4 we explain the connection between the Peterson and the Fomin-Stanley subalgebras, and the homology of the affine Grassmannian. The material in sections 3–4 are valid for the affine Grassmannian of any simply-connected simple complex algebraic group G .

Apart from Proposition 7.1, the remainder of the paper specializes to the case $G = Sp_{2n}(\mathbb{C})$. In section 5 we prove our main results (Theorems 1.2, 1.3 and 1.4) modulo two nilHecke algebra calculations – Theorems 5.1 and 5.5. Section 6 is devoted to the study of the Bruhat order of \tilde{C}_n restricted to \mathcal{Z} , and to the proof of Theorem 5.1. Section 7 presents a general formula (Proposition 7.1) for the coproduct in a nilHecke algebra and uses it to prove Theorem 5.5. Some data, in particular for the type C affine Stanley symmetric functions $Q_w^{(n)}$ and k -Schur functions $P_w^{(n)}$, is given in Appendices B and C.

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2. SYMMETRIC FUNCTIONS

In this section we study a subring $\Gamma_{(n)}$ and subquotient $\Gamma^{(n)}$ of the ring of symmetric functions. Let Λ be the Hopf algebra of symmetric functions over \mathbb{Z} . It has a number of bases indexed by partitions λ :

s_λ	Schur [20, I.3]
h_λ	homogeneous [20, I.1]
p_λ	power sums [20, I.1]
m_λ	monomial [20, I.1]
$P_\lambda[X; t]$	Hall-Littlewood P [20, III.2]
$Q_\lambda[X; t]$	Hall-Littlewood Q [20, III.2]

The power sums are a basis over \mathbb{Q} [20, I.2.12] and the Hall-Littlewood P - and Q -functions are a basis over $\mathbb{Q}(t)$ [20, III.2.7, 2.11].

Let $\langle \cdot, \cdot \rangle : \Lambda \otimes \Lambda \rightarrow \mathbb{Z}$ be the pairing defined by

$$(2.1) \quad \langle h_\lambda, m_\mu \rangle = \delta_{\lambda\mu}.$$

It has reproducing kernel [20, I.4.1, 4.2]

$$(2.2) \quad \begin{aligned} \Omega &:= \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} \\ &= \sum_{\lambda} h_\lambda[X] m_\lambda[Y] \\ &= \sum_{\lambda} z_\lambda^{-1} p_\lambda[X] p_\lambda[Y], \end{aligned}$$

where $z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$ and $m_i(\lambda)$ is the number of times the part i occurs in λ . Here and elsewhere, unless otherwise specified the sum runs over the set of all partitions.

The Schur P and Q functions are defined by [20, III.8]

$$(2.3) \quad \begin{aligned} P_\lambda[X] &= P_\lambda[X; -1] \\ Q_\lambda[X] &= Q_\lambda[X; -1] = 2^{\ell(\lambda)} P_\lambda[X] \end{aligned}$$

where $\ell(\lambda)$ is the number of nonzero parts of λ . We have [20, III.8.7]

$$(2.4) \quad P_\lambda[X] = Q_\lambda[X] = 0 \quad \text{if } \lambda \notin \mathcal{SP}$$

where \mathcal{SP} is the set of strict partitions λ , those with $\lambda_1 > \lambda_2 > \dots$; see (2.10) and (2.13).

2.1. Homology ring. Define the Hopf subalgebra $\Gamma_* \subset \Lambda$ by

$$(2.5) \quad \Gamma_* = \mathbb{Z}[P_1, P_3, P_5, \dots].$$

The P_i for i odd, are algebraically independent, so that

$$(2.6) \quad \Gamma_* = \bigoplus_{\lambda \in \mathcal{OP}} \mathbb{Z} P_{\lambda_1} P_{\lambda_2} \cdots$$

where \mathcal{OP} is the set of partitions with odd parts. The Hopf structure on Γ_* is given by

$$(2.7) \quad \Delta(P_r) = 1 \otimes P_r + P_r \otimes 1 + 2 \sum_{0 < s < r} P_s \otimes P_{r-s},$$

where the P_i for i even, satisfy only the relations [20, III.8.2']

$$(2.8) \quad P_{2i} = 2 (P_1 P_{2i-1} - P_2 P_{2i-2} + \cdots + (-1)^{i-2} P_{i-1} P_{i+1}) + (-1)^{i-1} P_i^2.$$

Iterating [20, III.8.15] yields the relation

$$(2.9) \quad P_{\lambda_1} P_{\lambda_2} \cdots = \sum_{\substack{\mu \in \mathcal{SP} \\ \mu \supseteq \lambda}} L_{\mu\lambda} P_\mu,$$

where $L_{\mu\lambda} \in \mathbb{Z}_{\geq 0}$ and $L_{\mu\mu} = 1$. Here \supseteq denotes the dominance partial order on partitions [20, I.1]. It follows that

$$(2.10) \quad \Gamma_* = \bigoplus_{\lambda \in \mathcal{SP}} \mathbb{Z} P_\lambda.$$

Define the Hopf subalgebra $\Gamma_{(n)} \subset \Gamma_*$ by

$$(2.11) \quad \begin{aligned} \Gamma_{(n)} &= \mathbb{Z}[P_1, P_2, \dots, P_{2n}] \\ &= \mathbb{Z}[P_1, P_3, \dots, P_{2n-1}] \\ &= \bigoplus_{\substack{\lambda \in \mathcal{OP} \\ \lambda_1 \leq 2n-1}} \mathbb{Z} P_{\lambda_1} P_{\lambda_2} \cdots \end{aligned}$$

2.2. Cohomology ring. Define

$$(2.12) \quad \Gamma^* = \mathbb{Z}[Q_1, Q_2, \dots] \subset \Lambda.$$

By (2.9) we have

$$(2.13) \quad \Gamma^* = \bigoplus_{\lambda \in \mathcal{SP}} \mathbb{Z}Q_\lambda.$$

Define the pairing $[\cdot, \cdot] : \Gamma_* \times \Gamma^* \rightarrow \mathbb{Z}$ by [20, III.8.12]

$$(2.14) \quad [P_\lambda, Q_\mu] = \delta_{\lambda\mu} \quad \text{for } \lambda, \mu \in \mathcal{SP}.$$

The pairing $[\cdot, \cdot]$ has reproducing kernel

$$(2.15) \quad \begin{aligned} \Omega_{-1} &:= \prod_{i,j \geq 1} \frac{1 + x_i y_j}{1 - x_i y_j} \\ &= \sum_{\lambda \in \mathcal{SP}} P_\lambda[X] Q_\lambda[Y] \\ &= \sum_{\lambda} P_{\lambda_1}[X] P_{\lambda_2}[X] \cdots M_\lambda[Y] \\ &= \sum_{\lambda \in \mathcal{OP}} z_\lambda^{-1} 2^{\ell(\lambda)} p_\lambda[X] p_\lambda[Y], \end{aligned}$$

where $M_\lambda = 2^{\ell(\lambda)} m_\lambda$. These equalities hold by definition, [20, III.8.13], setting $t = -1$ in [20, III.4.2], and [20, III.8.12].

Let $J_n \subset \Gamma^*$ be the ideal given by the annihilator of $\Gamma_{(n)} \subset \Gamma_*$ with respect to $[\cdot, \cdot]$. Define

$$(2.16) \quad \Gamma^{(n)} = \Gamma^* / J_n$$

which is a Hopf quotient algebra of Γ^* . The pairing $[\cdot, \cdot]$ descends to a perfect pairing $\Gamma_{(n)} \otimes \Gamma^{(n)} \rightarrow \mathbb{Z}$ which by (2.15) has reproducing kernel

$$(2.17) \quad \Omega_{-1}^{(n)} = \sum_{\lambda_1 \leq 2n} P_{\lambda_1}[X] P_{\lambda_2}[X] \cdots M_\lambda[Y].$$

2.3. Comparing Λ with Γ_* and Γ^* . Since $\Lambda = \bigoplus_{\lambda} \mathbb{Z}h_\lambda$ [20, I.2.8] one may define a surjective ring homomorphism $\theta : \Lambda \rightarrow \Gamma^*$ defined by $\theta(h_i) = Q_i$ for $i \in \mathbb{Z}_{>0}$. Over \mathbb{Q} it may be defined by $\theta(p_{2i}) = 0$ and $\theta(p_{2i-1}) = 2p_{2i-1}$ for $i \in \mathbb{Z}_{>0}$ [20, Ex. III.8.10].

Let $\iota : \Gamma_* \rightarrow \Lambda$ be the inclusion map.

Lemma 2.1.

$$(2.18) \quad \langle \iota(f), g \rangle = [f, \theta(g)] \quad \text{for } f \in \Gamma_*, g \in \Lambda.$$

Proof. By linearity one may reduce to the case $f = p_\lambda$ and $g = p_\mu$ for $\lambda, \mu \in \mathcal{OP}$. By (2.2) and (2.15) we have

$$\begin{aligned} [p_\lambda, \theta(p_\mu)] &= 2^{\ell(\mu)} [p_\lambda, p_\mu] \\ &= 2^{\ell(\mu) - \ell(\lambda)} z_\lambda \delta_{\lambda\mu} \\ &= z_\lambda \delta_{\lambda\mu} \\ &= \langle \iota(p_\lambda), p_\mu \rangle. \end{aligned}$$

□

Lemma 2.1 can be restated as

$$(2.19) \quad \theta^Y \Omega = \Omega_{-1}$$

where θ^Y means the operator θ applied to the Y variables.

Lemma 2.2. *For $\nu \in \mathcal{SP}$*

$$Q_\nu = \sum_{\lambda} L_{\nu\lambda} M_\lambda.$$

In particular $\Gamma^ \subset \bigoplus_{\lambda} \mathbb{Z}M_\lambda$.*

Proof. Since both Ω and Ω_{-1} are invariant under exchanging the X and Y variables, by (2.19) and (2.9) we have

$$\begin{aligned} \theta^Y \Omega &= \Omega_{-1} = \theta^X \Omega \\ &= \theta^X \sum_{\lambda} h_{\lambda}[X] m_{\lambda}[Y] \\ &= \sum_{\lambda} Q_{\lambda_1}[X] Q_{\lambda_2}[X] \cdots m_{\lambda}[Y] \\ &= \sum_{\lambda} P_{\lambda_1}[X] P_{\lambda_2}[X] \cdots M_{\lambda}[Y] \\ &= \sum_{\lambda} \sum_{\nu \in \mathcal{SP}} L_{\nu\lambda} P_{\nu}[X] M_{\lambda}[Y] \\ &= \sum_{\nu \in \mathcal{SP}} P_{\nu}[X] \sum_{\lambda} L_{\nu\lambda} M_{\lambda}[Y]. \end{aligned}$$

By (2.15) and (2.10), taking the coefficient of $P_{\nu}[X]$, the Lemma follows. \square

2.4. A monomial-like basis for $\Gamma^{(n)}$. For a partition λ , define $T_{\lambda} = \theta(m_{\lambda})$. We shall give a ‘‘monomial’’ basis of $\Gamma^{(n)}$ using the T_{λ} . Let $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$.

Lemma 2.3. *For every partition λ ,*

$$T_{\lambda} \in \chi(\lambda \in \mathcal{OP}) M_{\lambda} + \sum_{\mu \triangleright \lambda} \mathbb{Z} M_{\mu}.$$

Proof. Define $y_{\lambda} = \prod_{i \geq 1} m_i(\lambda)!$ where $m_i(\lambda)$ is the multiplicity of the part i in λ . By expanding p_{λ} it is easy to see that $p_{\lambda} \in y_{\lambda} m_{\lambda} + \sum_{\mu \triangleright \lambda} \mathbb{Z} y_{\mu} m_{\mu}$. It follows that $m_{\lambda} \in y_{\lambda}^{-1} p_{\lambda} + \sum_{\mu \triangleright \lambda} \mathbb{Q} p_{\mu}$ and

$$\begin{aligned} (2.20) \quad T_{\lambda} &\in y_{\lambda}^{-1} \theta(p_{\lambda}) + \sum_{\mu \triangleright \lambda} \mathbb{Q} \theta(p_{\mu}) \\ &= \chi(\lambda \in \mathcal{OP}) y_{\lambda}^{-1} 2^{\ell(\lambda)} p_{\lambda} + \sum_{\substack{\mu \triangleright \lambda \\ \mu \in \mathcal{OP}}} \mathbb{Q} p_{\mu} \\ &= \chi(\lambda \in \mathcal{OP}) M_{\lambda} + \sum_{\mu \triangleright \lambda} \mathbb{Q} m_{\mu}. \end{aligned}$$

By Lemma 2.2, $T_{\lambda} = \theta(m_{\lambda}) \in \Gamma^*$ is a \mathbb{Z} -linear combination of the M_{μ} . Since by definition the M_{μ} are integer multiples of the m_{μ} , (2.20) expresses T_{λ} as a \mathbb{Q} -linear combination of the M_{μ} . Since the M_{μ} are independent, the coefficients must then be integers. \square

Lemma 2.4. For $\lambda, \mu \in \mathcal{OP}$,

$$(2.21) \quad [P_{\lambda_1} P_{\lambda_2} \cdots, T_\mu] = 0 \quad \text{unless } \lambda \supseteq \mu,$$

$$(2.22) \quad [P_{\lambda_1} P_{\lambda_2} \cdots, T_\lambda] = 1.$$

Proof. Let $A = (a_{\lambda\nu})$ and $B = (b_{\lambda\nu})$ be the change of basis matrices

$$h_\lambda = \sum_{\nu \trianglelefteq \lambda} a_{\lambda\nu} p_\nu \quad \text{and} \quad p_\nu = \sum_{\rho \trianglelefteq \nu} b_{\nu\rho} h_\rho.$$

They are unitriangular and mutually inverse. We have

$$\begin{aligned} P_{\lambda_1} P_{\lambda_2} \cdots &= 2^{-\ell(\lambda)} \theta(h_\lambda) \\ &= \sum_{\substack{\nu \trianglelefteq \lambda \\ \nu \in \mathcal{OP}}} \sum_{\rho \trianglelefteq \nu} 2^{\ell(\nu) - \ell(\lambda)} a_{\lambda\nu} b_{\nu\rho} h_\rho. \end{aligned}$$

For any partition μ , by Lemmata 2.1 and 2.3 we have

$$\begin{aligned} [P_{\lambda_1} P_{\lambda_2} \cdots, T_\mu] &= \langle P_{\lambda_1} P_{\lambda_2} \cdots, m_\mu \rangle \\ &= \sum_{\substack{\nu \trianglelefteq \lambda \\ \nu \in \mathcal{OP}}} \sum_{\rho \trianglelefteq \nu} 2^{\ell(\nu) - \ell(\lambda)} a_{\lambda\nu} b_{\nu\rho} \delta_{\mu\rho} \end{aligned}$$

by (2.1). But this sum is zero unless $\lambda \supseteq \mu$, proving (2.21). When $\mu = \lambda$ we have $[P_{\lambda_1} P_{\lambda_2} \cdots, T_\lambda] = a_{\lambda\lambda} b_{\lambda\lambda} = 1$, since $AB = I$. \square

Proposition 2.5. We have

$$\Gamma^* = \bigoplus_{\lambda \in \mathcal{OP}} \mathbb{Z}T_\lambda, \quad \Gamma^{(n)} = \bigoplus_{\substack{\lambda \in \mathcal{OP} \\ \lambda_1 \leq 2n}} \mathbb{Z}T_\lambda, \quad J_n = \bigoplus_{\substack{\lambda \in \mathcal{OP} \\ \lambda_1 \geq 2n+1}} \mathbb{Z}T_\lambda.$$

Proof. This follows from Lemma 2.4, which says that $\{T_\mu \mid \mu \in \mathcal{OP}\}$ is unitriangularly related (over \mathbb{Z}) to the \mathbb{Z} -basis of Γ^* that is $[\cdot, \cdot]$ -dual to the \mathbb{Z} -basis of Γ_* given by $\{P_{\lambda_1} P_{\lambda_2} \cdots \mid \lambda \in \mathcal{OP}\}$. \square

2.5. Another realization of $\Gamma^{(n)}$. Let $I_k \subset \Lambda$ be the ideal generated by m_λ for $\lambda_1 \geq k$. There is a natural ring isomorphism

$$\Gamma^* / (\Gamma^* \cap I_{2n+1}) \cong (\Gamma^* + I_{2n+1}) / I_{2n+1}.$$

By Proposition 2.5, we have $\Gamma^* \cap I_{2n+1} = J_n$. Therefore

$$(2.23) \quad \Gamma^{(n)} \cong (\Gamma^* + I_{2n+1}) / I_{2n+1}.$$

It follows from (2.23) and Lemma 2.1 that

$$(2.24) \quad [P_{\lambda_1} P_{\lambda_2} \cdots, f] \text{ is the coefficient of } M_\lambda \text{ in } f$$

for $f \in \Gamma^{(n)}$ and λ satisfying $\lambda_1 \leq 2n$.

3. AFFINE ROOT SYSTEMS

3.1. Weyl group. A Cartan datum is a pair (I, A) where I is a finite set (the set of Dynkin nodes) and $A = (a_{ij} \mid i, j \in I)$ is a generalized Cartan matrix, which by definition satisfies $a_{ii} = 2$ for $i \in I$, $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} < 0$ if and only if $a_{ji} < 0$. The Cartan datum (I, A) is of finite type if A is nonsingular and of affine type if A has corank one.

Given a Cartan datum (I, A) , for $i, j \in I$ with $i \neq j$, define the integers $m_{ij} = 2, 3, 4, 6, \infty$ according as $a_{ij}a_{ji}$ is $0, 1, 2, 3$, or ≥ 4 . The Weyl group $W = W(I, A)$ is the Coxeter group with generators s_i for $i \in I$ such that $s_i^2 = 1$ for all $i \in I$ and braid relations

$$(3.1) \quad \underbrace{s_i s_j s_i \cdots}_{m_{ij} \text{ times}} = \underbrace{s_j s_i s_j \cdots}_{m_{ij} \text{ times}} \quad \text{for } i \neq j.$$

The length function $\ell : W \rightarrow \mathbb{Z}$ is given by $\ell(w) = l$ if a shortest expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ of w as a product of the s_i , is of length l . We call such an expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$ a *reduced expression*. The word $i_1 i_2 \cdots i_l$ consisting of the indices of a reduced expression is called a *reduced word* for w . We denote by $\mathcal{R}(w)$ the set of reduced words for w . We write $u \equiv u'$ if $u, u' \in \mathcal{R}(w)$ for some $w \in W$.

An element $s \in W$ is a reflection if $s = w s_i w^{-1}$ for some $i \in I$ and $w \in W$.

The Bruhat order on W_{af} is defined by $v \leq w$ if some (equivalently every) reduced word of w has a subword that is a reduced word for v . Alternatively $v \leq w$ if $v^{-1}w$ is a reflection and $\ell(w) = \ell(v) + 1$.

We now fix notation for an affine Cartan datum. Let (I, A) be the finite Cartan datum associated with the Lie algebra \mathfrak{g} of a simple simply-connected complex algebraic group G . Let $I = \{1, 2, \dots, n\}$ where n is the rank of \mathfrak{g} . Let $(I_{\text{af}}, A_{\text{af}})$ be the affine Cartan datum for the untwisted affine algebra $\mathfrak{g}_{\text{af}} = (\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}d$ [11, §7.2]. We write $I_{\text{af}} = \{0\} \sqcup I$ where $0 \in I_{\text{af}}$ is the distinguished Kac 0 node [11, §4.8]. Let $W = W(I, A)$ be the finite Weyl group and $W_{\text{af}} = W(I_{\text{af}}, A_{\text{af}})$ the affine Weyl group. We denote by $W_{\text{af}}^0 \subset W_{\text{af}}$ the set of *Grassmannian* elements, which by definition are the minimal length coset representatives of W_{af}/W .

Example 3.1. If $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$, the Cartan matrix for \mathfrak{g}_{af} is given by $a_{ii} = 2$ for $i \in I_{\text{af}}$, $a_{i, i+1} = a_{i+1, i} = -1$ for $1 \leq i \leq n-2$, $a_{01} = -1$, $a_{10} = -2$, $a_{n-1, n} = -2$, $a_{n, n-1} = -1$, and $a_{ij} = 0$ if $|i - j| \geq 2$. The affine Weyl group $W_{\text{af}} = \tilde{C}_n$ has generators $\{s_0, s_1, \dots, s_n\}$ and relations

$$(3.2) \quad \begin{aligned} s_i^2 &= 1 \\ s_i s_j &= s_j s_i && \text{if } |i - j| > 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{if } 1 \leq i \leq n-2 \\ s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0 \\ s_{n-1} s_n s_{n-1} s_n &= s_n s_{n-1} s_n s_{n-1}. \end{aligned}$$

For $W_{\text{af}} = \tilde{C}_n$, we use the notation $W_{\text{af}}^0 = \tilde{C}_n^0$ and $W = C_n$.

3.2. Affine root, coroot, and weight lattices. Let $P_{\text{af}} = \mathbb{Z}\delta \oplus \bigoplus_{i \in I_{\text{af}}} \mathbb{Z}\Lambda_i$ be the affine weight lattice, where δ is the null root and the Λ_i are the fundamental weights. Let $P_{\text{af}}^* = \text{Hom}_{\mathbb{Z}}(P_{\text{af}}, \mathbb{Z})$ be the dual weight lattice and $\langle \cdot, \cdot \rangle : P_{\text{af}}^* \times P_{\text{af}} \rightarrow \mathbb{Z}$

the natural perfect pairing. Let $\{d\} \cup \{\alpha_i^\vee \mid i \in I_{\text{af}}\}$ be the basis of P_{af}^* dual to the above basis of P_{af} ; in particular,

$$\begin{aligned} \langle \alpha_i^\vee, \Lambda_j \rangle &= \delta_{ij} && \text{for } i, j \in I_{\text{af}} \\ \langle \alpha_i^\vee, \delta \rangle &= 0 && \text{for } i \in I_{\text{af}} \end{aligned}$$

where δ_{ij} is the Kronecker delta. The α_i^\vee are called simple coroots. For $j \in I_{\text{af}}$ define the simple root $\alpha_j \in P_{\text{af}}$ by

$$(3.3) \quad \alpha_j = \sum_{i \in I_{\text{af}}} a_{ij} \Lambda_i + \delta_{j0} \delta.$$

Note that

$$(3.4) \quad \langle \alpha_i^\vee, \alpha_j \rangle = a_{ij} \quad \text{for all } i, j \in I_{\text{af}}.$$

Due to a linear dependence among the columns of the Cartan matrix, we have

$$(3.5) \quad \delta = \alpha_0 + \theta$$

where θ is the highest root of \mathfrak{g} . Let $Q_{\text{af}} = \bigoplus_{i \in I_{\text{af}}} \mathbb{Z} \alpha_i \subset P_{\text{af}}$ and $Q_{\text{af}}^\vee = \bigoplus_{i \in I_{\text{af}}} \mathbb{Z} \alpha_i^\vee$ be the affine root and coroot lattices. The nullroot satisfies

$$(3.6) \quad \langle \mu, \delta \rangle = 0 \quad \text{for all } \mu \in Q_{\text{af}}^\vee.$$

Similarly a dependence among the rows of the Cartan matrix, yields the *canonical central element* $K \in Q_{\text{af}}^\vee$ defined by

$$(3.7) \quad K = \alpha_0^\vee + \theta^\vee$$

where θ^\vee is the coroot associated to θ (defined in the next subsection). K satisfies

$$(3.8) \quad \langle K, \lambda \rangle = 0 \quad \text{for all } \lambda \in Q_{\text{af}}.$$

Example 3.2. For $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$, \mathfrak{g}_{af} has nullroot $\delta = \alpha_0 + 2(\alpha_1 + \cdots + \alpha_{n-1}) + \alpha_n$ and canonical central element $K = \alpha_0^\vee + \cdots + \alpha_n^\vee$.

The affine Weyl group W_{af} acts on P_{af} and P_{af}^\vee by

$$(3.9) \quad s_i \lambda = \lambda - \alpha_i \langle \alpha_i^\vee, \lambda \rangle \quad \text{for } \lambda \in P_{\text{af}}$$

$$(3.10) \quad s_i \mu = \mu - \alpha_i^\vee \langle \mu, \alpha_i \rangle \quad \text{for } \mu \in P_{\text{af}}^\vee.$$

One may show that

$$(3.11) \quad \langle w\mu, w\lambda \rangle = \langle \mu, \lambda \rangle \quad \text{for } w \in W_{\text{af}}, \lambda \in P_{\text{af}}, \mu \in P_{\text{af}}^\vee.$$

By (3.6) and (3.8) we have

$$(3.12) \quad w\delta = \delta, \quad wK = K \quad \text{for all } w \in W_{\text{af}}.$$

3.3. Finite root, coroot, and weight lattices. The finite coroot lattice is defined by $Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee \subset Q_{\text{af}}^\vee$. The finite root and weight lattices Q and P are quotients of their affine counterparts Q_{af} and P_{af} , but by abuse we will define them as sublattices. The finite root lattice is defined by $Q = \bigoplus_{i \in I} \mathbb{Z} \alpha_i \subset Q_{\text{af}} \subset P_{\text{af}}$. The finite weight lattice is defined by $P = \bigoplus_{i \in I} \mathbb{Z} \omega_i \subset P_{\text{af}}$ where

$$(3.13) \quad \omega_i = \Lambda_i - \langle K, \Lambda_i \rangle \Lambda_0$$

for $i \in I$; these are the fundamental weights of \mathfrak{g} . We have

$$(3.14) \quad \langle \alpha_i^\vee, \omega_j \rangle = \delta_{ij} \quad \text{for } i, j \in I.$$

3.4. Roots. The root system of \mathfrak{g} may be defined by

$$(3.15) \quad R = W \cdot \{\alpha_i \mid i \in I\}.$$

Given $\alpha \in R$ with $\alpha = u\alpha_i$ for some $u \in W$ and $i \in I$, its associated coroot is defined by $\alpha^\vee = u\alpha_i^\vee \in Q^\vee$. Its associated reflection is $s_\alpha = us_iu^{-1}$. Both are independent of the choice of u and i . There is a decomposition $R = R^+ \cup -R^+$ where $R^+ = R \cap \bigoplus_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i$ is the set of positive roots.

The set of affine roots $R_{\text{af}} \subset Q_{\text{af}}$ is given by the set of nonzero elements in the set $R + \mathbb{Z}\delta$. We have $R_{\text{af}} = R_{\text{af}}^+ \cup -R_{\text{af}}^+$ where R_{af}^+ is the set of positive affine roots, which have the form $\alpha + m\delta$ where either $m > 0$ or both $m = 0$ and $\alpha \in R^+$. Equivalently, $R_{\text{af}}^+ = R_{\text{af}} \cap \bigoplus_{i \in I_{\text{af}}} \mathbb{Z}_{\geq 0}\alpha_i$.

The set of real affine roots is defined by

$$R_{\text{af}}^{\text{re}} = W_{\text{af}} \cdot \{\alpha_i \mid i \in I_{\text{af}}\}.$$

For $\alpha = u\alpha_i \in R_{\text{af}}^{\text{re}}$ for $u \in W_{\text{af}}$ and $i \in I_{\text{af}}$ define the associated coroot by $\alpha^\vee = u\alpha_i^\vee$ and associated reflection $s_\alpha \in W_{\text{af}}$ by $s_\alpha = us_iu^{-1}$; as before one may show these definitions are independent of u and i .

Let $v < w$ in W_{af} . Then $s = v^{-1}w$ is a reflection $s = us_iu^{-1}$ for some $i \in I_{\text{af}}$ and $u \in W_{\text{af}}$. Let u be shortest so that $\alpha = u\alpha_i$ is a positive real root. For later use we denote this root α by α_{vw} and its associated coroot by α_{vw}^\vee .

Example 3.3. Let $W_{\text{af}} = \tilde{C}_3$, $w = s_1s_2s_3s_2s_1s_0$, and $v = s_1s_3s_2s_1s_0$; this defines a cover $v < w = vs_\alpha$. Then $s_\alpha = (s_0s_1s_2s_3)s_2(s_3s_2s_1s_0)$ and

$$\begin{aligned} \alpha_{vw}^\vee &= s_0s_1s_2s_3(\alpha_2^\vee) \\ &= 2\alpha_0^\vee + \alpha_1^\vee + \alpha_2^\vee + 2\alpha_3^\vee. \end{aligned}$$

3.5. Level 0 action. There is a surjective group homomorphism $W_{\text{af}} \rightarrow W$ given by $s_i \mapsto s_i$ for $i \in I$ and $s_0 \mapsto s_\theta$ where $\theta \in R^+$ is the highest root. Since W acts on P , W_{af} acts on P via the above homomorphism; this is called the level zero action. It is not faithful since s_0 and s_θ are different elements of W_{af} .

4. NILHECKE ALGEBRA AND AFFINE GRASSMANNIAN

4.1. (Co)homology of affine Grassmannian. For this section we fix G a simple and simply-connected complex algebraic group with Weyl group W , and Cartan datum (I, A) as in section 3.1. Let K denote a maximal compact subgroup of G and T denote a maximal torus in K .

Let $\mathbb{F} = \mathbb{C}((t))$ and $\mathbb{O} = \mathbb{C}[[t]]$. The affine Grassmannian Gr_G is the ind-scheme $G(\mathbb{F})/G(\mathbb{O})$ (see [13]). It is a homogeneous space for the affine Kac-Moody group \mathcal{G} associated to W_{af} . It is a classical result due to Quillen that the space Gr_G is homotopy-equivalent to the space ΩK of based loops in K ; see for example [7, 23].

The group \mathcal{G} possesses a *Bruhat decomposition* $\mathcal{G} = \bigcup_{w \in W_{\text{af}}} \mathcal{B}w\mathcal{B}$ where \mathcal{B} denotes the Iwahori subgroup. The Bruhat decomposition induces a decomposition of Gr_G into *Schubert cells* $\Omega_w = \mathcal{B}wG(\mathbb{O}) \subset G(\mathbb{F})/G(\mathbb{O})$. Thus the equivariant homology $H_T(\text{Gr}_G)$ and cohomology $H^T(\text{Gr}_G)$ of Gr_G are free $S = H^T(\text{pt})$ -modules with Schubert bases $\xi_x^T \in H_T(\text{Gr}_G)$ and $\xi_x^T \in H^T(\text{Gr}_G)$. Similarly, the homology $H_T(\text{Gr}_G)$ and cohomology $H^T(\text{Gr}_G)$ of Gr_G are free \mathbb{Z} -modules with Schubert bases $\xi_x \in H_T(\text{Gr}_G)$ and $\xi^x \in H^T(\text{Gr}_G)$. The index x varies over the Grassmannian elements W_{af}^0 . We refer the reader to [12, 13] for the general construction and properties of Schubert bases in the Kac-Moody setting.

The pointwise multiplication of loops on K induces the structure of dual Hopf algebras over \mathbb{Z} to $H_*(\text{Gr}_G)$ and $H^*(\text{Gr}_G)$, and the structure of dual Hopf algebras over S to $H_T(\text{Gr}_G)$ and $H^T(\text{Gr}_G)$. This is a special feature of the affine Grassmannian unavailable in the more general Kac-Moody setting.

4.2. NilCoxeter algebra. The nilCoxeter algebra \mathbb{A}_0 is the associative \mathbb{Z} -algebra with generators A_i for $i \in I$ and relations $A_i^2 = 0$ for $i \in I$ and braid relations

$$(4.1) \quad \underbrace{A_i A_j A_i \cdots}_{m_{ij} \text{ times}} = \underbrace{A_j A_i A_j \cdots}_{m_{ij} \text{ times}} \quad \text{for } i \neq j.$$

Since these are the same braid relations (3.1) satisfied by $s_i \in W$, for $w \in W$ one may define $A_w = A_{i_1} A_{i_2} \cdots A_{i_l}$ for any $i_1 i_2 \cdots i_l \in \mathcal{R}(w)$.

The algebra \mathbb{A}_0 is a free \mathbb{Z} -module with basis $\{A_w \mid w \in W\}$. In this basis, the multiplication is given by

$$A_v A_u = \begin{cases} A_{vu} & \text{if } \ell(v) + \ell(u) = \ell(vu) \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.1. For the affine Cartan datum of Example 3.1, \mathbb{A}_0 has generators A_i for $i \in I_{\text{af}}$ and relations

$$\begin{aligned} A_i^2 &= 0 \\ A_i A_j &= A_j A_i && \text{if } |i - j| > 1 \\ A_i A_{i+1} A_i &= A_{i+1} A_i A_{i+1} && \text{if } 1 \leq i \leq n - 2 \\ A_0 A_1 A_0 A_1 &= A_1 A_0 A_1 A_0 \\ A_{n-1} A_n A_{n-1} A_n &= A_n A_{n-1} A_n A_{n-1} \end{aligned}$$

4.3. Kostant and Kumar's NilHecke algebra. Let P be the weight lattice of \mathfrak{g} and $S = \text{Sym}(P)$ the symmetric algebra. The Peterson affine nilHecke algebra \mathbb{A} is by definition² the associative \mathbb{Z} -algebra generated by S and the nilCoxeter algebra \mathbb{A}_0 for the affine Cartan datum $(I_{\text{af}}, A_{\text{af}})$ with

$$(4.2) \quad A_i \lambda = (s_i \cdot \lambda) A_i + \langle \alpha_i^\vee, \lambda \rangle 1 \quad \text{for } i \in I_{\text{af}} \text{ and } \lambda \in P.$$

Consequently \mathbb{A} is a free left S -module with basis $\{A_w \mid w \in W_{\text{af}}\}$.

Iterating (4.2) produces the following relation.

Lemma 4.2. For $x \in W_{\text{af}}$ and $\lambda \in P$,

$$(4.3) \quad A_x \lambda = (x \cdot \lambda) A_x + \sum_{y \prec x} \langle \alpha_{y^x}^\vee, \lambda \rangle A_y$$

where $\alpha_{y^x}^\vee$ is defined in section 3.4.

Proposition 4.3. [22] Let M and N be left \mathbb{A} -modules. Define $M \otimes_S N = (M \otimes_{\mathbb{Z}} N) / \langle sm \otimes n - m \otimes sn \mid s \in S; m \in M; n \in N \rangle$. Then $M \otimes_S N$ is a left \mathbb{A} -module via

$$(4.4) \quad \begin{aligned} A_i \cdot (m \otimes n) &= (A_i \cdot m) \otimes n + m \otimes (A_i \cdot n) - \alpha_i(A_i \cdot m) \otimes (A_i \cdot n) \\ s \cdot (m \otimes n) &= sm \otimes n. \end{aligned}$$

²The nilHecke algebra of Kostant and Kumar [12] for the affine Cartan datum, uses a larger weight lattice than Peterson's nilHecke algebra. See [15] for a comparison of the two.

Since the proof of this proposition is not readily available in the literature, we include a proof.

Proof. We verify (4.2).

$$\begin{aligned}
& A_i \cdot (\lambda \cdot (m \otimes n)) \\
&= (A_i \cdot \lambda m) \otimes n + \lambda m \otimes (A_i \cdot n) - \alpha_i(A_i \cdot \lambda m) \otimes (A_i \cdot n) \\
&= (s_i \lambda)(A_i m \otimes n) + \langle \alpha_i^\vee, \lambda \rangle m \otimes n + \lambda m \otimes A_i n \\
&\quad - \alpha_i(s_i \lambda) A_i m \otimes A_i n - \alpha_i \langle \alpha_i^\vee, \lambda \rangle m \otimes A_i n \\
&= (s_i \lambda)(A_i m \otimes n) + (\lambda - \alpha_i \langle \alpha_i^\vee, \lambda \rangle) m \otimes A_i n \\
&\quad - (s_i \lambda) \alpha_i A_i m \otimes A_i n + \langle \alpha_i^\vee, \lambda \rangle m \otimes n \\
&= (s_i \lambda) \cdot A_i \cdot (m \otimes n) + \langle \alpha_i^\vee, \lambda \rangle m \otimes n.
\end{aligned}$$

We verify $A_i^2 = 0$.

$$\begin{aligned}
A_i \cdot (A_i \cdot (m \otimes n)) &= 2A_i m \otimes A_i n - (A_i \alpha_i A_i m) \otimes n \\
&= 2A_i m \otimes A_i n + \alpha_i A_i^2 m \otimes n - \langle \alpha_i^\vee, \alpha_i \rangle A_i m \otimes A_i n = 0.
\end{aligned}$$

To verify the braid relations, it is convenient to introduce the elements $r_i = 1 - \alpha_i A_i \in \mathbb{A}$. It is not difficult to see that given (4.2) the relation $(A_i A_j)^{m_{ij}} = (A_j A_i)^{m_{ij}}$ is equivalent to $(r_i r_j)^{m_{ij}} = (r_j r_i)^{m_{ij}}$. An easy calculation shows that the r_i act on $M \otimes_S N$ by $r_i \cdot (m \otimes n) = r_i m \otimes r_i n$. It is clear that this action satisfies the braid relations for the r_i . \square

Thus there is a left S -module homomorphism $\Delta : \mathbb{A} \rightarrow \mathbb{A} \otimes_S \mathbb{A}$ defined by

$$(4.5) \quad \Delta(a) = a \cdot (1 \otimes 1) \quad \text{for } a \in \mathbb{A}.$$

By (4.4) we have

$$(4.6) \quad \Delta(A_i) = A_i \otimes 1 + 1 \otimes A_i - A_i \otimes \alpha_i A_i$$

$$(4.7) \quad \Delta(s) = s \otimes 1.$$

The map Δ is injective so there is a linear map $\Delta(\mathbb{A}) \otimes (\mathbb{A} \otimes_S \mathbb{A}) \rightarrow (\mathbb{A} \otimes_S \mathbb{A})$ defined by

$$(4.8) \quad \Delta(a) \otimes (x \otimes y) \mapsto a \cdot (x \otimes y)$$

using the left \mathbb{A} -module structure on $\mathbb{A} \otimes_S \mathbb{A}$ afforded by Proposition 4.3. We deduce that this map yields a ring structure on $\Delta(\mathbb{A})$ and an action of $\Delta(\mathbb{A})$ on $\mathbb{A} \otimes_S \mathbb{A}$.

It follows by induction using Proposition 4.3 that this action is computed explicitly as follows. Let $a \in \mathbb{A}$ and $\Delta(a) = \sum_{w,v} A_w \otimes a_{wv} A_v$. Then

$$(4.9) \quad \Delta(a) \cdot (x \otimes y) = \sum_{w,v} A_w x \otimes a_{wv} A_v y.$$

In particular, if $b \in \mathbb{A}$ and $\Delta(b) = \sum_{w,v} A_w \otimes b_{wv} A_v$ for $b_{wv} \in S$, then

$$(4.10) \quad \Delta(ab) = \Delta(a) \Delta(b) = \sum_{w,v,w',v'} A_w A_{w'} \otimes a_{wv} A_v b_{w'v'} A_{v'}.$$

The ring structure on $\Delta(\mathbb{A})$ does not extend to all of $\mathbb{A} \otimes_S \mathbb{A}$ by the formula $(a \otimes b)(c \otimes d) = ac \otimes bd$, because if it did, then since $s \otimes 1 = 1 \otimes s$ we would have $fs \otimes g = (f \otimes g)(s \otimes 1) = (f \otimes g)(1 \otimes s) = f \otimes gs$, which is false in general (say, for $g = 1$ and $fs \neq sf$). Equation (4.10) says that when this ‘‘obvious’’ generally

ill-defined multiplication formula is applied to expressions coming from the action of $\Delta(\mathbb{A})$ on $\mathbb{A} \otimes_S \mathbb{A}$, the result is well-defined.

4.4. The Peterson subalgebra and equivariant cohomology of affine Grassmannian. The *Peterson subalgebra* of \mathbb{A} is the centralizer $Z_{\mathbb{A}}(S)$ of S . It is a Hopf algebra over S since the factorwise product on $Z_{\mathbb{A}}(S) \otimes_S Z_{\mathbb{A}}(S)$ gives it an S -algebra structure under which the restriction of Δ to $Z_{\mathbb{A}}(S)$, is an S -algebra homomorphism.

Theorem 4.4. [22] [15, Theorem 4.4] *There is an S -Hopf algebra isomorphism*

$$j : H_T(\mathrm{Gr}_G) \rightarrow Z_{\mathbb{A}}(S)$$

which is characterized by the property that for all $x \in W_{\mathrm{af}}^0$, $j(\xi_x^T)$ is the unique element of $Z_{\mathbb{A}}(S) \cap (A_x + \sum_{y \in W_{\mathrm{af}} \setminus W_{\mathrm{af}}^0} S A_y)$.

For $x \in W_{\mathrm{af}}^0$ and $y \in W_{\mathrm{af}}$ let $j_x^y \in S$ be defined by

$$(4.11) \quad j(\xi_x^T) = \sum_{y \in W_{\mathrm{af}}} j_x^y A_y.$$

Proposition 4.5. [22] [17, Theorem 6.3]

- (1) For $x \in W_{\mathrm{af}}^0$ and $y \in W_{\mathrm{af}}$, the polynomial j_x^y is either zero or homogeneous of degree $\ell(y) - \ell(x)$; in particular it is zero if $\ell(y) < \ell(x)$.
- (2) For $x, z \in W_{\mathrm{af}}^0$ we have

$$(4.12) \quad \xi_x^T \xi_z^T = \sum_y j_x^y \xi_{yz}^T$$

where y runs over the $y \in W_{\mathrm{af}}$ such that $yz \in W_{\mathrm{af}}^0$ and $\ell(yz) = \ell(y) + \ell(z)$.

We wish to compute j_x^y in the “nonequivariant case” $\ell(x) = \ell(y)$, when $j_x^y \in \mathbb{Z}_{\geq 0}$. For this purpose we consider the maps that forget the T -equivariance.

4.5. Affine Fomin-Stanley subalgebra. Let $\phi_0 : S \rightarrow \mathbb{Z}$ be the map that sends a polynomial to its evaluation at 0. By abuse of notation define $\phi_0 : \mathbb{A} \rightarrow \mathbb{A}_0$ by $\phi_0(\sum_w s_w A_w) = \sum_w \phi_0(s_w) A_w$ for $s_w \in S$. Peterson’s j -map induces an injective ring homomorphism $j_0 : H_*(\mathrm{Gr}_G) \rightarrow \mathbb{A}_0$ such that the diagram commutes:

$$(4.13) \quad \begin{array}{ccc} H_T(\mathrm{Gr}_G) & \xrightarrow{j} & \mathbb{A} \\ \epsilon \downarrow & & \downarrow \phi_0 \\ H_*(\mathrm{Gr}_G) & \xrightarrow{j_0} & \mathbb{A}_0 \end{array}$$

where $\epsilon : H_T(\mathrm{Gr}_G) \rightarrow H_*(\mathrm{Gr}_G)$ is obtained by $\xi_x^T \mapsto \xi_x$ and the evaluation ϕ_0 .

By (4.11) and (4.13) we have

$$(4.14) \quad j_0(\xi_w) = \sum_{\substack{u \in W_{\mathrm{af}} \\ \ell(u) = \ell(w)}} j_w^u A_u \quad \text{for } w \in W_{\mathrm{af}}^0.$$

The affine Fomin-Stanley subalgebra is defined in [15] by

$$(4.15) \quad \mathbb{B} = \{a \in \mathbb{A} \mid \phi_0(s)a = \phi_0(as) \text{ for every } s \in S\}.$$

Define $\phi_0^{(2)} : \mathbb{A} \otimes_S \mathbb{A} \rightarrow \mathbb{A}_0 \otimes_{\mathbb{Z}} \mathbb{A}_0$ by

$$\phi_0^{(2)} \left(\sum_{w,v \in W_{\text{af}}} a_{w,v} A_w \otimes A_v \right) = \sum_{w,v \in W_{\text{af}}} \phi_0(a_{w,v}) A_w \otimes A_v$$

for $a_{w,v} \in S$. Then \mathbb{B} is a Hopf algebra with coproduct given by the restriction of $\phi_0^{(2)} \circ \Delta$ to \mathbb{B} .

Theorem 4.6 ([15, Prop. 5.4, Thm. 5.5]). *The map j_0 is a Hopf algebra isomorphism $H_*(\text{Gr}_G) \cong \mathbb{B}$. Moreover, for every $w \in W_{\text{af}}^0$, $j_0(\xi_w)$ is the unique element of $\mathbb{B} \cap (A_w + \sum_{u \in W_{\text{af}} \setminus W_{\text{af}}^0} \mathbb{Z}A_u)$.*

\mathbb{B} has a basis $\{\mathbb{P}_w \mid w \in W_{\text{af}}^0\}$ defined by

$$(4.16) \quad \mathbb{P}_w = j_0(\xi_w) \quad \text{for } w \in W_{\text{af}}^0.$$

For $G = SL_n(\mathbb{C})$ these are the noncommutative k -Schur functions of [15]. The following Lemma is an aid for computing the elements \mathbb{P}_w .

Lemma 4.7. *Let $a = \sum_{w \in W_{\text{af}}} c_w A_w \in \mathbb{A}_0$ with $c_w \in \mathbb{Z}$. Then $a \in \mathbb{B}$ if and only if $\sum_{w \succ v} c_w \alpha_{vw}^\vee \in \mathbb{Z}K$ for all $v \in W_{\text{af}}$.*

Proof. The following are equivalent:

- (1) Equation (4.15) holds for a .
- (2) $\phi_0(a\lambda) = 0$ for all $\lambda \in P$.
- (3) $\sum_w c_w \sum_{v \prec w} \langle \alpha_{vw}^\vee, \lambda \rangle A_v = 0$ for all $\lambda \in P$.
- (4) $\sum_{w \succ v} c_w \langle \alpha_{vw}^\vee, \lambda \rangle = 0$ for all $v \in W_{\text{af}}$ and all $\lambda \in P$.
- (5) $\sum_{w \succ v} c_w \alpha_{vw}^\vee \in \mathbb{Z}K$ for all $v \in W_{\text{af}}$.

(1) and (2) are easily seen to be equivalent. The equivalence of (2) and (3) follows from equation (4.2). (3) and (4) are equivalent because the A_v form a basis of \mathbb{A}_0 . (4) and (5) are equivalent because $\mathbb{Z}K = \{\mu \in Q_{\text{af}}^\vee \mid \langle \mu, P \rangle = 0\}$. \square

5. SCHUBERT POLYNOMIALS FOR $H_*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ AND $H^*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$

In this section we outline the proofs of Theorems 1.2, 1.3 and 1.4, relegating two technical calculations to sections 6 and 7.

5.1. Special generators of Fomin-Stanley subalgebra. Recall the special elements ρ_i defined in (1.1). For $1 \leq i \leq 2n$ define

$$(5.1) \quad \mathbb{P}_i = \mathbb{P}_{\rho_i}$$

where \mathbb{P}_w is defined in (4.16).

We now state the explicit expansion of the elements $\mathbb{P}_i \in \mathbb{B}$ that correspond to homology generators. Recall the set \mathcal{Z} defined in section 1.5. Note that ρ_r is the unique Grassmannian element in \mathcal{Z}_r for $1 \leq r \leq 2n$.

Theorem 5.1. *For $1 \leq r \leq 2n$,*

$$(5.2) \quad \mathbb{P}_r = \sum_{w \in \mathcal{Z}_r} 2^{c(w)-1} A_w.$$

This result is proved in section 6. Some examples for \mathbb{P}_r are given in Appendix A.

Remark 5.2. It follows from Theorem 1.3 that the elements $\mathbb{P}_r \in \mathbb{B}$ generate the affine Fomin-Stanley subalgebra \mathbb{B} .

5.2. Relations among special generators. Let \mathcal{P}_C^n be the set of partitions λ with $\lambda_1 \leq 2n$, which have at most one part of size i for all $i \leq n$. We first note the following result which is essentially [4, Lemma 24]. The bijection of Lemma 5.3 was first brought to our attention by Morse [21] who discovered it independently.

Lemma 5.3. *Let $w \in \tilde{C}_0^n$. Then w has a unique length-additive factorization*

$$w = \rho_{\lambda_1} \cdots \rho_{\lambda_2} \rho_{\lambda_1}$$

into Grassmannian Z -s such that every left factor $\rho_{\lambda_1} \cdots \rho_{\lambda_i}$ is Grassmannian. Furthermore the map $w \mapsto \lambda(w)$ is a bijection $\tilde{C}_0^n \rightarrow \mathcal{P}_C^n$ such that $\ell(w) = |\lambda(w)|$.

Proof. The result follows nearly immediately from [4, Lemma 24]. In [4] one associates to $w \in \tilde{C}_n$ the window

$$[-w_n, \dots, -w_1, 0, w_1, \dots, w_n]$$

of an affine permutation. This corresponds to the embedding of \tilde{C}_n into the affine symmetric group \tilde{S}_{2n+2} . In [4] the parabolic subgroup is generated by $\{s_0, \dots, s_{n-1}\}$ rather than by $\{s_1, \dots, s_n\}$ so we must apply the notational involution $s_i \leftrightarrow s_{n-i}$ to be compatible with [4].

In any case, for $w \in \tilde{C}_n^0$ it is shown in [4, Lemma 24] that the window of w can be successively *sorted* to become the identity. Each sorting operation corresponds to right multiplication by a factor ρ_{λ_i} . The requirement that every left factor $\rho_{\lambda_1} \cdots \rho_{\lambda_i}$ is Grassmannian corresponds to asking for the window of w to be completely sorted at each step. The rest of the statement now follows from [4]. \square

Proposition 5.4. *The elements $\mathbb{P}_i \in \mathbb{B}$ satisfy*

$$\mathbb{P}_{2m} = 2(\mathbb{P}_1 \mathbb{P}_{2m-1} - \mathbb{P}_2 \mathbb{P}_{2m-2} + \cdots + (-1)^{m-2} \mathbb{P}_{m-1} \mathbb{P}_{m+1}) + (-1)^{m-1} \mathbb{P}_m^2$$

for $1 \leq m \leq n$.

Proof. We use the explicit computation of the \mathbb{P}_i given in Theorem 5.1. By evaluating the statement of Proposition 4.5 at 0, we observe that

$$(5.3) \quad \mathbb{P}_i \mathbb{P}_j = \sum_{w=v\rho_j} 2^{c(v)-1} \mathbb{P}_w$$

where the summation is over all $w = v\rho_j$ such that (a) $v \in \mathcal{Z}_i$, (b) $\ell(w) = i + j$, and (c) $w \in \tilde{C}_n^0$. Now any reduced expression for $v \in \mathcal{Z}_i$ can have at most one occurrence of s_0 , so by Lemma 5.3, we deduce that the set of w such that \mathbb{P}_w can occur in a product of the form $\mathbb{P}_i \mathbb{P}_j$ has the form $\rho_a \rho_b$ where $a + b = i + j$.

Now fix $1 \leq m \leq n$ and let us compute

$$S = 2(\mathbb{P}_1 \mathbb{P}_{2m-1} - \mathbb{P}_2 \mathbb{P}_{2m-2} + \cdots + (-1)^{m-2} \mathbb{P}_{m-1} \mathbb{P}_{m+1}) + (-1)^{m-1} \mathbb{P}_m^2.$$

First via a direct calculation we note that $\rho_m \rho_m \notin \tilde{C}_n^0$ for $1 \leq m \leq n$. We claim that for $1 \leq j \leq m$ and $w = \rho_i \rho_{2m-i}$ satisfying $w \in \tilde{C}_n^0$ and $\ell(w) = 2m$ we have

$$[\mathbb{P}_w] \mathbb{P}_j \mathbb{P}_{2m-j} = \begin{cases} 0 & \text{if } i > j \\ 1 & \text{if } i = j \\ 2 & \text{if } 0 < i < j \\ 1 & \text{if } i = 0. \end{cases}$$

where $[\mathbb{P}_w]b$ denotes the coefficient of \mathbb{P}_w in $b \in \mathbb{B}$. The case $i > j$ follows from Lemma 5.3. The case $i = j$ is immediate since $c(\rho_i) = c(\rho_{2m-i}) = 1$. For the case $i < j$ we must consider $v = \rho_i \rho_{2m-i} \rho_{2m-j}^{-1}$. We observe that

$$\text{Supp}(v) = [0, i-1] \cup \text{Supp}(\rho_{2m-i} \rho_{2m-j}^{-1})$$

and i is smaller than all elements of $\text{Supp}(\rho_{2m-i} \rho_{2m-j}^{-1})$ (we use the inequality $2m-j > i$). Thus v , being a product two “non-touching” Z -s, is itself a Z and we have $c(v) = 2$. The final case $i = 0$ is trivial.

Now it follows that the $S = \mathbb{P}_{2m}$, as required. \square

5.3. Coproduct formula for special generators. The following result is proved in section 7.

Theorem 5.5. *For $1 \leq r \leq 2n$*

$$\phi_0^{(2)}(\Delta(\mathbb{P}_r)) = 1 \otimes \mathbb{P}_r + \mathbb{P}_r \otimes 1 + 2 \sum_{0 < s < r} \mathbb{P}_s \otimes \mathbb{P}_{r-s}.$$

Remark 5.6. An alternative formulation of Theorem 5.5 is that the coefficient of ξ^{ρ_r} in $\xi^{\rho_s} \xi^{\rho_{r-s}} \in H^*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ is equal to 2, for $1 \leq s \leq r-1$.

5.4. Affine type C Cauchy kernel. Define $\Phi : \Gamma_{(n)} \rightarrow H_*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ by $P_i \mapsto \xi^{\rho_i}$ for $1 \leq i \leq 2n$ as in Theorem 1.3. By Proposition 5.4 and Theorem 4.6 this map is well-defined. Define $\Omega_{-1}^{\mathbb{B}} \in \mathbb{B} \hat{\otimes} \Gamma^{(n)}$ by taking the image of $\Omega_{-1}^{(n)}$ under the composition $\Phi_B = j_0 \circ \Phi : \Gamma_{(n)} \rightarrow \mathbb{B}$:

$$\begin{aligned} \Omega_{-1}^{\mathbb{B}} &= \sum_{\lambda_1 \leq 2n} \mathbb{P}_{\lambda_1} \mathbb{P}_{\lambda_2} \cdots \otimes M_{\lambda}[Y] \\ (5.4) \quad &= \sum_{\alpha_i \leq 2n} \mathbb{P}_{\alpha_1} \mathbb{P}_{\alpha_2} \cdots \otimes 2^{\ell(\alpha)} y^{\alpha} \end{aligned}$$

where α runs over compositions whose parts have size at most $2n$. The second equality holds since \mathbb{B} is a commutative ring. For $w \in \tilde{C}_n$, the type C affine Stanley function $Q_w^{(n)}$ is defined by

$$(5.5) \quad \Omega_{-1}^{\mathbb{B}} = \sum_{w \in \tilde{C}_n} A_w \otimes Q_w^{(n)}[Y].$$

A straightforward computation shows that this definition agrees with (1.2). Note that (1.2) defines an element of the ring $\Gamma^{(n)}$ via (2.23). By Theorem 4.6 we have

$$(5.6) \quad \Omega_{-1}^{\mathbb{B}} = \sum_{w \in \tilde{C}_n^0} \mathbb{P}_w \otimes Q_w^{(n)}[Y]$$

where \mathbb{P}_w is defined by (4.16).

5.5. Proof of Theorem 1.4. Theorem 1.4 follows immediately from applying the non-equivariant part of Proposition 4.5 to Theorem 5.1.

5.6. Proof of Theorem 1.3. It follows from Proposition 5.4, Theorem 5.5 and Theorem 4.6 that $\Phi : \Gamma_{(n)} \rightarrow H_*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})})$ is a bialgebra morphism. Since both $\Gamma_{(n)}$ and $H_*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})})$ are graded commutative and cocommutative Hopf algebras, Φ must in addition be a Hopf morphism. Recall that we define $\Psi : H^*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})}) \rightarrow \Gamma^{(n)}$ by the linear map $\xi^w \mapsto Q_w^{(n)}$ for $w \in \tilde{C}_n^0$.

We first show that $\Psi : H^*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})}) \rightarrow \Gamma^{(n)}$ and $\Phi : \Gamma_{(n)} \rightarrow H_*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})})$ are dual with respect to the pairing $\langle \cdot, \cdot \rangle : H_*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})}) \times H^*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})}) \rightarrow \mathbb{Z}$ induced by the cap product and the pairing $[\cdot, \cdot] : \Gamma_{(n)} \times \Gamma^{(n)} \rightarrow \mathbb{Z}$ of section 2.4. It suffices to show that for each $w \in \tilde{C}_n^0$ we have $\langle \Phi(f), \xi^w \rangle = [f, \Psi(\xi^w)]$ as f varies over the spanning set $\{P_{\lambda_1} \cdots P_{\lambda_l} \mid \lambda_1 \leq 2n\}$ of $\Gamma_{(n)}$. Identifying \mathbb{B} with $H_*(\mathrm{Gr}_{Sp_{2n}(\mathbb{C})})$ via the map j_0 of Theorem 4.6, we calculate

$$\begin{aligned} [P_{\lambda_1} \cdots P_{\lambda_l}, \Psi(\xi^w)] &= [P_{\lambda_1} \cdots P_{\lambda_l}, Q_w^{(n)}] \\ &= [P_{\lambda_1} \cdots P_{\lambda_l}, \langle \Omega_{-1}^{\mathbb{B}}, \xi^w \rangle] \\ &= \langle [P_{\lambda_1} \cdots P_{\lambda_l}, \Omega_{-1}^{\mathbb{B}}], \xi^w \rangle \\ &= \langle \mathbb{P}_{\lambda_1} \cdots \mathbb{P}_{\lambda_l}, \xi^w \rangle \\ &= \langle \Phi_{\mathbb{B}}(P_{\lambda_1} \cdots P_{\lambda_l}), \xi^w \rangle. \end{aligned}$$

The second equality holds by (5.6). The fourth holds by (5.4) and (2.24). The other equalities hold by definition.

Since Φ is a Hopf-morphism, we deduce that Ψ is also a Hopf-morphism. It only remains to prove that Ψ is a bijection. For surjectivity, since the Q_r generate $\Gamma^{(n)}$ as an algebra, it suffices to show that $Q_{c_r}^{(n)} = Q_r$ in $\Gamma_{(n)}$, where $c_r \in \tilde{C}_n^0$ is the length r element of the form

$$c_r = \cdots s_1 s_0 s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1 s_0.$$

It is easy to see that c_r has a unique reduced word. So a length-additive factorization of c_r into a product $c_r = \prod_i v^i$ with each $v^i \in \mathcal{Z}$, is equivalent to a composition $(\alpha_1, \alpha_2, \dots, \alpha_s)$ of r into parts of size less than $2n$, where each v^i is either the identity or has one component. After multiplying by 2^t where $t = \#\{i \mid \alpha_i > 0\}$, we see that $Q_{c_r}^{(n)}$ is the generating function of shifted tableaux T whose shape is a single row of length r where no letter can be used more than $2n$ times. The tableau T is obtained from the composition α by setting α_i letters equal to i . The factor 2^t comes from the two possible choices of marking for the leftmost occurrence of each letter. This matches $Q_{c_r}^{(n)}$ to the combinatorial definition of Q_r using tableaux [20, III.8.16].

For injectivity, it suffices to show that $\{Q_w^{(n)} \mid w \in \tilde{C}_n^0\}$ is linearly independent. We shall establish the triangularity property

$$Q_w^{(n)} = \sum_{\mu \leq \lambda(w)} a_{\mu, w} M_{\mu}$$

where $w \mapsto \lambda(w)$ is the bijection between \tilde{C}_n^0 and \mathcal{P}_C^n of Lemma 5.3, and \leq is the lexicographic order on partitions. Furthermore $a_{\mu, w}$ is unitriangular.

We first observe that if $w \in \tilde{C}_n^0$ and $w = v^s \cdots v^1$ is a factorization into \mathcal{Z} 's then v^1 must be Grassmannian, so it is one of the ρ_r 's for $r \in [1, 2n]$. But if $w = \rho_{\lambda_l} \cdots \rho_{\lambda_2} \rho_{\lambda_1}$ where $\rho_{\lambda_l} \cdots \rho_{\lambda_2}$ is Grassmannian then $w(\rho_r)^{-1}$ cannot be length subtractive for $2n \geq r > \lambda_1$. This is because every reduced expression for

$\rho_{\lambda_1} \cdots \rho_{\lambda_2}$ ends in s_0 . Repeating this, we see that the matrix of coefficients $a_{\mu,w}$ is triangular with respect to the lexicographic order. We are using the fact that if $a_{\mu,w} \neq 0$ then the factorization $w = v^s \cdots v^1$ can be chosen so that $\ell(v^i) = \mu_i$.

Finally, the factorization $w = \rho_{\lambda_1} \cdots \rho_{\lambda_2} \rho_{\lambda_1}$ of Lemma 5.3 shows that $a_{\lambda(w),w} = 1$ since $c(\rho_i) = 1$.

5.7. Proof of Theorem 1.2. The fact that $Q_w^{(n)}$ is symmetric and defines an element of $\Gamma^{(n)}$ follows from the definition (5.5) via the affine type C Cauchy kernel. The statement that $\{Q_w^{(n)} \mid w \in \tilde{C}_n^0\}$ forms a basis follows from Theorem 1.3 and the fact that $\{\xi^w \mid w \in \tilde{C}_n^0\}$ is a basis for $H^*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$. The positivity of the product structure constants is a general theorem due to Graham [9] and Kumar [13].

The coproduct structure constants of $\{Q_w^{(n)} \mid w \in \tilde{C}_n^0\}$ are the same as those of $\{\xi^w \mid w \in \tilde{C}_n^0\}$. By the duality of $H_*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ and $H^*(\text{Gr}_{Sp_{2n}(\mathbb{C})})$ and their Schubert bases, the above constants are the same as the product structure constants for the homology classes $\{\xi_w \mid w \in \tilde{C}_n^0\}$. Using the nonequivariant case $\ell(y) = \ell(x)$ of (4.12), these constants are given by the coefficients j_x^y of (4.11). But these are known to be nonnegative from the work of Peterson [22] and Lam and Shimozono [17]; they are equal to certain three-point genus zero Gromov-Witten invariants of the (finite) flag variety.

For the final positivity statement we claim that

$$(5.7) \quad \text{the coefficient of } Q_v^{(n)} \text{ where } v \in \tilde{C}_n^0 \text{ in } Q_w^{(n)} \text{ is equal to } j_v^w$$

that is, the coefficient of A_w in \mathbb{P}_v . But this follows from expanding (5.6) using the definition of \mathbb{P}_w .

6. THE COMBINATORICS OF ZEE-S

It is obvious that \mathcal{Z}_r contains a unique Grassmannian element, namely, ρ_r , and that $c(\rho_r) = 1$. To prove Theorem 5.1, by Theorem 4.6 it remains to show that the right hand side of (5.2) is an element of \mathbb{B} . By Lemma 4.7 and Example 3.2 it suffices to prove the following result, whose proof occupies the rest of this section.

Proposition 6.1. *For any $v \in \mathcal{Z}$ with $\ell(v) < 2n$, let $\mathcal{C}_v = \{w \in \mathcal{Z} \mid w \succ v\}$. Then*

$$(6.1) \quad \sum_{w \in \mathcal{C}_v} 2^{c(w)-1} \alpha_{vw}^\vee = 2^{c(v)} K.$$

Example 6.2. Let $n = 3$ and $v = s_0 s_2 s_3 s_2 \in \mathcal{Z}$. Every $w \in \mathcal{C}_v$ is obtained by putting a 1 into some reduced word for v . For each $w \in \mathcal{C}_v$, a reduced word and the coroot α_{vw}^\vee is given below. They may be computed as in Example 3.3.

red. word	α_{vw}^\vee
10232	$2\alpha_0^\vee + \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee$
01232	$\alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee$
23210	$2\alpha_0^\vee + \alpha_1^\vee$
23201	α_1^\vee

The sum of these coroots is $4K$, which agrees with the fact that $\text{Supp}(v)$ has two components, $\{0\}$ and $\{2, 3\}$.

Let $w \in \mathcal{Z}$. Since $s_i s_j = s_j s_i$ for i and j in different components of $\text{Supp}(w)$, there exists a factorization $w = w_{I_1} \cdots w_{I_c}$ where I_1, I_2, \dots, I_c are the components of $\text{Supp}(w)$ and $\text{Supp}(w_{I_p}) = I_p$. Let us index the components so their elements are ordered consistently with the total order on I_{af} . Then the above factorization is unique. For a component $C = I_p$ of $\text{Supp}(w)$ define $w_C = w_{I_p}$, which is called the C -component of w .

Example 6.3. Let $n = 9$ and $u = 4689852102$; we have $u \in \mathcal{R}(w)$ for some $w \in \mathcal{Z}$. We have $I_1 = \{0, 1, 2\}$, $I_2 = \{4, 5, 6\}$, and $I_3 = \{8, 9\}$, and $w_{I_1} = s_2 s_1 s_0 s_2$, $w_{I_2} = s_4 s_6 s_5$, and $w_{I_3} = s_8 s_9 s_8$.

6.1. Bruhat covers in \mathcal{Z} . To prove Proposition 6.1 we study in detail the Bruhat order of \tilde{C}_n when restricted to the subset \mathcal{Z} . The results in these subsections may be of independent combinatorial interest.

We construct the set of covers \mathcal{C}_v in \mathcal{Z} , of a fixed element $v \in \mathcal{Z}$. For $k', k \in I_{\text{af}}$ with $k' < k$ let

$$\begin{aligned} N_{k,k'} &= k(k+1) \cdots (n-1)n(n-1) \cdots 101 \cdots (k'-1)k' \\ \overleftarrow{N}_{k',k} &= k'(k'-1) \cdots 101 \cdots (n-1)n(n-1) \cdots (k+1)k. \end{aligned}$$

For $w \in \mathcal{Z}$, we define

$$\begin{aligned} \mathcal{R}^N(w) &= \{u \in \mathcal{R}(w) \mid u \subset N_{k,k-1} \text{ for some } 1 \leq k \leq n\} \\ \mathcal{R}^{\overleftarrow{N}}(w) &= \{u \in \mathcal{R}(w) \mid u \subset \overleftarrow{N}_{k-1,k} \text{ for some } 1 \leq k \leq n\} \end{aligned}$$

where $u \subset u'$ denotes a specific embedding of a word u as a subword of a word u' . Then by definition $w \in \mathcal{Z}$ if and only if $\mathcal{R}^N(w) \cup \mathcal{R}^{\overleftarrow{N}}(w) \neq \emptyset$.

Therefore $w \in \mathcal{C}_v$ if and only if either (1) there is a word $u \in \mathcal{R}^N(v)$ with an embedding of the form $u \subset N_{k,k-1}$ and a letter $j \in N_{k,k-1}$ that is missing from u , such that the word \tilde{u} obtained by inserting j into u , is a reduced word of w , or (2) there is a $u \in \mathcal{R}^{\overleftarrow{N}}(v)$ with an embedding of the form $u \subset \overleftarrow{N}_{k-1,k}$ and a letter $j \in \overleftarrow{N}_{k-1,k}$ missing from u , such that inserting j into u yields $\tilde{u} \in \mathcal{R}(w)$.

Lemma 6.4. *Let $v \in \mathcal{Z}$ and $u \in \mathcal{R}^N(v)$ with $u \subset N_{k,k-1}$ (resp. $u \in \mathcal{R}^{\overleftarrow{N}}(v)$ with $u \subset \overleftarrow{N}_{k-1,k}$). Let $j \in N_{k,k-1}$ (resp. $j \in \overleftarrow{N}_{k-1,k}$) be a letter that is not in u . Then adding this copy of j to u , produces a word in $\mathcal{R}^Z(w)$ for some $w \in \mathcal{C}_v$, if and only if (1) $j \notin \text{Supp}(u)$ or (2) $j+1 \in \text{Supp}(u)$ for $j \geq k$ or $j-1 \in \text{Supp}(u)$ for $j \leq k-1$.*

Proof. This follows directly from the Coxeter relations for \tilde{C}_n . \square

We define the reduced words

$$\begin{aligned} V^{k,k'} &= k(k-1) \cdots 101 \cdots (k'-1)k' && \text{for } k, k' < n \\ \Lambda_{k,k'} &= k(k+1) \cdots (n-1)n(n-1) \cdots (k'+1)k' && \text{for } k, k' > 0 \\ I\uparrow_{k'}^k &= k'(k'+1) \cdots (k-1)k && \text{for } k' \leq k \\ I\downarrow_{k'}^k &= k(k-1) \cdots (k'+1)k' && \text{for } k' \leq k \end{aligned}$$

A word is an N if it is a subword of $N_{k,k-1}$ for some $1 \leq k \leq n$ and a reverse N (abbreviated by the symbol \overleftarrow{N}) if it is a subword of $\overleftarrow{N}_{k-1,k}$ for some $1 \leq k \leq n$. The name N is suggested by the definition: the values in such a word go up, then down, and then up, like the letter N. A word v is a Z if it is an N or a \overleftarrow{N} . For

$w \in \tilde{C}_n$ let $\mathcal{R}^Z(w)$ be the set of reduced words for w that are Zs. Then by definition, $w \in \tilde{C}_n$ is a Z if and only if $\mathcal{R}^Z(w) \neq \emptyset$. Let $\mathcal{R}^N(w)$ (resp. $\mathcal{R}^{\overleftarrow{N}}(w)$) be the subset of reduced words of w that are Ns or (resp. \overleftarrow{N} s).

A *saturated* N (resp. \overleftarrow{N}) is a word of the form $N_{k,k'}$ (resp. $\overleftarrow{N}_{k',k}$). An N or \overleftarrow{N} is *proper* if it contains both the letters 0 and n . We emphasize the important fact that if u is a proper N, then $\text{first}(u) > \text{last}(u)$, where $\text{first}(u)$ and $\text{last}(u)$ are the first and last letters of u respectively. Similarly if u is a proper \overleftarrow{N} then $\text{first}(u) < \text{last}(u)$.

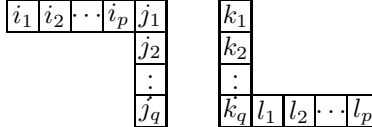
Let $u = i_1 i_2 \cdots i_M$ be a word with letters in I_{af} . We say that u has a *peak* at p if $1 < p < M$ and $i_{p-1} < i_p > i_{p+1}$ or if $p = 1$ and $i_1 > i_2$ or if $p = M$ and $p_{M-1} < p_M$ or if $M = 1$. We say that u has a *valley* at p if $1 < p < M$ and $i_{p-1} > i_p < i_{p+1}$ or if $p = 1$ and $i_1 < i_2$ or if $p = M$ and $p_{M-1} > p_M$ or if $M = 1$.

We say that a word is a V (resp. Λ) if it is either empty or has exactly one valley (resp. peak). Note that only the empty word has no valleys (resp. peaks). Note that Vs and Λ s are both Ns and \overleftarrow{N} s. Write $\mathcal{R}^V(w)$ and $\mathcal{R}^\Lambda(w)$ for the sets of reduced words of w that are respectively Vs and Λ s. A saturated V (resp. Λ) is one of the form $V^{k,k'}$ (resp. $\Lambda_{k,k'}$).

Example 6.5. Let $n = 4$. Then 234101 is a proper N , 20143 is a proper \overleftarrow{N} , 312 is a V , and 24321 is a Λ .

6.2. Equivalences for reduced words and rotation. The following Lemma is essentially a special case of Edelman-Greene insertion [5]. It says that a Λ with no $(n-1)n(n-1)$ is equivalent to a V . Similarly a V with no 101 is equivalent to a Λ .

Lemma 6.6. *Suppose $i_1 i_2 \cdots i_p j_1 j_2 \cdots j_q \in \mathcal{R}(w)$ for some $w \in \mathcal{Z}$ such that $i_1 < i_2 < \cdots < i_p < j_1 > j_2 > \cdots > j_q$ and $(n-1)n(n-1)$ is not a subword. Then there is a $k_1 k_2 \cdots k_q l_1 l_2 \cdots l_p \in \mathcal{R}(w)$ such that i_1 occurs in $k_1 k_2 \cdots k_q$, $k_1 > k_2 > \cdots > k_q < l_1 < l_2 < \cdots < l_p$ and $k_s \leq j_s$ for $1 \leq s \leq q$ and $i_s < l_s$ for $1 \leq s \leq p$.*



Proof. The result is trivial if $p = 0$ or $q = 0$. Suppose $p = 1$. If both $i_1 + 1$ and i_1 occur in $j_1 j_2 \cdots j_q$ then $i_1 > 0$ and $i_1 j_1 \cdots j_q \equiv j_1 \cdots j_q (i_1 + 1)$ using the braid relations. We take $k_s = j_s$ for $1 \leq s \leq q$ and $l_1 = i_1 + 1$, which satisfies $l_1 > i_1 \geq j_q$. Otherwise let r be maximal such that $i_1 < j_r$. It cannot be the case that $j_{r+1} = i_1$ for then $i_1 j_1 \cdots j_q$ is not reduced. We have $i_1 j_1 \cdots j_q \equiv (j_1 \cdots j_{r-1} i_1 j_{r+1} \cdots j_q) j_r$ and the latter word has the desired form. Note that in the case $p = 1$, i_1 occurs in $k_1 \cdots k_q$. Finally suppose $p > 1$. By induction $i_2 \cdots i_p j_1 \cdots j_q \equiv k'_1 \cdots k'_q l_2 \cdots l_p$ with $k'_1 > \cdots > k'_q < l_2 < \cdots < l_p$ with $j_s \leq k'_s$ for $1 \leq s \leq q$ and $i_s > l_s$ for $2 \leq s \leq p$. Since $i_1 < i_2$ and i_2 occurs in $k'_1 \cdots k'_q$, we may apply the $p = 1$ case and obtain $i_1 k'_1 \cdots k'_q \equiv k_1 \cdots k_q l_1$ with $k_1 > \cdots > k_q < l_1$ and $k_s \leq k'_s$ for $1 \leq s \leq q$. Since i_2 was in $k'_1 \cdots k'_q$ and $i_1 < i_2$, it follows by considering the $p = 1$ case that $l_1 \leq i_2 < l_2$. It follows that $k_1 \cdots k_q l_1 \cdots l_p$ is the desired reduced word. \square

Lemma 6.7. *Suppose u and u' are two Vs (resp. Λ s) such that all letters of u are greater than those in u' . Then uu' and $u'u$ are both equivalent to a V (resp. Λ).*

Proof. Let u and u' be Vs with $u = u_1mu_2$ where m is the valley of u . Then $u_1mu'u_2$ and $u_1u'mu_2$ are Vs that are equivalent to uu' and $u'u$ respectively. The proof for Λ s is similar. \square

Given a word u , let u^+ (resp. u^-) be the word obtained by adding (resp. subtracting) one from each letter in u .

Lemma 6.8. *Let $w \in \mathcal{Z}$ and $J = \text{Supp}(w)$.*

- (1) *Suppose $n \notin J$. Then $\mathcal{R}^V(w) \neq \emptyset$. Moreover if J is an interval then $\mathcal{R}^V(w)$ is a singleton.*
- (2) *Suppose $0 \notin J$. Then $\mathcal{R}^\Lambda(w) \neq \emptyset$. Moreover if J is an interval then $\mathcal{R}^\Lambda(w)$ is a singleton.*
- (3) *Suppose J is an interval $[m, M]$ with $0 < m \leq M < n$. Let $u_1Mu_2 \in \mathcal{R}^\Lambda(w)$ and $u'_2mu'_1 \in \mathcal{R}^V(w)$. Then $u'_2 = u_2^+$ and $u'_1 = u_1^+$.*

Proof. We shall prove (1) as (2) is similar. Let $u \in \mathcal{R}^Z(w)$. Suppose that $n \notin J$ and that u is an N ; the case of a \overleftarrow{N} is similar. Say u is embedded in $N_{k,k-1}$. Then $u = u_1u_2$ where u_1 is a Λ such that $\text{Supp}(u_1) \subset [k, n-1]$ and u_2 is a V with $\text{Supp}(u_2) \subset [0, k-1]$. Then $\mathcal{R}^V(w) \neq \emptyset$ by Lemmata 6.6 and 6.7.

Let $u \in \mathcal{R}^V(w)$ with J an interval. We prove its uniqueness by induction on $\ell(w)$. For $\ell(w) \leq 3$ this is evident from (3.2). Let $M = \max(J) < n$. Suppose first that u contains a single M . Then u has the form $u = M\hat{u}$ or $u = \hat{u}M$. We assume the former as the latter has an analogous proof. We have $\hat{u} \in \mathcal{R}^V(s_Mw)$ and $s_Mw \in \mathcal{Z}$. By induction \hat{u} is unique. Now let $u' \in \mathcal{R}^V(w)$. Since $\mathcal{R}(w)$ is connected by the braid relations (3.2), every reduced word for w (and in particular u') has a single M which precedes every $M-1$. Since u' is a V it must start with M . Therefore $u' = M\hat{u} = u$ by the uniqueness of \hat{u} .

Otherwise u must have the form $u = M\hat{u}M$. Let $u' \in \mathcal{R}^V(w)$. Clearly u' must contain an M which must be at the beginning or end. We suppose u' has the form $u' = Mu''$ as the case $u' = u''M$ is similar. By induction $\mathcal{R}^V(s_Mw)$ is a singleton. Therefore $u'' = \hat{u}M$ and $u' = u$ as desired.

(3) is proved by induction on the length of u_1Mu_2 . If either u_1 or u_2 is empty then the result certainly holds. Write $u_1 = u_3x$ and $u_2 = yu_4$ where x and y are letters. Since $\text{Supp}(u_1Mu_2) = [m, M]$, $x = M-1$ or $y = M-1$. Suppose $x = y = M-1$. By induction we have $u_1Mu_2 \equiv u_3(M-1)M(M-1)u_4 \equiv u_3M(M-1)Mu_4 \equiv Mu_3(M-1)u_4M \equiv Mu_4^+mu_3^+M = u_2^+mu_1^+$. Suppose next that $x = M-1 > y$. Then again by induction we have $u_1Mu_2 = u_3(M-1)Mu_2 \equiv u_3(M-1)u_2M \equiv u_2^+mu_3^+M = u_2^+mu_1^+$. The case $y = M-1 > x$ is similar. \square

Example 6.9. For $n > 7$ the N 676545 is equivalent to a V : $676545 \equiv 767545 \equiv 765457$.

Suppose $u \subset N_{k,k-1}$ is a subword and $\ell \subset N_{k,k-1}$ is a subletter (resp. $u \subset \overleftarrow{N}_{k-1,k}$ is a subword and $\ell \subset \overleftarrow{N}_{k-1,k}$ is a subletter) with ℓ missing from u . We give an explicit way to obtain another embedded word $u' \in \mathcal{R}^Z(v)$ such that ℓ is at the beginning or end of the ambient N or \overleftarrow{N} . We call this process *rotation*. The only cases not treated in Lemma 6.10 are $\ell = 0$ or $\ell = n$, in which case we may use Lemma 6.8 to obtain an equivalent reduced word that is a Λ or V respectively, and these can be embedded into an N or \overleftarrow{N} with the missing letter at the beginning or end.

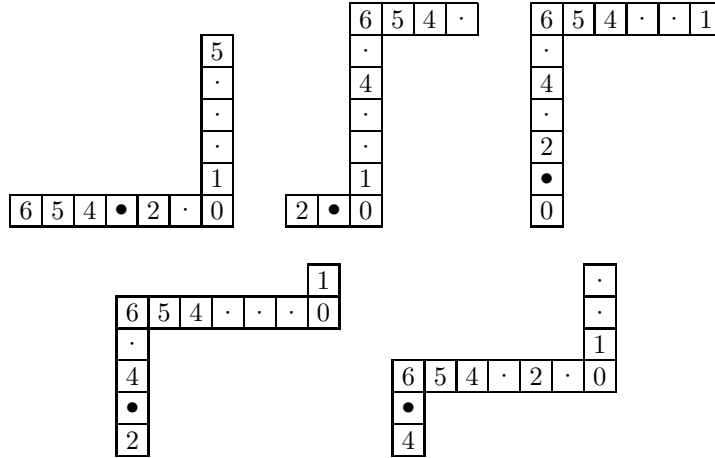
Lemma 6.10. *Suppose $w \in \mathcal{Z}$ and $u \in \mathcal{R}^N(w)$ (resp. $u \in \mathcal{R}^{\overleftarrow{N}}(w)$) with $u \subset N_{k,k-1}$ (resp. $u \subset \overleftarrow{N}_{k-1,k}$).*

- (1) *If there is an ℓ such that $k \leq \ell < n$ and ℓ does not appear in the part of u that is embedded in $I_k^n \subset N_{k,k-1}$ (resp. $I_k^n \subset \overleftarrow{N}_{k-1,k}$), then there is a $u' \in \mathcal{R}^N(w)$ (resp. $u' \in \mathcal{R}^{\overleftarrow{N}}(w)$) such that $u' \subset N_{\ell+1,\ell}$ (resp. $u' \subset \overleftarrow{N}_{\ell,\ell+1}$).*
- (2) *If there is an ℓ such that $0 < \ell \leq k-1$ and ℓ does not appear in the part of u that is embedded in $I_0^{k-1} \subset N_{k,k-1}$ (resp. $I_0^{k-1} \subset \overleftarrow{N}_{k-1,k}$), then there is a $u' \in \mathcal{R}^N(w)$ (resp. $u' \in \mathcal{R}^{\overleftarrow{N}}(w)$) such that $u' \subset N_{\ell,\ell-1}$ (resp. $u' \subset \overleftarrow{N}_{\ell-1,\ell}$).*
- (3) *If there is an ℓ such that $0 < \ell < n$ and ℓ does not appear in the part of u that is embedded in $I_0^n \subset N_{k,k-1}$ (resp. $I_0^n \subset \overleftarrow{N}_{k-1,k}$) then there is a $u' \in \mathcal{R}^{\overleftarrow{N}}(w)$ (resp. $u' \in \mathcal{R}^N(w)$) such that $u' \subset \overleftarrow{N}_{\ell-1,\ell}$ (resp. $u' \subset N_{\ell,\ell-1}$). Moreover $k-1$ or k is missing in the increasing (resp. decreasing) part of u' , according as $\ell < k$ or $\ell \geq k$.*

Proof. We prove (1) for $u \in \mathcal{R}^N(w)$; the other cases of (1) and (2) are similar. Let $u = u_1 u_2 u_3 u_4$ where $u_1 \subset I_k^{\ell-1}$, $u_2 \subset \Lambda_{\ell+1,\ell+1}$, $u_3 \subset I_k^\ell$, and $u_4 \subset V^{k-1,k-1}$. We have $u \equiv u_2 u_1 u_3 u_4$. $u_1 u_3$ is reduced since it is a factor of a reduced word. Since $u_1 u_3$ is a Λ with no n , by Lemma 6.8 it is equivalent to a V : $u_1 u_3 \equiv u'_3 u'_1$ where $u'_3 u'_1$ is a V with valley last(u'_3) such that $u'_3 \subset I_k^\ell$ and $u'_1 = u_1^+ \subset I_{k+1}^\ell$. Then $u \equiv u_2 u_1 u_3 u_4 \equiv u_2 u'_3 u'_1 u_4 \equiv u_2 u'_3 u_4 u'_1 \subset N_{\ell+1,\ell}$.

We prove (3) for $u \in \mathcal{R}^N(w)$ and $\ell < k$; the cases that $\ell \geq k$ and $u \in \mathcal{R}^{\overleftarrow{N}}(w)$, are similar. Let $u = u_1 u_2 u_3 u_4$ where $u_1 \subset \Lambda_{k,k}$, $u_2 \subset I_{\ell+1}^{k-1}$, $u_3 \subset V^{\ell-1,\ell-1}$, and $u_4 \subset I_\ell^{k-1}$. We have $u \equiv u_1 u_3 u_2 u_4$. $u_2 u_4$ is a reduced word supported on $[\ell, k-1]$ that is a V . By Lemma 6.8 there is an equivalent Λ : $u_2 u_4 \equiv u'_4 u'_2$ where $u'_4 u'_2$ is a Λ with peak first(u'_2) such that $u'_4 \subset I_\ell^{k-2}$ and $u'_2 \subset I_\ell^{k-1}$. Then $u \equiv u_1 u_3 u_2 u_4 \equiv u_1 u_3 u'_4 u'_2 \equiv u_3 u'_4 u_1 u'_2 \subset \overleftarrow{N}_{\ell-1,\ell}$. \square

Example 6.11. Let $n = 6$. We start with a reduced word for an element of $w \in \mathcal{Z}$ and apply rotations, choosing ℓ to be the first break from the left, indicated by the symbol \bullet , in the given reduced word.



then u_1 and u'_1 are reduced words and V s for the same element of \mathcal{Z} , so by Lemma 6.8, $u_1 = u'_1$ and therefore $u = u'$. \square

6.4. Special words. In this section we assume that $v \in \mathcal{Z}$ is such that $\text{Supp}(v) = I_{\text{af}}$. By Lemma 6.13, v has a unique normal word v_{nor} . We say that an embedded subword $u \subset N_{k,k-1}$ (resp. $u \subset \overleftarrow{N}_{k-1,k}$) is *normally embedded* if $u = v_{\text{nor}}$ for some $v \in \mathcal{Z}$ with $\text{Supp}(v) = I_{\text{af}}$ and $u = I\uparrow_k^n \cdots \subset N_{k,k-1}$ (resp. $u = I\downarrow_1^{k-1} I\uparrow_0^n \cdots \subset \overleftarrow{N}_{k-1,k}$).

Suppose $u \subset u'$ and $u \neq u'$. Define $\text{firstgap}(u \subset u')$ (resp. $\text{lastgap}(u \subset u')$) to be the first (resp. last) letter $j \subset u'$ that is not in u .

We say that $u \in \mathcal{R}^{\mathcal{Z}}(v)$ is *special* if it has a *special embedding*, that is, an embedding of the form $u \subset u'$ where $u' = N_{a,a-1}$ or $u' = \overleftarrow{N}_{a-1,a}$ for some $1 \leq a \leq n$ such that, if $j = \text{firstgap}(u \subset u')$, then adding j to u produces the normally embedded word $w_{\text{nor}} \subset u'$ for some $w \in \mathcal{C}_v$. More specifically, one of the following holds:

- (1) $u \in \mathcal{R}^N(v)$ and $u \subset N_{a,a-1}$ for some $1 \leq a \leq n$ such that u contains all but exactly one of the letters in $I\uparrow_a^n \subset N_{a,a-1}$, or
- (2) $u \in \mathcal{R}^{\overleftarrow{N}}(v)$ with $u \subset \overleftarrow{N}_{a-1,a}$ for some $1 \leq a \leq n$ and u contains all but exactly one of the letters in $I\downarrow_1^{a-1} I\uparrow_0^n \subset \overleftarrow{N}_{a-1,a}$.

Lemma 6.14. *Let $v \in \mathcal{Z}$ with $\text{Supp}(v) = I_{\text{af}}$ and $\ell(v) < 2n$. Then v has a unique specially embedded word, denoted $v_{\text{sp}} \subset u''$, which is obtained by rotating the normal embedding $v_{\text{nor}} \subset u'$ at $p = \text{lastgap}(v_{\text{nor}} \subset u')$. This given, we define the special cover $v^* \in \mathcal{C}_v$ of v , to be the unique cover $w \in \mathcal{C}_v$ such that the normally embedded word w_{nor} is obtained from the specially embedded word $v_{\text{sp}} \subset u''$ by inserting $\text{firstgap}(v_{\text{sp}} \subset u'')$. Moreover, if $\ell = \text{firstgap}(v_{\text{nor}} \subset u')$ and $\ell(v) < 2n - 1$ then $\ell = \text{firstgap}(v^*_{\text{nor}} \subset u'')$, except when $u' = N_{k,k-1}$ and $\text{lastgap}(v_{\text{nor}} \subset u') < \ell \leq k - 1$, in which case $\text{firstgap}(v^*_{\text{nor}} \subset u'') = \ell - 1$.*

Proof. Suppose $v_{\text{nor}} \subset N_{k,k-1}$. Let $\ell = \text{firstgap}(v_{\text{nor}} \subset N_{k,k-1})$.

Suppose $p \subset I\uparrow_1^{k-1} \subset N_{k,k-1}$ is missing from v_{nor} . Let $v_{\text{nor}} = u_1 u_2 p u_3 u_4 \subset N_{k,k-1}$ where $u_1 \subset \Lambda_{k,k}$, $u_2 \subset I\downarrow_{p+1}^{k-1}$, $u_3 \subset V^{p-1,p-1}$, and $u_4 \subset I\uparrow_{p+1}^{k-1}$. Then using Lemma 6.8(3) we have $v_{\text{nor}} \equiv u_1 u_2 p u_4 u_3 \equiv u_1 u_4^-(k-1) u_2^- u_3 \equiv u_4^- u_1 (k-1) u_2^- u_3 =: u \subset N_{p,p-1}$. Now $u_4^- \subset I\uparrow_p^{k-2}$. In this case, u is special if and only if $u_4 = I\uparrow_{p+1}^{k-1}$, that is, $p = \text{lastgap}(v_{\text{nor}} \subset N_{k,k-1})$. Suppose so. Then $\text{firstgap}(v^*_{\text{nor}} \subset N_{p,p-1})$ is ℓ unless $\ell \subset I\downarrow_{p+1}^{k-1} \subset N_{k,k-1}$, in which case the answer is $\ell - 1$.

Suppose $p \subset I\downarrow_1^{k-1} \subset N_{k,k-1}$ is missing from v_{nor} . Let $v_{\text{nor}} = u_1 u_2 u_3 p u_4$ where $u_1 \subset \Lambda_{k,k}$, $u_2 \subset I\downarrow_{p+1}^{k-1}$, $u_3 \subset V^{p-1,p-1}$, and $u_4 \subset I\uparrow_{p+1}^{k-1}$. Then $v_{\text{nor}} \equiv u_3 u_1 u_2 p u_4 \equiv u_3 u_1 u_4^-(k-1) u_2^- \equiv u_3 u_4^- u_1 (k-1) u_2^- =: u \subset \overleftarrow{N}_{p-1,p}$. In this case u is special if and only if $u_3 = V^{p-1,p-1}$ and $u_4 = I\uparrow_{p+1}^{k-1}$, that is, $p = \text{lastgap}(v_{\text{nor}} \subset N_{k,k-1})$. Suppose so. Then $\text{firstgap}(v^*_{\text{nor}} \subset \overleftarrow{N}_{p-1,p})$ is ℓ unless $\ell \subset I\downarrow_1^{k-1} \subset N_{k,k-1}$ and $\ell > p$ ($\ell = p$ cannot happen if $\ell(v) < 2n - 1$), in which case the answer is $\ell - 1$.

Suppose $p \subset I\downarrow_k^{n-1} \subset N_{k,k-1}$ is missing from v_{nor} . Let $v_{\text{nor}} = I\uparrow_k^{p-1} p u_1 u_2 u_3 \subset N_{k,k-1}$ where $u_1 \subset \Lambda_{p+1,p+1}$, $u_2 \subset I\downarrow_k^{p-1}$, and $u_3 \subset V^{k-1,k-1}$. We have $v_{\text{nor}} \equiv I\uparrow_k^{p-1} p u_2 u_3 u_1 \equiv u_2^+ k I\uparrow_{k+1}^p u_3 u_1 \equiv u_2^+ k u_3 I\uparrow_{k+1}^p u_1 =: u \subset \overleftarrow{N}_{p,p+1}$. In this case u is special if and only if $u_2 = I\downarrow_k^{p-1}$ and $u_3 = V^{k-1,k-1}$, that is, $p = \text{lastgap}(v_{\text{nor}} \subset N_{k,k-1})$. Suppose so. Then $\ell \subset I\downarrow_{p+1}^{n-1} \subset N_{k,k-1}$, and $\ell = \text{firstgap}(v^*_{\text{nor}} \subset \overleftarrow{N}_{p,p+1})$.

Suppose $v_{\text{nor}} \subset \overleftarrow{N}_{k-1,k}$. Let $\ell = \text{firstgap}(v_{\text{nor}} \subset \overleftarrow{N}_{k-1,k})$. Let $p \subset \overleftarrow{N}_{k-1,k}$ be missing for v_{nor} . Then from the definitions we have $p \subset I \downarrow_k^{n-1} \subset \overleftarrow{N}_{k-1,k}$. Write $v_{\text{nor}} = I \downarrow_1^{k-1} I \uparrow_0^n u_1 u_2$ where $u_1 \subset I \downarrow_{p+1}^{n-1}$ and $u_2 \subset I \downarrow_k^{p-1}$. Therefore $v_{\text{nor}} \equiv u_2^+ I \downarrow_1^{k-1} I \uparrow_0^n u_1 =: u \subset \overleftarrow{N}_{p,p+1}$. u is special if and only if $u_2 = I \downarrow_k^{p-1}$, that is, $p = \text{lastgap}(v_{\text{nor}} \subset \overleftarrow{N}_{k-1,k})$. Suppose so. Then $\ell = \text{firstgap}(v_{\text{nor}}^* \subset \overleftarrow{N}_{p,p+1})$.

Thus rotation at $\text{lastgap}(v_{\text{nor}} \subset u')$ creates a particular specially embedded word which we shall denote by $v_{\text{sp}} \subset u'$. It remains to show that v_{sp} is unique. Suppose $u \in \mathcal{R}(v)$ is such that $u \subset u'$ is a special embedding. Rotating $u \subset u'$ at $\text{firstgap}(u \subset u')$, we obtain the normal embedding of v_{nor} , which is unique. The explicit computation of this rotation shows that it is the inverse of the rotation at the last gap of the normal embedding of v_{nor} (which was given above explicitly in all cases). It follows that there is a unique specially embedded word for v . \square

For later use we summarize the construction of Lemma 6.14 in the following table, where p is the last gap. We have indicated the form of v_{sp} , and used the symbol $*$ to indicate where a letter (either k or $k-1$) can be added to obtain v_{nor}^* .

$p \subset$	v_{sp}	$u_1 \subset$	$u_2 \subset$	$u_3 \subset$
$I \uparrow_1^{k-1} \subset N_{k,k-1}$	$I \uparrow_p^{k-2} * u_1(k-1) u_2^- u_3$	$\Lambda_{k,k}$	$I \downarrow_{p+1}^{k-1}$	$V^{p-1,p-1}$
$I \downarrow_1^{k-1} \subset N_{k,k-1}$	$I \downarrow_1^{p-1} I \uparrow_0^{k-2} * u_1(k-1) u_2^-$	$\Lambda_{k,k}$	$I \downarrow_{p+1}^{k-1}$	
$I \downarrow_k^{n-1} \subset N_{k,k-1}$	$I \downarrow_1^p I \uparrow_0^{k-1} * I \uparrow_{k+1}^p u_1$	$\Lambda_{p+1,p+1}$		
$I \downarrow_k^{n-1} \subset \overleftarrow{N}_{k-1,k}$	$I \downarrow_{k+1}^{p-1} * I \downarrow_1^{k-1} I \uparrow_0^n u_1$	$I \downarrow_{p+1}^{n-1}$		

Example 6.15. Take $n = 7$ and $v_{\text{nor}} = 56754310124 \subset N_{5,4}$. In this case $p = 3$, $v_{\text{sp}} = 35675431012$, and $v_{\text{nor}}^* = 345675431012$.

6.5. Kinds of covers. Let $v \in \mathcal{Z}$ be fixed. The set I_{af} is divided into four kinds of letters. Let j be v -internal if $j \in \text{Supp}(v)$. If $j \notin \text{Supp}(v)$, let j be v -isolated, v -adjoining, and v -merging if the number of components of $\text{Supp}(v)$ adjacent to j is 0, 1, or 2, that is, $|\{j-1, j+1\} \cap \text{Supp}(v)|$ is 0, 1, or 2.

Let $w \in \mathcal{C}_v$ with a reduced word $\tilde{u} \in \mathcal{R}^Z(w)$ and a letter $j \subset \tilde{u}$ whose omission leaves a reduced word $u \in \mathcal{R}^Z(v)$. Then we call the cover w internal, isolated, adjoining, or merging, according as j is (with respect to v). Such w have $c(w)$ equal to $c(v)$, $c(v) + 1$, $c(v)$, and $c(v) - 1$ respectively.

In the case of an internal cover the omitted letter j may vary if the reduced word \tilde{u} is changed; however the component C of $j \in \text{Supp}(v)$ depends only on w . Moreover $w_C \succ v_C$ and $w_{C'} = v_{C'}$ for components C' of $\text{Supp}(v)$ with $C' \neq C$.

If $j \notin \text{Supp}(v)$ then the omitted letter j is uniquely determined by w .

Lemma 6.16. *Let $v \in \mathcal{Z}$ with $\ell(v) < 2n$.*

- (1) *For each v -isolated letter $j \in I_{\text{af}}$ there is a unique cover w in \mathcal{C}_v that omits j , namely, $s_j v$.*
- (2) *For each v -adjoining letter $j \in I_{\text{af}}$ there are exactly two covers $w \in \mathcal{C}_v$ that omit j , namely, $s_j v$ and vs_j .*
- (3) *For each v -merging letter $j \in I_{\text{af}}$ there are exactly four covers $w \in \mathcal{C}_v$ that omit j . Let $u \in \mathcal{R}(v)$ and u_+ and u_- the subwords of u given by the restriction to the letters greater and less than j respectively and let v_+ and v_- be the corresponding elements of \mathcal{Z} . Then the four covers of v that omit j are $s_j v_+ v_-$, $v_+ s_j v_-$, $v_+ v_- s_j$, and $v_- s_j v_+$.*

Proof. We prove (3) as the other cases are easier. We observe that v_+ and v_- are defined independent of the reduced word u . The four given elements of \tilde{C}_n are all covers of v that omit j , and are distinct since $j-1 \in \text{Supp}(v_-)$ and $j+1 \in \text{Supp}(v_+)$. We now realize each of them by reduced words that are Zs. By Lemma 6.8 let $u_+ \in \mathcal{R}^\Lambda(v_+)$ and $u_- \in \mathcal{R}^V(v_-)$. Then ju_+u_- , u_+ju_- , and u_+u_-j are all Ns, and u_-ju_+ is a \overleftarrow{N} , and they are reduced words for the above elements of \tilde{C}_n . It remains to show that if $w \in \mathcal{C}_v$ omits j then w is one of the four given covers. Let $\tilde{u} \in \mathcal{R}^Z(w)$ and $j \in \tilde{u}$ such that the omission of j from \tilde{u} leaves $u \in \mathcal{R}^Z(v)$. Suppose $\tilde{u} \subset N_{k,k-1}$. Suppose $j \geq k$. Let $u = u_1u_2u_3$ where $u_2 \subset \Lambda_{j+1,j+1}$, so that $\tilde{u} = u_1ju_2u_3$ or $u_1u_2ju_3$. Here $\text{Supp}(u_1) \subset [0, j-1]$ and $\text{Supp}(u_3) \subset [0, j-1]$ while $\text{Supp}(u_2) \subset [j+1, n]$. If $\tilde{u} = u_1ju_2u_3$ then $\tilde{u} \equiv u_1ju_3u_2$. But not both u_1 and u_3 can contain $j-1$, for if they did then u_1 ends with $j-1$ and u_3 starts with $j-1$ and $u \equiv u_1u_3u_2$ is not reduced. If u_1 does not contain $j-1$ then $\tilde{u} \equiv ju_1u_3u_2$ and $w = s_jv_-v_+$. If u_2 does not contain $j-1$ then $\tilde{u} \equiv u_1u_3ju_2$ and $w = v_-s_jv_+$. The cases that $\tilde{u} = u_1u_2ju_3$, $j < k$ and $\tilde{u} \subset \overleftarrow{N}_{k-1,k}$ are similar. \square

We now classify the internal covers of v . For this purpose we may assume $\text{Supp}(v)$ has a single component. For $k, \ell \leq M$ let $\Lambda_{k,\ell}^M = I \uparrow_k^M I \downarrow_\ell^{M-1}$ and for $m \leq k, \ell$ let $V_m^{k,\ell} = I \downarrow_m^k I \uparrow_{m+1}^\ell$.

Lemma 6.17. *Suppose $v \in \mathcal{Z}$ is such that $\text{Supp}(v)$ consists of a single component $[m, M]$.*

- (1) *If $M < n$ (resp. $m > 0$) then the internal covers of v are precisely those obtained by inserting missing letters into $u \subset V_m^{M,M}$ (resp. $u \subset \Lambda_{m,m}^M$) where u is the unique element of $\mathcal{R}^V(v)$ (resp. $\mathcal{R}^\Lambda(v)$).*
- (2) *If $m = 0$ and $M = n$, consider the normal embedding $v_{\text{nor}} \subset u'$. Then the internal covers of v are precisely those obtained by inserting missing letters into $v_{\text{nor}} \subset u'$ (normal covers), plus the special cover, which is obtained from the special embedding of v_{sp} by inserting the first missing letter.*

Proof. Since internal covers do not change the support and the support is assumed to be an interval, by Lemma 6.4 adding any missing letter of $[m, M]$ creates a cover. Any internal cover $w \in \mathcal{C}_v$ has the same support as v . If $M < n$ then w has a reduced word that is a V, and removing one of its letters yields a reduced word for v that is a V. By uniqueness this word must be u . This proves (1) for $M < n$, and $m > 0$ is similar. For (2) suppose $\text{Supp}(v) = I_{\text{af}}$. Let $w \in \mathcal{C}_v$. Consider the normal embedding of w_{nor} , which is unique since $\text{Supp}(w) = I_{\text{af}}$. There is a unique letter in w_{nor} whose removal yields an embedded reduced word u for v . It is easy to check that u is either v_{nor} normally embedded or v_{sp} specially embedded. \square

6.6. Associated coroots. Let $v \leq v'$ with $v, v' \in \tilde{C}_n$ and let $u \in \mathcal{R}(v)$ and $u' \in \mathcal{R}(v')$ be such that $u \subset u'$. For $j \subset u'$, define $\alpha^\vee(u \subset u', j)$ to be α_{vw}^\vee if adding the given occurrence of j to u creates a reduced word for a cover $w \triangleright v$, and 0 otherwise. In particular the value is 0 if $j \subset u$. Define $\alpha^\vee(u \subset u') = \sum_{j \subset u'} \alpha^\vee(u \subset u', j)$. The following Lemma holds by the definitions.

Lemma 6.18. *Let $v_1 \leq v'_1$, $v_2 \leq v'_2$, u_1, u'_1, u_2, u'_2 reduced words for v_1, v'_1, v_2, v'_2 such that $u_1 \subset u'_1$ and $u_2 \subset u'_2$. Then*

$$\alpha^\vee(u_1u_2 \subset u'_1u'_2) = v_2^{-1} \alpha^\vee(u_1 \subset u'_1) + \alpha^\vee(u_2 \subset u'_2).$$

It is straightforward to compute sums of associated coroots for subwords of increasing or decreasing reduced words.

Lemma 6.19. *Let $0 \leq m \leq M \leq n$ and $u \subset I \uparrow_m^M$ or $u \subset I \downarrow_m^M$. Then*

$$(6.2) \quad \begin{aligned} \alpha^\vee(u \subset I \uparrow_m^M) &= \alpha_k^\vee + \alpha_{k+1}^\vee + \cdots + \alpha_M^\vee && \text{if } M < n \\ \alpha^\vee(u \subset I \uparrow_m^n) &= \alpha_k^\vee + \alpha_{k+1}^\vee + \cdots + \alpha_{n-1}^\vee + 2\alpha_n^\vee \\ \alpha^\vee(u \subset I \downarrow_m^M) &= \alpha_k^\vee + \alpha_{k-1}^\vee + \cdots + \alpha_m^\vee && \text{if } m > 0 \\ \alpha^\vee(u \subset I \downarrow_0^M) &= \alpha_k^\vee + \alpha_{k-1}^\vee + \cdots + \alpha_1^\vee + 2\alpha_0^\vee \end{aligned}$$

where $k = \text{firstgap}(u \subset u')$ for $u' = I \uparrow_m^M$ or $u' = I \downarrow_m^M$. If k does not exist (that is, $u = u'$) then the sum is 0.

Next we compute sums of associated coroots for subwords of Vs and As whose support are intervals. We assume there is a letter missing in the initial monotonic part of the embedded word; otherwise the result is given by Lemma 6.19.

Lemma 6.20. *Let $v \in \mathcal{Z}$ have $\text{Supp}(v) = [m, M] \subsetneq I_{\text{af}}$. Suppose $u \in \mathcal{R}^V(v)$ with $M < n$ (resp. $u \in \mathcal{R}^\Lambda(v)$ with $m > 0$) of the form $u = u_1 u_2$ with $u_1 \subset I \downarrow_m^M$ (resp. $u_1 \subset I \uparrow_m^M$) and $u_2 \subset I \uparrow_{m+1}^M$ (resp. $u_2 \subset I \downarrow_m^{M-1}$) so that $u \subset V_m^{M,M}$ (resp. $u \subset \Lambda_{m,m}^M$). Suppose that $u_1 \neq u'$ for $u' = I \downarrow_m^M$ (resp. $u' = I \uparrow_m^M$) so that $k = \text{firstgap}(u_1 \subset u')$ is well-defined. Let $k' = \text{firstgap}(u_2 \subset u'')$ where $u'' = I \uparrow_{m+1}^M$ (resp. $u'' = I \downarrow_m^{M-1}$); if $u_2 = u''$ then set $k' = M + 1$ (resp. $k' = m - 1$). Then*

$$(6.3) \quad \begin{aligned} \alpha^\vee(u \subset V_m^{M,M}) &= (\alpha_m^\vee + \cdots + \alpha_{k-1}^\vee) + (\alpha_{k'}^\vee + \cdots + \alpha_M^\vee) && \text{if } m > 0 \\ \alpha^\vee(u \subset V_0^{M,M}) &= 2(\alpha_0^\vee + \cdots + \alpha_{k-1}^\vee) + (\alpha_k^\vee + \cdots + \alpha_M^\vee) \\ \alpha^\vee(u \subset \Lambda_{m,m}^M) &= (\alpha_m^\vee + \cdots + \alpha_{k'}^\vee) + (\alpha_{k+1}^\vee + \cdots + \alpha_M^\vee) && \text{if } M < n \\ \alpha^\vee(u \subset \Lambda_{m,m}^n) &= (\alpha_m^\vee + \cdots + \alpha_k^\vee) + 2(\alpha_{k+1}^\vee + \cdots + \alpha_n^\vee) \end{aligned}$$

Proof. Since $\text{Supp}(v)$ is an interval and we are adding letters in that same interval, adding any missing letter creates a reduced word by Lemma 6.4. Let $v_2 \in \mathcal{Z}$ be such that $u_2 \in \mathcal{R}(v_2)$.

Let $M < n$ and $u \subset V_m^{M,M}$. Suppose $m > 0$. We have

$$\alpha^\vee(u_1 \subset I \downarrow_m^M) = \alpha_m^\vee + \cdots + \alpha_{k-1}^\vee + \alpha_k^\vee.$$

By the assumption on support, since $k \notin \text{Supp}(u_1)$ we have $k \in \text{Supp}(u_2)$ and $k \neq k'$. Therefore

$$v_2^{-1} \alpha^\vee(u_1 \subset I \downarrow_m^M) = \alpha_m^\vee + \cdots + \alpha_{k-1}^\vee.$$

By Lemma 6.18 the desired expression is obtained.

Suppose $m = 0$. Since $0 \in \text{Supp}(w)$ we have

$$\begin{aligned} \alpha^\vee(u_1 \subset I \downarrow_0^M) &= \alpha_k^\vee + \alpha_{k-1}^\vee + \cdots + \alpha_1^\vee + 2\alpha_0^\vee \\ v_2^{-1} \alpha^\vee(u_1 \subset I \downarrow_0^M) &= \begin{cases} 2(\alpha_0^\vee + \cdots + \alpha_{k'-1}^\vee) + (\alpha_{k'}^\vee + \cdots + \alpha_{k-1}^\vee) & \text{if } k > k' \\ 2(\alpha_0^\vee + \cdots + \alpha_{k-1}^\vee) + (\alpha_k^\vee + \cdots + \alpha_{k'-1}^\vee) & \text{if } k < k'. \end{cases} \end{aligned}$$

By Lemma 6.18 we obtain the desired formula.

The other computations are similar. \square

Lemma 6.21. *Suppose $v \in \mathcal{Z}$ is such that $\text{Supp}(v) = I_{\text{af}}$ with normal embedding $v_{\text{nor}} \subset N_{k,k-1}$, v_{nor} does not contain $I_{\downarrow 0}^n \subset N_{k,k-1}$, and $\ell = \text{firstgap}(v_{\text{nor}} \subset N_{k,k-1})$. Then*

$$\alpha^\vee(u \subset N_{k,k-1}) = \begin{cases} 2(\alpha_0^\vee + \cdots + \alpha_{k-1}^\vee) + (\alpha_k^\vee + \cdots + \alpha_\ell^\vee) & \text{if } \ell \geq k \\ 2(\alpha_0^\vee + \cdots + \alpha_{\ell-1}^\vee) + (\alpha_\ell^\vee + \cdots + \alpha_{k-1}^\vee) & \text{if } \ell < k. \end{cases}$$

Proof. Follows from Lemmata 6.18, 6.19 and 6.20. \square

Lemma 6.22. *Suppose $v \in \mathcal{Z}$ is such that $\ell(v) < 2n$ and $\text{Supp}(v) = [m, M]$ is an interval. Then the sum of α_{vw}^\vee as w runs over the internal covers in \mathcal{C}_v , is given by*

$$\begin{aligned} & 2(\alpha_m^\vee + \alpha_{m+1}^\vee + \cdots + \alpha_M^\vee) \\ & - \chi(M < n)(-\alpha_{M+1}^\vee + v^{-1}\alpha_{M+1}^\vee) \\ & - \chi(m > 0)(-\alpha_{m-1}^\vee + v^{-1}\alpha_{m-1}^\vee). \end{aligned}$$

Proof. We begin with the most involved case, when $\text{Supp}(v) = I_{\text{af}}$. In this case we must show that $\sum_{w \in \mathcal{C}_v} \alpha_{vw}^\vee = 2(\alpha_0^\vee + \cdots + \alpha_n^\vee) = 2K$. By Lemma 6.17, $\sum_{w \in \mathcal{C}_v} \alpha_{vw}^\vee = \alpha^\vee(v_{\text{nor}} \subset u') + \alpha_{vv^*}^\vee$ where $v_{\text{nor}} \subset u'$ is the normal embedding. For the computation of the special coroot $\alpha_{vv^*}^\vee$ we shall refer back to the proof of Lemma 6.14 without further mention, for the explicit computations of the special embedding $v_{\text{sp}} \subset u''$ given by rotating the normal embedding $v_{\text{nor}} \subset u'$ at $p = \text{lastgap}(v_{\text{nor}} \subset u')$. The reader may find the table after Lemma 6.14 helpful.

Suppose that $v_{\text{nor}} \subset N_{k,k-1}$ is the normal embedding for some $1 \leq k \leq n$. Let $\ell = \text{firstgap}(v_{\text{nor}} \subset N_{k,k-1})$ and $p = \text{lastgap}(v_{\text{nor}} \subset N_{k,k-1})$.

Suppose $\ell \subset I_{\uparrow 1}^{k-1} \subset N_{k,k-1}$. We have $p \subset I_{\uparrow \ell}^{k-1} \subset N_{k,k-1}$, $v_{\text{nor}}^* \subset N_{p,p-1}$ is normally embedded, and $\ell = \text{firstgap}(v_{\text{nor}}^* \subset N_{p,p-1})$. We compute $\alpha_{vv^*}^\vee = s_{\ell-1} \cdots s_1 s_0 s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_k (\alpha_{k-1}^\vee) = 2(\alpha_0^\vee + \cdots + \alpha_n^\vee) - (\alpha_\ell^\vee + \cdots + \alpha_{k-1}^\vee)$. Combining this with $\alpha^\vee(v_{\text{nor}} \subset N_{k,k-1}) = \alpha_\ell^\vee + \cdots + \alpha_{k-1}^\vee$ from Lemma 6.19 we obtain the total $2K$.

Suppose $\ell \subset I_{\downarrow 1}^{k-1} \subset N_{k,k-1}$. Since $\text{Supp}(v) = I_{\text{af}}$, $\ell \subset I_{\uparrow 1}^{k-1} \subset N_{k,k-1}$ appears in v_{nor} . Suppose $p \subset I_{\uparrow 1}^{k-1} \subset N_{k,k-1}$. Suppose first that $p > \ell$. Then $v_{\text{nor}}^* \subset N_{p,p-1}$ is normally embedded with $\text{firstgap}(v_{\text{nor}}^* \subset N_{p,p-1}) = \ell$. We have $\alpha_{vv^*}^\vee = s_\ell \cdots s_{\ell+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_k \alpha_{k-1}^\vee = 2(\alpha_\ell^\vee + \cdots + \alpha_n^\vee) - (\alpha_\ell^\vee + \cdots + \alpha_{k-1}^\vee)$. By Lemma 6.21 for $\ell < k$ we have $\alpha^\vee(v_{\text{nor}} \subset N_{k,k-1}) = 2(\alpha_0^\vee + \cdots + \alpha_{\ell-1}^\vee) + (\alpha_\ell^\vee + \cdots + \alpha_{k-1}^\vee)$, and the total is $2K$. Suppose next that $p < \ell$. Again $v_{\text{nor}}^* \subset N_{p,p-1}$ is normally embedded and $\text{firstgap}(v_{\text{nor}}^* \subset N_{p,p-1}) = \ell - 1$. The coroot computation is similar to the previous case. By definition p occurs after ℓ in $N_{k,k-1}$ so the remaining subcase is $p \subset I_{\downarrow 1}^\ell \subset N_{k,k-1}$. Then $v_{\text{nor}}^* \subset \overleftarrow{N}_{p-1,p}$ is normally embedded and $\text{firstgap}(v_{\text{nor}}^* \subset \overleftarrow{N}_{p-1,p}) = \ell - 1 \subset I_{\downarrow p}^{n-1}$. The coroot computation is similar.

Suppose $\ell \subset I_{\downarrow k}^{n-1} \subset N_{k,k-1}$. Since $\text{Supp}(v) = I_{\text{af}}$, $k-1$ must occur in v_{sp} after ℓ . In all cases $\text{firstgap}(v_{\text{nor}}^* \subset u'') = \ell$. If $p \subset I_{\uparrow 1}^{k-1} \subset N_{k,k-1}$ then $v_{\text{nor}}^* \subset N_{p,p-1}$ is normally embedded with $\alpha_{vv^*}^\vee = s_{k-1} s_{\ell+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_k (\alpha_{k-1}^\vee) = s_{k-1} (2(\alpha_{\ell+1}^\vee + \cdots + \alpha_n^\vee) + (\alpha_{k-1}^\vee + \cdots + \alpha_\ell^\vee)) = 2(\alpha_{\ell+1}^\vee + \cdots + \alpha_n^\vee) + (\alpha_k^\vee + \cdots + \alpha_\ell^\vee)$. Combined with $\alpha^\vee(v_{\text{nor}} \subset N_{k,k-1}) = 2(\alpha_0^\vee + \cdots + \alpha_{k-1}^\vee) + (\alpha_k^\vee + \cdots + \alpha_\ell^\vee)$ from Lemma 6.21 we obtain a total of $2K$. If $p \subset I_{\downarrow 1}^{k-1} \subset N_{k,k-1}$ then $v_{\text{nor}}^* \subset \overleftarrow{N}_{p-1,p}$ is the normal embedding with coroot computation proceeding as in the previous case. If $p \subset I_{\downarrow k}^\ell \subset N_{k,k-1}$ then $v_{\text{nor}}^* \subset \overleftarrow{N}_{p,p+1}$ and the coroot computation proceeds in the same way.

The case $v_{\text{nor}} \subset \overleftarrow{N}_{k-1,k}$ is very similar to the case $v_{\text{nor}} \subset N_{k,k-1}$ with $\ell \subset I_1^{k-1} \subset N_{k,k-1}$ and $p \subset I_\ell^{k-1}$.

This finishes the case $\text{Supp}(v) = I_{\text{af}}$.

Next we consider the case $m = 0$ and $M < n$. Let $u \in \mathcal{R}^V(v)$ with $u \subset V^{M,M}$. In this case the sum of α_{vw}^\vee for $w \in \mathcal{C}_v$ an internal cover of v , is equal to $\alpha^\vee(u \subset V^{M,M})$. Let $u = u_1 0 u_2$ where $u_1 \subset I_1^M$ and $u_2 \subset I_1^M$. Let $a \subset I_1^M$ (resp. $b \subset I_1^M$) be the first missing letter from u_1 (resp. u_2), which exists if $u_1 \neq I_1^M$ (resp. $u_2 \neq I_1^M$). By Lemma 6.20 we have

$$\alpha^\vee(u \subset V^{M,M}) = \begin{cases} 2(\alpha_0^\vee + \cdots + \alpha_{a-1}^\vee) + (\alpha_a^\vee + \cdots + \alpha_M^\vee) & \text{if } u_1 \neq I_1^M \\ \alpha_b^\vee + \cdots + \alpha_M^\vee & \text{if } u_1 = I_1^M \text{ and } u_2 \neq I_1^M \\ 0 & \text{if } u_1 = I_1^M \text{ and } u_2 = I_1^M. \end{cases}$$

Consider $\beta = -\alpha_{M+1}^\vee + v^{-1}\alpha_{M+1}^\vee$. Suppose first that $u_1 \neq I_1^M$. Since a is missing from u_1 and $\text{Supp}(v) = [0, M]$ is an interval, $a \in u_2$. Therefore $\beta = -\alpha_{M+1}^\vee + s_a s_{a+1} \cdots s_M \alpha_{M+1}^\vee = \alpha_a^\vee + \cdots + \alpha_M^\vee$, which yields the desired total. Suppose $u_1 = I_1^M$ and $u_2 \neq I_1^M$. Then $\beta = -\alpha_{M+1}^\vee + s_{b-1} \cdots s_1 s_0 s_1 \cdots s_M \alpha_{M+1}^\vee = 2(\alpha_0^\vee + \cdots + \alpha_{b-1}^\vee) + (\alpha_b^\vee + \cdots + \alpha_M^\vee)$ as desired. If $u_1 = I_1^M$ and $u_2 = I_1^M$ then $\beta = -\alpha_{M+1}^\vee + s_M \cdots s_1 s_0 s_1 \cdots s_M \alpha_{M+1}^\vee = 2(\alpha_0^\vee + \cdots + \alpha_M^\vee)$ as desired.

The case that $m > 0$ and $M = n$ is entirely similar to the previous case. The remaining case is $0 < m$ and $M < n$. Using $u \in \mathcal{R}^V(v)$ and $u_1 m u_2 = u \subset V_m^{M,M}$, the proof is similar to the case for $m = 0$ and $M = n$ except that one must also compute $-\alpha_{m-1}^\vee + v^{-1}\alpha_{m-1}^\vee$, which equals $\alpha_m^\vee + \cdots + \alpha_{b-1}^\vee$ if $u_2 \neq I_{m+1}^M$ and equals $\alpha_m^\vee + \cdots + \alpha_M^\vee$ if $u_2 = I_{m+1}^M$. \square

6.7. Proof of Proposition 6.1. We fix $v \in \mathcal{Z}$ with $\ell(v) < 2n$ and $i \in I_{\text{af}}$. Let

$$\mathcal{C}'_v = \{w \in \mathcal{C}_v \mid \alpha_i^\vee \text{ occurs in } \alpha_{vw}^\vee\}.$$

Case 1. $i \notin \text{Supp}(v)$. Let $w \in \mathcal{C}'_v$. Since α_i^\vee occurs in α_{vw}^\vee and $i \notin \text{Supp}(v)$ it follows that $i \in \text{Supp}(w)$. It is easy to check that α_i^\vee occurs in α_{vw}^\vee with coefficient 1. The desired multiplicity is obtained by Lemma 6.16.

Case 2. $i \in \text{Supp}(v)$. Let $C = [m, M]$ be the component of i in $\text{Supp}(v)$. The covers in \mathcal{C}'_v add letters that are either in C or adjacent to C .

Case 2a. $C = I_{\text{af}}$. In this case there are only internal covers. Therefore $\sum_{w \in \mathcal{C}'_v} \alpha_{vw}^\vee = 2(\alpha_0^\vee + \cdots + \alpha_n^\vee)$ by Lemma 6.22. Since $c(w) = c(v)$ for all such w , Proposition 6.1 is verified in this case.

Case 2b. $C = [0, M]$ with $M < n$. (The case $C = [m, n]$ with $m > 0$ is similar.) Write $v = v_C v'$ where v' is the product of the components of v other than v_C . Then the internal covers in \mathcal{C}'_v consist of the $w \in \mathcal{C}_v$ such that $w_C \succ v_C$ and $w_{C'} = v_{C'}$ for components C' of $\text{Supp}(v)$ with $C' \neq C$. The sum of α_{vw}^\vee for internal covers of v in \mathcal{C}'_v , is given by Lemma 6.22. For such w we have $c(w) = c(v)$. Suppose $M + 2 \notin \text{Supp}(v)$, so that $M + 1$ is v -adjoining. Then all the noninternal covers $w \in \mathcal{C}'_v$ adjoin the letter $M + 1$ to C ; such w satisfy $c(w) = c(v)$ also. By Lemma 6.16 there are exactly two adjoining covers in \mathcal{C}_v , namely, $s_{M+1}v = v' s_{M+1} v_C$ and $v s_{M+1}$. The latter has associated coroot α_{M+1}^\vee and therefore does not contribute α_i^\vee for $i \in C$. For $w = s_{M+1}v$ we have $\alpha_{vw}^\vee = v_C^{-1} \alpha_{M+1}^\vee$. Combining this with the sum of coroots for internal covers associated to the component C of $\text{Supp}(v)$, by Lemma 6.22 the coefficient of α_i^\vee is 2 as desired. Suppose $M + 2 \in \text{Supp}(v)$. Then

$M + 1$ is v -merging. By Lemma 6.16 there are four covers $w \in \mathcal{C}_v$ that add $M + 1$; each has $c(w) = c(v) - 1$. Their associated coroots are

$$\begin{aligned}\alpha_{v,vs_{M+1}}^\vee &= \alpha_{M+1}^\vee \\ \alpha_{v,v's_{M+1}v_C}^\vee &= v_C^{-1}\alpha_{M+1}^\vee \\ \alpha_{v,s_{M+1}v'v_C}^\vee &= -\alpha_{M+1}^\vee + v_C^{-1}\alpha_{M+1}^\vee + (v')^{-1}\alpha_{M+1}^\vee \\ \alpha_{v,v_Cs_{M+1}v'}^\vee &= (v')^{-1}\alpha_{M+1}^\vee.\end{aligned}$$

The sum of these coroots, forgetting the α_j^\vee for $j \notin C$, is $2(-\alpha_{M+1}^\vee + v_C^{-1}\alpha_{M+1}^\vee)$. Together with the coroots corresponding to internal covers given by Lemma 6.22, which receive a relative factor of 2 since $c(w) = c(v)$ for internal covers and $c(w) = c(v) - 1$ for merging covers, gives the desired result.

Case 2c. $0 < m < M < n$. The computations for this case are similar to those above.

This completes the proof of Proposition 6.1.

7. HOPF PROPERTY OF Φ

In this section we prove Theorem 5.5.

7.1. A coproduct formula for nilHecke algebras. In Proposition 7.1 below, we give a complicated but explicit formula for $\phi_0^{(2)}(\Delta(A_w))$ for $w \in W_{\text{af}}$. This formula is valid for the nilHecke algebra for any Cartan datum.

Let $v \in \mathcal{R}(w)$ and consider the tuples $\mathbf{v} = [v^{(1)}, v^{(2)}, \dots, v^{(k)}]$ consisting of subwords $v^{(i)} \subset v$ (the embedding of the $v^{(i)}$ are fixed). Let x_i (resp. y_i) be the first (resp. last) letters of $v^{(i)}$, considered as subletters of v via the embedding $x_i \subset v^{(i)} \subset v$. Define \mathcal{S}_v to be the set of (possibly empty) tuples $\mathbf{v} = [v^{(1)}, \dots, v^{(k)}]$ such that:

- (1) $v^{(i)}$ is a subword of length at least two of $v \setminus \{x_1, \dots, x_{i-1}\}$, which is the word v with the letters x_1, \dots, x_{i-1} removed;
- (2) $y = y_1y_2 \cdots y_k$ is a subword of v ; and
- (3) the letters x_i are distinct from the letters y_j as subwords in v .

For a word $u = u_1u_2 \cdots u_\ell$ and a tuple $\mathbf{v} = [v^{(1)}, \dots, v^{(k)}]$ let

$$b_u = \prod_{i=1}^{\ell-1} b_{u_iu_{i+1}} \quad \text{and} \quad b_{\mathbf{v}} = \prod_{i=1}^k b_{v^{(i)}},$$

where $b_{ij} = -\langle \alpha_i^\vee, \alpha_j \rangle = -a_{ij}$ is the negative of the entry of the Cartan matrix.

For a given $\mathbf{v} = [v^{(1)}, \dots, v^{(k)}] \in \mathcal{S}_v$ let $x = \{x_1, \dots, x_k\}$ and $y = \{y_1, \dots, y_k\}$. Then set $v \setminus (x \cup y)$ to be the word v with the letters in x and y removed. For a subword $u \subset v \setminus (x \cup y)$ (again with a fixed embedding), define $u \cdot y$ to be the word u with the letters in y added in the correct order of v .

Proposition 7.1. *For $w \in W_{\text{af}}$ and $v \in \mathcal{R}(w)$,*

$$(7.1) \quad \phi_0^{(2)}\Delta(A_w) = \sum_{\mathbf{v}=[v^{(1)}, \dots, v^{(k)}] \in \mathcal{S}_v} b_{\mathbf{v}} \sum_{u \subset v \setminus (x \cup y)} A_{u \cdot y} \otimes A_{u^\perp \cdot y}$$

where u^\perp is the complement word of u in $v \setminus (x \cup y)$.

Example 7.2. Take $v = ijkl$ and $\mathbf{v} = [v^{(1)}]$ with $v^{(1)} = v$, so that $v \setminus (x \cup y) = jk$. Take the subword $u = j$ of $v \setminus (x \cup y)$. Then $u.y = jl$ and $u^\perp.y = kl$, so that the term $A_{u.y} \otimes A_{u^\perp.y} = A_{jl} \otimes A_{kl}$ appears with coefficient $b_{ij}b_{jk}b_{kl}$ for this particular \mathbf{v} and u in the sum. Of course such a term can appear in other summands. For example taking $\mathbf{v} = [ikl]$, we also have $v \setminus (x \cup y) = jk$. Taking again $u = j$, we get the term $A_{jl} \otimes A_{kl}$ with a coefficient of $b_{ik}b_{kl}$.

Proof of Proposition 7.1. The proof proceeds by induction on $\ell(w)$. For $\ell(w) = 1$, let $v = i \in \mathcal{R}(w)$. We have $\mathcal{S}_v = \{[\]\}$, so that $\phi_0^{(2)}\Delta(A_w) = \sum_{u \subset v} A_u \otimes A_{u^\perp} = A_i \otimes 1 + 1 \otimes A_i$.

Now suppose $\ell(w) > 1$ and let $v = v'i \in \mathcal{R}(w)$ where $i \in I_{\text{af}}$. By induction, (4.6) and (4.10) we have

$$\begin{aligned}
 (7.2) \quad \phi_0^{(2)}(\Delta A_v) &= \phi_0^{(2)}(\phi_0^{(2)}(\Delta A_{v'})\Delta A_i) \\
 &= \phi_0^{(2)} \left[\left(\sum_{\mathbf{v}'=[v^{(1)}, \dots, v^{(k)}] \in \mathcal{S}_{v'}} b_{\mathbf{v}'} \sum_{u' \subset v' \setminus (x' \cup y')} A_{u'.y'} \otimes A_{(u')^\perp.y'} \right) \right. \\
 &\quad \left. (A_i \otimes 1 + 1 \otimes A_i - A_i \otimes \alpha_i A_i) \right] \\
 &= \sum_{\mathbf{v}'=[v^{(1)}, \dots, v^{(k)}] \in \mathcal{S}_{v'}} b_{\mathbf{v}'} \sum_{u' \subset v' \setminus (x' \cup y')} A_{u'.y'} \otimes A_{u^\perp.y'} \\
 &\quad - \phi_0^{(2)} \left[\sum_{\mathbf{v}'=[v^{(1)}, \dots, v^{(k)}] \in \mathcal{S}_{v'}} b_{\mathbf{v}'} \sum_{u' \subset v' \setminus (x' \cup y')} A_{u'.y'i} \otimes A_{(u')^\perp.y'} \alpha_i A_i \right]
 \end{aligned}$$

where to obtain the first term in the last equation we have merged the terms obtained from $A_i \otimes 1$ and $1 \otimes A_i$ which correspond to $i \in u$ and $i \notin u$ respectively. From (4.3) and (3.9) we have, for an element w with reduced word $z = z_1 z_2 \cdots z_k$

$$\begin{aligned}
 \phi_0 [A_w(-\alpha_i)] &= \sum_{ws_\beta \leq w} -\langle \beta^\vee, \alpha_i \rangle A_{ws_\beta} \\
 &= -\sum_{j=1}^k \langle z_k \cdots z_{j+1} \cdot \alpha_{z_j}^\vee, \alpha_i \rangle A_{z \setminus z_j} \\
 &= \sum_{j=1}^k \left(\sum_{r_1 \cdots r_\ell \subset z_{j+1} \cdots z_k} b_{j,r_1} b_{r_1,r_2} \cdots b_{r_\ell,i} A_{z \setminus z_j} \right) \\
 &= \sum_{j=1}^k \sum_{p \subset zi} b_p A_{z \setminus z_j}
 \end{aligned}$$

where in the last equation $p = p_1 \cdots p_\ell$ is a subword of zi satisfying: (a) $\ell \geq 2$, (b) $p_\ell = i$, and (c) $p_1 = z_j$. Applying this equation to the last summand of (7.2) with $z = (u')^\perp.y$ we see that it suffices to find a bijection Φ from the set of triples

$$(\mathbf{v}' = [v^{(1)}, \dots, v^{(k)}], u', p)$$

such that (a) $\mathbf{v}' \in \mathcal{S}_{v'}$, (b) $u' \subset v' \setminus (x' \cup y')$, and (c) $p = p_1 \cdots p_\ell$ is a subword of $((u')^\perp.y')i$ satisfying $p_\ell = i$ and $\ell \geq 2$, to the set of pairs

$$(\mathbf{v} = [v^{(1)}, \dots, v^{(r)}], u)$$

such that (a) $\mathbf{v} \in \mathcal{S}_v$, (b) $u \subset v \setminus (x \cup y)$, and (c) $y_r = i$. Furthermore under Φ we must have (a) $b_{\mathbf{v}} = b_{\mathbf{v}'}$, (b) $u.y = u'.y'i$ and (c) $u^\perp.y$ equal to the word $(u')^\perp.y'i$ with the letter p_1 removed.

Given $(\mathbf{v}' = [v'^{(1)}, \dots, v'^{(k)}], u', p)$ we consider two cases. If $p_1 \notin y'$ we define $\mathbf{v} = [v'^{(1)}, \dots, v'^{(k)}, p]$ and $u = u'$. It is clear then that $\mathbf{v} \in \mathcal{S}_v$ and we have $x = x' \cup \{p_1\}$ and $y = y' \cup \{i\}$. Since $p_1 \in (u')^\perp$ we see that $u \subset v \setminus (x \cup y)$ and that u^\perp is $(u')^\perp$ with p_1 removed.

Now suppose $p_1 = y_j$ for some $1 \leq j \leq k$. We define the *fusion* of two words zi and iz' , where z, z' are words and i is a letter to be $zi \star iz' = ziz'$. Here, all words and letters are considered subwords of v . Let $\tilde{v} = v'^{(j)} \star p$ be the fusion of $v'^{(j)}$ and p , which is defined since the last letter of $v'^{(j)}$ and the first letter of p are the same. Define $\tilde{\mathbf{v}} = [v'^{(1)}, \dots, \widehat{v'^{(j)}}, \dots, v'^{(k)}, \tilde{v}]$ where the hat denotes omission. Now $\tilde{\mathbf{v}}$ satisfies all the conditions of \mathcal{S}_v except possibly condition (1). We produce \mathbf{v} from $\tilde{\mathbf{v}}$ by the following *shuffling* procedure. Suppose $\tilde{v} = \tilde{v}_1 \cdots \tilde{v}_s$ and let $t \in (1, s)$ be the maximal index (if it exists) such that $\tilde{v}_t \in x'$, say $\tilde{v}_t = x'_m$ where $m \in (j, k)$. We now define $v^{(k)} = \tilde{v}_t \tilde{v}_{t+1} \cdots \tilde{v}_s$ and replace $v'^{(m)}$ with $\tilde{v} = (\tilde{v}_1 \cdots \tilde{v}_{t-1} \tilde{v}_t) \star v'^{(m)}$. Now repeat the procedure with the new \tilde{v} , searching for some $m' \in (j, m)$ such that $\tilde{v}_{t'} = x'_{m'}$ for $t' \in (1, t-1)$. When no more shuffling occurs, we label the subwords $v^{(1)}, \dots, v^{(k)}$ in order. Note that the y'_r for $r \neq j$ are always kept in order. By construction $\mathbf{v} \in \mathcal{S}_v$ and we have $x = x'$ and $y = (y' \setminus \{y_j\}) \cup \{i\}$. We define $(u = u' \cup \{y_j\}) \subset v \setminus (x \cup y)$ and check that $u.y = u'.y'i$ and $u^\perp = (u')^\perp$. We now make the crucial observation: *shuffling is invertible if the letter $y_j \in v^{(k)}$ is given* – we will call this “performing inverse shuffling at y_j ”. This completes the definition of Φ .

We now define Φ^{-1} . Given $(\mathbf{v} = [v^{(1)}, \dots, v^{(r)}], u)$ we consider again two cases. If $v^{(r)} \cap u = \emptyset$ we proceed by defining $(\mathbf{v}' = [v'^{(1)}, \dots, v'^{(r-1)}], u' = u, p = v^{(r)})$. Otherwise, suppose $v^{(r)} = v_1^{(r)} \cdots v_s^{(r)}$ and let $t \in (1, s)$ be the maximal index such that $v_t^{(r)} \in u$. We define $p = v_t^{(r)} v_{t+1}^{(r)} \cdots v_s^{(r)}$, $u' = u \setminus \{v_t^{(r)}\}$ and to produce \mathbf{v}' we perform inverse shuffling at $v_t^{(r)}$. It is straightforward to show that this process well-defines a map that is inverse to Φ . \square

7.2. Proof of Theorem 5.5. Recall that by Theorem 5.1,

$$\mathbb{P}_r = A_{\rho_r} + \text{non-Grassmannian terms.}$$

By Theorems 4.6 and 5.1, in order to prove Theorem 5.5 it suffices to show that

$$\phi_0^{(2)}(\Delta(A_{\rho_r})) = 1 \otimes A_{\rho_r} + A_{\rho_r} \otimes 1 + 2 \sum_{1 \leq s < r} A_{\rho_s} \otimes A_{\rho_{r-s}} + \text{non-Grassmannian terms.}$$

We have used the fact that if w is not Grassmannian then any term $A_x \otimes A_y$ occurring in $\phi_0^{(2)}(\Delta(A_w))$ has either x or y non-Grassmannian.

We apply Proposition 7.1 to the case $w = \rho_r$ and v the unique reduced word of ρ_r , which by (1.1) is given by

$$v = \begin{cases} (r-1)(r-2)\cdots 1 0 & \text{for } 1 \leq r \leq n, \\ \overline{2n+1-r} \overline{2n+2-r} \cdots \overline{n-1} n n-1 \cdots 1 0 & \text{for } n < r \leq 2n, \end{cases}$$

where, by convention, if a letter occurs twice in ρ_r the left occurrence is distinguished by a bar.

The terms $1 \otimes A_{\rho_r} + A_{\rho_r} \otimes 1$ come from $\mathbf{v} = [] \in \mathcal{S}_v$ and $u = \emptyset$ and $u = v$ in (7.1). All other $u \subset v$ yield non-Grassmannian terms.

Now we calculate the coefficient of the term $A_{\rho_s} \otimes A_{\rho_{r-s}}$ for $s \geq 1$. Since the operation $\phi_0^{(2)} \circ \Delta$ is cocommutative, it suffices to consider the case $s \leq r - s$.

Define R to be the set of letters occurring in $\rho_r \rho_{r-s}^{-1}$ (together with the bars, if any). If $\overline{r-s-1} \in R$ (in particular $r-s-1 < n$) define \overline{R} to be R with $\overline{r-s-1}$ replaced by $r-s-1$; otherwise set $\overline{R} = R$. If $\overline{s-1} \in R$ define R_- to be R with $\overline{s-1}$ removed. If $\overline{r-s-1} \in R_-$ define $\overline{R_-}$ to be R_- with $\overline{r-s-1}$ replaced by $r-s-1$; otherwise set $\overline{R_-} = R_-$.

Lemma 7.3. *Suppose $s \leq r - s$. The terms in Proposition 7.1 which give $A_{\rho_s} \otimes A_{\rho_{r-s}}$ are exactly the following tuples $\mathbf{v} \in \mathcal{S}_v$ and $u \subset v \setminus (x \cup y)$:*

Case 1: $r \leq n$ or $r > n$ and $s \leq 2n + 1 - r$:

$y = s - 1 \ s - 2 \cdots 1 \ 0$, $x = x_1 x_2 \cdots x_s$ is a permutation of the letters in R or \overline{R} , and $u = \emptyset$;

Case 2: $r > n$, $s < r - s$ and $s > 2n + 1 - r$:

In addition to the possibilities in Case 1 we may also have $y = \overline{s-2} \cdots 1 \ 0$, $x = x_1 x_2 \cdots x_{s-1}$ is a permutation of the letters in R_- or $\overline{R_-}$, and $u = \overline{s-1}$;

Case 3: $r > n$, $s = r - s$ and $s > 2n + 1 - r$:

We have either $y = s - 1 \ s - 2 \cdots 1 \ 0$, $u = \emptyset$, and x is a permutation of R ; or $y = \overline{s-1} \ s - 2 \cdots 1 \ 0$, $u = \emptyset$, and x is a permutation of \overline{R} ; or $y = s - 2 \cdots 1 \ 0$, $u = s - 1$, $u^\perp = \overline{s-1}$, and x is a permutation of $R_- = \overline{R_-}$; or $y = \overline{s-2} \cdots 1 \ 0$, $u = \overline{s-1}$, $u^\perp = s - 1$, and x is a permutation of $R_- = \overline{R_-}$.

Proof. $u.y = \rho_s$ is embedded as a subword of ρ_r in two ways: $s - 1 \ s - 2 \cdots 1 \ 0$ or $\overline{s-1} \ s - 2 \cdots 1 \ 0$. Similarly if $r - s > n$ then $u^\perp.y = \rho_{r-s} \subset \rho_r$ and if $r - s \leq n$ then $u^\perp.y$ is either $r - s - 1 \ r - s - 2 \cdots 1 \ 0$ or $\overline{r-s-1} \ r - s - 2 \cdots 1 \ 0$.

Suppose $0 \leq p \leq s - 2$ is such that $p \notin y$. Then $p \in u$ and $p \notin u^\perp$, so that $p \notin u^\perp.y$, a contradiction. Therefore $y \supset s - 2 \cdots 1 \ 0$. This gives four cases.

(1) $u = \emptyset$ and $y = s - 1 \ s - 2 \cdots 1 \ 0$.

Here x can be a permutation of R or also of \overline{R} provided $\overline{r-s-1} \in R$.

(2) $u = \emptyset$ and $y = \overline{s-1} \ s - 2 \cdots 1 \ 0$.

Suppose $r - s > n$. Since $u^\perp.y$ contains $\overline{s-1}$ we have $2n + 1 - (r - s) < s$, giving the contradiction $2n + 1 < r$. Therefore $r - s \leq n$. Again since $u^\perp.y$ contains $\overline{s-1}$ we must have $r - s = s$ and $u^\perp = \emptyset$. Then x is a permutation of \overline{R} .

(3) $u = s - 1$ and $y = s - 2 \cdots 1 \ 0$.

$s - 1 \notin u^\perp.y$. This can only occur if $r - s = s$, $u^\perp.y = \overline{s-1} \ s - 2 \cdots 1 \ 0$, and $u^\perp = \overline{s-1}$. Then x is a permutation of $R_- = \overline{R_-}$.

(4) $u = \overline{s-1}$ and $y = s - 2 \cdots 1 \ 0$.

If $r - s > n$ then x is a permutation of R_- . Suppose $s < r - s \leq n$. If $u^\perp.y = r - s - 1 \ r - s - 2 \cdots 1 \ 0$ then x is a permutation of R_- and if $u^\perp.y = \overline{r-s-1} \ r - s - 2 \cdots 1 \ 0$ then x is a permutation of $\overline{R_-}$. If $s = r - s$ then $u^\perp = s - 1$ and x is a permutation of $R_- = \overline{R_-}$.

In particular the last three cases only occur if $\overline{s-1} \in \rho_r$, that is, $2n + 1 - r < s$. This given, the Lemma follows. \square

By Lemma 7.3 the possibilities for the tuples $\mathbf{v} \in \mathcal{S}_v$ which contribute to the coefficient of $A_{\rho_s} \otimes A_{\rho_{r-s}}$ in $\phi_0^{(2)}(\Delta(A_{\rho_r}))$ are determined by whether $u = \emptyset$, $u = s-1$ and $u = \overline{s-1}$. We denote the corresponding subsets of \mathcal{S}_{ρ_r} by \mathcal{S}_u . Note that $\mathcal{S}_{s-1} = \mathcal{S}_{\overline{s-1}}$. Define

$$T'_u = \sum_{\mathbf{v} \in \mathcal{S}_u} b_{\mathbf{v}}.$$

Proposition 7.4. *Depending on the case of Lemma 7.3, we have $T'_\emptyset = 2$ or $T'_\emptyset + T'_{s-1} = 2$ or $T'_\emptyset + T'_{s-1} + T'_{\overline{s-1}} = 2$.*

Proposition 7.4 shows that the coefficient of $A_{\rho_s} \otimes A_{\rho_{r-s}}$ in $\phi_0^{(2)}(\Delta(A_{\rho_r}))$ is equal to 2, thereby proving Theorem 5.5.

The proof of Proposition 7.4 is given in Section 7.9, after first preparing some technical preliminary results in Sections 7.3-7.8.

7.3. Notation. For the evaluation of T'_u we require slightly more general functions.

Let $\hat{x}I\hat{y}$ be an embedded subword of ρ_r with \hat{x} and \hat{y} subletters, and I a subword. Let $\mathcal{S}_I^{\hat{x}\hat{y}}$ be the set of all subwords p of ρ_r with first letter \hat{x} and last letter \hat{y} such that $p \cap I = \emptyset$. Then define

$$T_I^{\hat{x}\hat{y}} := \sum_{p \in \mathcal{S}_I^{\hat{x}\hat{y}}} b_p.$$

Given sequences $x = x_1x_2 \cdots x_k$ and $y = y_1y_2 \cdots y_k$ of subletters of ρ_r we define

$$T'(x, y) = \prod_{i=1}^k T_{\{x_1, \dots, x_{i-1}\} \cap (x_i, y_i)}^{x_i, y_i}$$

where (x_i, y_i) denotes the subword of ρ_r occurring between the letters x_i and y_i . Finally, let $T(x, y)$ be obtained from $T'(x, y)$ by ignoring the extra power of 2 (if any) which arises when $y_k = 0$.

Given an interval $[t, q]$ of unbarred letters and a set X of barred letters, let $f(t, q, X)$ denote the sum of $T(x, y)$ as x varies over all permutations of X and $y = q(q-1) \cdots t$. We always assume $|[t, q]| = |X|$, and write $k = |X|$. Given (t, q, X) , we partition X into subsets A, B, C where

- (1) $A \subset X$ consists of letters greater than q ,
- (2) $B \subset X$ is a subset of $[t, q]$,
- (3) $C \subset X$ consists of letters less than t .

Thus X is the disjoint union of A, B , and C . In the following we will write $X - a$ to mean the set $X - \{a\}$ with the element $a \in X$ removed. For a set S of (barred) integers let S^- denote S with its maximum element removed.

Now let us suppose that we are given a set X of barred and unbarred letters, such that X is the disjoint union of sets A, B, C, U , and U' , satisfying:

- (1) U' is a set of unbarred letters including n ,
- (2) A consists of some barred letters in X greater than q ,
- (3) B consists of the barred letters in X in the interval $[t, q]$,
- (4) C consists of the barred letters in X less than t ,
- (5) U consists of some barred letters in X greater than q ,
- (6) every letter in U or in U' is greater than every letter in A, B , and C ,
- (7) the minimum element of U is smaller than or equal to the minimum element of U' .

Denote by $g(t, q, X)$ the sum of $T(x, y)$ as x varies over all permutations of X such that all letters in U occur to the left of all letters in U' and $y = q(q-1) \cdots t$. Let us call X *balanced* if $\min(U') = \min(U)$. Note that if X is unbalanced it will stay unbalanced if $\min(U')$ is removed from it.

Let us now suppose that $y_1 = \bar{q}$ (instead of q), but the rest of y is as before. Denote the answers by $f'(t, q, X)$ and $g'(t, q, X)$ and use all the same conventions as before.

Results about these various functions are proven in Sections 7.4-7.7. We are ultimately interested in the case that X is one of $R, \bar{R}, R_-,$ or \bar{R}_- and y is one of the words described in Lemma 7.3. In Section 7.8 we explain how to construct the various subsets $A, B, C, U,$ and U' , before proving Proposition 7.4 in Section 7.9.

7.4. Results for $T_I^{\hat{x}\hat{y}}$. In this section we calculate $T_I^{\hat{x}\hat{y}}$, which gives the contribution from a single pair of letters \hat{x} and \hat{y} .

Lemma 7.5. *Let $\hat{x}I\hat{y}$ be an embedded subword of ρ_r with \hat{x} and \hat{y} subletters, I a subword, and $\hat{y} < n$. Then*

$$T_I^{\hat{x}\hat{y}} = \sum_{p \in \mathcal{S}_I^{\hat{x}\hat{y}}} b_p = -\chi(\hat{x} = \bar{\hat{y}}) + 2^{\chi(\hat{y}=0)} \begin{cases} 1 & \text{if } I = \emptyset \text{ or } \min(I) = \max(\hat{x}, \hat{y}) \\ -1 & \text{if } I = \{n\} \text{ or } \min(I) = \{i, \bar{i}\} \\ 0 & \text{else} \end{cases}$$

where $\min(I)$ consists of the 0, 1, or 2 smallest letters of I and in the comparison $\min(I) = \max(\hat{x}, \hat{y})$ we ignore bars.

Proof. Let $p = p_1 p_2 \cdots p_\ell \in \mathcal{S}_I^{\hat{x}\hat{y}}$ be such that $b_p \neq 0$. Then $p_i - p_{i+1} \in \{\pm 1, 0\}$ for all $1 \leq i < \ell$. Since $\bar{0}$ never occurs in ρ_r , $\hat{y} = 0$ and $\hat{x} = \bar{\hat{y}}$ cannot both hold. If $\hat{y} = 0$ then since $b_{10} = 2$, b_p contains a factor of two, giving rise to the overall factor $2^{\chi(\hat{y}=0)}$.

Since p is a subword of the unique reduced word of ρ_r and the latter word is a Λ (see subsection 6.1), p crosses at most once from barred to unbarred letters. Suppose $\bar{i}j$ are consecutive letters in p . Then $j \in \{i-1, i, i+1\}$ and the corresponding values of b_{ij} are 1, -2, 1 provided that $j \neq n$.

The contributions of the paths that only differ by $\dots \bar{i} i - 1 \dots, \dots \bar{i} i i - 1 \dots$ and $\dots \bar{i} i + 1 i i - 1 \dots$ all cancel out. For $\hat{y} > \hat{x}$, the contributions of the paths which differ in $\dots \hat{y} - 1 \hat{y} \dots, \dots \hat{y} - 1 \hat{y} \hat{y} \dots,$ and $\dots \hat{y} - 1 \hat{y} \hat{y} + 1 \hat{y} \dots$ all cancel out. For $\hat{x} = \hat{y}$ the paths $\hat{y} \hat{y}$ and $\hat{y} \hat{y} + 1 \hat{y}$ leave a net contribution of -1.

The case $\bar{i}j = \overline{n-1}n$ is special since $b_{n-1n} = 2$. Given the above discussion, it is tedious but not difficult to check all cases claimed in the lemma explicitly. \square

In the following sections, we will use Lemma 7.5 repeatedly without mention.

7.5. Results for $f(t, q, X)$: Barred letters only. In this section we derive results for $f(t, q, X)$, which is defined for a pair of unbarred letters t, q and a set X of barred letters. It follows from its definition that $f(t, q, X)$ only depends on B and the sizes $|A|$ and $|C|$. By definition $f(q, q, \emptyset) = 0$.

Let $\epsilon(n)$ denote the function which is 1 when n is even and 0 when n is odd.

Lemma 7.6. *If $|B| = 0$ then $f(t, q, X) = 1$.*

Proof. By Lemma 7.5 the only nonzero contributions to $f(t, q, X)$ occur when $I = \{x_1, \dots, x_{i-1}\} \cap (x_i, y_i) = \emptyset$ for all i . For this to hold for all $i \in [1, q-t+1]$, the

letters in $x = (x_1, x_2, \dots, x_{q-t+1})$ must occur in the same order as in ρ_r . This term gives contribution 1 to $f(t, q, X)$. \square

Lemma 7.7. *Suppose $X \neq \emptyset$. If $t \notin B$ then $f(t, q, X) = f(t+1, q, X^-)$.*

Lemma 7.8. *Suppose $|C| > 0$. Then $f(t, q, X) = f(t+1, q, X^-)$.*

Proof. By Lemma 7.7, this holds if $t \notin B$. Suppose $t \in B$. We have two cases $t = \max(X)$ or $t \neq \max(X)$. In the first case, a sequence x can only contribute to $f(t, q, X)$ if $x_k = \max(C)$. But $f(t+1, q, X - \max(C)) = f(t+1, q, X^-) = 1$ by Lemma 7.6. In the second case x_k may be $\max(X)$, t or $\max(C)$, all of which are distinct. Hence we have $f(t, q, X) = f(t+1, q, X^-) + f(t+1, q, X - \max(C)) - f(t+1, q, X - t)$ by Lemma 7.5. Since $f(t+1, q, X - t) = f(t+1, q, X - \max(C))$, we conclude that $f(t, q, X) = f(t+1, q, X^-)$. \square

Lemma 7.9. *Suppose $|C| > 0$. Then $f(t, q, X) = 1$.*

Proof. This follows from Lemmas 7.6 and 7.8 by induction. \square

Lemma 7.10. *Suppose $|C| = 0$. Then $f(t, q, X) = \epsilon(|B|)$.*

Proof. We proceed by induction on the size of X . Suppose $t \notin B$. Then $|A|$ must be non-empty. The inductive step holds by Lemma 7.7. Suppose $t \in B$. If $t = \max(X)$ then $|X| = 1$ and the result is trivial. Thus we may assume that $t \neq \max(X)$. Then $f(t, q, X) = f(t+1, q, X^-) - f(t+1, q, X - t)$. By Lemma 7.9, $f(t+1, q, X^-) = 1$. Also by the induction hypothesis $f(t+1, q, X - t) = \epsilon(|B| - 1)$. The result follows. \square

7.6. Results for $g(t, q, X)$: Unbarred letters as well. In this section we prove results for the function $g(t, q, X)$, where X is a set of barred and unbarred letters.

Lemma 7.11. *Suppose $|A| = |B| = |C| = 0$. Then $g(t, q, X) = 1$.*

Proof. Since by assumption all letters in U occur to the left of U' , again by Lemma 7.5 the only contribution to $g(t, q, X)$ occurs when all letters in x occur in the same order as in ρ_r . \square

Lemma 7.12. *Suppose X is unbalanced and $|C| > 0$. Then $g(t, q, X) = 1$.*

Proof. We proceed by induction on $|X|$. If $t \notin B$ then for a non-zero contribution we must have $x_k = \min(U')$. If $|U'| = 1$ the result follows from Lemma 7.9. Otherwise it is the inductive hypothesis. If $t \in B$ then x_k may be equal to $\max(C)$, t , or $\min(U')$, which are distinct. Hence we have $g(t, q, X) = g(t+1, q, X - \min(U')) + g(t+1, q, X - \max(C)) - g(t+1, q, X - t)$ by Lemma 7.5. Since $g(t+1, q, X - t) = g(t+1, q, X - \max(C))$, we conclude that $g(t, q, X) = g(t+1, q, X - \min(U'))$. The argument now proceeds as for $t \notin B$. \square

Lemma 7.13. *Suppose X is unbalanced and $|C| = 0$. Then $g(t, q, X) = \epsilon(|B|)$.*

Proof. We proceed by induction on $|X|$. If $t \notin B$ the argument is as for Lemma 7.12. If $t \in B$, we may have $x_k \in \{t, \min(U')\}$. Thus $g(t, q, X) = g(t+1, q, X - \min(U')) - g(t+1, q, X - t)$. By induction and Lemma 7.12 this is equal to $1 - \epsilon(|B| - 1)$, as required. \square

Lemma 7.14. *Suppose X is balanced and $|C| > 0$. Then $g(t, q, X) = \epsilon(|A| + |B| + |C|)$.*

Proof. We proceed by induction on $|X|$. Let $a = \max(A \cup B \cup C)$. If $t \notin B$ then $g(t, q, X) = g(t+1, q, X - \min(U')) - g(t+1, q, X - a)$. By Lemma 7.12, $g(t+1, q, X - \min(U')) = 1$. If $a \in C$ and $|C| = 1$ then the result follows from Lemma 7.11, otherwise it follows by induction. If $t \in B$ then $g(t, q, X) = g(t+1, q, X - \min(U')) - g(t+1, q, X - a) - g(t+1, q, X - t) + g(t+1, q, X - \max(C))$. Note that this formula holds even if $t = a$. Clearly, the last two terms cancel, so the result follows by induction and Lemma 7.12. \square

Let $\theta(n)$ be the function with values $1, -1, 2, -2, 3, -3, \dots$ on the nonnegative integers.

Lemma 7.15. *Suppose X is balanced, $|C| = 0 = |A|$, and $t \in B$. Then $g(t, q, X) = \theta(|B|)$.*

Proof. We proceed by induction on $|X|$. We have

$$g(t, q, X) = g(t+1, q, X - \min(U')) - g(t+1, q, X - t) - g(t+1, q, X - b)$$

where $b = \max(B)$. Again this holds even if $t = b$. By Lemma 7.12, $g(t+1, q, X - \min(U')) = 1$. The result follows from induction if $t = b$ since $g(t+1, q, X - t) = \theta(0) = 1$. Assume $t \neq b$. By Lemma 7.14, $g(t+1, q, X - b) = \epsilon(|B| - 1)$. By the inductive hypothesis, $g(t+1, q, X - t) = \theta(|B| - 1)$. But $\theta(|B|) = 1 - \theta(|B| - 1) - \epsilon(|B| - 1)$, proving the lemma. \square

Lemma 7.16. *Suppose X is balanced and $|C| = 0$. Furthermore suppose that either we have $|B| = 0$ or we have $B = [t', t'']$ for some $t' \geq t$ and $|A| \leq t' - t$. Then $g(t, q, X) = (-1)^{|A|} \theta(|B|) + \epsilon(|B|) \epsilon(|A| - 1)$.*

Proof. First suppose $|A| = 0$. If $|B| = 0$ then $g(t, q, X) = g(t+1, q, X - \min(U'))$ and we are done by Lemma 7.13. Otherwise let $t' = \min(B)$. If $t = t'$ we are done by Lemma 7.15. So let $t' > t$. We proceed by induction on $t' - t \geq 1$. Thus $t \notin B$ and $g(t, q, X) = g(t+1, q, X - \min(U')) - g(t+1, q, X - b)$ where $b = \max(B)$. By inductive hypothesis and Lemma 7.13, $g(t, q, X) = \epsilon(|B|) - \theta(|B| - 1) = \theta(|B|)$, as required.

Now suppose that $|A| > 0$. Then $t \notin B$ and by Lemma 7.13 and induction we have

$$\begin{aligned} g(t, q, X) &= g(t+1, q, X - \min(U')) - g(t+1, q, X - \max(A)) \\ &= \epsilon(|B|) - \left((-1)^{|A|-1} \theta(|B|) + \epsilon(|B|) \epsilon(|A| - 2) \right) \\ &= (-1)^{|A|} \theta(|B|) + \epsilon(|B|) \epsilon(|A| - 1) \end{aligned}$$

as required. \square

For the next lemma we assume that $\min(U) > \min(U')$. Suppose that $\min(U) \in U'$ (though one is barred and the other unbarred). Define $U'_- = \{u \in U' \mid u < \min(U)\}$.

Lemma 7.17. *If $|C| = 0$, $B = [t', t'']$ and $|U'_-| + |A| \leq t' - t$, then Lemma 7.16 holds as stated.*

Proof. The statements follow from the fact that a non-zero contribution only occurs with $x_k = \min(U'_-)$. When $|U'_-|$ becomes zero, X is balanced and we are in the situation of Lemma 7.16. \square

7.7. Results for $f'(t, q, X)$ and $g'(t, q, X)$: One barred letter in y . Let us now suppose that $y_1 = \bar{q}$ (instead of q), but the rest of y is as before. Denote the answers by $f'(t, q, X)$ and $g'(t, q, X)$ and use all the same conventions as before. We state the relevant results. The proofs are identical to before. Let $\epsilon'(n) = 1 - \epsilon(n)$ and $\theta'(n)$ be defined on nonnegative integers by the sequence $0, 1, -1, 2, -2, \dots$

Lemma 7.18. *Suppose $|C| > 0$. Then $f'(t, q, X) = 1$.*

Lemma 7.19. *Suppose $|C| = 0$. Then $f'(t, q, X) = \epsilon'(|B|)$.*

Lemma 7.20. *Suppose X is unbalanced and $|C| > 0$. Then $g'(t, q, X) = 1$.*

Lemma 7.21. *Suppose X is unbalanced and $|C| = 0$. Then $g'(t, q, X) = \epsilon'(|B|)$.*

Lemma 7.22. *Suppose X is balanced and $|C| > 0$. Then $g'(t, q, X) = \epsilon'(|A| + |B| + |C|)$.*

Lemma 7.23. *Suppose X is balanced and $|C| = 0 = |A|$. Then $g'(t, q, X) = \theta'(|B|)$.*

7.8. Defining U, U', A, B, C . Let us set X to be one of $R, \bar{R}, R_-, \bar{R}_-$ and pick y to be one of the words described in Lemma 7.3. Thus $t = 0$ and $q \in \{s-1, s-2\}$. We describe how to construct U, U', A, B , and C . First we must have $C = \emptyset$ and there is no choice for B and U' . We let A be the set of barred letters in X which are greater than q and less than all unbarred letters in X . We let U be any remaining barred letters in X .

For example, in Case 1 of 7.3 when $X = R$, $r > n$, and $2n + 1 - r \leq r - s$, we would have $B = \emptyset$ and

$$\begin{aligned} U' &= \{r-s, r-s+1, \dots, n\} \\ U &= \{\overline{r-s}, \overline{r-s+1}, \dots, \overline{n-1}\} \\ A &= \{\overline{r-s-1}, \overline{r-s-2}, \dots, \overline{2n+1-r}\}. \end{aligned}$$

We claim that the sum $g(t, q, X)$ of subsection 7.6 is equal to the same sum, but without the assumption that letters in U occur to the left of letters in U' .

This is proved as follows. In the case that $X \in \{R, \bar{R}, R_-, \bar{R}_-\}$, the letters in U are all present in U' . Let us take a permutation x of X such that $T(x, y) \neq 0$. Then it is clear that unbarred letters of X occur in x in the same order as in ρ_r . Suppose $x_i = n$ and there is a barred letter $\bar{j} \in U$ occurring to the right of x_i , say $x_p = \bar{j}$ where $p > i$. Then one checks that in x , (a) barred letters greater than \bar{j} occur before x_i , (b) letters in $\{n-1, \dots, j+1\}$ occur between x_i and x_p and (c) no unbarred letter less than j occurs between x_i and x_p . We now define a sign-reversing involution: let \tilde{x} be obtained from x by swapping the locations of j with \bar{j} . By Lemma 7.5, $T(x, y) = -T(\tilde{x}, y)$. In other words, to calculate the sum of $T(x, y)$ as x varies over all permutations of X we need only consider permutations x such that letters in U occur to the left of letters in U' .

7.9. Proof of Proposition 7.4.

7.9.1. Notation. Since $t = 0$ and $q \in \{s-1, s-2\}$ are known for the cases of Lemma 7.3, from now on we will denote the total contributions by $f(X), g(X), f'(X), g'(X)$, where $X \in \{R, \bar{R}, R_-, \bar{R}_-\}$ is one of the sets in Lemma 7.3. We prove Proposition 7.4 by splitting into the cases of Lemma 7.3. Note that the terms T'_u of Proposition 7.4 always differ from the corresponding numbers $f(X)$ and $g(X)$ by a factor of 2 (arising from $y_k = 0$).

7.9.2. *Case 1.* If $R = \overline{R}$ then $T'_\emptyset = 2f(R)$. The result follows from Lemma 7.17.

Suppose $R \neq \overline{R}$. By Lemma 7.17 we have $g(R) = (-1)^a + \epsilon(a-1)$, where $a = |A|$ for $X = R$. Similarly, we have $g(\overline{R}) = (-1)^{a-1} + \epsilon(a)$. Thus $g(R) + g(\overline{R}) = 1$, so $T'_\emptyset = 2$.

7.9.3. *Case 2.* If R consists only of barred letters, then we have $f(R) + f(R_-) = 1$ by Lemma 7.10.

Otherwise we need to calculate $g(R) + g(R_-) + g(\overline{R}) + g(\overline{R_-})$. Let a and b be the sizes of $|A|$ and $|B|$ when $X = R$. Then by Lemma 7.17

$$\begin{aligned} g(R) &= (-1)^a \theta(b) + \epsilon(b) \epsilon(a-1) \\ g(\overline{R}) &= (-1)^{a-1} \theta(b) + \epsilon(b) \epsilon(a) \\ g(R_-) &= (-1)^a \theta(b-1) + \epsilon(b-1) \epsilon(a-1) \\ g(\overline{R_-}) &= (-1)^{a-1} \theta(b-1) + \epsilon(b-1) \epsilon(a) \end{aligned}$$

and the sum is 1.

In both cases $T'_\emptyset + T'_{s-1} = 2$ as required.

7.9.4. *Case 3.* Let $b = 3s - 2n - 1$ be the size of B in R . Then $g(R) = \theta(b)$ and $g(R_-) = \theta(b-1)$ by Lemma 7.17. For the case $y = \overline{s-1s-2 \cdots 0}$ we have a contribution of $g'(\overline{R}) = \theta'(b-1)$ by Lemma 7.23. We calculate $T'_\emptyset + T'_{s-1} + T'_{s-1} = 2(\theta(b) + \theta'(b-1)) + 2\theta(b-1) + 2\theta(b-1) = 2$.

APPENDIX A. \mathbb{P}_i

In the data that follows, elements of \tilde{C}_n are indicated by reduced words. Some \mathbb{P}_i are expressed in the A_w basis of \mathbb{A}_0 .

A.1. $n = 2$.

$$\begin{aligned} \mathbb{P}_1 &= A_0 + A_1 + A_2 \\ \mathbb{P}_2 &= A_{01} + A_{10} + A_{12} + 2A_{20} + A_{21} \\ \mathbb{P}_3 &= A_{012} + A_{101} + A_{120} + A_{121} + A_{201} + A_{210} \\ \mathbb{P}_4 &= A_{0121} + A_{1012} + A_{1210} + A_{2101} \end{aligned}$$

A.2. $n = 3$.

$$\begin{aligned} \mathbb{P}_1 &= A_0 + A_1 + A_2 + A_3 \\ \mathbb{P}_2 &= A_{01} + A_{10} + A_{12} + 2A_{20} + A_{21} + A_{23} + 2A_{30} + 2A_{31} + A_{32} \\ \mathbb{P}_3 &= A_{012} + A_{101} + A_{120} + A_{121} + A_{123} + A_{201} + A_{210} + 2A_{230} \\ &\quad + A_{231} + A_{232} + 2A_{301} + 2A_{310} + A_{312} + 2A_{320} + A_{321} \\ \mathbb{P}_4 &= A_{0121} + A_{0123} + A_{1012} + A_{1210} + A_{1230} + A_{1231} \\ &\quad + A_{1232} + A_{2101} + A_{2301} + A_{2310} + 2A_{2320} + A_{2321} \\ &\quad + A_{3012} + 2A_{3101} + A_{3120} + A_{3121} + A_{3201} + A_{3210} \\ \mathbb{P}_5 &= A_{01231} + A_{01232} + A_{10123} + A_{12310} + A_{12320} + A_{12321} + A_{21012} \\ &\quad + A_{23101} + A_{23201} + A_{23210} + A_{30121} + A_{31012} + A_{31210} + A_{32101} \\ \mathbb{P}_6 &= A_{012321} + A_{101232} + A_{123210} + A_{210123} + A_{232101} + A_{321012} \end{aligned}$$

B.2. $n = 3$.

	1		2	11		3	21	111		4	31	22	211	1 ⁴
0	1	10	1	1	010		1	1	0210		1	2	2	2
					210	1	1	1	3210	1	1	1	1	1

	5	41	32	311	221	21 ³	1 ⁵
10210			1	1	2	2	2
03210		1	2	2	3	3	3
23210	1	1	1	1	1	1	1

	6	51	42	411	33	321	31 ³	222	2211	21 ⁴	1 ⁶
010210						1	1	2	2	2	2
103210			1	1	2	3	3	5	5	5	5
023210		1	2	2	2	3	3	4	4	4	4
123210	1	1	1	1	1	1	1	1	1	1	1

	61	52	511	43	421	41 ³	331	322	3211	31 ⁴	2 ³ 1	221 ³	21 ⁵	1 ⁷
0103210					1	1	2	4	4	4	7	7	7	7
2103210				1	1	1	2	3	3	3	5	5	5	5
1023210		1	1	1	2	2	2	3	3	3	4	4	4	4
0123210	1	1	1	1	1	1	1	1	1	1	1	1	1	1

APPENDIX C. $P_w^{(n)}$

In the following tables, for $w \in \tilde{C}_n^0$ and λ a strict partition, the (w, λ) -th entry is the coefficient of the Schur P -function P_λ in $P_w^{(n)}$. Again w is given as a reduced word.

C.1. $n = 2$.

	1		2		3	21		4	31		5	41	32
0	1	10	1	010		1	0210		1	10210		1	1
				210	1		1210	1		01210	1	1	

	6	51	42	321		7	61	52	43	421	
010210			1	1	1	0210210			1	1	1
210210				1		0101210		1	1		1
101210	1	2	1			2101210	1	2	2	1	

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