Log Structures on Generalized Semi-Stable Varieties

Ting Li
Directed by Professor Zhao Chunlai

Abstract

In this paper we study the log structures on generalized semistable varieties, generalize the result by F. Kato and M. Olsson, and prove the canonicity of log structure when it can be expected.

In out text we first give the definitions of local chart and weakly normal crossing morphism. Then we study the invariants of complete noetherian local ring coming from weakly normal crossing morphisms. These invariants enable us to further define the refined local charts and prove that all log structures induced by refined local charts are locally isomorphic. Let $f\colon X \to S$ be a surjective, proper and weakly normal crossing morphism of locally noetherian schemes which satisfies the conditions (†) and (‡) in 3.3 and certain local conditions stated at the beginning of 5. Then the obstructions for the existence of semistable log structures on X is an invertible sheaf $\mathscr{L}(f)$ on a finite X-scheme $E = E(f)$. The main result of local case with respect to base schemes is:

Theorem.

(1) There exists a semistable log structure on X if and only if $\mathscr{L}(f) \cong \mathcal{O}_E$.

(2) The semistable log structure on X is unique up to (not necessarily canonical) isomorphisms if it exists.

The main result of global case with respect to base schemes is:

Theorem. Let X and S be locally noetherian schemes, $f\colon X \to S$ a surjective proper weakly normal crossing morphism without powers. If f satisfies the condition (†) in 3.3 and every fiber of f is geometrically connected, then

(1) There exists a semistable log structure for f if and only if for every point $y \in S$, $\mathscr{L}_{\bar{y}}$ is trivial on $E_{\bar{y}}$.

(2) Let $(\mathscr{M}_1, \mathscr{N}_1, \sigma_1, \tau_1, \phi_1)$ and $(\mathscr{M}_2, \mathscr{N}_2, \sigma_2, \tau_2, \phi_2)$ be two semistable log structures for f. Then there exist isomorphisms of log structures $\varphi\colon \mathscr{M}_1 \xrightarrow{\sim} \mathscr{M}_2$ and $\psi\colon \mathscr{N}_1 \xrightarrow{\sim} \mathscr{N}_2$ such that $\varphi \circ \phi_1 = \phi_2 \circ f^*\psi$, $\sigma_2 \circ \varphi = \sigma_1$ and $\tau_2 \circ \psi = \tau_1$. Moreover such pair (φ, ψ) is unique.

We further prove that the existence of semistable log structures remains under fibred products, base extension, inverse limits, flat descent. Finally we study the semistable curves. The main result is:

Theorem. Any semistable curve over a locally noetherian scheme is a weakly normal crossing morphism without powers and has a canonical semistable log structure.

Key Words. Log structure, normal crossing singularity, semistable variety.

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Department of Mathematics, Peking University, Beijing 100871, P. R. China
EMail: moduli@live.cn

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INTRODUCTION

A major advantage of logarithmic geometry is that it enables us to treat some kind of singularity as smooth case. To achieve this, we must equip a singular morphism with suitable log structure so that it becomes *smooth* in the sense of logarithmic geometry.

In this paper, we study the existence and uniqueness of semistable log structures on the morphisms of schemes which locally have the form

$$\mathrm{Spec}\,\frac{A[T_{11},\ldots,T_{1q_1},\ldots,T_{p1},\ldots,T_{pq_p},T_1,\ldots,T_m]}{\Big(\prod_{j_1=1}^{q_1}T_{1j_1}^{e_{1j_1}}-a_1,\ldots,\prod_{j_p=1}^{q_p}T_{pj_p}^{e_{pj_p}}-a_p\Big)}\,.$$

If p and all e_{ij_i} are equal to 1, then the singularity is the so called *normal crossing* in the classical sense.

The study of normal crossing singularities began with Deligne and Mumford [2], where they showed that any curve with normal crossing singularities deforms to a smooth curve. For higher dimensional spaces, Friedman [4] discovered that an obstruction for the existence of smoothenings with regular total space is an invertible sheaf on the singular locus. In [9, §11-12] and [8], F. Kato introduced log structures for normal crossing varieties over fields. And in [16], M. Olsson generalizes them to morphisms $f\colon X\to S$, where X is locally isomorphic to

$$\mathrm{Spec}\,\mathcal{O}_S[T_1,\ldots,T_l]/(T_1\cdots T_r-t)\,,$$

with $t\in\Gamma(S,\mathcal{O}_S)$ a fixed section. Also in [10], F. Kato considered the existence of log structures on pointed stable curves.

In this paper we generalize the results in [8] and [16], mainly add nontrivial powers and remove the fixed section t in [16]. Roughly speaking, we construct an obstruction at every morphism $X\times_S\mathrm{Spec}\,\mathcal{O}_{S,\bar{y}}\to\mathrm{Spec}\,\mathcal{O}_{S,\bar{y}}$. Then we prove that the semistable log structure exists if and only if all these obstructions vanish (see Theorem 5.6 and 6.7). In the case of no power, we shall see that this kind of semistable log structures is canonical (i.e. unique up to a unique isomorphism), which was not discussed in [8] and [16].

In Section 1 we generalize the concept of normal crossing to the so called "weakly normal crossing". In Section 2 we study the invariants of complete local rings, which is of fundamental importance. In Section 3 we define the concept of refined local chart. On each refined local chart, we may define a log structure, which is the tile for building the global semistable log structures. In Section 4 we list some technique and notations in cohomology theory which are needed in later sections.

In Section 5 we study the local case. In other words, for a weakly normal crossing morphism $f\colon X\to S$, we focus on morphisms $X_V\to V$ for every étale neighborhood V on S which is *small enough*, especially the case when the base scheme S is the spectrum of a strictly Henselian local ring. For weakly normal crossing morphisms $f\colon X\to S$ with nontrivial power, the theory can only be built on local cases, because semistable log structures on $X_V\to V$ may not be unique (up to isomorphism). In Section 6 we prove that for a weakly normal crossing morphism without powers, the semistable log structures exist if and only if all local obstructions vanish. If so, then it must be canonical.

In Section 7 we study properties of weakly normal crossing morphisms under base change. We shall show that the semistable log structures constructed in §6 have good functorial properties. In Section 8 we show that on semistable curves, our constructed obstructions are always trivial. So there exists a canonical log structure on any semistable curve which make it log smooth.

Notation and Conventions. Throughout the paper, rings, algebras and monoids are all assumed to be commutative and have multiplicative identity elements. A homomorphism of rings (resp. monoids) is assumed to preserve identity element. A subring (resp. monoid) is assumed to contain the identity of the total ring (resp. monoid).

If n is a positive integer, we use S_n to denote the symmetric group on $\{1, 2, \ldots, n\}$.

For every pair of integers m and n, we define a set $[m, n]$ as

$$[m,n] \coloneqq \begin{cases} \{m, m+1, \ldots, n\}\,, & \text{if } m \leqslant n, \\ \varnothing\,, & \text{if } m > n. \end{cases}$$

For a field k, we use $\bar{k}$ to denote the algebraic closure of k and k_s the separable closure of k.

If X is a scheme, $f \in \Gamma(X, \mathcal{O}_X)$ is a section and $x \in X$ is a point, we use $f(x)$ to denote the image of the stalk f_x in the residue field $\kappa(x)$.

If X is a scheme, a geometric point on X is a morphism of schemes $\operatorname{Spec} K \to X$ where K is a separably closed field; if x is a point on X, we use $\bar{x}$ to denote the geometric point $\operatorname{Spec} \kappa(x)_s \to X$.

If S is a scheme, $f\colon X \to S$ and $T \to S$ are two S-schemes, then we define $X_T \coloneqq X \times_S T$ and let $f_T\colon X_T \to T$ denote the second projection.

For every morphism $f\colon X \to S$ of schemes, we use $X_{[f]}$ to denote the S-scheme X via f.

If X is a scheme and G a monoid (resp. abelian group), we use G_X to denote the constant sheaf of monoids (resp. of abelian groups) on X_{et} associated to G.

If $\mathscr{M}$ is a log structure on a scheme X, we write $\overline{\mathscr{M}} \coloneqq \mathscr{M}/\mathcal{O}_X^*$.

1. DEFINITION

Definition 1.1. Let $f\colon X \to S$ be a morphism of finite type of locally noetherian schemes, x a point on X and $y \coloneqq f(x)$. A *local chart* of f at x consists of the following data:

(1) an étale S-scheme $V = \operatorname{Spec} R$ which is a connected affine scheme;
(2) a point y' on V which maps onto y;
(3) an étale $V \times_S X$-scheme $U = \operatorname{Spec} A$ which is a connected affine scheme;
(4) a point x' on U which maps onto x and y';
(5) a finitely generated R-algebra P such that $\Omega_{P/R}$ is a free P-module, $\operatorname{Spec} P$ is connected and is smooth over V;
(6) a point $\mathfrak{p}$ in $\operatorname{Spec} P$ which maps onto y';
(7) a subset
$$\left\{\, T_{ij_i} \,\middle|\, i \in [1,p],\ j_i \in [1,q_i] \,\right\}$$
of P such that $T_{ij_i} \in \mathfrak{p}$ for all i and j_i, and
$$\left\{\, d_{P/R}(T_{ij_i}) \,\middle|\, i \in [1,p],\ j_i \in [1,q_i] \,\right\}$$
is a part of a basis of $\Omega_{P/R}$;
(8) a closed V-immersion $U \hookrightarrow \operatorname{Spec} P$ which maps x' onto $\mathfrak{p}$ and is defined by the ideal
$$\Big(\prod_{j_1=1}^{q_1} T_{1j_1}^{e_{1j_1}} - a_1,\ \prod_{j_2=1}^{q_2} T_{2j_2}^{e_{2j_2}} - a_2, \ldots, \prod_{j_p=1}^{q_p} T_{pj_p}^{e_{pj_p}} - a_p\Big)\,,$$
where $a_i \in R$ such that $a_i(y') = 0$, and $e_{ij_i} \geqslant 1$ are integers which are invertible in R, such that for every $i \in [1,p]$, $\sum_{j_i=1}^{q_i} e_{ij_i} > 1$ and
$$D_i(U/V) \coloneqq \operatorname{Spec}\Big(P\Big/ \sum_{j_i=1}^{q_i} P \cdot \big(T_{i1}^{e_{i1}} \cdots T_{i,j_i-1}^{e_{i,j_i-1}} T_{ij_i}^{e_{ij_i}-1} T_{i,j_i+1}^{e_{i,j_i+1}} \cdots T_{i,q_i}^{e_{i,q_i}}\big)\Big) \tag{1.1}$$
is connected.

We use

$$U \to \operatorname{Spec}\Big(P\Big/\Big(\dots,\prod_{j_i=1}^{q_i} T_{ij_i}^{e_{ij_i}} - a_i,\dots\Big)\Big) \tag{1.2}$$

or U/V, or simply U to denote the local chart.

Remark 1.2. Note that all these *connectedness* can be satisfied by contracting $\operatorname{Spec} P$, U and V suitably, so they are not essential restriction.

The following theorem shows that if a point has a local chart, then all points in some of its open neighborhood have local charts.

Theorem 1.3. *Let R be a noetherian ring, P a finitely generated R-algebra,*

$$\left\{ T_{ij_i} \,\middle|\, i \in [1,p],\ j_i \in [1,q_i] \right\}$$

a subset of P, $a_1, a_2, \dots, a_p \in R$,

$$\left\{ e_{ij_i} \,\middle|\, i \in [1,p],\ j_i \in [1,q_i] \right\}$$

a set of positive integers which are invertible in R. For each $i \in [1,p]$, put

$$b_i \coloneqq \prod_{j=1}^{q_i} T_{ij}^{e_{ij}} - a_i\,.$$

Put $A \coloneqq P/(b_1, b_2, \dots, b_p)$, $S \coloneqq \operatorname{Spec} R$ and $X \coloneqq \operatorname{Spec} A$. Assume that

(a) *P is smooth over R;*
(b) *$\Omega_{P/R}$ is a free P-module;*
(c) *$\left\{ d_{P/R}(T_{ij_i}) \,\middle|\, i \in [1,p],\ j_i \in [1,q_i] \right\}$ is a part of a basis of $\Omega_{P/R}$;*
(d) *for any $i \in [1,q_i]$, $\sum_{j_i=1}^{q_i} e_{ij_i} > 1$.*

Then we have

(1) *$b_1, b_2, \dots, b_p$ is a P-regular sequence.*
(2) *$X \to S$ is a flat and local complete intersection morphism.*
(3) *For every point x on X, there is a local chart at x.*

Proof. (1) and (2). Since P is smooth over R and $\{\cdots, d(T_{ij_i}), \cdots\}$ is a part of a basis of $\Omega_{P/R}$, $\{\dots, T_{ij_i}, \dots\}$ are algebraically independent over R and P is smooth over $R[\cdots, T_{ij_i}, \cdots]$. So we may assume that

$$P = R[\cdots, T_{ij_i}, \cdots]$$

is a polynomial algebra over R with indeterminates $\{\dots, T_{ij_i}, \dots\}$. Then (1) is by [11, (20.F), COROLLARY 2] and induction on p. So $X \to S$ is a local complete intersection morphism. By [12, Corollary of Theorem 22.5], X is flat over S.

(3) x defines a prime ideal $\mathfrak{P}$ of P. Put $\mathfrak{p} \coloneqq R \cap \mathfrak{P}$. Assume that $a_i \in \mathfrak{p}$ for $i \in [1,l]$ and $a_i \notin \mathfrak{p}$ for $i \in [l+1,p]$. And for each $i \in [1,l]$, we assume that $T_{ij_i} \in \mathfrak{P}$ for $j_i \in [1,s_i]$ and $T_{ij_i} \notin \mathfrak{P}$ for $j_i \in [s_i+1,q_i]$. Obviously, for all $i \in [1,l]$ we have $s_i \geqslant 1$. Assume that $\sum_{j=1}^{s_i} e_{ij} > 1$ when $i \in [1,r]$, and $\sum_{j=1}^{s_i} e_{ij} = 1$ when $i \in [r+1,l]$. By taking an affine open neighborhood of $\mathfrak{P}$ in $\operatorname{Spec} P$ and an affine open neighborhood of $\mathfrak{p}$ in $\operatorname{Spec} R$, we may assume that $a_i \in R^*$ whenever $i \in [l+1,p]$, and $T_{ij_i} \in P^*$ whenever

$$i \in [l+1,p] \vee \big(i \in [1,l] \wedge j \in [s_i+1,q_i]\big)$$

is valid. Then

$$P' \coloneqq P/(b_{r+1}, \dots, b_p)$$

is smooth over R. For each $i \in [1,r]$, since e_{i1} is invertible in $P^{\mathrm{sh}}_{\mathfrak{P}}$, there exists an element $u_i \in P^{\mathrm{sh}}_{\mathfrak{P}}$ such that

$$u_i^{e_{i1}} = \prod_{j_i=s_i+1}^{q_i} T_{ij_i}^{e_{ij_i}} \qquad (\text{if } s_i = q_i, \text{ we let } u_i = 1)\,.$$

By taking an affine étale neighborhood of $\mathfrak{P}$ in $\operatorname{Spec} P$, we may assume that $u_i \in P$ for all $i \in [1,r]$. For each $i \in [1,r]$, let T'_{i1} be the image of $u_i T_{i1}$ in P', and for each $j_i \in [2, s_i]$, T'_{ij_i} be the image of T_{ij_i} in P'. Then we have

$$A = P' \Big/ \Big(\prod_{j_1=1}^{s_1} (T'_{1j_1})^{e_{1j_1}} - a_1, \prod_{j_2=1}^{s_2} (T'_{2j_2})^{e_{2j_2}} - a_2, \dots, \prod_{j_r=1}^{s_r} (T'_{rj_r})^{e_{rj_r}} - a_r \Big)\,.$$

Moreover, P' is smooth over R and $\{\cdots, d(T'_{ij_i}), \cdots\}$ is a part of basis of $\Omega_{P'/R}$. $\square$

Definition 1.4. Let $f\colon X \to S$ be a morphism of locally noetherian schemes. We say that f is *weakly normal crossing* if it is of finite type, and for every point $x \in X$, either f is smooth at x or there exists a local chart at x.

A weakly normal crossing morphism $f\colon X \to S$ is said to be *without powers* if in every local chart of f as (1.2), all the powers e_{ij_i} are equal to 1.

By Theorem 1.3, if $f\colon X \to S$ is weakly normal crossing, then f is a flat and local complete intersection morphism.

The following lemma is obvious.

Theorem 1.5. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ \downarrow & \square & \downarrow \\ X & \xrightarrow[f]{} & S \end{array}$$

be a Cartesian square of locally noetherian schemes. If f is weakly normal crossing, so is f'.

2. Invariants of Complete Local Rings

In this section we study the invariants of complete noetherian local ring coming from weakly normal crossing morphisms, which ensure that all log structures induced by local charts are locally isomorphic.

Let R be a complete noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

Let P and Q be rings of power series over R in variables

$$\big\{\, X_{ij} \,\big|\, i \in [1,p],\ j \in [1,q_i] \,\big\} \bigcup \big\{ X_1, X_2, \dots, X_m \big\}$$

and

$$\big\{\, Y_{i'j'} \,\big|\, i' \in [1,p'],\ j' \in [1,q'_{i'}] \,\big\} \bigcup \big\{ Y_1, Y_2, \dots, Y_{m'} \big\}$$

respectively.

For each $i \in [1,p]$, let $e_{i1}, e_{i2}, \dots, e_{iq_i}$ be positive integers which are invertible in R with $\sum_{j=1}^{q_i} e_{ij} > 1$, a_i an element in $\mathfrak{m}$, and

$$F_i \coloneqq \prod_{j=1}^{q_i} X_{ij}^{e_{ij}} - a_i \in P\,.$$

For each $i' \in [1, p']$, let $e'_{i'1}, e'_{i'2}, \dots, e'_{i'q'_{i'}}$ be positive integers which are invertible in R with $\sum_{j'=1}^{q'_{i'}} e'_{i'j'} > 1$, $b_{i'}$ an element in $\mathfrak{m}$, and

$$G_{i'} \coloneqq \prod_{j'=1}^{q'_{i'}} Y_{i'j'}^{e'_{i'j'}} - b_{i'} \in Q\,.$$

Put

$$A \coloneqq P/(F_1, F_2, \dots, F_p) \qquad \text{and} \qquad B \coloneqq Q/(G_1, G_2, \dots, G_{p'})\,.$$

Let x_{ij}, x_k and $y_{i'j'}$, $y_{k'}$ be the images of X_{ij}, X_k and $Y_{i'j'}$, $Y_{k'}$ in A and B respectively. Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be the maximal ideals of A and B, $\mathfrak{N}_1$ and $\mathfrak{N}_2$ the nilradicals of A and B.

The following theorem is the main result of this section.

Theorem 2.1. *Let $\varphi\colon A \xrightarrow{\sim} B$ be an isomorphism of R-algebras. Then $p = p'$, $m = m'$; and there exists a $\sigma \in S_p$ such that for each $i \in [1, p]$, we have*

(1) $q_i = q'_{\sigma(i)}$,
(2) $a_i = u_i b_{\sigma(i)}$ *for some* $u_i \in R^*$,
(3) *there exists a* $\tau_i \in S_{q_i}$ *such that for each* $j \in [1, q_i]$, *we have* $e_{ij} = e'_{\sigma(i),\tau_i(j)}$ *and* $\varphi(x_{ij}) = v_{ij} y_{\sigma(i),\tau_i(j)}$ *for some* $v_{ij} \in B^*$.

To prove Theorem 2.1, we note the following simple fact.

Lemma 2.2. *Every element in A can be uniquely written as a power series:*

$$\sum c(\dots, \alpha_{ij}, \dots; \dots, \beta_k, \dots) \cdot \left(\prod_{i=1}^{p} \prod_{j=1}^{q_i} x_{ij}^{\alpha_{ij}} \right) \cdot \left(\prod_{k=1}^{m} x_k^{\beta_k} \right)\,, \tag{2.1}$$

where α_{ij}, β_k are in $\mathbb{N}$, $c(\cdots)$ are in R satisfying the following conditions: for every $i \in [1, p]$, there exists a $j \in [1, q_i]$ such that $\alpha_{ij} < e_{ij}$. (So we may talk monomials *and* coefficients *etc.) Furthermore, if*

$$a_1 = a_2 = \cdots = a_p = 0\,,$$

then $A = \bigoplus A_n$ is a graded ring, where A_n consists of homogeneous polynomials of degree n.

We first prove Theorem 2.1 in a simple but fundamental case.

Lemma 2.3. *If $R = K$ is a field, then Theorem* 2.1 *is valid.*

Proof. Without lose of generality, we may assume that

$$q_i \begin{cases} > 1\,, & i \in [1, r]\,, \\ = 1\,, & i \in [r+1, p]\,; \end{cases} \qquad \text{and} \qquad q'_{i'} \begin{cases} > 1\,, & i' \in [1, r']\,, \\ = 1\,, & i' \in [r'+1, p']\,, \end{cases}$$

for some $r \in [0, p]$ and $r' \in [0, p']$. Firstly it easy to see that

$$\mathfrak{N}_1 = \left(\prod_{j=1}^{q_1} x_{1j}, \prod_{j=1}^{q_2} x_{2j}, \dots, \prod_{j=1}^{q_p} x_{pj} \right)\,,$$

$$\mathfrak{N}_2 = \left(\prod_{j'=1}^{q'_1} y_{1j'}, \prod_{j'=1}^{q'_2} y_{2j'}, \dots, \prod_{j'=1}^{q'_{p'}} y_{p'j'} \right)\,.$$

Note that φ induces an isomorphism of vector spaces over K:

$$\bar{\varphi}\colon \mathfrak{N}_1/(\mathfrak{N}_1 \cap \mathfrak{M}_1^2) \xrightarrow{\sim} \mathfrak{N}_2/(\mathfrak{N}_2 \cap \mathfrak{M}_2^2)\,.$$

As $\mathfrak{N}_1/(\mathfrak{N}_1 \cap \mathfrak{M}_1^2)$ has a base $\bar{x}_{r+1,1}, \ldots, \bar{x}_{p1}$ and $\mathfrak{N}_2/(\mathfrak{N}_2 \cap \mathfrak{M}_2^2)$ has a base $\bar{y}_{r'+1,1}, \ldots, \bar{y}_{p'1}$, we have

$$p - r = p' - r' = f \tag{2.2}$$

and there is a $D = (d_{i'i}) \in \mathrm{GL}_f(K)$ such that

$$\big(\bar{\varphi}(\bar{x}_{r+1,1}), \ldots, \bar{\varphi}(\bar{x}_{p1})\big) = \big(\bar{y}_{r'+1,1}, \ldots, \bar{y}_{p'1}\big) \cdot D\,.$$

For each $i \in [r+1, p]$, put

$$g_i \coloneqq \max\left\{ e'_{i'1} \,\middle|\, i' \in [r'+1, p'] \text{ such that } d_{i'-r',i-r} \neq 0 \right\} \tag{2.3}$$

and let $\sigma(i)$ be the smallest number i' in $[r'+1, p']$ such that $d_{i'-r',i-r} \neq 0$ and $e'_{i'1} = g_i$. Then for every $i \in [r+1, p]$, we may write $\varphi(x_i)$ as

$$\varphi(x_{i1}) = v_i y_{\sigma(i),1} + w_i\,,$$

where if we write v_i and w_i as the form (2.1), then the constant term of v_i is

$$d_{\sigma(i)-r',i-r} \in K^* \qquad (\text{so } v_i \in B^*)$$

and w_i does not contain $y_{\sigma(i),1}$. For any $h \in [0, e'_{\sigma(i),1} - 1]$, by considering the coefficient of $y^h_{\sigma(i),1}$ in $(v_i y_{\sigma(i),1} + w_i)^h$, we know that $\varphi(x_{i1})^h \neq 0$. Hence $e_{i1} \geqslant e'_{\sigma(i),1}$.

Suppose that $w_i \neq 0$. We write w_i as

$$w_i = c_{i1}L_{i1} + c_{i2}L_{i2} + \cdots + c_{il_i}L_{il_i} + H_i\,,$$

where $L_{i1}, L_{i2}, \ldots, L_{il_i}$ are monic monomials occurred in w_i with lowest degree n ($\geqslant 1$), $c_{i1}, c_{i2}, \ldots, c_{il_i}$ are nonzero elements in K, and H_i is the sum of monomials of degree greater than n in w_i. The coefficient of $y_{\sigma(i),1}^{e'_{\sigma(i),1}-1} L_{i1}$ in $(v_i y_{\sigma(i),1} + w_i)^{e'_{\sigma(i),1}}$ is equal to

$$e'_{\sigma(i),1} \cdot d_{\sigma(i)-r',i-r} \cdot c_{i1} \neq 0\,.$$

So $\varphi(x_{i1})^{e'_{\sigma(i),1}} \neq 0$. Hence $e_{i1} > e'_{\sigma(i),1}$.

As D is an invertible matrix over K, there exists a $\sigma' \in S_f$ such that

$$d_{\sigma'(1),1}, d_{\sigma'(2),2}, \ldots, d_{\sigma'(f),r}$$

are all nonzero. By (2.3), for every $i \in [r+1, p]$ we have

$$e'_{\sigma'(i-r)+r',1} \leqslant g_i = e'_{\sigma(i),1} \leqslant e_{i1}\,.$$

Thus

$$\sum_{i'=r'+1}^{p'} e'_{i'1} = \sum_{i=r+1}^{p} e'_{\sigma'(i-r)+r',1} \leqslant \sum_{i=r+1}^{p} e_{i1}\,.$$

Applying above analysis to the homomorphism $\varphi^{-1}\colon B \to A$, we have

$$\sum_{i=r+1}^{p} e_{i1} \leqslant \sum_{i'=r'+1}^{p'} e'_{i'1}\,.$$

Hence for each $i \in [r+1, p]$, we have

$$e'_{\sigma'(i-r)+r',1} = e'_{\sigma(i),1} = e_{i1}$$

and $w_i = 0$; and $\sigma(i) = \sigma'(i-r) + r'$ is a bijective from $[r+1, p]$ to $[r'+1, p']$.

In the following we prove that $p = p'$ and extend σ to an element in S_p. Put

$$J \coloneqq [1, q_1] \times [1, q_2] \times \cdots \times [1, q_p]\,,$$
$$J' \coloneqq [1, q'_1] \times [1, q'_2] \times \cdots \times [1, q'_{p'}]\,.$$

For each $j. = (j_1, j_2, \dots, j_p) \in J$, put

$$\mathfrak{a}_{j.} = \mathfrak{a}_{j_1, j_2, \dots, j_p} \coloneqq \left(x_{1j_1}^{e_{1j_1}}, x_{2j_2}^{e_{2j_2}}, \dots, x_{pj_p}^{e_{pj_p}}\right),$$
$$\mathfrak{p}_{j.} = \mathfrak{p}_{j_1, j_2, \dots, j_p} \coloneqq \left(x_{1j_1}, x_{2j_2}, \dots, x_{pj_p}\right);$$

and for each $j'_. = (j'_1, j'_2, \dots, j'_{p'}) \in J'$, put

$$\mathfrak{b}_{j'_.} = \mathfrak{b}_{j'_1, j'_2, \dots, j'_{p'}} \coloneqq \left(y_{1j'_1}^{e'_{1j'_1}}, y_{2j'_2}^{e'_{2j'_2}}, \dots, y_{p'j'_{p'}}^{e'_{p'j'_{p'}}}\right),$$
$$\mathfrak{q}_{j'_.} = \mathfrak{q}_{j'_1, j'_2, \dots, j'_{p'}} \coloneqq \left(y_{1j'_1}, y_{2j'_2}, \dots, y_{p'j'_{p'}}\right).$$

Then $\mathfrak{p}_{j.} = \sqrt{\mathfrak{a}_{j.}}$ is a prime ideal of A and $\mathfrak{q}_{j'_.} = \sqrt{\mathfrak{b}_{j'_.}}$ is a prime ideal of B. Moreover

$$(0) = \bigcap_{j. \in J} \mathfrak{a}_{j.} \qquad \text{and} \qquad (0) = \bigcap_{j'_. \in J'} \mathfrak{b}_{j'_.}$$

are the primary decompositions of $(0) \subseteq A$ and $(0) \subseteq B$ respectively. Note that for every $j. \in J$,

$$\dim(A/\mathfrak{p}_{j.}) = \sum_{i=1}^{p} q_i - p + m\,, \tag{2.4}$$

which does not depend on $j.$. So all $\mathfrak{p}_{j.}$ are isolated prime ideals belonging to (0). Similarly we have

$$\dim(B/\mathfrak{q}_{j'_.}) = \sum_{i'=1}^{p'} q'_{i'} - p' + m'\,, \tag{2.5}$$

and all $\mathfrak{q}_{j'_.}$ are isolated prime ideals belonging to (0). By the uniqueness of primary decomposition of ideals, there is a bijective $\alpha\colon J \to J'$ such that for every $j. \in J$, $\varphi(\mathfrak{a}_{j.}) = \mathfrak{b}_{\alpha(j.)}$ and $\varphi(\mathfrak{p}_{j.}) = \mathfrak{q}_{\alpha(j.)}$. By (2.4) and (2.5), we have

$$\sum_{i=1}^{p} q_i - p + m = \dim(A/\mathfrak{p}_{j.}) = \dim(B/\mathfrak{q}_{\alpha(j.)}) = \sum_{i'=1}^{p'} q'_{i'} - p' + m'\,. \tag{2.6}$$

Note that φ induces an isomorphism of rings:

$$A\Big/\sum_{j. \in J} \mathfrak{p}_{j.} \xrightarrow{\sim} B\Big/\sum_{j'_. \in J'} \mathfrak{q}_{j'_.}\,.$$

By comparing the dimensions of both sides, we get $m = m'$. For any $j. = (j_1, j_2, \dots, j_p)$, $l. = (l_1, l_2, \dots, l_p) \in J$, put

$$\mathfrak{d}(j., l.) \coloneqq \#\{\, i \in [1, p] \mid j_i \neq l_i \,\};$$

and for each $j'_. = (j'_1, j'_2, \dots, j'_{p'})$, $l'_. = (l'_1, l'_2, \dots, l'_{p'}) \in J'$, put

$$\mathfrak{d}'(j'_., l'_.) \coloneqq \#\{\, i' \in [1, p'] \mid j'_{i'} \neq l'_{i'} \,\}.$$

For each $j., l. \in J$, we have

$$\begin{aligned}\sum_{i=1}^{p} q_i - p + m - \mathfrak{d}\big(j., l.\big) &= \dim A/(\mathfrak{p}_{j.} + \mathfrak{p}_{l.}) \\ &= \dim B/(\mathfrak{q}_{\alpha(j.)} + \mathfrak{q}_{\alpha(l.)}) \\ &= \sum_{i'=1}^{p'} q'_{i'} - p' + m - \mathfrak{d}'\big(\alpha(j.), \alpha(l.)\big)\,.\end{aligned}$$

By (2.6), we have

$$\mathfrak{d}\big(j.,l.\big)=\mathfrak{d}'\big(\alpha(j.),\alpha(l.)\big)\,. \tag{2.7}$$

So we get

$$\begin{aligned} r&=\mathfrak{d}\big((1,\dots,1),(2,\dots,2,1,\dots,1)\big)\\ &=\mathfrak{d}'\big(\alpha(1,\dots,1),\alpha(2,\dots,2,1,\dots,1)\big)\\ &\leqslant r'\,. \end{aligned}$$

Applying above argument to α^{-1}, we get $r'\leqslant r$ and hence $r=r'$. By (2.2) we obtain $p=p'$.

For each $j.\in J$, we put

$$\alpha(j.)=\big(\alpha_1(j.),\alpha_2(j.),\dots,\alpha_p(j.)\big)\,.$$

For each $h\in[1,r]$, let s and t be two different numbers in $[1,q_h]$, and

$$(j_1,\dots,\hat{j}_h,\dots,j_p)\in[1,q_1]\times\cdots\times[1,q_{h-1}]\times[1,q_{h+1}]\times\cdots\times[1,q_p]\,.$$

By (2.7), there is a unique integer

$$\sigma=\sigma(h,s,t;j_1,\dots,\hat{j}_h,\dots,j_p)\in[1,r]$$

such that

$$\alpha_\sigma(j_1,\dots,j_{h-1},s,j_{h+1},\dots,j_p)\neq\alpha_\sigma(j_1,\dots,j_{h-1},t,j_{h+1},\dots,j_p)\,,$$

and for all $l\in[1,p]-\{\sigma\}$,

$$\alpha_l(j_1,\dots,j_{h-1},s,j_{h+1},\dots,j_p)=\alpha_l(j_1,\dots,j_{h-1},t,j_{h+1},\dots,j_p)\,.$$

First we prove that $\sigma(h,s,t;j_1,\dots,\hat{j}_h,\dots,j_p)$ does not depend on $j_1,\dots,\hat{j}_h,\dots,j_p$. For simplicity we assume that $h=1$ and $\gamma,\delta\in[1,q_2]$ are two different numbers. Put

$$\begin{aligned} n_1&\coloneqq\sigma(1,s,t;\gamma,j_3,\dots,j_p)\,,\\ n_2&\coloneqq\sigma(1,s,t;\delta,j_3,\dots,j_p)\,. \end{aligned}$$

Suppose that $n_1\neq n_2$. Then we have

$$\begin{aligned} \alpha_{n_2}(s,\gamma,j_3,\dots,j_p)&=\alpha_{n_2}(t,\gamma,j_3,\dots,j_p)\,,\\ \alpha_{n_2}(s,\delta,j_3,\dots,j_p)&\neq\alpha_{n_2}(t,\delta,j_3,\dots,j_p)\,. \end{aligned}$$

So either

$$\alpha_{n_2}(s,\gamma,j_3,\dots,j_p)\neq\alpha_{n_2}(s,\delta,j_3,\dots,j_p)\,, \tag{2.8}$$

or

$$\alpha_{n_2}(t,\gamma,j_3,\dots,j_p)\neq\alpha_{n_2}(t,\delta,j_3,\dots,j_p)\,. \tag{2.9}$$

Assume that (2.8) is valid, then

$$\sigma(2,\gamma,\delta;s,j_3,\dots,j_p)=n_2\,.$$

Thus for all $l\in[1,p]-\{n_2\}$, we have

$$\alpha_l(s,\gamma,j_3,\dots,j_p)=\alpha_l(s,\delta,j_3,\dots,j_p)=\alpha_l(t,\delta,j_3,\dots,j_p)\,,$$

i.e.,

$$\mathfrak{d}'\big(\alpha(s,\gamma,j_3,\dots,j_p),\alpha(t,\delta,j_3,\dots,j_p)\big)\leqslant 1\,,$$

which contradicts to (2.7). Similarly the validity of (2.9) leads to a contradiction. Hence $n_1=n_2$. So $\sigma(h,s,t;j_1,\dots,\hat{j}_h,\dots,j_p)$ depends only on h,s,t; thus we may write it as $\sigma(h,s,t)$. Clearly $\sigma(h,s,t)=\sigma(h,t,s)$.

Second we prove that $\sigma(h,s,t)$ does not depend on s and t. We also assume that $h=1$ and let s_1,s_2,s_3 be three different numbers in $[1,q_h]$. Put $n_1 \coloneqq \sigma(1,s_1,s_2)$ and $n_2 \coloneqq \sigma(1,s_1,s_3)$. Suppose that $n_1 \neq n_2$. Since

$$\alpha_{n_1}(s_3,1,\dots,1)=\alpha_{n_1}(s_1,1,\dots,1)\neq \alpha_{n_1}(s_2,1,\dots,1)\,,$$

we have $\sigma(1,s_2,s_3)=n_1$. So

$$\alpha_{n_2}(s_3,1,\dots,1)=\alpha_{n_2}(s_2,1,\dots,1)=\alpha_{n_2}(s_1,1,\dots,1)\,,$$

which contradicts to the fact that $\sigma(1,s_1,s_3)=n_2$. Thus $\sigma(h,s,t)$ depends only on h, so we may write it as $\sigma(h)$.

We shall prove that $\sigma\colon [1,r]\to[1,r]$ is injective. Suppose that $\sigma(1)=\sigma(2)=n$. Let

$$1\leqslant s_1<s_2\leqslant q_1,\ 1\leqslant t_1<t_2\leqslant q_2, 1\leqslant j_3\leqslant q_3,\dots,\ 1\leqslant j_p\leqslant q_p$$

be integers. Then for any $l\neq n$,

$$\alpha_l(s_1,t_1,j_3,\dots,j_p)=\alpha_l(s_2,t_1,j_3,\dots,j_p)=\alpha_l(s_2,t_2,j_3,\dots,j_p)\,.$$

Thus

$$\mathfrak{d}'\big(\alpha(s_1,t_1,j_3,\dots,j_p),\alpha(s_2,t_2,j_3,\dots,j_p)\big)\leqslant 1\,,$$

which contradicts to (2.7).

Therefore we obtain an element $\sigma\in S_p$.

From above discussion we see that for any

$$(h;j_1,\dots,\hat{j}_h,\dots,j_p)\in[1,r]\times[1,q_1]\times\cdots\times[1,q_{h-1}]\times[1,q_{h+1}]\times\cdots\times[1,q_p]\,,$$

there exists an injective map

$$\tau_h(j_1,\dots,\hat{j}_h,\dots,j_p)\colon [1,q_h]\to[1,q'_{\sigma(h)}]$$

such that for all $s\in[1,q_h]$,

$$\alpha_{\sigma(h)}(j_1,\dots,j_{h-1},s,j_{h+1},\dots,j_h)=\tau_h(j_1,\dots,\hat{j}_h,\dots,j_p)(s)\,,$$

and for all $l\in[1,p]-\{\sigma(h)\}$, the value of $\alpha_l(j_1,\dots,j_{h-1},s,j_{h+1},\dots,j_h)$ does not depend on s. Now we prove that $\tau_h(j_1,\dots,\hat{j}_h,\dots,j_p)$ does not depend on $j_1,\dots,\hat{j}_h,\dots,j_p$. We assume that $h=1$ and $t_1,t_2\in[1,q_2]$ are two different numbers. Suppose that there is an $s\in[1,q_1]$ such that

$$n_1\coloneqq\tau_1(t_1,j_3,\dots,j_p)(s)\neq n_2\coloneqq\tau_1(t_2,j_3,\dots,j_p)(s)\,.$$

Then

$$\alpha_{\sigma(1)}(s,t_1,j_2,\dots,j_p)=n_1\neq n_2=\alpha_{\sigma(1)}(s,t_2,j_2,\dots,j_p)\,.$$

Thus $\sigma(2)=\sigma(1)$, which contradicts to the injectivity of σ. So we have a well-defined injection $\tau_h\colon[1,q_h]\to[1,q'_{\sigma(h)}]$. Hence $q_h\leqslant q'_{\sigma(h)}$. So we get

$$\sum_{i=1}^{p}q_i\leqslant\sum_{i=1}^{p}q'_{\sigma(i)}=\sum_{i'=1}^{p}q'_{i'}\,.$$

Applying above argument to φ^{-1}, we obtain $\sum\limits_{i'=1}^{p}q'_{i'}\leqslant\sum\limits_{i=1}^{p}q_i$. Hence for every $i\in[1,p]$, we have $q_i=q'_{\sigma(i)}$ and τ_h is a bijective.

Put

$$\tau_{r+1}=\cdots=\tau_p=\mathrm{id}\colon\{1\}\to\{1\}\,.$$

Then for every $(j_1,j_2,\dots,j_p)\in J$, we have

$$\alpha(j_1,j_2,\dots,j_p)=\big(\tau_{\sigma^{-1}(1)}(j_{\sigma^{-1}(1)}),\tau_{\sigma^{-1}(2)}(j_{\sigma^{-1}(2)}),\dots,\tau_{\sigma^{-1}(p)}(j_{\sigma^{-1}(p)})\big)\,.$$

In other words,

$$\varphi(\mathfrak{a}_{j_1,j_2,\dots,j_p}) = \mathfrak{b}_{\tau_{\sigma^{-1}(1)}(j_{\sigma^{-1}(1)}),\tau_{\sigma^{-1}(2)}(j_{\sigma^{-1}(2)}),\dots,\tau_{\sigma^{-1}(p)}(j_{\sigma^{-1}(p)})} \tag{2.10}$$

and

$$\varphi(\mathfrak{p}_{j_1,j_2,\dots,j_p}) = \mathfrak{q}_{\tau_{\sigma^{-1}(1)}(j_{\sigma^{-1}(1)}),\tau_{\sigma^{-1}(2)}(j_{\sigma^{-1}(2)}),\dots,\tau_{\sigma^{-1}(p)}(j_{\sigma^{-1}(p)})} \,. \tag{2.11}$$

Let $h \in [1, r]$ and $s \in [1, q_h]$. Put

$$x \coloneqq x_{hs}\,, \qquad y \coloneqq y_{\sigma(h),\tau_h(s)}\,, \qquad e \coloneqq e_{hs}\,, \qquad e' \coloneqq e'_{\sigma(h),\tau_h(s)}$$

for shortness. Then we have

$$(x^e) = \bigcap_{\substack{j_1\in[1,q_1],\dots,j_{h-1}\in[1,q_{h-1}],\\ j_{h+1}\in[1,q_{h+1}],\dots,j_p\in[1,q_p]}} \mathfrak{a}_{j_1,\dots,j_{h-1},s,j_{h+1},\dots,j_p}\,,$$

By (2.10), we have $\varphi\big((x^e)\big) = \big(y^{e'}\big)$. So $\varphi(x^e) = uy^{e'}$ for some $u \in B^*$. Note that $x^e \in \mathfrak{M}_1^e - \mathfrak{M}_1^{e+1}$ and $uy^{e'} \in \mathfrak{M}_2^{e'} - \mathfrak{M}_2^{e'+1}$. So $e = e'$. On the other hand, we have

$$(x) + \mathfrak{N}_1 = \bigcap_{\substack{j_1\in[1,q_1],\dots,j_{h-1}\in[1,q_{h-1}],\\ j_{h+1}\in[1,q_{h+1}],\dots,j_p\in[1,q_p]}} \mathfrak{p}_{j_1,\dots,j_{h-1},s,j_{h+1},\dots,j_p}\,,$$

By (2.11), we have

$$\varphi\big((x) + \mathfrak{N}_1\big) = (y) + \mathfrak{N}_2\,.$$

So $\varphi(x) = vy + w$ for some $v \in B$ and $w \in \mathfrak{N}_2$. We write v and w as the form (2.1) and assume that w does not contain y. Suppose that $v \in \mathfrak{M}_2$. Then $\varphi(x) \in \mathfrak{M}_2^2 + \mathfrak{N}_2$. So $x \in \mathfrak{M}_1^2 + \mathfrak{N}_1$, a contradiction. Thus $v \in B^*$, i.e., the constant term c_0 of v is nonzero.

Suppose that $w \neq 0$. We write w as

$$w = c_1L_1 + c_2L_2 + \cdots + c_sL_s + H\,,$$

where $L_1, L_2, \dots, L_s$ are monic monomials occurred in w with lowest degree n ($\geqslant 1$), $c_1, c_2, \dots, c_s$ are nonzero elements in K, and H is the sum of monomials of degree greater that n in w. Note that

$$uy^e = \varphi(x^e) = \varphi(x)^e = (vy + w)^e\,.$$

By Comparing the coefficients of $y^{e-1}L_1$ in the above equality, we get $0 = ec_0c_1$, a contradiction. So $w = 0$, i.e., $x = vy$. □

Proof of Theorem 2.1. By Lemma 2.3, $p = p'$, $m = m'$, and for every $i \in [1, p]$ and $j \in [1, q_i]$,

$$\varphi(x_{ij}) = u_{ij}y_{ij} + w_{ij}$$

for some $u_{ij} \in B^*$ and $w_{ij} \in \mathfrak{m}B$. (Here to without loss of generality, we assume that $\sigma, \tau_1, \tau_2, \dots, \tau_p$ are identities.) We express u_{ij} and w_{ij} in the form of (2.1) and assume that w_{ij} does not contain y_{ij}. For every integer $h \geqslant 1$, put $\mathfrak{a}_{ih} \coloneqq \mathfrak{m}^h + (b_i)$. Assume that we have proved that $a_i \in \mathfrak{a}_{ih}$ and $w_{ij} \in \mathfrak{a}_{ih}B$. Then we have

$$\begin{aligned} a_i &= \varphi\Big(\prod_{j=1}^{q_i} x_{ij}^{e_{ij}}\Big) = \prod_{j=1}^{q_i}(u_{ij}y_{ij} + w_{ij})^{e_{ij}} \\ &\equiv \sum_{j=1}^{q_i} e_{ij}u'_{ij}y_{i1}^{e_{ij}} \cdots y_{i,j-1}^{e_{i,j-1}} y_{ij}^{e_{ij}-1} y_{i,j+1}^{e_{i,j+1}} \cdots y_{i,q_i}^{e_{i,q_i}} w_{ij} \pmod{\mathfrak{a}_{i,h+1}B}\,, \end{aligned} \tag{2.12}$$

where $u'_{ij} \in B^*$. Now we apply Lemma 2.2 to the $R/\mathfrak{a}_{i,h+1}$-algebra $B/\mathfrak{a}_{i,h+1}B$. By comparing the constant terms in (2.12), we have $a_i \in \mathfrak{a}_{i,h+1}$. Suppose that $w_{ij} \notin \mathfrak{a}_{i,h+1}B$. Then we have

$$w_{ij} \equiv \sum_{l=1}^{s_{ij}} c_{ijl} L_{ijl} + H_{ij} \pmod{\mathfrak{a}_{i,h+1}B},$$

where $c_{ij1}, c_{ij2}, \dots, c_{ij,s_{ij}} \in \mathfrak{a}_{ih} - \mathfrak{a}_{i,h+1}$, $L_{ij1}, L_{ij2}, \dots, L_{ij,s_{ij}}$ are different monoic monomials in w_i with lowest degree t_{ij}, and H_{ij} are sums of monomials of degree greater t_{ij} in w_i. By comparing the coefficients of the term

$$y_{i1}^{e_{ij}} \cdots y_{i,j-1}^{e_{i,j-1}} y_{ij}^{e_{ij}-1} y_{i,j+1}^{e_{i,j+1}} \cdots y_{i,q_i}^{e_{i,q_i}} L_{ij1}$$

in (2.12), we get a contradiction. So we have

$$a_i \in \bigcap_{h=1}^{\infty} \big(\mathfrak{m}^h + (b_i)\big) = (b_i)$$

and

$$w_{ij} \in \bigcap_{h=1}^{\infty} \big(\mathfrak{m}^h B + B b_i\big) = B b_i \,.$$

The same reasoning for φ^{-1} shows that $b_i \in (a_i)$. So $a_i = u_i b_i$ for some $u_i \in R^*$. Put $w_{ij} \coloneqq w'_{ij} b_i$ and

$$v_{ij} \coloneqq u_{ij} + y_{i1}^{e_{ij}} \cdots y_{i,j-1}^{e_{i,j-1}} y_{ij}^{e_{ij}-1} y_{i,j+1}^{e_{i,j+1}} \cdots y_{i,q_i}^{e_{i,q_i}} w'_{ij} \,.$$

Then $v_{ij} \in B^*$ and $\varphi(x_{ij}) = v_{ij} y_{ij}$. This complete the proof of Theorem 2.1. □

The following Theorem is easy to prove.

Theorem 2.4. *For each $i \in [1,p]$, let $\mathfrak{a}_i$ denote the kernel of multiplication by a_i on R; and for each $j \in [1,q_i]$, let $\mathfrak{I}_{ij}$ denote the kernel of multiplication by x_{ij} on A. Then*

(1) *for each $i \in [1,p]$ and $j \in [1,q_i]$, we have*

$$\mathfrak{I}_{ij} = \mathfrak{a}_i \cdot \big(x_{i1}^{e_{i1}} \cdots x_{i,j-1}^{e_{i,j-1}} x_{ij}^{e_{ij}-1} x_{i,j+1}^{e_{i,j+1}} \cdots x_{iq_i}^{e_{iq_i}}\big)\,;$$

(2) *for each $i \in [1,p]$, the canonical homomorphism of A-modules*

$$\bigoplus_{j=1}^{q_i} \mathfrak{I}_{ij} \to \sum_{j=1}^{q_i} \mathfrak{I}_{ij}$$

is an isomorphism.

3. Refined Local Charts

In this section we define the concept of refined local chart, which is more delicate than local charts. Log structures induced by refined local charts are all locally isomorphic. But it is not true for local charts. Also we introduce two assumptions on which the main results of this paper is built.

Let $f\colon X \to Y$ be a surjective, proper and weakly normal crossing morphism of locally noetherian schemes.

3.1. **The singular locus.**

Lemma 3.1. *Let $\varphi\colon A \to B$ be a flat local homomorphism of noetherian local rings, x and y two nonzero elements in A. If there is a $v \in B^*$ such that $\varphi(y) = v\varphi(x)$, then there exists a $u \in A^*$ such that $y = ux$.*

Proof. Since A and B are local rings and φ is flat, we see that φ is faithfully flat. So $xA = \varphi^{-1}(xB)$ and $yA = \varphi^{-1}(yB)$. Thus $xA = yA$ if and only if $xB = yB$. □

Lemma 3.2. *Let $x \in X$, $y \coloneqq f(x)$,*

$$U \to \operatorname{Spec} P\Big/\Big(\prod_{j_1=1}^{q_1} T_{1j_1}^{e_{1j_1}} - a_1,\ \prod_{j_2=1}^{q_2} T_{2j_2}^{e_{2j_2}} - a_2, \dots,\ \prod_{j_p=1}^{q_p} T_{pj_p}^{e_{pj_p}} - a_p\Big)$$

and

$$U' \to \operatorname{Spec} P'\Big/\Big(\prod_{j_1'=1}^{q_1'} (T'_{1j_1'})^{e'_{1j_1'}} - a_1',\ \prod_{j_2'=1}^{q_2'} (T'_{2j_2'})^{e'_{2j_2'}} - a_2', \dots,\ \prod_{j_{p'}'=1}^{q_{p'}'} (T'_{p'j_{p'}'})^{e'_{p'j_{p'}'}} - a_{p'}'\Big)$$

be two local charts of f at x. Let t_{ij_i} and $t'_{i'j_{i'}'}$ be the image of T_{ij_i} and $T'_{i'j_{i'}'}$ in $\mathcal{O}_{X,\bar{x}}$ respectively. Then $p = p'$ and there exists a $\sigma \in S_p$ such that for each $i \in [1, p]$, we have

1. $q_i = q'_{\sigma(i)}$,
2. $a_i = u_i a'_{\sigma(i)}$ *for some* $u_i \in \mathcal{O}^*_{Y,\bar{y}}$,
3. *there exists a* $\tau_i \in S_{q_i}$ *such that for each* $j \in [1, q_i]$, $e_{ij} = e'_{\sigma(i),\tau_i(j)}$ *and* $t_{ij} = v_{ij} t'_{\sigma(i),\tau_i(j)}$ *for some* $v_{ij} \in \mathcal{O}^*_{X,\bar{x}}$.

Proof. We use notations in Definition 1.1. Let $x' \in U$ and $x'' \in U'$ be the points as in Definition 1.1 (4) and $y' \in V$ the point as in Definition 1.1 (2). Put $U'' \coloneqq U \times_X U'$. Then there is a point $x_0 \in U''$ which maps onto both x' and x''. Let x_1' be a closed point in $\overline{\{x_0\}} \subseteq U''$ and let x_1 be the image of x_1' on X. Then x_1 is a closed point in $\overline{\{x\}}$. So by considering the cospecialization map $\mathcal{O}_{X,\bar{x}_1} \to \mathcal{O}_{X,\bar{x}}$, we may assume that $x = x_1$ is a closed point. Then $\kappa(x)/\kappa(y)$ is a finite extension of fields. By [6, Ch. 0, (10.3.1)], there is a complete noetherian local ring R' whose residue field is algebraically closed and a flat local homomorphism $\mathcal{O}_{S,\bar{y}} \to R'$. By taking base extension $\operatorname{Spec} R' \to S$ and applying Lemma 3.1, we may assume that $\kappa(y)$ is algebraic closed. As x is a closed point, $\kappa(x) = \kappa(y)$ is algebraic closed. Thus $\kappa(x) = \kappa(x')$. Let $\mathfrak{M}$ and $\mathfrak{m}$ be the maximal ideals of $\mathcal{O}_{\operatorname{Spec} P, x'}$ and $\mathcal{O}_{V,y'}$ respectively. There there is a canonical isomorphism:

$$L \coloneqq \mathfrak{M}/(\mathfrak{M}^2 + \mathfrak{m}P) \xrightarrow{\sim} \Omega_{P/R} \otimes_P P/\mathfrak{M}\,.$$

As $\dots, \overline{T}_{ij_i}, \dots \in L$ are linearly independent over $\kappa(x')$, we may select $T_1, T_2, \dots, T_n \in \mathfrak{M}$ such that $\{\dots, \overline{T}_{ij_i}, \dots, \overline{T}_k, \dots\}$ is a basis of L. By taking a connected affine open neighborhood of x' in $\operatorname{Spec} P$, we may assume that $\{\dots, d(T_{ij_i}), \dots, d(T_k), \dots\}$ is a basis of $\Omega_{P/R}$. Then $\{\dots, T_{ij_i}, \dots, T_k, \dots\}$ is algebraically independent over R, and P is étale over $R[\dots, T_{ij_i}, \dots, T_k, \dots]$. So we have an isomorphism of $\widehat{\mathcal{O}}_{S,\bar{y}}$-algebras:

$$\widehat{\mathcal{O}}_{X,\bar{x}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{S,\bar{y}}[[\dots, T_{ij_i}, \dots, T_k, \dots]]\Big/\Big(\dots, \prod_{j_i=1}^{q_i} T_{ij_i}^{e_{ij_i}} - a_i, \dots\Big)\,.$$

Similarly we have

$$\widehat{\mathcal{O}}_{X,\bar{x}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{S,\bar{y}}[[\dots, T'_{i'j_{i'}'}, \dots, T'_{k'}, \dots]]\Big/\Big(\dots, \prod_{j_{i'}'=1}^{q_{i'}'} (T'_{i'j_{i'}'})^{e'_{i'j_{i'}'}} - a'_{i'}, \dots\Big)\,.$$

So the lemma is valid by Theorem 2.1. □

Let x be a point on X equipped with a local chart of the form (1.2). For each $i \in [1, p]$, let $\mathscr{I}_i$ be the ideal of $\mathcal{O}_U$ generated by

$$\left\{ T_{i1}^{e_{i1}} \cdots T_{i,j-1}^{e_{i,j-1}} T_{ij}^{e_{ij}-1} T_{i,j+1}^{e_{i,j+1}} \cdots T_{i,q_i}^{e_{i,q_i}} \,\middle|\, j \in [1, q_i] \right\} . \tag{3.1}$$

By Lemma 3.2, we see that

$$\big(\mathscr{I}_1, \mathscr{I}_2, \ldots, \mathscr{I}_p\big)$$

are independent of the choice of local charts (up to a unique permutation of the subscripts in S_p). From (1.1), we know

$$D_i(U/V) = \operatorname{Spec}\big(\mathcal{O}_U/\mathscr{I}_i\big)$$

for each $i \in [1, p]$. So for U/V we have a finite V-morphism

$$D_{U/V} \coloneqq \coprod_{i=1}^{p} D_i(U/V) \to U .$$

Clearly

$$\left\{ D_{U/V} \to U \,\middle|\, U/V \text{ is a local chart for } f \right\}$$

can be glued to a global finite S-morphism $g\colon D \to X$. To consider the properties under base extension, we also use $D(f)$ or $D(X/S)$ to denote the scheme D for preciseness.

Obviously the set-theoretic image of the finite morphism $D(f) \to X$ is the set of all points at which f are not smooth.

Clearly we have

Theorem 3.3. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ \downarrow & \square & \downarrow \\ X & \xrightarrow[f]{} & S \end{array}$$

be a Cartesian square of locally noetherian schemes. Then we have

$$D(f') = D(f) \times_S S' .$$

For a point $y \in Y$, we define

$$D_{\bar{y}} \coloneqq D\big((X \times_S \operatorname{Spec} \mathcal{O}_{Y,\bar{y}})/\operatorname{Spec} \mathcal{O}_{Y,\bar{y}}\big) = D(X/S) \times_S \operatorname{Spec} \mathcal{O}_{Y,\bar{y}} ,$$

and let $\mathrm{CP}(y)$ denote the set of connected components of $D_{\bar{y}}$.

3.2. **Reduced to local cases.** In this subsection, we assume that $S = \operatorname{Spec} R$, where R is a strictly Henselian noetherian local ring, and y_1 is the closed point of S.

Lemma 3.4. *Let T be the spectrum of a Henselian local ring, t the closed point of T, Y a connected scheme, $g\colon Y \to T$ a étale morphism, y a point on Y such that $g(y) = t$ and $\kappa(t) \to \kappa(y)$ is isomorphic. Then $g\colon Y \to T$ is an isomorphism.*

Proof. See [7], (18.5.11) a) $\Longrightarrow$ c) and (18.5.18). □

Lemma 3.5. *Let T be the spectrum of a Henselian local ring, t the closed point of T, $Y \to T$ a proper morphism. Then $Z \mapsto Z_t$ defines a bijection from the set of connected components of Y to the set of connected components of Y_t.*

Proof. See [7, (18.5.19)]. □

By Lemma 3.4, if $x \in f^{-1}(y_1)$ and U/V is a local chart of f at x, then $V = S$. By Lemma 3.2, there is a canonical map

$$\omega\colon \mathrm{CP}(y_1) \to R/R^* \tag{3.2}$$

(here "/" means taking quotient of monoids) such that if $x \in f^{-1}(y_1)$, U a local chart of the form (1.2) at x, $i \in [1, p]$, and $C \in \mathrm{CP}(y_1)$ is the connected component of $D(X/S)$ which contains the image of $D_i(U/S)$ on $D(X/S)$, then

$$\overline{a_i} = \omega(C)\,.$$

Now we consider the following conditions.

($*$) For each point $x \in f^{-1}(y_1)$ and for each local chart U at x, the images of $D_1(U/S), D_2(U/S), \dots, D_p(U/S)$ in $D(X/S)$ are contained in different connected components of $D(X/S)$.

Lemma 3.6. *If* ($*$) *holds, then every connected components of $D(X/S)$ is a closed subscheme of X.*

3.3. **Construction of refined local chart.** We return to the general case that S is only a locally noetherian scheme. For each point $y \in S$, we use

$$\omega_{\bar{y}}\colon \mathrm{CP}(y) \to \mathcal{O}_{S,\bar{y}}/\mathcal{O}^*_{S,\bar{y}}$$

to denote the map as in (3.2).

In the following, we require that $f\colon X \to S$ satisfies the following conditions.

(†) For each point $y \in S$, $X \times_S \operatorname{Spec} \mathcal{O}_{S,\bar{y}} \to \operatorname{Spec} \mathcal{O}_{S,\bar{y}}$ satisfies the condition ($*$).

Lemma 3.7. *Let y be a point on S. Fix an open affine neighborhood W_y of y. We define a full subcategory $N(y)$ of the category of étale neighborhoods of $\bar{y}$ as follows: for an étale neighborhood V of $\bar{y}$, $V \in N(y)$ if and only if it satisfies the following conditions:*

(a) *the image of V in S is contained in W_y;*
(b) *the inverse image of y in V contains only one point $y' \in V$;*
(c) *V is an affine scheme and every irreducible component of V contains y';*
(d) *for every irreducible component F of D_V, the image of F on V contains y'.*

Then we have

(1) *For any pair of objects V and V' in $N(y)$, there exists at most one morphism from V' to V;*
(2) *$N(y)$ is a* local base *of $\bar{y}$, i.e., for every étale neighborhood V of $\bar{y}$, there is an object V' in $N(y)$ and a morphism $V' \to V$ of étale neighborhoods of $\bar{y}$.*
(3) *for any morphism $V' \to V$ in $N(y)$, $D_{V'} \to D_V$ is dominant.*

Proof. (1) is by [3, I 5.4].

(2) For every étale neighborhood V of $\bar{y}$, we may contract V under the Zariski topology to obtain an object in $N(y)$.

(3) For each object V in $N(y)$, since the image of $\operatorname{Spec} \mathcal{O}_{S,\bar{y}} \to V$ contains all generizations of y' in V, the morphism $D_{\bar{y}} \to D_V$ is dominant. So for any morphism $V' \to V$ in $N(y)$, $D_{V'} \to D_V$ is dominant. □

Remark 3.8. By (1), we may define a partial order on $N(y)$ as follows. For any pair of objects V and V' in $N(y)$, $V' \geqslant V$ if and only if there is a morphism from V' to V in $N(y)$. Obviously $N(y)$ is directly ordered, i.e., for any pair of objects V and V' in $N(y)$, there exists an object V'' in $N(y)$ such that $V'' \geqslant V$ and $V'' \geqslant V'$.

Let y be a point on S. For each object V in $N(y)$, let $q_V\colon D_{\bar{y}} \to D_V$ be the canonical morphism. By [7, (8.4.2)], there exists an object V_0 in $N(y)$ such that for all $V \geqslant V_0$, $F \mapsto \overline{q_V(F)}$ defines a bijection between $\mathrm{CP}(y)$ and the set of connected components of D_V. By the definition of $N(y)$, the inverse image of y in V_0 contains only one point y'. For every closed point x in $f_{V_0}^{-1}(y')$, if f_{V_0} is smooth at x, we select open neighborhoods U_x of x and V_x of y respectively such that $f_{V_0}(U_x) \subseteq V_x$ and $f_{V_0}\colon U_x \to V_x$ is smooth; otherwise we select a local chart U_x/V_x at x. For every U_x, let U'_x be its image in X_{V_0}. As

$$f_{V_0}^{-1}(y') \subseteq \bigcup U'_x$$

and $f_{V_0}^{-1}(y')$ is quasi-compact, there exists a finite number of closed points $x_1, x_2, \ldots, x_m$ in $f_{V_0}^{-1}(y')$ such that

$$f_{V_0}^{-1}(y') \subseteq \bigcup_{i=1}^{m} U'_{x_i} \, .$$

Put

$$V' := V_0 - f_{V_0}\left(X_{V_0} - \bigcup_{i=1}^{m} U'_{x_i} \right) .$$

As $f_{V_0}\colon X_{V_0} \to V_0$ is proper, V' is an open neighborhood of y'. There exists an object V_1 in $N(y)$ and a morphism of étale neighborhoods of $\bar{y}'$,

$$V_1 \to V' \times_{V_0} V_{x_1} \times_{V_0} \cdots \times_{V_0} V_{x_m} \, .$$

Let $C_1, C_2, \ldots, C_n$ be all connected components of $D_{\bar{y}}$. For each $i \in [1, n]$ and each $V \in N(y)$ with $V \geqslant V_1$, C_i defines a connected component $C_i(V)$ of D_V. By Lemma 3.2, there exists an element

$$b_i(V) \in \Gamma(V, \mathcal{O}_S/\mathcal{O}_S^*)$$

such that for every point $z \in V$ and for every connected component F of $D_{\bar{z}}$ which maps into $C_i(V)$,

$$\omega_{\bar{z}}(F) = b_i(V)_{\bar{z}} \, .$$

Obviously $b_i(V)$ depends only on y, C_i and V. Let $Z_i(V)$ be the closed subscheme of V defined by the ideal generated by $b_i(V)$. Clearly the inverse image of y' ($\in V_0$) in V is contained in all these subschemes $Z_i(V)$.

The following lemma can be directly verified.

Lemma 3.9. *Let $V' \geqslant V$ ($\geqslant V_1$) be two elements in $N(y)$. Then $C_i(V') = C_i(V) \times_V V'$ and $Z_i(V') = Z_i(V) \times_V V'$.*

Lemma 3.10. *$C_i(V) \to V$ factors through $Z_i(V)$ and $C_i(V)$ is faithfully flat over $Z_i(V)$.*

Proof. It is by the following lemma. □

Lemma 3.11. *Let R be a ring, $A = R[T_1, T_2, \ldots, T_n]$ a polynomial ring over R, $e_1, e_2, \ldots, e_n$ positive integers. Then*

$$B := A/(T_1^{e_1-1}T_2^{e_2}\cdots T_n^{e_n}, T_1^{e_1}T_2^{e_2-1}\cdots T_n^{e_n}, \ldots, T_1^{e_1}T_2^{e_2}\cdots T_n^{e_n-1})$$

is flat over R.

Proof. Note that B is a free R-module with basis

$$\left\{ T_1^{i_1}T_2^{i_2}\cdots T_n^{i_n} \,\middle|\, \begin{array}{l}\text{either there exists an integer } k \in [1,n] \text{ such that } i_k < e_k - 1 \text{ or} \\ \text{there exist at least two integers } k \in [1,n] \text{ such that } i_k \leqslant e_k - 1\end{array} \right\} .$$ □

Notation 3.12. We defines a subset $N_0(y)$ of $N(y)$ as follows: for $V \in N(y)$, $V \in N_0(y)$ if and only if it satisfies that

(1) $V \geqslant V_1$.
(2) For each $i \in [1, n]$, there exists a section $a_i \in \Gamma(V, \mathcal{O}_V)$ such that

$$a_i \equiv b_i(V) \pmod{\mathcal{O}_V^*} .$$

(3) If z is the inverse image of y in V, then for any $i \in [1, n]$ and any irreducible component F of $Z_i(V)$, F contains z.

It is easy to show that for each $V \in N(y)$, there exists an object V' in $N_0(y)$ such that $V' \geqslant V$; and we have the following easy lemma.

Lemma 3.13. *Let z be a generization of y, $u\colon \mathcal{O}_{S,\bar{y}} \to \mathcal{O}_{S,\bar{z}}$ a cospecialization map, and $v\colon D_{\bar{z}} \to D_{\bar{y}}$ the morphism induced by u.*

(1) *For each i, $v^{-1}(C_i) = \varnothing$ if and only if $\bar{u}\big(\omega_{\bar{y}}(C_i)\big) = \bar{1}$ in $\mathcal{O}_{S,\bar{z}}/\mathcal{O}^*_{S,\bar{z}}$.*
(2) *If $v^{-1}(C_i) \neq \varnothing$, then every connected component of $v^{-1}(C_i)$ is a connected component of $D_{\bar{z}}$.*
(3) *All connected components of $D_{\bar{z}}$ can be obtained as in* (2)*.*

We consider the following conditions about f:

(‡) Let y_1 be a point on S, y_0 a generization of y_1, $u\colon \mathcal{O}_{S,\bar{y}_1} \to \mathcal{O}_{S,\bar{y}_0}$ a cospecialization map, and $v\colon D_{\bar{y}_0} \to D_{\bar{y}_1}$ the morphism induced by u. Then for every connected component C of $D_{\bar{y}_1}$, $v^{-1}(C)$ is connected. (Here empty set is also regarded to be connected.)

Lemma 3.14. (‡) *is satisfied if one of the following conditions holds:*

(1) *S is a spectrum of a field.*
(2) *There exists a finite set L of closed points in S such that f is smooth outside L.*
(3) *S is a spectrum of a discrete valuative ring and f is smooth at the generic fiber.*
(4) *f is a weakly normal crossing morphism without powers.*

Proof. (1), (2) and (3) are trivial.

(4) Obviously for each $y \in S$, X_y is geometrically reduced over $\kappa(y)$. Let C be a connected component of $D_{\bar{y}_1}$ such that $v^{-1}(C) \neq \varnothing$. Fix an object $V \in N_0(y_1)$. Let y'_1 be the inverse image of y_1 in V. Then the cospecialization map $u\colon \mathcal{O}_{S,\bar{y}_1} \to \mathcal{O}_{S,\bar{y}_0}$ defines a point y'_0 on V which maps to y_0 and is a generization of y'_1. C defines a connected component $\bar{C}$ of D_V. By Lemma 3.10, there is a closed subscheme Z of V such that $\bar{C}$ factors through Z and $\bar{C}$ is proper and flat over Z. As $y'_1 \in Z$ and $v^{-1}(C) \neq \varnothing$, by Lemma 3.13 $y'_0 \in Z$. Obviously $\bar{C}_{y'_1}$ is geometrically connected and geometrically reduced over $\kappa(y'_1)$. Hence

$$\dim_{\kappa(y'_1)} \mathit{\Gamma}\big(\bar{C}_{y'_1}, \mathcal{O}_{\bar{C}_{y'_1}}\big) = 1\,.$$

By [6, (7.7.5)],

$$\big\{\, z \in V \;\big|\; \dim_{\kappa(z)} \mathit{\Gamma}\big(\bar{C}_z, \mathcal{O}_{\bar{C}_z}\big) \leqslant 1 \,\big\}$$

is an open neighborhood of y'_1. As y'_0 is a generization of y'_1,

$$\dim_{\kappa(y_0)} \mathit{\Gamma}\big(\bar{C}_{y_0}, \mathcal{O}_{\bar{C}_{y_0}}\big) \leqslant 1\,.$$

So $\bar{C}_{y_0}$ is geometrically connected. By Lemma 3.5, $v^{-1}(C)$ is connected. □

Lemma 3.15. *Assume that f satisfies the condition* (‡)*. Let $y \in S$, $V \in N_0(y)$, $C_1, C_2, \dots, C_n$ the connected components of $D_{\bar{y}}$. For each $i \in [1,n]$, let $\bar{C}_i$ be the connected component of D_V defined by C_i and let $a_i \in \mathit{\Gamma}(V, \mathcal{O}_S)$ be a representative element of $b_i(V)$. Let z' be a point on V and z its image on S. Then*

(1) *Let $i \in [1,n]$ such that $(a_i)_{z'} \in \mathfrak{m}_{V,z'}$. Then $\bar{C}_i \times_V \operatorname{Spec} \mathcal{O}_{S,\bar{z}}$ is a connected component of $D_{\bar{z}}$ and its image under the map $\omega_{\bar{z}}$ is equal to $\overline{(a_i)_{z'}}$.*
(2) *$\big\{\, \bar{C}_i \times_V \operatorname{Spec} \mathcal{O}_{S,\bar{z}} \;\big|\; i \in [1,n]$ and $(a_i)_{z'} \in \mathfrak{m}_{V,z'} \,\big\}$ is the set of all connected components of $D_{\bar{z}}$.*

Proof. (1) Obviously $z' \in Z_i(V)$. Let F be an irreducible component of $Z_i(V)$ containing z'. Let w' be the generic point of F and w its image on S. Suppose that $\bar{C}_i \times_V \operatorname{Spec} \mathcal{O}_{S,\bar{z}}$ is disconnected. Then by Lemma 3.13, $\bar{C}_i \times_V \operatorname{Spec} \mathcal{O}_{S,\bar{w}}$ is disconnected. By the definition of $N_0(y)$, we see that $y \in F$. So w is a generization of y. Let

$$u\colon \mathcal{O}_{S,\bar{y}} = \mathcal{O}_{S,\bar{y}'} \to \mathcal{O}_{S,\bar{w}'} = \mathcal{O}_{S,\bar{w}}$$

be a cospecialization map and $v\colon D_{\bar{w}} \to D_{\bar{y}}$ the induced morphism. Then

$$v^{-1}(C_i) = \bar{C}_i \times_V \operatorname{Spec} \mathcal{O}_{S,\bar{w}}\,.$$

Since f satisfies the condition (‡), $v^{-1}(C_i)$ is connected, a contradition.

(2) is a consequence of (1). □

Definition 3.16. Let $x \in X$ be a point and $y \coloneqq f(x)$. A *refined local chart* of f at x is of the form

$$(U, V; T_{11}, \ldots, T_{1q_1}; \ldots; T_{p1}, \ldots, T_{pq_p}; a_1, \ldots, a_n)$$

where

$$U \to \operatorname{Spec} P \Big/ \Big(\ldots, \prod_{j_i=1}^{q_i} T_{ij_i}^{e_{ij_i}} - a_i, \ldots \Big)$$

is a local chart at x, $V \in N_0(y)$, $n \coloneqq \#\mathrm{CP}(y) \geqslant p$ and $a_i \equiv b_i(V) \pmod{\mathcal{O}_V^*}$ for all $i \in [1, n]$. We also simply use U/V or U to denote a refined local chart.

Remark 3.17. If f is smooth at x, then $p = 0$; and if f is smooth at the fiber X_y, then $p = n = 0$.

Remark 3.18. Obviously every local chart can be contracted in the sense of étale topology to become a refined local chart.

Remark 3.19. Let $\mathscr{M}_U$ be the log structure on U associating to $\alpha_U\colon \mathbb{N}_U^m \to \mathcal{O}_U$ with $m \coloneqq \sum_{i=1}^{p} q_i + n - p$, where if

$$\eta_{11}, \ldots, \eta_{1q_1}, \ldots, \eta_{p1}, \ldots, \eta_{pq_p}, \eta_{p+1}, \ldots, \eta_n$$

is a basis of $\mathbb{N}^m$, then $\alpha_U(\eta_{ij_i}) = \bar{T}_{ij_i}$ for $i \in [1, p]$ and $j_i \in [1, q_i]$, and $\alpha_U(\eta_i) = a_i$ for $i \in [p+1, n]$. Let $\mathscr{N}_V$ be the log structrue on V associating to $\beta_V\colon \mathbb{N}_V^n \to \mathcal{O}_V$, where if $\varepsilon_1, \ldots, \varepsilon_n$ is a basis of $\mathbb{N}^n$, then $\beta_V(\varepsilon_i) = a_i$ for all $i \in [1, n]$. Let $g\colon U \to V$ be the canonical morphism. Then there is a canonical morphism

$$\varphi_{U/V}\colon g^*\mathscr{N}_V \to \mathscr{M}_U$$

defined by the map $\gamma\colon \mathbb{N}^n \to \mathbb{N}^m$, where $\gamma(\varepsilon_i) = \sum_{j=1}^{q_i} e_{ij}\eta_{ij}$ for $i \in [1, p]$ and $\gamma(\varepsilon_i) = \eta_i$ for $i \in [p+1, n]$.

Remark 3.20. As $\overline{\mathscr{M}}_U = \mathbb{N}_U^m / \alpha_U^{-1}(\mathcal{O}_U^*)$ does not depend on the choice of a_i and T_{ij_i}, we may glue the sheaves $\overline{\mathscr{M}}_U$ to obtain a global sheaf $\mathscr{P}$ of monoids on X_{et} and there is a canonical morphism

$$\theta\colon \mathscr{P} \to \mathcal{O}_X/\mathcal{O}_X^*\,.$$

Similarly we may glue the sheaves $\overline{\mathscr{N}}_V$ to obtain a global sheaf $\mathscr{Q}$ of monoids on S_{et} and there is a canonical morphism

$$\vartheta\colon \mathscr{Q} \to \mathcal{O}_S/\mathcal{O}_S^*\,.$$

Moreover, there is a canonical morphism $\mathfrak{d}\colon f^{-1}\mathscr{Q} \to \mathscr{P}$ defined by γ which makes the following diagram commutative:

$$\begin{array}{ccc} f^{-1}\mathscr{Q} & \xrightarrow{f^{-1}\vartheta} & f^{-1}\big(\mathcal{O}_S/\mathcal{O}_S^*\big) \\ {\scriptstyle \mathfrak{d}}\downarrow & & \downarrow \\ \mathscr{P} & \xrightarrow[\theta]{} & \mathcal{O}_X/\mathcal{O}_X^* \end{array}$$

Lemma 3.21. *$\mathscr{Q}$ is canonically isomorphic to the direct image of $\mathbb{N}_{D(f)}$ under morphism $D(f) \to S$.*

4. COHOMOLOGY AND HYPERCOVERINGS

In this section, we review some technique in [5]. A brief version can be found in [17, §2 and §3].

4.1. **Cohomology.** Let X be a scheme. We define a category $\mathfrak{U}(X)$ as follows: an object in $\mathfrak{U}(X)$ is a diagram

$$V \underset{v_2}{\overset{v_1}{\rightrightarrows}} U \xrightarrow{u} X \tag{4.1}$$

where U and V are schemes, $u\colon U \to X$ and $v_1, v_2\colon V \rightrightarrows U$ are surjective étale morphisms such that $u \circ v_1 = u \circ v_2$ and the induced morphism

$$(v_1, v_2)_X\colon V \to U \times_X U$$

is surjective (and obviously étale); we also simply use U/V to denote the object (4.1); a morphism in $\mathfrak{U}(X)$ is a pair of morphisms

$$(f, g)\colon U'/V' \to U/V$$

which makes a commutative diagram

$$\begin{array}{ccccc} V' & \underset{v_2'}{\overset{v_1'}{\rightrightarrows}} & U' & \xrightarrow{u'} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \| \\ V & \underset{v_2}{\overset{v_1}{\rightrightarrows}} & U & \xrightarrow[u]{} & X \end{array}$$

Given an object of form (4.1) in $\mathfrak{U}(X)$. Put $(V/U)_0 \coloneqq U$, $(V/U)_1 \coloneqq V$, $p_{00} \coloneqq u$, $p_{10} \coloneqq v_1$, $p_{11} \coloneqq v_2$. Assume that for some integer $n \geqslant 2$, we have schemes $(V/U)_k$ for $k \in [1, n-1]$ and étale morphisms

$$p_{ki}\colon (V/U)_k \to (V/U)_{k-1}$$

for $i \in [0, k]$ such that whenever $0 \leqslant i < j \leqslant k$, we have

$$p_{k-1,i} \circ p_{kj} = p_{k-1,j-1} \circ p_{ki}\,. \tag{4.2}$$

Put

$$P_n \coloneqq \underbrace{(V/U)_{n-1} \times_X (V/U)_{n-1} \times_X \cdots \times_X (V/U)_{n-1}}_{n+1 \text{ copies of } (V/U)_{n-1}}$$

and let $q_{ni}\colon P_n \to (V/U)_{n-1}$ be the $(i+1)$-th projection. For each $0 \leqslant i < j \leqslant n$, let $K(n, i, j)$ be the equalizer of $p_{n-1,i} \circ q_{nj}$ and $p_{n-1,j-1} \circ q_{ni}$ in the category of schemes. As $(V/U)_{n-1}$ is étale over X, $K(n, i, j)$ is an open subscheme of P_n. Put

$$(V/U)_n \coloneqq \bigcap_{0 \leqslant i < j \leqslant n} K(n, i, j)$$

and

$$p_{ni} \coloneqq q_{ni}|_{(V/U)_n}\colon (V/U)_n \to (V/U)_{n-1}\,.$$

Let $\mathscr{F}$ be an abelian sheaf on X_{et}. We define a cochain complex of abelian groups as follows: for each $n \in \mathbb{N}$, put

$$C^n(V/U, \mathscr{F}) \coloneqq \Gamma\big((V/U)_n, \mathscr{F}\big)$$

and let

$$d^n \coloneqq \sum_{i=0}^{n+1} (-1)^i p_{n+1,i}^*\colon C^n(V/U, \mathscr{F}) \to C^{n+1}(V/U, \mathscr{F})$$

be the differential. Let $H^n(V/U, \mathscr{F})$ be the corresponding cohomology group. We define

$$\mathbb{H}^n(X, \mathscr{F}) \coloneqq \varinjlim H^n(V/U, \mathscr{F})\,,$$

where the colimit runs through all elements in $\mathfrak{U}(X)$.

Let

$$0 \to \mathscr{F}' \xrightarrow{f} \mathscr{F} \xrightarrow{g} \mathscr{F}'' \to 0$$

be an exact sequence of abelian sheaves on X_{et}. For $i = 0, 1$, we define

$$\delta^i \colon \mathbb{H}^i(X, \mathscr{F}'') \to \mathbb{H}^{i+1}(X, \mathscr{F}')$$

as follows.

Let $s \in Z^0(V/U, \mathscr{F}'')$ be a 0-cocycle. By refining U, we may choose a lifting $\tilde{s} \in \mathscr{F}(U)$ of s. Then we define

$$\delta^0(s) = p_{10}^*(\tilde{s}) - p_{11}^*(\tilde{s})\,.$$

Let $t \in Z^1(V/U, \mathscr{F}'')$ be a 1-cocycle. By refining V, we may choose a lifting $\tilde{t} \in \mathscr{F}(V)$ of t. Then we define

$$\delta^1(t) = p_{20}^*(\tilde{t}) - p_{21}^*(\tilde{t}) + p_{22}^*(\tilde{t})\,.$$

Theorem 4.1. *For each $n = 0, 1, 2$, we have a natural equivalence*

$$\mathbb{H}^n(X, \bullet\,) \xrightarrow{\sim} H^n(X_{\mathrm{et}}, \bullet\,)\,;$$

and for each short exact sequence the natural equivalences commute with the connecting functors δ.

4.2. **Gerbe.** Now we fix an abelian sheaf $\mathscr{F}$ on X_{et}.

Definition 4.2. An $\mathscr{F}$-gerbe $(\mathfrak{X}, \omega)$ consists of the following two data:

(a) a stack $\mathfrak{X}$ over X_{et};
(b) for each étale X-scheme U and for each object A in $\mathfrak{X}(U)$, an isomorphism of sheaves of groups:

$$\omega(A) \colon \mathscr{F}|_U \xrightarrow{\sim} \mathscr{A}ut_U(A)\,.$$

These data satisfy the following conditions:

(1) for any étale X-scheme U, there exists an étale covering $\{U_i \to U\}_{i \in I}$ in X_{et} such that $\mathfrak{X}(U_i) \neq \varnothing$ for all $i \in I$;
(2) for any étale X-scheme U and any pair of objects A and B in $\mathfrak{X}(U)$, there exists an étale covering $\{U_i \to U\}_{i \in I}$ in X_{et} such that $A|_{U_i}$ and $B|_{U_i}$ are isomorphic in $\mathfrak{X}(U_i)$ for all $i \in I$;
(3) for any étale X-scheme U, any element $g \in \mathscr{F}(U)$, and any isomorphim $\varphi \colon A \xrightarrow{\sim} B$ in $\mathfrak{X}(U)$, we have

$$\varphi \circ \omega(A)(g) = \omega(B)(g) \circ \varphi\,.$$

(So we may simply write $\varphi \circ g$ or $g \circ \varphi$ or even $g \cdot \varphi$ for above morphism.)

Fix an $\mathscr{F}$-gerbe $\mathfrak{X}$. Choose an étale covering $U \to X$ which admits an object $A \in \mathfrak{X}(U)$, and an étale covering $V \to U \times_X U$ which admits an isomorphism

$$\phi \colon p_{10}^*(A) \xrightarrow{\sim} p_{11}^*(A)$$

in $\mathfrak{X}(V)$. Then there exists a cocycle $g \in Z^2(V/U, \mathscr{F})$ such that

$$g \circ p_{21}^*(\phi) = p_{22}^*(\phi) \circ p_{20}^*(\phi) \colon (p_{10} \circ p_{20})^*(A) \xrightarrow{\sim} (p_{11} \circ p_{22})^*(A)\,.$$

We define

$$[\mathfrak{X}] \coloneqq [g] \in H^2(X_{\mathrm{et}}, \mathscr{F})\,.$$

Lemma 4.3. *Let V/U be an object in $\mathfrak{A}(X)$, A an object in $\mathfrak{X}(U)$, and $\phi \colon p_{10}^*(A) \xrightarrow{\sim} p_{11}^*(A)$ an isomorphism in $\mathfrak{X}(V)$ satisfying the cocycle condition:*

$$p_{21}^*(\phi) = p_{22}^*(\phi) \circ p_{20}^*(\phi) \colon (p_{10} \circ p_{20})^*(A) \xrightarrow{\sim} (p_{11} \circ p_{22})^*(A)\,.$$

Then there exists an object B in $\mathfrak{X}(X)$ and an isomorphism $\varphi \colon B|_U \xrightarrow{\sim} A$ in $\mathfrak{X}(U)$ such that $\phi \circ p_{10}^*(\varphi) = p_{11}^*(\varphi)$. And (B, φ) is unique up to isomorphism.

Theorem 4.4. *$\mathfrak{X}(X) \neq \varnothing$ if and only if $[\mathfrak{X}] = 0$ in $H^2(X_{\mathrm{et}}, \mathscr{F})$.*

Let $g \in Z^1(V/U, \mathscr{F})$, A an object in $\mathfrak{X}(X)$. By Lemma 4.3, there is an object $g(A)$ in $\mathfrak{X}(X)$ and an isomorphism $\phi_g \colon g(A)|_U \to A|_U$ in $\mathfrak{X}(U)$ such that $g \circ p_{10}^*(\phi_g) = p_{11}^*(\phi_g)$. It is easy to show that $(g, A) \mapsto g(A)$ defines an action of the group $H^1(X_{\mathrm{et}}, \mathscr{F})$ on the set of isomorphic classes of objects in $\mathfrak{X}(X)$.

Let A and B be objects in $\mathfrak{X}(X)$, $\phi \colon A|_U \xrightarrow{\sim} B|_U$ an isomorphism in $\mathfrak{X}(U)$ and $g \in \mathscr{F}(V)$ such that $g \circ p_{10}^*(\phi) = p_{11}^*(\phi)$. Then $g \in Z^1(V/U, \mathscr{F})$ and $g(B) \cong A$ in $\mathfrak{X}(X)$.

Theorem 4.5. *If $\mathfrak{X}(X) \neq \varnothing$, then the group $H^1(X_{\mathrm{et}}, \mathscr{F})$ acts canonically and simply transitively on the set of isomorphic classes of objects in $\mathfrak{X}(X)$.*

5. Local Cases

Let $f \colon X \to S$ be a surjective, proper and weakly normal crossing morphism of locally noetherian schemes which satisfies the conditions (†) and (‡) in §3.3, $D_1, D_2, \dots, D_n$ the connected components of $D(f)$. Assume that there exist global sections

$$a_1, a_2, \dots, a_n \in \Gamma(S, \mathcal{O}_S)$$

such that for any point $y \in S$ and any $i \in [1, n]$, the following two conditions holds:

(1) if $a_{i,\bar{y}} \in \mathcal{O}_{S,\bar{y}}^*$, then $D_i \times_S \operatorname{Spec} \mathcal{O}_{S,\bar{y}} = \varnothing$;

(2) if $a_{i,\bar{y}} \in \mathfrak{m}_{S,\bar{y}}$, then

$$D_{i,\bar{y}} \coloneqq D_i \times_S \operatorname{Spec} \mathcal{O}_{S,\bar{y}}$$

is a connected component of $D_{\bar{y}}$ and $\omega_{\bar{y}}(D_{i,\bar{y}}) = \overline{a_{i,\bar{y}}}$.

Thus

$$\mathrm{CP}(y) = \left\{ D_{i,\bar{y}} \,\middle|\, i \in [1, n],\ a_{i,\bar{y}} \in \mathfrak{m}_{S,\bar{y}} \right\}.$$

Obviously when S is a spectrum of a strictly Henselian local ring, then above conditions hold. Furthermore, for any point $y \in S$ and any $V \in N_0(y)$, $X_V \to V$ satisfies above conditions.

Let $\mathscr{N}$ be the log structure on S defined by

$$\mathbb{N}_S^n \to \mathcal{O}_S, \qquad \varepsilon_i \mapsto a_i,$$

where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ is a basis of $\mathbb{N}^n$.

For each $i \in [1, n]$, let $\mathscr{I}_i$ be the ideal sheaf on X corresponding to the closed subscheme D_i and let $\mathscr{K}_i$ denote the kernel of the multiplication by a_i on $\mathcal{O}_S$. As f is flat, the kernel of the multiplication by a_i on $\mathcal{O}_X$ is equal to $\mathscr{K}_i \cdot \mathcal{O}_X$. Let E_i be the closed subscheme of X defined by $\mathscr{K}_i \cdot \mathscr{I}_i$ and put $E \coloneqq \coprod_{i=1}^{n} E_i$. We also use $E(f)$ or $E(X/S)$ to denote the scheme E.

Let $\mathscr{F}_i$ be the kernel of the morphism $\mathcal{O}_X^* \to \mathcal{O}_{E_i}^*$ on X_{et}. Then $\mathscr{F}_i = 1 + \mathscr{K}_i \cdot \mathscr{I}_i$. Put $\mathscr{F} \coloneqq \prod_{i=1}^{n} \mathscr{F}_i$. Then we have an exact sequence of abelian sheaves:

$$0 \to \mathscr{F} \to (\mathcal{O}_X^*)^n \to \mathcal{O}_E^* \to 0.$$

We also use $\mathscr{F}(f)$ or $\mathscr{F}(X/S)$ to denote this abelian sheaf $\mathscr{F}$.

Let $\mathscr{P}$, $\mathscr{Q}$, θ, ϑ and $\mathfrak{d}$ be the notations defined in Remark 3.20. Obviously there is a canonical morphism $\mathbb{N}_S^n \to \mathscr{Q}$. Let γ denote the composite

$$\mathbb{N}_X^n \to f^{-1}\mathscr{Q} \xrightarrow{\mathfrak{d}} \mathscr{P}.$$

We define a stack $\mathfrak{X}$ on X_{et} as follows. For each étale X-scheme U, an object in $\mathfrak{X}(U)$ is a pair $(\mathscr{M}, \sigma)$, where $\mathscr{M}$ is a fine saturated log structure on U and $\sigma \colon \mathscr{M} \to \mathscr{P}|_U$ is a morphism of sheaves of monoids which induces an isomorphism $\overline{\mathscr{M}} \xrightarrow{\sim} \mathscr{P}|_U$ and makes the following diagram commutative:

$$\begin{array}{ccc} \mathscr{M} & \longrightarrow & \mathcal{O}_U \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ \mathscr{P}|_U & \xrightarrow[\theta|_U]{} & \mathcal{O}_U/\mathcal{O}_U^* \end{array}$$

If $U' \to U$ is a morphism of étale X-schemes, $(\mathscr{M}, \sigma) \in \mathfrak{X}(U)$ and $(\mathscr{M}', \sigma') \in \mathfrak{X}(U')$, then a morphism of $(\mathscr{M}', \sigma')$ to $(\mathscr{M}, \sigma)$ in $\mathfrak{X}$ lying above $U' \to U$ is an isomorphism $\varphi\colon \mathscr{M}' \xrightarrow{\sim} \mathscr{M}|_{U'}$ of log structures such that $\sigma|_{U'} \circ \varphi = \sigma'$.

We shall prove that $\mathfrak{X}$ is an $\mathscr{F}$-gerbe (see Lemma 5.4). The proof needs the following three simple lemmas.

Lemma 5.1. *Let X be a scheme, $\mathscr{M}$ a fine saturated log structure on X and $\bar{x}$ a geometric point on X. Then there exists an étale neighborhood U of $\bar{x}$ and a fine saturated chart $P_U \to \mathscr{M}|_U$ such that the induced map $P \to \overline{\mathscr{M}}_{\bar{x}}$ is a bijection.*

Lemma 5.2. *Let X be a scheme and $\alpha\colon \mathscr{M} \to \mathcal{O}_X$ a fine log structure on X. Put $\mathscr{P} := \overline{\mathscr{M}}$ and let $\bar{\alpha}\colon \mathscr{P} \to \mathcal{O}_X/\mathcal{O}_X^*$ be the morphism induced by α. We define an abelian sheaf $\mathscr{A}$ on X_{et} as follows: for every étale X-scheme U, $\mathscr{A}(U)$ is the set of morphisms $\sigma\colon \mathscr{P}^{\mathrm{gp}}|_U \to \mathcal{O}_U^*$ of abelian sheaves such that for any étale U-scheme V and any section $s \in \mathscr{P}(V)$, we have $\sigma_V(s) \cdot t = t$, where $t \in \Gamma(V, \mathcal{O}_X)$ is a lifting of $\bar{\alpha}(s)$. Then there is a canonical isomorphism from $\mathscr{A}$ to the sheaf of automorphisms of log structures of $\mathscr{M}$ which induce identities on $\mathscr{P}$ defined as follows: for any étale X-scheme U and any section $\sigma \in \Gamma(U, \mathscr{A})$, $\omega_U(\sigma)_V(s) = s \cdot \sigma_V(\bar{s})$, where V is an étale U-scheme and $s \in \mathscr{M}(V)$.*

Lemma 5.3. *Let X be a scheme and P a fine monoid. For each $i = 1, 2$, let $\alpha_i\colon P_X \to \mathcal{O}_X$ be a morphism of sheaves of monoids, $\mathscr{M}_i$ the log structure associating to α_i, $\iota_i\colon P_X \to \mathscr{M}_i$ the induced morphism. Assume that there exists a morphism $\delta\colon P_X \to \mathcal{O}_X^*$ of sheaves of monoids such that $\alpha_1 = \delta \cdot \alpha_2$. Then there exists a unique isomorphism $\rho\colon \mathscr{M}_1 \xrightarrow{\sim} \mathscr{M}_2$ of log structures such that $\rho \circ \iota_1 = \delta \cdot \iota_2$.*

Lemma 5.4. *$\mathfrak{X}$ is an $\mathscr{F}$-gerbe.*

Proof. We have to verify the conditions in Definition 4.2. (1) is obvious. Datum (b) and Condition (3) is by Lemma 5.2 and Theorem 2.4.

For the condition (2), let U be an étale X-scheme, $\alpha_1\colon \mathscr{M}_1 \to \mathcal{O}_U$ and $\alpha_2\colon \mathscr{M}_2 \to \mathcal{O}_U$ two objects in $\mathfrak{X}(U)$, x a point on U. Put $P := \mathscr{P}_{\bar{x}}$. By Lemma 5.1, for each $i = 1, 2$, there exists an étale neighborhood V_i, and a chart $\beta_i\colon P_{V_i} \to \mathscr{M}_i|_{V_i}$ which inducs identity on $P = \mathscr{P}_{\bar{x}}$. Put $V_3 := V_1 \times_U V_2$. Since both

$$(\alpha_i \circ \beta_i)_{\bar{x}}\colon P \to \mathcal{O}_{U,\bar{x}}$$

are liftings of

$$\theta_{\bar{x}}\colon P \to \mathcal{O}_{U,\bar{x}}/\mathcal{O}_{U,\bar{x}}^*\,,$$

we have an étale neighborhood $V \to V_3$ of $\bar{x}$ and a morphism $u\colon P_V \to \mathcal{O}_V^*$ of sheaves of monoids such that $\delta_1 = u \cdot \delta_2$, where

$$\delta_i := (\alpha_i \circ \beta_i)|_V\colon P_V \to \mathcal{O}_V\,.$$

As $P_V \to \mathscr{M}_i|_V$ are charts, by Lemma 5.3 there exists an isomorphism $\varphi\colon \mathscr{M}_1 \xrightarrow{\sim} \mathscr{M}_2$ of log structures such that $\varphi \circ \beta_1|_V = u \cdot \beta_2|_V$. Thus φ induces identity on $\mathscr{P}|_V$. So φ is an isomorphism in $\mathfrak{X}(V)$. ∎

Obviously there exists an étale covering $U \to X$, an object $\mathscr{M}$ in $\mathfrak{X}(U)$, and a morphism $\rho\colon \mathbb{N}_U^n \to \mathscr{M}$ which is a lifting of $\gamma|_U\colon \mathbb{N}_U^n \to \mathscr{P}|_U$. So there exists an étale covering $V \to U \times_X U$ and an isomorphism $\phi\colon p_{10}^*(\mathscr{M}) \xrightarrow{\sim} p_{11}^*(\mathscr{M})$ of log structures on V (here the notations $(V/U)_k$ and $p_{ki}\colon (V/U)_k \to (V/U)_{k-1}$ are defined as in §4.1). Hence there exists an element

$$u = (u_1, u_2, \dots, u_n) \in (\mathcal{O}_X^*)^n(V)$$

such that

$$\phi \circ p_{10}^*(\rho) = u \cdot p_{11}^*(\rho)$$

and a cocycle $g \in Z^2(V/U, \mathscr{F})$ such that

$$g \circ p_{21}^*(\phi) = p_{22}^*(\phi) \circ p_{20}^*(\phi)\,.$$

We have

$$[\mathfrak{X}] = [g] \in H^2(X_{\mathrm{et}}, \mathscr{F})\,.$$

By (4.2), we have

$$\begin{aligned} p_{10} \circ p_{21} &= p_{10} \circ p_{20}\,,\\ p_{10} \circ p_{22} &= p_{11} \circ p_{20}\,,\\ p_{11} \circ p_{22} &= p_{11} \circ p_{21}\,. \end{aligned}$$

We also have

$$\begin{aligned} & g \cdot p_{21}^*(u) \cdot (p_{11} \circ p_{21})^*(\rho)\\ &= g \cdot p_{21}^*(\phi) \circ (p_{10} \circ p_{21})^*(\rho)\\ &= p_{22}^*(\phi) \circ p_{20}^*(\phi) \circ (p_{10} \circ p_{20})^*(\rho)\\ &= p_{20}^*(u) \cdot p_{22}^*(\phi) \circ (p_{11} \circ p_{20})^*(\rho)\\ &= p_{20}^*(u) \cdot p_{22}^*(\phi) \circ (p_{10} \circ p_{22})^*(\rho)\\ &= p_{20}^*(u) \cdot p_{22}^*(u) \cdot (p_{11} \circ p_{22})^*(\rho)\\ &= p_{20}^*(u) \cdot p_{22}^*(u) \cdot (p_{11} \circ p_{21})^*(\rho)\,. \end{aligned}$$

Thus $g = p_{20}^*(u) \cdot p_{22}^*(u) \cdot p_{21}^*(u)^{-1}$. Since the image of g in $\mathcal{O}_E^*$ is equal to 1, we see that u determinates an element in $H^1(E_{\mathrm{et}}, \mathcal{O}_E^*)$. So we obtain an invertible $\mathcal{O}_E$-module, which depends only on the morphism $f\colon X \to S$. We denote it by $\mathscr{L}(f)$ or $\mathscr{L}(X/S)$.

Definition 5.5. A *semistable log structure* on X is an object $(\mathscr{M}, \sigma)$ in $\mathfrak{X}(X)$ such that there is a morphism $\rho\colon \mathbb{N}_X^n \to \mathscr{M}$ which lifts the morphism $\gamma\colon \mathbb{N}_X^n \to \mathscr{P}$.

Theorem 5.6.

(1) *There exists a semistable log structure on X if and only if $\mathscr{L}(f) \cong \mathcal{O}_E$.*
(2) *The semistable log structure on X is unique (up to isomorphism) if it exists.*

Proof. (1) If semistable log structures on X exist, obviously $\mathscr{L}(f) \cong \mathcal{O}_E$.

Assume that $\mathscr{L}(f) \cong \mathcal{O}_E$. By above argument, $[\mathfrak{X}]$ is the image of $\mathscr{L}(f)$ under the connecting map

$$H^1(E_{\mathrm{et}}, \mathcal{O}_E^*) \to H^2(X_{\mathrm{et}}, \mathscr{F})\,.$$

Thus $[\mathfrak{X}] = 0$. By Theorem 4.4, there exists an element $\mathscr{M} \in \mathfrak{X}(X)$. Let $U \to X$ be an étale covering such that there exists a lifting $\rho\colon \mathbb{N}_U^n \to \mathscr{M}|_U$ of $\gamma|_U\colon \mathbb{N}_U^n \to \mathscr{P}|_U$. Put $V \coloneqq U \times_S U$ and let $u \in (\mathcal{O}_X^*)^n(V)$ such that $p_{10}^*(\rho) = u \cdot p_{11}^*(\rho)$. Let $\bar{u}$ be the image of u in $\mathcal{O}_E^*(V)$. As $\mathscr{L}(f) \cong \mathcal{O}_E$ is represented by $[\bar{u}]$, there exists an element $v' \in \mathcal{O}_E(U)$ such that $\bar{u} = p_{10}^*(v') \cdot p_{11}^*(v')^{-1}$. By contracting U suitably, we may choose a lifting $v \in (\mathcal{O}_X^*)^n(U)$ of v'. Then

$$u = p_{10}^*(v) \cdot p_{11}^*(v)^{-1} \cdot g$$

for some $g \in \mathscr{F}(V)$. As $\partial^2(g) = \partial^2(u) = 1$, g is a cocycle. Put

$$\rho_1 \coloneqq v^{-1} \cdot \rho \colon \mathbb{N}_U^n \to \mathscr{M}|_U \,.$$

Then

$$p_{10}^*(\rho_1) = g \cdot p_{11}^*(\rho_1) \,.$$

By Lemma 4.3, there exists an object $\mathscr{M}'$ in $\mathfrak{X}(X)$ and an isomorphism $\phi\colon \mathscr{M}'|_U \to \mathscr{M}|_U$ in $\mathfrak{X}(U)$ such that $g^{-1} \cdot p_{10}^*(\phi) = p_{11}^*(\phi)$. Put $\rho_2 \coloneqq \phi^{-1} \circ \rho_1$. Then $p_{10}^*(\rho_2) = p_{11}^*(\rho_2)$. So there exists a morphism $\rho' \colon \mathbb{N}_X^n \to \mathscr{M}'$ such that $\rho'|_U = \rho_2$. Obviously ρ' is a lift of $\gamma\colon \mathbb{N}_X^n \to \mathscr{P}$. Thus $\mathscr{M}'$ is a semistable log structure on X.

(2) Let $\mathscr{M}_1$ and $\mathscr{M}_2$ be two semistable log structures on X. For $i = 1, 2$, let $\rho_i \colon \mathbb{N}_X^n \to \mathscr{M}_i$ be liftings of $\gamma\colon \mathbb{N}_X^n \to \mathscr{P}$. By Theorem 4.5, there exist étale coverings $U \to X$ and $V \to U \times_X U$, a cocycle $g \in Z^1(V/U, \mathscr{F})$ and an isomorphism $\phi\colon \mathscr{M}_2|_U \xrightarrow{\sim} \mathscr{M}_1|_U$ such that $g \cdot p_{10}^*(\phi) = p_{11}^*(\phi)$. Let $\delta \in (\mathcal{O}_X^*)^n(U)$ such that $\phi \circ \rho_2|_U = \delta \cdot \rho_1|_U$. Then we have

$$\begin{aligned} g \cdot p_{10}^*(\delta) \cdot \rho_1|_V &= g \cdot p_{10}^*(\delta \cdot \rho_1|_U) \\ &= g \cdot p_{10}^*(\phi \circ \rho_2|_U) \\ &= \big(g \cdot p_{10}^*(\phi)\big) \circ \rho_2|_V \\ &= p_{11}^*(\phi) \circ \rho_2|_V \\ &= p_{11}^*(\delta) \cdot \rho_1|_V \,. \end{aligned}$$

So $g = p_{11}^*(\delta) \cdot p_{10}^*(\delta)^{-1}$, i.e., $[g] = 0$. Therefore $\mathscr{M}_1 \cong \mathscr{M}_2$ in $\mathfrak{X}(X)$. □

Remark 5.7. Note that the isomorphisms between semistable log structures may not be unique. So this kind of structure is not canonical.

Theorem 5.8. *Assume that all $a_1, a_2, \ldots, a_n$ are regular elements in $\Gamma(S, \mathcal{O}_S)$ (i.e., $(0 : a_i) = 0$ for all $i \in [1, n]$). Then $\mathscr{L}(f) \cong \mathcal{O}_E$, i.e., there exists a semistable log structure on X.*

Proof. Note that

$$\mathscr{F} = \prod_{i=1}^{n} (1 + \mathscr{K}_i \cdot \mathscr{I}_i) = 0 \,.$$

So there exists an object $\mathscr{M}$ in $\mathfrak{X}(X)$. Obviouly there exists an étale covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of X such that for each $\lambda \in \Lambda$, there exists a lifting $\rho_\lambda \colon \mathbb{N}_{U_\lambda}^n \to \mathscr{M}|_{U_\lambda}$ of $\gamma|_{U_\lambda} \colon \mathbb{N}_{U_\lambda}^n \to \mathscr{P}|_{U_\lambda}$ such that the composite morphism

$$\mathbb{N}_{U_\lambda}^n \xrightarrow{\rho_\lambda} \mathscr{M}|_{U_\lambda} \to \mathcal{O}_{U_\lambda}$$

is equal to

$$\mathbb{N}_{U_\lambda}^n \to \mathcal{O}_{U_\lambda} \,, \qquad \varepsilon_i \to a_i \,.$$

Since $a_1, a_2, \ldots, a_n$ are regular elements in $\Gamma(S, \mathcal{O}_S)$ and $f\colon X \to S$ is flat, $a_1, a_2, \ldots, a_n$ are regular elements in $\Gamma(X, \mathcal{O}_X)$ too. So on each $U_{\lambda\mu}$, we have $\rho_\lambda|_{U_{\lambda\mu}} = \rho_\mu|_{U_{\lambda\mu}}$. Thus $\{\rho_\lambda\}$ can be glued to obtain a global lifting $\rho\colon \mathbb{N}_X^n \to \mathscr{M}$ of $\gamma\colon \mathbb{N}_X^n \to \mathscr{P}$. □

Theorem 5.9. *Let $\mathscr{M}$ be a semistable log structure on X and $\rho\colon \mathbb{N}^n \to \mathscr{M}$ a lifting of $\gamma\colon \mathbb{N}_X^n \to \mathscr{P}$. By Lemma 5.3, ρ induces a morphism $\varphi\colon f^*\mathscr{N} \to \mathscr{M}$ of log structures. Then*

$$(f, \varphi) \colon (X, \mathscr{M}) \to (Y, \mathscr{N})$$

is log smooth and integral.

Proof. Let x be a point on X and $y \coloneqq f(x)$. Let

$$(U, V; T_{11}, \ldots, T_{1q_1}; \ldots; T_{p1}, \ldots, T_{pq_p}; a_1, \ldots, a_l)$$

be a refined local chart of f at x and put $m \coloneqq \sum_{i=1}^{p} q_i + l - p$. Let $g\colon U \to V$ be the induced morphism. Then

$$(g, \varphi_{U/V})\colon (U, \mathscr{M}_U) \to (V, \mathscr{N}_V)$$

is log smooth, where $\mathscr{M}_U$, $\mathscr{N}_V$ and $\varphi_{U/V}$ are defined in Remark 3.19. By Lemma 5.3, we may contract U/V suitably such that there exists two isomorphism $\sigma\colon \mathscr{M}_U \xrightarrow{\sim} \mathscr{M}|_U$ and $\tau\colon \mathscr{N}_V \xrightarrow{\sim} \mathscr{N}|_V$ of log structures. Put $\mathscr{M}_0 \coloneqq \mathscr{M}_U$, $\mathscr{N}_0 \coloneqq \mathscr{N}_V$, $\varphi_1 \coloneqq \varphi_{U/V}$, $\varphi_2 \coloneqq \sigma^{-1} \circ \varphi|_U \circ f^*(\tau)$. Let $\alpha \coloneqq \alpha_U\colon \mathbb{N}_U^m \to \mathscr{M}_0$, $\beta \coloneqq \beta_V\colon \mathbb{N}_V^l \to \mathscr{N}_0$ and $\gamma\colon \mathbb{N}^l \to \mathbb{N}^m$ be the notations as in Remark 3.19. Now (g, φ_1) is log smooth. We shall use the definition of log smoothness to prove that (g, φ_2) is also log smooth. Let $(T_0, \mathscr{T}_0)$ and $(T, \mathscr{T})$ be fine log schemes, $(T_0, \mathscr{T}_0) \to (T, \mathscr{T})$ a thickening of order one (Cf. [9, 3.5]), $(t_0, \psi_0)\colon (T_0, \mathscr{T}_0) \to (U, \mathscr{M}_0)$ and $(t, \psi)\colon (T, \mathscr{T}) \to (V, \mathscr{N}_0)$ be morphisms of log schemes which makes a commutative diagram:

$$\begin{array}{ccc} (T_0, \mathscr{T}_0) & \xrightarrow{(t_0, \psi_0)} & (U, \mathscr{M}_0) \\ \downarrow & & \downarrow{\scriptstyle (g, \varphi_2)} \\ (T, \mathscr{T}) & \xrightarrow[(t, \psi)]{} & (V, \mathscr{N}_0) \end{array}$$

For each $i = 1, 2$, let ρ_i denote the composite morphism

$$\mathbb{N}_U^l \xrightarrow{g^*(\beta)} g^* \mathscr{N}_0 \xrightarrow{\varphi_i} \mathscr{M}_0 \,.$$

Then there exists a section $u \in (\mathcal{O}_U^*)^n(U)$ such that $\rho_2 = u \cdot \rho_1$. The composite morphism

$$\mathbb{N}_T^l \xrightarrow{t^*(\beta)} t^* \mathscr{N}_0 \xrightarrow{\psi} \mathscr{T} \to \mathscr{T}_0$$

is equal to $\psi_0 \circ t_0^*(\rho_2)$. Let $v \in (\mathcal{O}_T^*)^n(T)$ be a lift of

$$t_0^*(u) \in (\mathcal{O}_{T_0}^*)^n(T_0) \,.$$

Then there is a morphism $\psi'\colon t^* \mathscr{N}_0 \to \mathscr{T}$ of log structures such that

$$\psi' \circ t^*(\beta) = v^{-1} \cdot \big(\psi \circ t^*(\beta)\big) \,.$$

So the composite morphism

$$\mathbb{N}_T^l \xrightarrow{t^*(\beta)} t^* \mathscr{N}_0 \xrightarrow{\psi'} \mathscr{T} \to \mathscr{T}_0$$

is equal to $\psi_0 \circ t_0^*(\rho_1)$, which shows that

$$\begin{array}{ccc} (T_0, \mathscr{T}_0) & \xrightarrow{(t_0, \psi_0)} & (U, \mathscr{M}_0) \\ \downarrow & & \downarrow{\scriptstyle (g, \varphi_1)} \\ (T, \mathscr{T}) & \xrightarrow[(t, \psi')]{} & (V, \mathscr{N}_0) \end{array}$$

is commutative. As (g, ψ_1) is log smooth, by replace T with an étale covering, we may assume that there is a morphism

$$(h, \xi_1)\colon (T, \mathscr{T}) \to (U, \mathscr{M}_0)$$

of log schemes which makes the following diagram

$$\begin{array}{ccc} (T_0, \mathscr{T}_0) & \xrightarrow{(t_0,\psi_0)} & (U, \mathscr{M}_0) \\ \downarrow & \overset{(h,\xi_1)}{\nearrow} & \downarrow {\scriptstyle (g,\varphi_1)} \\ (T, \mathscr{T}) & \xrightarrow[(t,\psi')]{} & (V, \mathscr{N}_0) \end{array}$$

commutative. By the following Lemma 5.10, there exists a section $w \in (\mathcal{O}_T^*)^m(T)$ which makes a commutative diagram:

$$\begin{array}{ccc} \mathbb{N}_T^l & \xrightarrow{v \cdot h^*(u)^{-1}} & \mathcal{O}_T^* \\ {\scriptstyle \gamma_T} \downarrow & \underset{w}{\nearrow} & \\ \mathbb{N}_T^m & & \end{array}$$

Let $\xi_2 \colon h^*\mathscr{M}_0 \to \mathscr{T}$ be the morphism of log structures satisfying that $\xi_2 \circ h^*(\alpha) = w \cdot \big(\xi_1 \cdot h^*(\alpha)\big)$. Then we have a commutative diagram:

$$\begin{array}{ccc} (T_0, \mathscr{T}_0) & \xrightarrow{(t_0,\psi_0)} & (U, \mathscr{M}_0) \\ \downarrow & \overset{(h,\xi_2)}{\nearrow} & \downarrow {\scriptstyle (g,\varphi_2)} \\ (T, \mathscr{T}) & \xrightarrow[(t,\psi)]{} & (V, \mathscr{N}_0) \end{array}$$

Thus (g, ψ_2) is log smooth. □

Lemma 5.10. *Let A be a ring, I an ideal of A such that $I^2 = (0)$, $u \in 1 + I$, $e_1, e_2, \dots, e_n$ be positive integers which are invertible in A. Then there exists elements $v_1, v_2, \dots, v_n \in 1 + I$ such that $u = \prod_{i=1}^{n} v_i^{e_i}$.*

The following theorem is obvious.

Theorem 5.11. *Let S' be a locally noetherian scheme and $S' \to S$ a flat morphism. Put $X' \coloneqq X \times_S S'$ and let $f' \colon X' \to S'$ be the projection. Then*

(1) *f' satisfies all these conditions of f mentioned at the beginning of this section.*
(2) *For each $i \in [1, n]$, let b_i be the image of a_i in $\Gamma(S', \mathcal{O}_{S'})$. Assume that b_i is invertible for $i \in [m + 1, n]$ and b_i is not invertible for $[1, m]$. Then*
$$E(f') = \Big(\coprod_{i=1}^{m} E_i\Big) \times_S S' .$$
(3) *$\mathscr{L}(f')$ is isomorphic to the inverse image of $\mathscr{L}$ under the canonical morphism $E(f') \to E(f)$.*

6. Global Cases

Let X and S be locally noetherian schemes, $f \colon X \to S$ a surjective proper weakly normal crossing morphism without powers such that f satisfies the condition (†) in §3.3 and every fiber of f is geometrically connected. By Lemma 3.14 (4), f also satisfies the condition (‡) in §3.3.

Let $\mathscr{P}$, $\mathscr{Q}$, θ, ϑ and $\mathfrak{d}$ be the notations defined in Remark 3.20.

For every point $y \in S$, we write

$$E_{\bar{y}} \coloneqq E\big(X \times_S \operatorname{Spec} \mathcal{O}_{S,\bar{y}} / \operatorname{Spec} \mathcal{O}_{S,\bar{y}}\big) ,$$
$$\mathscr{L}_{\bar{y}} \coloneqq \mathscr{L}\big(X \times_S \operatorname{Spec} \mathcal{O}_{S,\bar{y}} / \operatorname{Spec} \mathcal{O}_{S,\bar{y}}\big) .$$

Lemma 6.1. *Let $y \in S$. If $\mathscr{L}_{\bar{y}}$ is trivial, then there exists an element $V_0 \in N_0(y)$ such that for all elements $V \geqslant V_0$ in $N_0(y)$, $\mathscr{L}(X_V/V)$ is trivial.*

Proof. See [7, (8.5.2.5)]. □

Corollary 6.2. *Let $y \in S$. If $\mathscr{L}_{\bar{y}}$ is trivial, then there exists an open neighborhood V of y such that for all $z \in V$, $\mathscr{L}_{\bar{z}}$ is trivial.*

Lemma 6.3. *Let $f\colon X \to Y$ be a proper and flat morphism of locally noetherian schemes such that every fiber of f is geometrically reduced and geometrically connected. Then the canonical morphism $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is isomorphic.*

Proof. See [6, (7.8.7) and (7.8.8)]. □

Lemma 6.4. *Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$, $a \in \mathfrak{m}$, $\mathfrak{a} = (0 : a)$, $n \geqslant 2$ an integer,*

$$A = R[[T_1, T_2, \dots, T_n]]$$

a ring of power series over R, I the ideal of A generated by

$$T_2 \cdots T_n, \dots, T_1 \cdots \hat{T}_i \cdots T_n, \dots, T_1 \cdots T_{n-1}$$

and

$$J \coloneqq (T_1 T_2 \cdots T_n - a) + \mathfrak{a} \cdot I\,.$$

Then $J \cap R = (0)$.

Proof. Let $b \in J \cap R$ and put

$$b = (T_1 T_2 \cdots T_n - a) \cdot F_0 + \sum_{i=1}^{n} T_1 \cdots \hat{T}_i \cdots T_n \cdot F_i\,,$$

where $F_i \in A$ for all $i \in [0, n]$ and for every $i \in [1, n]$, all coefficients of F_i are contained in $\mathfrak{a}$. For each $i \in [1, n]$, put $F_i = G_i + T_i \cdot N_i$, where all monomials in G_i do not contain T_i and all coefficients of G_i and N_i are contained in $\mathfrak{a}$. Then we have

$$T_1 \cdots \hat{T}_i \cdots T_n \cdot F_i = T_1 \cdots \hat{T}_i \cdots T_n \cdot G_i + (T_1 T_2 \cdots T_n - a) \cdot N_i\,.$$

Put $G_0 \coloneqq F_0 + \sum_{i=1}^{n} N_i$. Then

$$b = (T_1 T_2 \cdots T_n - a) \cdot G_0 + \sum_{i=1}^{n} T_1 \cdots \hat{T}_i \cdots T_n \cdot G_i\,. \tag{6.1}$$

For each $q \in \mathbb{N}$, let c_q denote the coefficient of $(T_1 T_2 \cdots T_n)^q$ in G_0. By comparing the coefficient of $(T_1 T_2 \cdots T_n)^q$ in (6.1), we have $b = -a c_0$ and $c_{q-1} = a c_q$ for all $q \geqslant 1$. Thus

$$b \in \bigcap_{q=1}^{\infty} (a^q) \subseteq \bigcap_{q=1}^{\infty} \mathfrak{m}^q = (0)\,.$$ □

Lemma 6.5. *For all points $y \in S$ and $V \in N_0(y)$, we have $(f_V)_*\big(\mathscr{F}(X_V/V)\big) = 1$.*

Proof. Let $D_1, D_2, \dots, D_n$ be the connected components of D_V, $a_1, a_2, \dots, a_n \in \mathcal{O}_S(V)$ the corresponding sections. For each $i \in [1, n]$, let $\mathscr{K}_i$ and $\mathscr{I}_i$ be ideal sheaves on V and X_V respectively defined in §5, and put $\mathscr{J}_i \coloneqq \mathscr{K}_i \cdot \mathscr{I}_i$. Then $\mathscr{F}(X_V/V) = \prod_{i=1}^{n} (1 + \mathscr{J}_i)$. So we have only to prove $(f_V)_*\mathscr{J}_i = (0)$. Let W be an open subset of V and $b \in \Gamma(X_W, \mathscr{I}_i)$. By Lemma 6.3, we have $b \in \Gamma(W, \mathcal{O}_S)$. Suppose that $W_b \neq \varnothing$. As $b|_{W_b} = b|_{f^{-1}(W_b)}$ is invertible, $\mathscr{J}_i|_{f^{-1}(W_b)} = (1)$, so $\mathscr{K}_i|_{W_b} = (1)$ and $\mathscr{I}_i|_{f^{-1}(W_b)} = (1)$. As $\mathscr{K}_i = (0 : a)^{\sim}$, $a_i = 0$. Hence $(D_i)_{W_b} \neq \varnothing$, i.e., $\mathscr{I}_i|_{f^{-1}(W_b)} \neq (1)$, a contradiction. Hence $W_b = \varnothing$. Thus for any $w \in W$, $b_{\bar{w}}$ is contained in the maximal ideal of $\mathcal{O}_{S,\bar{w}}$. By Lemma 6.4, $b_{\bar{w}} = 0$. So $b = 0$. □

Definition 6.6. A *semistable log structure* for f is of the form $(\mathscr{M}, \mathscr{N}, \sigma, \tau, \varphi)$, where $\mathscr{M}$ and $\mathscr{N}$ are fine saturated log structures on X and S respectively, $\varphi\colon f^*\mathscr{N} \to \mathscr{M}$ is a morphism of log structures on X, $\sigma\colon \mathscr{M} \to \mathscr{P}$ and $\tau\colon \mathscr{N} \to \mathscr{Q}$ are morphisms of sheaves of monoids, such that σ and τ induce isomorphisms $\bar{\sigma}\colon \overline{\mathscr{M}} \xrightarrow{\sim} \mathscr{P}$ and $\bar{\tau}\colon \overline{\mathscr{N}} \xrightarrow{\sim} \mathscr{Q}$, and the following three diagrams are commutative:

$$\begin{array}{ccc} \mathscr{N} & \longrightarrow & \mathcal{O}_S \\ {\scriptstyle\tau}\downarrow & & \downarrow \\ \mathscr{Q} & \xrightarrow[\vartheta]{} & \mathcal{O}_S/\mathcal{O}_S^* \end{array} \qquad \begin{array}{ccc} \mathscr{M} & \longrightarrow & \mathcal{O}_X \\ {\scriptstyle\sigma}\downarrow & & \downarrow \\ \mathscr{P} & \xrightarrow[\theta]{} & \mathcal{O}_X/\mathcal{O}_X^* \end{array} \qquad \begin{array}{ccc} f^{-1}\mathscr{N} & \longrightarrow & \mathscr{M} \\ {\scriptstyle f^{-1}\tau}\downarrow & & \downarrow{\scriptstyle\sigma} \\ f^{-1}\mathscr{Q} & \xrightarrow[\mathfrak{d}]{} & \mathscr{P} \end{array}$$

The following two theorems are the main results of this papers.

Theorem 6.7.

(1) *There exists a semistable log structure for f if and only if for every point $y \in S$, $\mathscr{L}_{\bar{y}}$ is trivial on $E_{\bar{y}}$.*

(2) *Let $(\mathscr{M}_1, \mathscr{N}_1, \sigma_1, \tau_1, \phi_1)$ and $(\mathscr{M}_2, \mathscr{N}_2, \sigma_2, \tau_2, \phi_2)$ be two semistable log structures for f. Then there exists isomorphism $\varphi\colon \mathscr{M}_1 \xrightarrow{\sim} \mathscr{M}_2$ and $\psi\colon \mathscr{N}_1 \xrightarrow{\sim} \mathscr{N}_2$ of log structures such that $\varphi \circ \phi_1 = \phi_2 \circ f^*\psi$, $\sigma_2 \circ \varphi = \sigma_1$ and $\tau_2 \circ \psi = \tau_1$. Furthermore, such pair (φ, ψ) is unique.*

Proof. (2) Let y be a point on S, $V \in N_0(y)$, and let

$$a_1, a_2, \dots, a_n \in \Gamma(V, \mathcal{O}_S)$$

be sections satisfying the condition (2) in Notation 3.12. Clearly $\mathscr{N}_{\bar{y}} = \mathbb{N}^n$. By Lemma 5.1, we may contract V suitably to make both $\mathscr{N}_1$ and $\mathscr{N}_2$ isomorphic to the log structure associated to

$$\beta_0\colon \mathbb{N}^n_V \to \mathcal{O}_V\,, \qquad \varepsilon_i \mapsto a_i\,,$$

where $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n\}$ is a basis of $\mathbb{N}^n$. In other words, we have an isomorphisms $\psi_0\colon \mathscr{N}_1|_V \xrightarrow{\sim} \mathscr{N}_2|_V$ of log structures, and charts $\beta_i\colon \mathbb{N}^n \to \mathscr{N}_i|_V$ such that $\psi_0 \circ \beta_1 = \beta_2$, $\tau_2|_V \circ \psi_0 = \tau_1|_V$, and the diagram

$$\begin{array}{ccccc} & & \mathbb{N}^n & & \\ & \swarrow & \downarrow{\scriptstyle\beta_i} & \searrow{\scriptstyle\beta_0} & \\ \mathscr{Q}|_V & & & & \mathcal{O}_V \\ & \nwarrow{\scriptstyle\tau_i|_V} & & \nearrow & \\ & & \mathscr{N}_i|_V & & \end{array}$$

is commutative. Since the composite morphisms

$$\mathbb{N}^n_{X_V} \xrightarrow{f_V^*(\beta_i)} f_V^*(\mathscr{N}_i|_V) \xrightarrow{\phi_i|_{X_V}} \mathscr{M}_i|_{X_V}$$

are liftings of

$$\mathbb{N}^n_{X_V} \to f^{-1}\mathscr{Q}|_{X_V} \xrightarrow{\mathfrak{d}|_{X_V}} \mathscr{P}|_{X_V}\,,$$

we see that $(\mathscr{M}_i|_{X_V}, \sigma_i|_{X_V})$ are semistable log structures on X_V. So, by Theorem 5.6 (2), there exists an isomorphism $\varphi_V\colon \mathscr{M}_1|_{X_V} \xrightarrow{\sim} \mathscr{M}_2|_{X_V}$ such that $\varphi_V \circ \sigma_1|_{X_V} = \sigma_2|_{X_V}$. Obviously there exists a section

$$v = (v_1, v_2, \dots, v_n) \in (\mathcal{O}_X^*)^n(X_V)$$

such that

$$\varphi_V \circ \big((\phi_1|_{X_V}) \circ f_V^*(\beta_1)\big) = v \cdot \big((\phi_2|_{X_V}) \circ f_V^*(\beta_2)\big)\,.$$

Taking composite of both sides of above equality with the morphism $\mathscr{M}_2|_{X_V} \to \mathcal{O}_{X_V}$, we have $v_i a_i = a_i$ for each $i \in [1, n]$. Applying Lemma 6.3 to the morphism $X_V \to V$, we have $v_i \in \mathcal{O}_S^*(V)$. By Lemma 5.3, there is an isomorphism $\psi_V : \mathscr{N}_1|_V \xrightarrow{\sim} \mathscr{N}_2|_V$ of log structures such that $\psi_V \circ \beta_1 = v \cdot \beta_2$. Thus $\varphi_V \circ (\phi_1|_{X_V}) = (\phi_2|_{X_V}) \circ f_V^*(\psi_V)$.

Suppose that there exists another pair of isomorphisms $\varphi' : \mathscr{M}_1|_{X_V} \xrightarrow{\sim} \mathscr{M}_2|_{X_V}$ and $\psi' : \mathscr{N}_1|_V \xrightarrow{\sim} \mathscr{N}_2|_V$ of log structures such that $\varphi' \circ (\phi_1|_{X_V}) = (\phi_2|_{X_V}) \circ f_V^*(\psi')$, $(\sigma_2|_{X_V}) \circ \varphi' = \sigma_1|_{X_V}$ and $\tau_2|_V \circ \psi' = \tau_1|_{X_V}$. By Lemma 5.4,

$$\varphi'^{-1} \circ \varphi|_{X_V} \in \Gamma\big(X_V, \mathscr{F}(X_V/V)\big) .$$

By Lemma 6.5, $\varphi'^{-1} \circ \varphi|_{X_V} = \mathrm{id}$, i.e., $\varphi_V = \varphi'$. Thus $(\phi_2|_{X_V}) \circ f_V^*(\psi_V) = (\phi_2|_{X_V}) \circ f_V^*(\psi')$. It is easy to show that ϕ_2 is injective. So $f_V^*(\psi_V) = f_V^*(\psi')$. Since f_V is faithfully flat, we get $\psi_V = \psi'$.

Now we can glue these (φ_V, ψ_V) to a pair of isomorphism of log structures (φ, ψ).

(1) is by (2) and Theorem 5.6. □

Theorem 6.8. *Let $(\mathscr{M}, \mathscr{N}, \sigma, \tau, \phi)$ be a semistable log structure for f. Then*

$$(f, \phi) : (X, \mathscr{M}) \to (Y, \mathscr{N})$$

is log smooth and integral.

Proof. The conclusion is a consequence of Theorem 5.9. □

7. Properties under Base Change

Definition 7.1. Let $f : X \to S$ be a morphism of locally noetherian schemes.

(1) We say that f satisfies (N_1) if it is surjective, proper, weakly normal crossing without powers, and all fibers of f are geometrically connected.
(2) We say that f satisfies (N_2) if it satisfies (N_1) and the condition (†) in §3.3.
(3) We say that f satisfies (N_3) if it satisfies (N_2) and for every point $y \in S$, the invertible sheaf $\mathscr{L}_{\bar{y}}$ on $E_{\bar{y}}$ defined in §6 is trivial.

7.1. **Properties under fibred products.** Let S, X and Y be locally noetherian schemes, $f : X \to S$ and $g : Y \to S$ two morphisms. For an S-scheme Z which satisfies (N_2) and a point s on S, we use $E_{\bar{s}}(Z/S)$ and $\mathscr{L}_{\bar{s}}(Z/S)$ to denote the notations $E_{\bar{s}}$ and $\mathscr{L}_{\bar{s}}$ defined in §6 for preciseness, and write

$$Z(\bar{s}) := Z \times_S \operatorname{Spec} \mathcal{O}_{S,\bar{s}} .$$

Theorem 7.2. *Assume that f and g satisfies (N_1). Then $X \times_S Y \to S$ satisfies (N_1). Furthermore we have*

$$D\big((X \times_S Y)/S\big) = \big(D(X/S) \times_S Y\big) \coprod \big(X \times_S D(Y/S)\big) .$$

Theorem 7.3. *Assume that f and g satisfies (N_2). Then $X \times_S Y \to S$ satisfies (N_2). Furthermore if $s \in S$, then*

$$E_{\bar{s}}(X \times_S Y) = \big(E_{\bar{s}}(X) \times_S Y(\bar{s})\big) \coprod \big(X(\bar{s}) \times_S E_{\bar{s}}(Y)\big) ,$$
$$\mathscr{L}_{\bar{s}}(X \times_S Y) = \big(\mathscr{L}_{\bar{s}}(X) \otimes_S \mathcal{O}_{Y(\bar{s})}\big) \coprod \big(\mathcal{O}_{X(\bar{s})} \otimes_S \mathscr{L}_{\bar{s}}(Y)\big) .$$

Theorem 7.4. *If f and g satisfies (N_3), so does $X \times_S Y \to S$.*

7.2. **Properties under base extension.** Let

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ {\scriptstyle f'}\downarrow & \square & \downarrow{\scriptstyle f} \\ S' & \xrightarrow[q]{} & S \end{array}$$

be a Cartesian square of locally noetherian schemes.

Theorem 7.5. *If f satisfies (N_1), so does f'.*

Lemma 7.6. *Assume that f satisfies (N_1). Let y' be a point on S', $y := q(y')$. Fix a $\kappa(y)$-embedding of $\kappa(y)_s$ into $\kappa(y')_s$. By [7, (18.8.8) (2)], it induces a local homomorphism $u\colon \mathcal{O}_{S,\bar{y}} \to \mathcal{O}_{S',\bar{y}'}$ which makes a commutative diagram*

$$\begin{array}{ccc} \mathcal{O}_{S,y} & \xrightarrow{q_y^{\#}} & \mathcal{O}_{S',y'} \\ \downarrow & & \downarrow \\ \mathcal{O}_{S,\bar{y}} & \xrightarrow[u]{} & \mathcal{O}_{S',\bar{y}'} \end{array}$$

Let $v\colon \operatorname{Spec}\mathcal{O}_{S',\bar{y}'} \to \operatorname{Spec}\mathcal{O}_{S,\bar{y}}$ be the morphism induced by u. Then

(1) *The diagram*

$$\begin{array}{ccc} \operatorname{Spec}\mathcal{O}_{S',\bar{y}'} & \xrightarrow{v} & \operatorname{Spec}\mathcal{O}_{S,\bar{y}} \\ \downarrow & & \downarrow \\ S' & \xrightarrow[q]{} & S \end{array}$$

is commutative.

(2) *The square*

$$\begin{array}{ccc} D_{\bar{y}'} & \xrightarrow{\mathrm{id}\times v} & D_{\bar{y}} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathcal{O}_{S',\bar{y}'} & \xrightarrow[v]{} & \operatorname{Spec}\mathcal{O}_{S,\bar{y}} \end{array}$$

is Cartesian.

(3) *If $X \times_S \operatorname{Spec}\mathcal{O}_{S,\bar{y}} \to \operatorname{Spec}\mathcal{O}_{S,\bar{y}}$ satisfies the condition $(*)$ in §3.2, so does $X' \times_{S'} \operatorname{Spec}\mathcal{O}_{S',\bar{y}'} \to \operatorname{Spec}\mathcal{O}_{S',\bar{y}'}$.*

We assume that $X \times_S \operatorname{Spec}\mathcal{O}_{S,\bar{y}} \to \operatorname{Spec}\mathcal{O}_{S,\bar{y}}$ satisfies the condition $(*)$ in §3.2.

(4) *v induces a canonical bijection*

$$\varphi\colon \mathrm{CP}(y) \xrightarrow{\sim} \mathrm{CP}(y'), \qquad C \mapsto (\mathrm{id}\times v)^{-1}(C).$$

(5) *We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{CP}(y) & \xrightarrow[\sim]{\varphi} & \mathrm{CP}(y') \\ {\scriptstyle \omega_{\bar{y}}}\downarrow & & \downarrow{\scriptstyle \omega_{\bar{y}'}} \\ \mathcal{O}_{S,\bar{y}}/\mathcal{O}^*_{S,\bar{y}} & \xrightarrow[\bar{u}]{} & \mathcal{O}_{S',\bar{y}'}/\mathcal{O}^*_{S',\bar{y}'} \end{array}$$

(6) *There is a canonical closed immersion*

$$E_{\bar{y}'} \hookrightarrow E_{\bar{y}} \times_{\operatorname{Spec}\mathcal{O}_{S,\bar{y}}} \operatorname{Spec}\mathcal{O}_{S',\bar{y}'}.$$

(7) *If q is flat at y', then above morphism is an isomorphism.*

(8) *$\mathscr{L}_{\bar{y}'}$ is isomorphic the inverse image of $\mathscr{L}_{\bar{y}}$ under the morphism $E_{\bar{y}'} \to E_{\bar{y}}$.*

Theorem 7.7. *If f satisfies (N_2) (resp. (N_3)), so does f'.*

Theorem 7.8.

(1) *Assume that f satisfies (N_2). Let $\mathscr{P}$, $\mathscr{Q}$, θ, ϑ and $\mathfrak{d}$ be the notations for f defined in Remark* 3.20*, and $\mathscr{P}'$, $\mathscr{Q}'$, θ', ϑ' and $\mathfrak{d}'$ the corresponding notations for f'. Then there are two canonical isomorphisms of sheaves of monoids $\lambda\colon p^{-1}\mathscr{P} \xrightarrow{\sim} \mathscr{P}'$ and $\mu\colon q^{-1}\mathscr{Q} \xrightarrow{\sim} \mathscr{Q}'$ which makes the following three diagrams commutative.*

$$\begin{tikzcd} p^{-1}\mathscr{P} \arrow[r, "\lambda", "\sim"'] \arrow[d, "p^{-1}\theta"'] & \mathscr{P}' \arrow[d, "\theta'"] \\ p^{-1}\big(\mathcal{O}_X/\mathcal{O}_X^*\big) \arrow[r] & \mathcal{O}_{X'}/\mathcal{O}_{X'}^* \end{tikzcd} \qquad \begin{tikzcd} q^{-1}\mathscr{Q} \arrow[r, "\mu", "\sim"'] \arrow[d, "q^{-1}\vartheta"'] & \mathscr{Q}' \arrow[d, "\vartheta'"] \\ q^{-1}\big(\mathcal{O}_S/\mathcal{O}_S^*\big) \arrow[r] & \mathcal{O}_{S'}/\mathcal{O}_{S'}^* \end{tikzcd}$$

$$\begin{tikzcd} p^{-1}\big(f^{-1}\mathscr{Q}\big) = f'^{-1}\big(q^{-1}\mathscr{Q}\big) \arrow[r, "f'^{-1}\mu", "\sim"'] \arrow[d, "p^{-1}\mathfrak{d}"'] & \mathscr{Q}' \arrow[d, "\mathfrak{d}'"] \\ p^{-1}\mathscr{P} \arrow[r, "\lambda", "\sim"'] & \mathscr{P}' \end{tikzcd}$$

(2) *Assume that f satisfies (N_3). Let $(\mathscr{M}, \mathscr{N}, \sigma, \tau, \varphi)$ and $(\mathscr{M}', \mathscr{N}', \sigma', \tau', \varphi')$ be the semistable log structures for f and f' respectively. Then there exists two isomorphisms $\zeta\colon p^*\mathscr{M} \xrightarrow{\sim} \mathscr{M}'$ and $\eta\colon q^*\mathscr{N} \xrightarrow{\sim} \mathscr{N}'$ of log structures which make the following three diagrams commutative.*

$$\begin{tikzcd} p^*\big(f^*\mathscr{N}\big) = f'^*\big(q^*\mathscr{N}\big) \arrow[r, "f'^*\eta", "\sim"'] \arrow[d, "p^*\varphi"'] & \mathscr{N}' \arrow[d, "\varphi'"] \\ p^*\mathscr{M} \arrow[r, "\zeta", "\sim"'] & \mathscr{M}' \end{tikzcd}$$

$$\begin{tikzcd} p^{-1}\mathscr{M} \arrow[r] \arrow[d, "p^{-1}\sigma"'] & p^*\mathscr{M} \arrow[r, "\zeta", "\sim"'] \arrow[dl, dashed] & \mathscr{M}' \arrow[d, "\sigma'"] \\ p^{-1}\mathscr{P} \arrow[rr, "\lambda", "\sim"'] & & \mathscr{P}' \end{tikzcd} \qquad \begin{tikzcd} q^{-1}\mathscr{N} \arrow[r] \arrow[d, "q^{-1}\tau"'] & q^*\mathscr{N} \arrow[r, "\eta", "\sim"'] \arrow[dl, dashed] & \mathscr{N}' \arrow[d, "\tau'"] \\ q^{-1}\mathscr{Q} \arrow[rr, "\mu", "\sim"'] & & \mathscr{Q}' \end{tikzcd}$$

Moreover the pair (ζ, η) is unique. Simply speaking, the semistable log structure of f' may be viewed as the inverse image of that of f.

The following theorem shows that above isomorphisms are functorial.

Theorem 7.9. *Let*

$$\begin{tikzcd} X_2 \arrow[r, "p_2"] \arrow[d, "f_2"'] \arrow[dr, phantom, "\square"] & X_1 \arrow[r, "p_1"] \arrow[d, "f_1"] \arrow[dr, phantom, "\square"] & X_0 \arrow[d, "f_0"] \\ S_2 \arrow[r, "q_2"'] & S_1 \arrow[r, "q_1"'] & S_0 \end{tikzcd}$$

be a commutative diagram of locally noetherian schemes with both squares Cartesian. Put $p_0 \coloneqq p_1 \circ p_2$ and $q_0 \coloneqq q_1 \circ q_2$.

(1) *Assume that f satisfies (N_2). For each $i = 1, 2, 3$, let $\mathscr{P}_i$, $\mathscr{Q}_i$, θ_i, ϑ_i and $\mathfrak{d}_i$ be the notations for f_i defined in Remark* 3.20*. Let*

$$\lambda_1\colon p_1^{-1}\mathscr{P}_0 \xrightarrow{\sim} \mathscr{P}_1\,, \qquad \lambda_2\colon p_2^{-1}\mathscr{P}_1 \xrightarrow{\sim} \mathscr{P}_2\,, \qquad \lambda_0\colon p_0^{-1}\mathscr{P}_0 \xrightarrow{\sim} \mathscr{P}_2\,,$$
$$\mu_1\colon q_1^{-1}\mathscr{Q}_0 \xrightarrow{\sim} \mathscr{Q}_1\,, \qquad \mu_2\colon q_2^{-1}\mathscr{Q}_1 \xrightarrow{\sim} \mathscr{Q}_2\,, \qquad \mu_0\colon q_0^{-1}\mathscr{Q}_0 \xrightarrow{\sim} \mathscr{Q}_2\,,$$

be the isomorphisms defined in Theorem 7.8 (1)*. Then*

$$\lambda_2 \circ p_2^{-1}(\lambda_1) = \lambda_0 \qquad \textit{and} \qquad \mu_2 \circ q_2^{-1}(\mu_1) = \mu_0\,.$$

(2) *Assume that f satisfies* (N_3). *For each $i = 1, 2, 3$, let $(\mathscr{M}_i, \mathscr{N}_i, \sigma_i, \tau_i, \varphi_i)$ be the semistable log structure for f_i. Let*

$$\zeta_1 \colon p_1^*\mathscr{M}_0 \xrightarrow{\sim} \mathscr{M}_1\,, \qquad \zeta_2 \colon p_2^*\mathscr{M}_1 \xrightarrow{\sim} \mathscr{M}_2\,, \qquad \zeta_0 \colon p_0^*\mathscr{M}_0 \xrightarrow{\sim} \mathscr{M}_2\,,$$
$$\eta_1 \colon q_1^*\mathscr{N}_0 \xrightarrow{\sim} \mathscr{N}_1\,, \qquad \eta_2 \colon q_2^*\mathscr{N}_1 \xrightarrow{\sim} \mathscr{N}_2\,, \qquad \eta_0 \colon q_0^*\mathscr{N}_0 \xrightarrow{\sim} \mathscr{N}_2\,,$$

be the isomorphisms defined in Theorem 7.8 (2). *Then*

$$\zeta_2 \circ p_2^*(\zeta_1) = \zeta_0 \qquad \textit{and} \qquad \eta_2 \circ q_2^*(\eta_1) = \eta_0\,.$$

Theorem 7.10. *Let S_0, S_1 and X_0 be locally noetherian schemes, $f_0 \colon X_0 \to S_0$ and $q \colon S_1 \to S_0$ be two morphisms. Put $S_2 \coloneqq S_1 \times_{S_0} S_1$ and $S_3 \coloneqq S_1 \times_{S_0} S_1 \times_{S_0} S_1$. For each $i = 1, 2, 3$, let $X_i \coloneqq X_0 \times_{S_0} S_i$ and $f_i \colon X_i \to S_i$ the second projections. Assume that both S_2 and S_3 are locally noetherian.*

$$\begin{array}{ccccccc} X_3 & \Rrightarrow & X_2 & \rightrightarrows & X_1 & \longrightarrow & X_0 \\ \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\ S_3 & \Rrightarrow & S_2 & \rightrightarrows & S_1 & \xrightarrow[q]{} & S_0 \end{array}$$

(1) *Assume that f_1 satisfies* (N_2). *Then f_2 and f_3 also satisfy* (N_2). *For each $i = 1, 2, 3$, let $\mathscr{P}_i$, $\mathscr{Q}_i$, θ_i, ϑ_i and $\mathfrak{d}_i$ be the notations for f_i defined in Remark* 3.20. *For each $i = 1, 2$, let $\lambda_i \colon \mathrm{pr}_i^{-1}\mathscr{P}_1 \xrightarrow{\sim} \mathscr{P}_2$ and $\mu_i \colon \mathrm{pr}_i^{-1}\mathscr{Q}_1 \xrightarrow{\sim} \mathscr{Q}_2$ be the isomorphisms corresponding to the i-th projections defined in Theorem* 7.9 (1). *Put*

$$\lambda \coloneqq \lambda_2^{-1} \circ \lambda_1 \colon \mathrm{pr}_1^{-1}\mathscr{P}_1 \to \mathrm{pr}_2^{-1}\mathscr{P}_1\,,$$
$$\mu \coloneqq \mu_2^{-1} \circ \mu_1 \colon \mathrm{pr}_1^{-1}\mathscr{Q}_1 \to \mathrm{pr}_2^{-1}\mathscr{Q}_1\,.$$

Then

$$\mathrm{pr}_{23}^{-1}(\lambda) \circ \mathrm{pr}_{12}^{-1}(\lambda) = \mathrm{pr}_{13}^{-1}(\lambda) \qquad \textit{and} \qquad \mathrm{pr}_{23}^{-1}(\mu) \circ \mathrm{pr}_{12}^{-1}(\mu) = \mathrm{pr}_{13}^{-1}(\mu)\,.$$

(2) *Assume that f_1 satisfies* (N_3). *Then f_2 and f_3 also satisfy* (N_3). *For each $i = 1, 2, 3$, let $(\mathscr{M}_i, \mathscr{N}_i, \sigma_i, \tau_i, \varphi_i)$ be the semistable log structure for f_i. For each $i = 1, 2$, let $\zeta_i \colon \mathrm{pr}_i^*\mathscr{M}_1 \xrightarrow{\sim} \mathscr{M}_2$ and $\eta_i \colon \mathrm{pr}_i^*\mathscr{N}_1 \xrightarrow{\sim} \mathscr{N}_2$ be the isomorphisms corresponding to the i-th projections defined in Theorem* 7.9 (2). *Put*

$$\zeta \coloneqq \zeta_2^{-1} \circ \zeta_1 \colon \mathrm{pr}_1^*\mathscr{M}_1 \xrightarrow{\sim} \mathrm{pr}_2^*\mathscr{M}_1\,,$$
$$\mu \coloneqq \mu_2^{-1} \circ \mu_1 \colon \mathrm{pr}_1^*\mathscr{N}_1 \xrightarrow{\sim} \mathrm{pr}_2^*\mathscr{N}_1\,.$$

Then

$$\mathrm{pr}_{23}^*(\zeta) \circ \mathrm{pr}_{12}^*(\zeta) = \mathrm{pr}_{13}^*(\zeta) \qquad \textit{and} \qquad \mathrm{pr}_{23}^*(\eta) \circ \mathrm{pr}_{12}^*(\eta) = \mathrm{pr}_{13}^*(\eta)\,.$$

Proof. For each $i = 1, 2, 3$, let $\lambda'_i \colon \mathrm{pr}_i^{-1}\mathscr{P}_1 \xrightarrow{\sim} \mathscr{P}_3$ be the isomorphism corresponding to the i-th projection. For each $1 \leqslant i < j \leqslant 3$, let $\lambda''_{ij} \colon \mathrm{pr}_{ij}^{-1}\mathscr{P}_2 \xrightarrow{\sim} \mathscr{P}_3$ be the isomorphism corresponding to $\mathrm{pr}_{ij} \colon X_3 \to X_2$. By Theorem 7.9, we have

$$\lambda''_{ij} \circ \mathrm{pr}_{ij}^{-1}(\lambda_1) = \lambda'_i \qquad \text{and} \qquad \lambda''_{ij} \circ \mathrm{pr}_{ij}^{-1}(\lambda_2) = \lambda'_j\,.$$

Thus

$$\mathrm{pr}_{ij}^{-1}(\lambda) = \mathrm{pr}_{ij}^{-1}(\lambda_2^{-1}) \circ \mathrm{pr}_{ij}^{-1}(\lambda_1) = (\lambda'_j)^{-1} \circ \lambda'_j\,.$$

The other three equations are similar. □

7.3. **Properties under inverse limit.**

Theorem 7.11. *Let S_0 be a noetherian scheme, $(S_\lambda, s_{\mu\lambda})_{\lambda,\mu\in\Lambda}$ an inverse system of noetherian affine S_0-schemes such that all $s_{\mu\lambda}$ are affine morphisms. Let (S, s_λ) be its inverse limit. Assume that S is also noetherian.*

(1) *Let $f_0 \colon X_0 \to S_0$ be a morphism of finite type. Put $X := X_0 \times_{S_0} S$ and $f := (f_0)_S \colon X \to S$. For each $\lambda \in \Lambda$, put $X_\lambda := X_0 \times_{S_0} S_\lambda$ and $f_\lambda := (f_0)_{S_\lambda} \colon X_\lambda \to S_\lambda$. Then f satisfies (N_1) (resp. (N_2) or (N_3)) if and only if there exists an index $\lambda_0 \in \Lambda$ such that for any $\lambda \geqslant \lambda_0$, f_λ satisfies (N_1) (resp. (N_2) or (N_3)).*

(2) *Let $f \colon X \to S$ be a morphism which satisfies (N_1) (resp. (N_2) or (N_3)). Then there exists an index $\lambda_0 \in \Lambda$ and a morphism $f_{\lambda_0} \colon X_{\lambda_0} \to S_{\lambda_0}$ which satisfies (N_1) (resp. (N_2) or (N_3)) such that X is S-isomorphic to $X_{\lambda_0} \times_{S_{\lambda_0}} S$.*

Proof. See [7, §8]. Note that every local chart (resp. refined local chart) of f can be descended to some index $\lambda_0 \in \Lambda$ and X can be covered by a finite number of local chart (resp. refined local chart) of f. □

Corollary 7.12. *Let $f \colon X \to S$ be a morphism of finite type of locally noetherian schemes. Then f satisfies (N_1) (resp. (N_2) or (N_3)) if and only if for every point $y \in S$, $X \times_S \operatorname{Spec} \mathcal{O}_{S,\bar{y}} \to \operatorname{Spec} \mathcal{O}_{S,\bar{y}}$ satisfies (N_1) (resp. (N_2) or (N_3)).*

7.4. **Properties under flat descent.**

Lemma 7.13. *Let $X' \to X$ be a faithfully flat morphism locally of finite presentation of schemes, $U \to X$ an étale morphism of schemes, $\mathscr{M}$ a fine saturated log structure on X. Put $X'' := X' \times_X X'$, and let $\mathscr{M}'$ and $\mathscr{M}''$ be the pull-back of $\mathscr{M}$ on X' and X'' respectively. Then*

$$\mathscr{M}(U) \longrightarrow \mathscr{M}'(U \times_X X') \rightrightarrows \mathscr{M}''(U \times_X X'')$$

is exact.

Proof. [15, Lemma 1.1.3]. □

Lemma 7.14. *Let $p \colon X' \to X$ be a faithfully flat morphism locally of finite presentation of schemes. Put $X'' := X' \times_X X'$ and $X''' := X' \times_X X' \times_X X'$. Let $\mathscr{M}$ be a fine saturated log structure on X' and $\phi \colon \mathrm{pr}_1^*\mathscr{M} \xrightarrow{\sim} \mathrm{pr}_2^*\mathscr{M}$ an isomorphism of log structures on X'' such that on X''' we have*

$$\mathrm{pr}_{13}^*(\phi) = \mathrm{pr}_{23}^*(\phi) \circ \mathrm{pr}_{12}^*(\phi)\,.$$

Then there exists a unique (up to isomorphism) pair $(\mathscr{N}, s)$ on X, where $\mathscr{N}$ is a fine saturated log structure on X and $s \colon p^*\mathscr{N} \xrightarrow{\sim} \mathscr{M}$ is an isomorphism of log structures on X', such that the following diagram

$$\begin{array}{ccc} \mathrm{pr}_1^* p^*\mathscr{N} & \xrightarrow{\mathrm{pr}_1^*(s)} & \mathrm{pr}_1^*\mathscr{M} \\ \| & & \downarrow \phi \\ \mathrm{pr}_2^* p^*\mathscr{N} & \xrightarrow{\mathrm{pr}_2^*(s)} & \mathrm{pr}_2^*\mathscr{M} \end{array}$$

is commutative.

Proof. [15, Theorem 1.1.5]. □

Lemma 7.15. *Let $f \colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ be a morphism of fine saturated log schemes, $\mathscr{C}$ the cokernel of $f^*\mathscr{N} \to \mathscr{M}$, $\bar{x}$ a geometric point on X, $\bar{y} := f(\bar{x})$, $Q \to \mathscr{N}$ a chart with Q fine saturated. Assume that*

(1) *$Q \to \overline{\mathscr{N}}_{\bar{y}}$ is isomorphic;*
(2) *$\overline{\mathscr{N}}_{\bar{y}} \to \overline{\mathscr{M}}_{\bar{x}}$ is injective;*

(3) *the torsion part of* $\mathscr{C}_{\bar{x}}^{\mathrm{gp}}$ *is a finite group of order invertible in* $\kappa(\bar{x})$.

Then there exists a chart

$$(P \to \mathscr{M}|_U, Q \to \mathscr{N}|_V, Q \to P)$$

of f *at* $\bar{x}$ *such that* $P \to \overline{\mathscr{M}}_{\bar{x}}$ *is isomorphic.*

Proof. [14, Theorem 2.13]. □

Lemma 7.16. *If* S *is a scheme,* x *is a point on* S*, and* $\mathscr{F}$ *is an* $\mathcal{O}_S$*-module, we define* $\mathscr{F}(x) \coloneqq \mathscr{F}_{\bar{x}} \otimes \kappa(\bar{x})$.

Let $X^\dagger = (X, \mathscr{M})$ *and* $Y^\dagger = (Y, \mathscr{N})$ *be two fine log schemes,* $f\colon X^\dagger \to Y^\dagger$ *a morphism of log schemes,* $x \in X$ *a point,* $\mathscr{C}$ *the cokernel of the morphism* $f^*\mathscr{N} \to \mathscr{M}$*. Then there is a commutative diagram*

$$\begin{array}{ccccc} & & \mathscr{M}_{\bar{x}}^{\mathrm{gp}} & \longrightarrow & \mathscr{C}_{\bar{x}}^{\mathrm{gp}} \\ & & \downarrow{\scriptstyle d\log} & \searrow^{\pi} & \downarrow{\scriptstyle \bar{\pi}} \\ \Omega_{X/Y}(x) & \longrightarrow & \Omega_{X^\dagger/Y^\dagger}(x) & \xrightarrow[\rho_{X^\dagger/Y^\dagger,\bar{x}}]{} & \kappa(\bar{x}) \otimes_{\mathbb{Z}} \mathscr{C}_{\bar{x}}^{\mathrm{gp}}, \end{array}$$

where $\bar{\pi}(m) = 1 \otimes m$ *for each* $m \in \mathscr{C}_{\bar{x}}^{\mathrm{gp}}$*, and the bottom row is an exact sequence of linear space over* $\kappa(\bar{x})$.

The homomorphism $\rho_{X^\dagger/Y^\dagger,\bar{x}}$ *is sometimes called the* Poincaré residue mapping *at* x.

Proof. See [14, Proposition 2.22]. □

Lemma 7.17. *Let* $f\colon (X, \mathscr{M}) \to (Y, \mathscr{N})$ *be a log smooth morphism of fine saturated log schemes,* $x \in X$*,* $y \coloneqq f(x)$*, and let*

$$(P \to \mathscr{M}, Q \to \mathscr{N}, Q \xrightarrow{\iota} P)$$

be a chart of f*. Assume that* $\iota\colon Q \to P$ *is injective,* $P \xrightarrow{\sim} \overline{\mathscr{M}}_{\bar{x}}$ *and* $Q \xrightarrow{\sim} \overline{\mathscr{N}}_{\bar{y}}$ *are isomorphisms. Put* $Z \coloneqq Y \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P]$ *and* $g\colon X \to Z$ *the induced morphism. Then* g *is smooth at* $\bar{x}$.

Proof. In this proof, for a log scheme $(X, \mathscr{M})$, we use simply $X^\dagger$ to denote it. First we have

$$\Omega_{Z^\dagger/Y^\dagger} = \mathcal{O}_Y \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}).$$

Hence

$$\kappa(\bar{x}) \otimes (g^*\Omega_{Z^\dagger/Y^\dagger}) \xrightarrow{\sim} \kappa(\bar{x}) \otimes_{\mathbb{Z}} (P^{\mathrm{gp}}/Q^{\mathrm{gp}}).$$

Let $\mathscr{C}$ denote the cokernel of the morphism $f^*\mathscr{N} \to \mathscr{M}$. Then the composite

$$\kappa(\bar{x}) \otimes (g^*\Omega_{Z^\dagger/Y^\dagger}) \to \kappa(\bar{x}) \otimes \Omega_{X^\dagger/Y^\dagger} \xrightarrow{\rho_{X^\dagger/Y^\dagger,\bar{x}}} \kappa(\bar{x}) \otimes_{\mathbb{Z}} \mathscr{C}_{\bar{x}}^{\mathrm{gp}}$$

is isomorphic, where $\rho_{X^\dagger/Y^\dagger,\bar{x}}$ is defined in Lemma 7.16. Hence

$$\kappa(\bar{x}) \otimes (g^*\Omega_{Z^\dagger/Y^\dagger}) \to \kappa(\bar{x}) \otimes \Omega_{X^\dagger/Y^\dagger}$$

is injective. Hence $g^*\Omega_{Z^\dagger/Y^\dagger} \to \Omega_{X^\dagger/Y^\dagger}$ has a left inverse in some open neighborhood U of x. So $g|_U\colon U \to Z$ is smooth. □

Theorem 7.18. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & S \end{array}$$

be a Cartesian square of locally noetherian schemes such that q *is faithfully flat and locally of finite type, and* f' *satisfies* (N_3)*. Then* f *satisfies* (N_3) *too.*

Proof. Let $\mathscr{P}'$, $\mathscr{Q}'$, θ', ϑ' and $\mathfrak{d}'$ be the notations for f' defined in Remark 3.20; and let $(\mathscr{M}', \mathscr{N}', \sigma', \tau', \varphi')$ be the semistable log structure of f'. By Theorem 7.10 and Lemma 7.14, there exist fine saturated log structures $\mathscr{M}$ on X and $\mathscr{N}$ on S, a morphism of log structures $\varphi\colon f^*\mathscr{N} \to \mathscr{M}$, two isomorphisms of log structures $p^*\mathscr{M} \xrightarrow{\sim} \mathscr{M}'$ and $q^*\mathscr{N} \xrightarrow{\sim} \mathscr{N}'$ which makes a commutative diagram.

$$\begin{array}{ccc} f'^*q^*\mathscr{N} = p^*f^*\mathscr{N} & \xrightarrow{p^*(\varphi)} & p^*\mathscr{M} \\ \cong\downarrow & & \downarrow\cong \\ f'^*\mathscr{N}' & \xrightarrow[\varphi']{} & \mathscr{M}' \end{array}$$

Let x be a point on X and $y := f(x)$. Let $x' \in p^{-1}(x)$ and put $y' := f'(x')$. Then $q(y') = y$. We have

$$\overline{\mathscr{M}}_{\bar{x}} \cong \overline{\mathscr{M}'}_{\bar{x}'} \cong \mathbb{N}^m \qquad \text{and} \qquad \overline{\mathscr{N}}_{\bar{y}} \cong \overline{\mathscr{N}'}_{\bar{y}'} \cong \mathbb{N}^n$$

for some $m, n \in \mathbb{N}$. Furthermore the homomorphism

$$d := \bar{\varphi}_{\bar{x}}\colon \overline{\mathscr{N}}_{\bar{y}} \to \overline{\mathscr{M}}_{\bar{x}}$$

is defined as $d(\varepsilon_i) = \sum_{j=1}^{s_i} \eta_{ij}$ for $i \in [1, r]$ and $d(\varepsilon_i) = \eta_i$ for $i \in [r+1, n]$, where $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ is a basis of $\mathbb{N}^n$,

$$\{\eta_{11}, \ldots, \eta_{1s_1}, \ldots, \eta_{r1}, \ldots, \eta_{rs_r}, \eta_{r+1}, \ldots, \eta_n\}$$

is a basis of $\mathbb{N}^m$, and $m = \sum_{i=1}^{r} s_i + n - r$. By Lemma 7.15, there exists a chart

$$(\mathbb{N}^m \to \mathscr{M}|_U, \mathbb{N}^n \to \mathscr{N}|_V, \mathbb{N}^n \xrightarrow{d} \mathbb{N}^m)$$

of f at $\bar{x}$ such that $\mathbb{N}^m \xrightarrow{\sim} \overline{\mathscr{M}}_{\bar{x}}$ and $\mathbb{N}^n \xrightarrow{\sim} \overline{\mathscr{N}}_{\bar{y}}$ are isomorphic. Put $U' := U \times_X X'$ and $V' := V \times_S S'$. By Lemma 7.17,

$$U' \to V' \times_{\operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]} \operatorname{Spec}\mathbb{Z}[\mathbb{N}^m]$$

is smooth at $\bar{x}'$. By [7, (17.7.1)],

$$U \to V \times_{\operatorname{Spec}\mathbb{Z}[\mathbb{N}^n]} \operatorname{Spec}\mathbb{Z}[\mathbb{N}^m]$$

is smooth at $\bar{x}$. Therefore we may contract U/V suitable to obtain a local chart at x. Furthermore it is easy to verify that $(\mathscr{M}, \mathscr{N}, \varphi)$ is just the semistable log structure of f. □

8. Semistable Curves

Definition 8.1. Let k be a separably closed field. A *semistable curve* over k is a connected proper 1-equidimensional k-scheme X such that for any closed point $x \in X$, either X is smooth at x over k, or $\widehat{\mathcal{O}}_{X,x}$ is k-isomorphic to $k[[T_1, T_2]]/(T_1T_2)$.

Lemma 8.2. *Let k be a separably closed field and X a semistable curve over k. Then*

(1) *X is reduced.*
(2) *X has only a finite number of singular points and all singular points are k-rational.*

Definition 8.3. Let S be a scheme. A *semistable curve* over S is an S-scheme $f\colon X \to S$ such that f is proper, faithfully flat, of finite presentation, and every geometric fiber is a semistable curve in the sense of Definition 8.1.

Remark 8.4. The notation of *semistable curve* here is slightly weaker than the notation of *stable curve* in [2].

Lemma 8.5. *Let S be a locally noetherian scheme, $f\colon X \to S$ a morphism which satisfies* (N_1) *in Definition* 7.1 *and is of relative dimension* 1. *Then f also satisfies* (N_3) *in Definition* 7.1.

Proof. By Corollary 7.12, we may assume that S is the spectrum of a strictly Henselian ring R. Let $\mathfrak{m}$ be the maximal ideal of R and y_1 the closed point of S. If f is smooth, the question is trivial. Assume that f is not smooth. Then $D(f) \neq \varnothing$. Let $D_1, D_2, \ldots, D_n$ be the connected components of $D(f)$. For each $i \in [1,n]$, select an element $a_i \in \mathfrak{m}$ such that $\bar{a}_i = \omega_{\bar{y}_1}(D_i)$. Then $D_i = \operatorname{Spec} R/(a_i)$. Let z_i be the closed point of D_i and let $l_i \colon D_i \to X$ be the inclusion. Then there exists a refined local chart at $l_i(z_i)$ of the form

$$(U_i, S; T_{i1}, T_{i2}; a_1, a_2, \ldots, a_n)$$

satisfying that $U_i \times_X D_j = \varnothing$ for all $j \in [1,n] - \{i\}$. Let $\mathscr{M}_i$ be the log structure on U_i associate to $\alpha'_i \colon \mathbb{N}^{n+1}_{U_i} \to \mathcal{O}_{U_i}$, where if

$$\eta_{i1}, \eta_{i2}, \eta_1, \ldots, \hat{\eta}_i, \ldots, \eta_n$$

is a basis of $\mathbb{N}^{n+1}$, then $\alpha'_i(\eta_{ij}) = T_{ij}$ for $j = 1, 2$, and $\alpha'_i(\eta_k) = a_k$ for $k \in [1,n] - \{i\}$. Let $\alpha_i \colon \mathbb{N}^{n+1}_{U_i} \to \mathscr{M}_i$ be the induced morphism. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be a basis of $\mathbb{N}^n$. We define three homomorphisms of monoids

$$\partial_i, \partial_{i1}, \partial_{i2} \colon \mathbb{N}^n \to \mathbb{N}^{n+1}$$

as follows: for $k \in [1,n] - \{i\}$,

$$\partial_i(\varepsilon_k) = \partial_{i1}(\varepsilon_k) = \partial_{i2}(\varepsilon_k) = \eta_k \,,$$

and

$$\partial_i(\varepsilon_i) = \eta_{i1} + \eta_{i2} \,, \qquad \partial_{i1}(\varepsilon_i) = \eta_{i1} \,, \qquad \partial_{i2}(\varepsilon_i) = \eta_{i2} \,.$$

Then

$$\rho_i \coloneqq \alpha_i \circ \partial_i \colon \mathbb{N}^n_{U_i} \to \mathscr{M}_i$$

is a lifting of $\gamma|_{U_i} \colon \mathbb{N}^n_{U_i} \to \mathscr{P}|_{U_i}$, where Let $\gamma \colon \mathbb{N}^n_X \to \mathscr{P}$ be the notation defined in §5. Put $U_0 \coloneqq X - D(f)$ and $\mathscr{M}_0$ the log structure on U_0 induced by

$$\mathbb{N}^n_{U_0} \to \mathcal{O}_{U_0} \,, \qquad \varepsilon_i \mapsto a_i \,.$$

Let $\rho_0 \colon \mathbb{N}^n_{U_0} \to \mathscr{M}_0$ be the induced morphism. Then ρ_0 is a lifting of $\gamma|_{U_0} \colon \mathbb{N}^n_{U_0} \to \mathscr{P}|_{U_0}$.

For any $i \in [1,n]$ and any point x on $U_0 \times_X U_i$, there exists an étale neighborhood W of $\bar{x}$ such that $T_{i1}|_W$ or $T_{i2}|_W$ is invertible. Without lose of generality, we may assume that $T_{i2}|_W$ is invertible. Then $T_{i1}|_W = (T_{i2}|_W)^{-1} \cdot a_i$ and

$$\alpha_i|_W \circ \partial_{i1} \colon \mathbb{N}^n \to \mathscr{M}_i|_W$$

is a chart of $\mathscr{M}_i|_W$. So there exists an isomorphism $\phi \colon \mathscr{M}_i|_W \xrightarrow{\sim} \mathscr{M}_0|_W$ of log structures such that

$$\phi \circ \alpha_i|_W \circ \partial_{i1} = (u_1, u_2, \ldots, u_n) \cdot \rho_0|_W \,,$$

where $u_i = (T_{i2}|_W)^{-1}$ and $u_k = 1$ for $k \in [1,n] - \{i\}$. Thus $\phi \circ \rho_i|_W = \rho_0|_W$.

For any pair of integers $1 \leqslant i < j \leqslant n$ and any point x on $U_i \times_X U_j$, there exists an étale neighborhood W of $\bar{x}$ such that $T_{i1}|_W$ or $T_{i2}|_W$ is invertible, and $T_{j1}|_W$ or $T_{j2}|_W$ is invertible. Without lose of generality, we may assume that $T_{i2}|_W$ and $T_{j2}|_W$ are invertible. Then for $s = i, j$, $T_{s1}|_W = (T_{s2}|_W)^{-1} \cdot a_s$ and

$$\alpha_s|_W \circ \partial_{s1} \colon \mathbb{N}^n \to \mathscr{M}_s|_W$$

is a chart of $\mathscr{M}_s|_W$. So there exists an isomorphism $\phi \colon \mathscr{M}_i|_W \xrightarrow{\sim} \mathscr{M}_j|_W$ of log structures such that

$$\phi \circ \alpha_i|_W \circ \partial_{i1} = (u_1, u_2, \ldots, u_n) \cdot (\alpha_j|_W \circ \partial_{j1}) \,,$$

where $u_i = (T_{i2}|_W)^{-1}$, $u_j = T_{j2}|_W$, and $u_k = 1$ for $k \in [1,n] - \{i,j\}$. Thus $\phi \circ \rho_i|_W = \rho_j|_W$.

Now we translate above analysis into the language in §5. We obtains an étale covering $U \to X$, an object $\mathscr{M}$ in $\mathfrak{X}(U)$, and a morphism $\rho \colon \mathbb{N}^n_U \to \mathscr{M}$ which is a lifting of $\gamma|_U \colon \mathbb{N}^n_U \to \mathscr{P}|_U$, an étale covering $V \to U \times_X U$, and an isomorphism $\phi \colon p^*_{10}(\mathscr{M}) \xrightarrow{\sim} p^*_{11}(\mathscr{M})$ of log structures on V, such that $\phi \circ p^*_{10}(\rho) = p^*_{11}(\rho)$. So $\mathscr{L}_{\bar{y}_1}$ is trivial. □

Lemma 8.6. *Let R, A be two complete noetherian local rings with maximal ideals $\mathfrak{m}, \mathfrak{M}$ and residue fields k, K respectively, $R \to A$ a flat local homomorphism, $R[[T_1, T_2]]$ a ring of power series over R with variables T_1 and T_2. Assume that exists two elements $x_{11}, x_{12} \in \mathfrak{M}$ such that the homomorphism of k-algebras*

$$k[[T_1, T_2]]/(T_1T_2) \xrightarrow{\sim} A/\mathfrak{m}A\,, \qquad T_i \mapsto \bar{x}_{1i}\ (i = 1, 2)$$

is an isomorphism. Then there exists an element $a \in \mathfrak{m}$, two elements $x_1, x_2 \in \mathfrak{M}$ such that the homomorphism of R-algebras

$$R[[T_1, T_2]]/(T_1T_2 - a) \xrightarrow{\sim} A\,, \qquad T_i \mapsto x_i\ (i = 1, 2)$$

is an isomorphism.

Proof. Put $a_1 := 0$ and $P_1 := k[[T_1, T_2]]/(T_1T_2)$. For each $n \in \mathbb{N}$, put $R_n := R/\mathfrak{m}^n$ and $A_n := A/\mathfrak{m}^n A$. Let $\psi_1\colon P_1 \xrightarrow{\sim} A_1$ be the isomorphism defined in the lemma. Assume that we have found $a_n \in R$ and $x_{n1}, x_{n2} \in A$ such that

$$\psi_n\colon P_n := (R/\mathfrak{m}^n)[[T_1, T_2]]/(T_1T_2 - \bar{a}_n) \xrightarrow{\sim} A_n\,, \qquad T_i \mapsto \bar{x}_{ni}\ (i = 1, 2)$$

is an isomorphism. Then

$$z := x_{n1}x_{n2} - a_n \in \mathfrak{m}^n A\,.$$

Put $z = \sum_{j=1}^{n} b_j z_j$, where $b_j \in \mathfrak{m}^n$ and $z_j \in A$. Obviously we have $A = R + \mathfrak{M}$ and $\mathfrak{M} = \mathfrak{m}A + x_{n1}A + x_{n2}A$. So for each $j \in [1, n]$, we may write z_j as

$$z_j = c_j + d_j u_j + x_{n1} v_j + x_{n2} w_j\,,$$

where $c_j \in R$, $d_j \in \mathfrak{m}$, $u_j, v_j, w_j \in A$. Put

$$\begin{aligned} x_{n+1,1} &:= x_{n1} - \sum_{j=1}^{n} b_j w_j\,, \\ x_{n+1,2} &:= x_{n2} - \sum_{j=1}^{n} b_j v_j\,, \\ a_{n+1} &:= a_n + \sum_{j=1}^{n} b_j c_j\,. \end{aligned}$$

Then we have

$$x_{n+1,1}x_{n+1,2} - a_{n+1} = \sum_{j=1}^{n} b_j d_j u_j + \Big(\sum_{j=1}^{n} b_j v_j\Big)\Big(\sum_{j=1}^{n} b_j w_j\Big) \in \mathfrak{m}^{n+1}A\,.$$

Put

$$P_{n+1} := (R/\mathfrak{m}^{n+1})[[T_1, T_2]]/(T_1T_2 - \bar{a}_{n+1})$$

and let $\psi_{n+1}\colon P_{n+1} \to A_{n+1}$ be the homomorphism of R-algebras defined by $\psi_{n+1}(T_i) = \bar{x}_{n+1,i}$ for $i = 1, 2$. Obviously ψ_{n+1} is surjective. We shall prove that ψ_{n+1} is injective. Assume that $\mathfrak{I} := \mathrm{Ker}(\psi_{n+1}) \neq 0$. If $\mathfrak{m} = \mathfrak{m}^{n+1}$, then $\psi_{n+1} = \psi_1$ is an isomorphism. So we may assume that $\mathfrak{m}^{n+1} \neq \mathfrak{m}$. Since the diagram

$$\begin{array}{ccc} P_{n+1} & \xrightarrow{\psi_{n+1}} & A_{n+1} \\ \downarrow & & \downarrow \\ P_n & \xrightarrow{\psi_n} & A_n \end{array}$$

is commutative, $\mathfrak{I}$ is contained in $(\mathfrak{m}^n/\mathfrak{m}^{n+1})\cdot P_{n+1}$. Since both P_{n+1} and A_{n+1} are flat over R_{n+1}, so is $\mathfrak{I}$. As $\mathfrak{I}$ is a finitely generated P_{n+1}-module, by [11, Ch. 2, COROLLARY of (4.A)], $\mathfrak{I}$ is faithfully flat over R_{n+1}. And because $\mathfrak{m}/\mathfrak{m}^{n+1}\neq 0$,

$$(\mathfrak{m}/\mathfrak{m}^{n+1})\cdot\mathfrak{I}\cong(\mathfrak{m}/\mathfrak{m}^{n+1})\otimes_R\mathfrak{I}\neq 0$$

by [11, Ch. 2, (4.A)]. But it contradicts that

$$(\mathfrak{m}/\mathfrak{m}^{n+1})\cdot\mathfrak{I}\subseteq(\mathfrak{m}/\mathfrak{m}^{n+1})\cdot(\mathfrak{m}^n/\mathfrak{m}^{n+1})\cdot P_{n+1}=0\,.$$

Thus ψ_{n+1} is an isomorphism.

Clearly $\{a_n\}$ is a Cauchy sequence in R, and $\{x_{n1}\}$ and $\{x_{n2}\}$ are Cauchy sequences in A. Since R and A are complete, we may let $a\coloneqq\lim a_n$ and $x_i\coloneqq\lim x_{ni}$ for $i=1,2$. □

Lemma 8.7. *Let R be a ring, A and B two R-algebras, $\mathfrak{a}$, I and J be ideals of R, A and B respectively such that $\mathfrak{a}A\subseteq I$ and $\mathfrak{a}B\subseteq J$. Let C and D denote the topological R-algebras $A\otimes_R B$ and $\widehat{A}\otimes_{\widehat{R}}\widehat{B}$ equipped with $(IC+JC)$-adic and $(ID+JD)$-adic topologies. Then $\widehat{C}\cong\widehat{D}$.*

Proof. Note that for all $n\in\mathbb{N}$,

$$I^nC+J^nC\subseteq(IC+JC)^n$$

and

$$(IC+JC)^{2n}\subseteq I^nC+J^nC\,.$$

Thus

$$\widehat{C}\cong\varprojlim C/(I^nC+J^nC)\cong\varprojlim(A/I^n)\otimes_{R/\mathfrak{a}^n}(B/J^n)\,.$$

Similarly we have

$$\widehat{D}\cong\varprojlim(\widehat{A}/I^n\widehat{A})\otimes_{\widehat{R}/\mathfrak{a}^n\widehat{R}}(\widehat{B}/J^n\widehat{B})\cong\varprojlim(A/I^n)\otimes_{R/\mathfrak{a}^n}(B/J^n)\,.$$

Hence $\widehat{C}\cong\widehat{D}$. □

Lemma 8.8. *Let S be a scheme of finite type over a field or an excellent dedekind domain, X_1 and X_2 two S-schemes of finite type, $x_1\in X_1$ and $x_2\in X_2$ two points which map onto the same point s on S. Assume that $\widehat{\mathcal{O}}_{X_1,x_1}$ and $\widehat{\mathcal{O}}_{X_2,x_2}$ are $\mathcal{O}_{S,s}$-isomorphic. Then there exists an S-scheme U, a point $u\in U$, two étale S-morphisms $\varphi_1\colon U\to X_1$ and $\varphi_2\colon U\to X_2$, such that $\varphi_i(u)=x_i$ and $\kappa(x_i)\xrightarrow{\sim}\kappa(u)$ for $i=1,2$.*

Proof. See [1, (2.6)]. □

Remark 8.9. Note that $\mathbb{Z}$ is an excellent dedekind domain. So to use this lemma, we usually apply the inverse limit of schemes to descend the base scheme to become of finite type over $\mathbb{Z}$.

Lemma 8.10. *Let A be a strictly Henselian noetherian local ring, $S\coloneqq\operatorname{Spec}A$, s the closed point of S, $f\colon X\to S$ a faithfully flat, proper morphism such that X_s is a semistable curve over $\kappa(s)$. Then X is a semistable curve over S and satisfies* (N_3) *in Definition* 7.1.

Proof. Let $\mathfrak{m}$ be the maximal ideal of A and $k\coloneqq A/\mathfrak{m}$ the residue field. If X_s is smooth over k, then X is smooth over S and the lemma is valid. So we may assume that X_s is not smooth over k. Let $x_1,x_2,\dots,x_n$ be all singular points of X_s. By Lemma 8.2, all x_i are k-rational. So x_i defines a closed immersion $\gamma_i\colon\operatorname{Spec}k\to X_s$. As $\widehat{\mathcal{O}}_{X_s,x_i}\cong k[[T_1,T_2]]/(T_1T_2)$, by Lemma 8.8, there exists a k-scheme V_i of finite type, a point y_i on V_i, two étale k-morphisms $p_i\colon V_i\to X_s$ and

$$q_i\colon V_i\to\operatorname{Spec}k[T_1,T_2]/(T_1T_2)$$

such that $p_i(y_i)=x_i$ and $q_i(y_i)=0$, and $\kappa(y_i)=\kappa(x_i)=k$. So y_i is a k-rational point on V_i, and it defines a closed immersion $\delta_i\colon\operatorname{Spec}k\to V_i$. Let U denote the set of points at which X is smooth over S. Then U is open in X and

$$U_s=X_s-\{x_1,x_2,\dots,x_n\}\,.$$

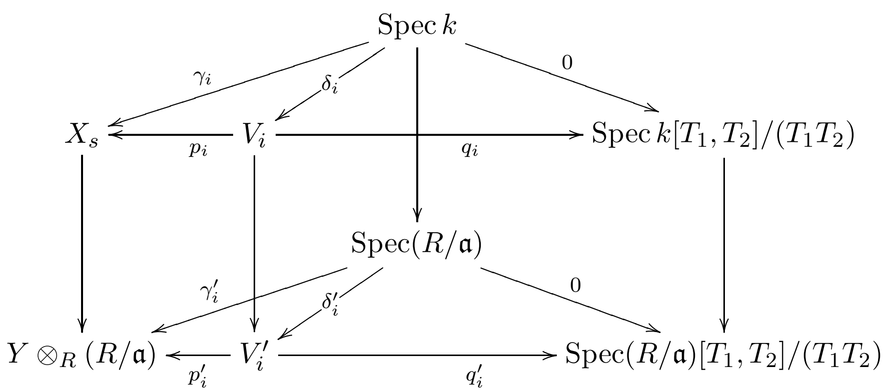
By [7, §(8.8), §(8.6), (8.10.5), (11.2.6), (17.7.8)], there exists a finitely generated $\mathbb{Z}$-subalgebra R of A, a proper and faithfully flat R-scheme Y such that X is S-isomorphic to $Y \otimes_R A$, an open subscheme U' of Y such that U' is smooth over R and U is the inverse image of U' under the morphism $X \to Y$, an ideal $\mathfrak{a}$ of R such that $\mathfrak{a}A = \mathfrak{m}$; and for each $i \in [1, n]$, an $(R/\mathfrak{a})$-scheme V_i' of finite type, two closed $(R/\mathfrak{a})$-immersions $\gamma_i' \colon \operatorname{Spec}(R/\mathfrak{a}) \to Y \otimes_R (R/\mathfrak{a})$ and $\delta_i' \colon \operatorname{Spec}(R/\mathfrak{a}) \to V_i'$ such that

$$\big(U' \otimes_R (R/\mathfrak{a})\big) \coprod \underbrace{\operatorname{Spec}(R/\mathfrak{a}) \coprod \cdots \coprod \operatorname{Spec}(R/\mathfrak{a})}_{n} \xrightarrow{\gamma_0 \coprod \gamma_1' \coprod \cdots \coprod \gamma_n'} Y \otimes_R (R/\mathfrak{a}) \tag{8.1}$$

is surjective where $\gamma_0 \colon U' \otimes_R (R/\mathfrak{a}) \to Y \otimes_R (R/\mathfrak{a})$ is the inclusion, two étale $(R/\mathfrak{a})$-morphisms $p_i' \colon V_i' \to Y \otimes_R (R/\mathfrak{a})$ and

$$q_i' \colon V_i' \to \operatorname{Spec}(R/\mathfrak{a})[T_1, T_2]/(T_1T_2)\,,$$

finally a commutative diagram



(8.2)

with all vertical squares Cartesian.

Put $\mathfrak{p} \coloneqq \mathfrak{m} \cap R$ and $k' \coloneqq \kappa(\mathfrak{p})$. Then $\mathfrak{a} \subseteq \mathfrak{p}$. Put $A_0 \coloneqq (R_\mathfrak{p})^{\mathrm{h}}$, $S_0 \coloneqq \operatorname{Spec} A_0$, $\hat{S} \coloneqq \operatorname{Spec} \widehat{A}_0$, $X_0 \coloneqq Y \times_{\operatorname{Spec} R} S_0$, $\hat{X} \coloneqq Y \times_{\operatorname{Spec} R} \hat{S}$. Let s' be the point on $\operatorname{Spec} R$ defined by $\mathfrak{p}$. Then s' is the image of s under $S \to \operatorname{Spec} R$. Let s_0 and $\hat{s}$ be the closed points of S_0 and $\hat{S}$ respectively. As

$$\kappa(\hat{s}) = \kappa(s_0) = \kappa(s') = k'\,,$$

we may regard that

$$\hat{X}_{\hat{s}} = (X_0)_{s_0} = Y_{s'}\,.$$

For each $i \in [1, n]$, γ_i' induces a k'-rational point x_i' on $Y_{s'}$ which is the image of x_i under $X \to Y$. By the bottom part of Diagram (8.2), we have

$$\widehat{\mathcal{O}}_{Y_{s'}, x_i'} \cong k'[[T_1, T_2]]/(T_1T_2)\,. \tag{8.3}$$

By the surjective morphism (8.1), we know that $x_1', x_2', \ldots, x_n'$ are all singular points of $Y_{s'}$ over k'. By Lemma 8.6 and Lemma 8.7,

$$\widehat{\mathcal{O}}_{Y, x_i'} = \widehat{\mathcal{O}}_{\hat{X}, x_i'} = \widehat{\mathcal{O}}_{X_0, x_i'} \cong \widehat{A}_0[[T_1, T_2]]/(T_1T_2 - a_i) \tag{8.4}$$

for some $a_i \in \mathfrak{p}\widehat{A}_0$. Let R' be the R-subalgebra of $\widehat{A}_0$ generated by $a_1, a_2, \ldots, a_n$. Put $\mathfrak{q} \coloneqq \mathfrak{p}\widehat{A}_0 \cap R'$, $T \coloneqq \operatorname{Spec} R'$, $Y' \coloneqq Y \times_{\operatorname{Spec} R} T$. Let $t \in T$ be the point defined by $\mathfrak{q}$. Then $\kappa(t) = k'$ and $Y_t' = Y_{s'}$. By (8.4) and Lemma 8.7, we have

$$\widehat{\mathcal{O}}_{Y', x_i'} \cong \widehat{\mathcal{O}}_{T,t}[[T_1, T_2]]/(T_1T_2 - a_i)\,.$$

By Lemma 8.8, there exists a T-scheme W_i' of finite type, two étale T-morphisms $p_i''\colon W_i' \to Y'$ and

$$q_i''\colon W_i' \to \operatorname{Spec} R'[T_1,T_2]/(T_1T_2 - a_i)\,,$$

a point z_i' on W_i' such that $p_i''(z_i') = x_i'$, $q_i''(z_i')$ is the point defined by the prime ideal generated by $\mathfrak{q} \cup \{T_1, T_2\}$, and $\kappa(z_i') = k'$. Put $W_i \coloneqq W_i' \times_T \hat{S}$,

$$\begin{aligned}\hat{p}_i &\coloneqq p_i'' \times \mathrm{id}_{\hat{S}}\colon W_i \to X_1\,,\\ \hat{q}_i &\coloneqq q_i'' \times \mathrm{id}_{\hat{S}}\colon W_i \to \operatorname{Spec} A_1[T_1,T_2]/(T_1T_2 - a_i)\,.\end{aligned}$$

As $z_i' \in W_i'$ and $\hat{s} \in \hat{S}$ both map onto $t \in T$, there is a point $z_i \in W_i$ which maps onto both z_i' and $\hat{s}$. Then $\hat{p}_i(z_i) = x_i'$ and $\hat{q}_i(z_i)$ is the point defined by the prime ideal generated by $\mathfrak{p} \cup \{T_1, T_2\}$. Thus W_i may be contracted to a local chart of x_i'. Therefore $\hat{X} \to \hat{S}$ satisfies (N_1). By Lemma 8.5 $\hat{X} \to \hat{S}$ satisfies (N_3). Let $\hat{D}_i$ denote the connected component of

$$\operatorname{Spec} \mathcal{O}_{W_i}/\big((\hat{q}_i)^{\#}(T_1), (\hat{q}_i)^{\#}(T_2)\big)$$

containing z_i. Then $\hat{D}_i$ is étale over $\operatorname{Spec} \widehat{A}_0/(a_i)$. Since $\kappa(z_i) = k'$ and $\widehat{A}_0/(a_i)$ is complete, a fortiori Henselian, by Lemma 3.4 $\hat{D}_i = \operatorname{Spec} \widehat{A}_0/(a_i)$. Thus $\hat{D}_i \to \hat{S}$ is a closed immersion. Since $\hat{X} \to \hat{S}$ is separated, the composite morphism

$$\hat{D}_i \hookrightarrow W_i \xrightarrow{p_i'''} \hat{X}$$

is a closed immersion. So we may regard $\hat{D}_i$ as a closed subscheme of $\hat{X}$. Since $x_1', x_2', \dots, x_n'$ are all singular points of $\hat{X}_{\hat{s}}$ over k', we have

$$D(\hat{X}/\hat{S}) = \coprod_{i=1}^{n} \hat{D}_i\,.$$

Note that as subsets of $\hat{X}$, $\hat{D}_i \cap \hat{D}_j = \varnothing$ for all $1 \leqslant i < j \leqslant n$. Thus $D(\hat{X}/\hat{S})$ is also a closed subscheme of $\hat{X}$.

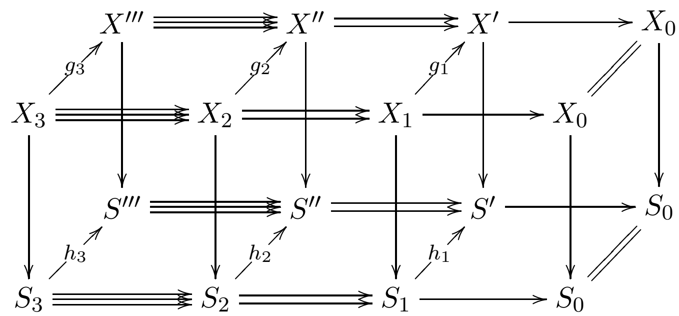
In the following we shall descend a_i to elements in A_0. Put

$$A_1 \coloneqq \widehat{(R_{\mathfrak{p}})^{\mathrm{sh}}}\,, \qquad A_2 \coloneqq A_1 \otimes_{A_0} A_1\,, \qquad A_3 \coloneqq A_1 \otimes_{A_0} A_1 \otimes_{A_0} A_1\,.$$

For each $i \in [1,3]$, put $S_i \coloneqq \operatorname{Spec} A_i$ and $X_i \coloneqq Y \times_{\operatorname{Spec} R} S_i$. Obviously we may regard $\hat{A}_0$ as a subring of A_1. So there is a canonical morphism $S_1 \to \hat{S}$. Thus $X_1 \to S_1$ satisfies (N_3). For each $i \in [1,n]$, put $D_i \coloneqq \hat{D}_i \times_{\hat{S}} S_1$. Then $D_1, D_2, \dots, D_n$ are all connected components of $D(X_1/S_1)$. Note that S_2 and S_3 might not be noetherian. We shall use the trick of inverse limits of schemes to avoid this difficult. By Theorem 7.11 and [7, §(8.6)], there exists a finitely generated A_0-subalgebra A' of A_1 which contains $a_1, a_2, \dots, a_n$ such that if let $S' \coloneqq \operatorname{Spec} A'$ and $X' \coloneqq X_0 \times_{S_0} S'$, then $X' \to S'$ satisfies (N_3); and closed subschemes $D_1', D_2', \dots, D_n'$ of X' such that $D_i' \times_{S'} S_1 = D_i$ for all $i \in [1,n]$ and $D(X'/S') = \coprod_{i=1}^{n} D_i'$. Put

$$\begin{aligned}A'' &\coloneqq A' \otimes_{A_0} A' & S'' &\coloneqq \operatorname{Spec} A'' & X'' &\coloneqq X_0 \times_{S_0} S''\\ A''' &\coloneqq A' \otimes_{A_0} A' \otimes_{A_0} A' & S''' &\coloneqq \operatorname{Spec} A''' & X''' &\coloneqq X_0 \times_{S_0} S'''\,.\end{aligned}$$

Let $g_1\colon X_1 \to X'$, $g_2\colon X_2 \to X''$, $g_3\colon X_3 \to X'''$, $h_1\colon S_1 \to S'$, $h_2\colon S_2 \to S''$, $h_3\colon S_3 \to S'''$ be the canonical morphisms. Then we have a commutative diagram with all squares Cartesian.



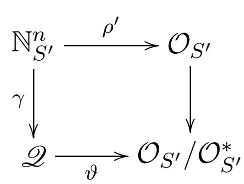
Let s_1 be the closed point of S_1 and put $s' \coloneqq h_1(s_1) \in S'$. By Theorem 3.3, we have

$$\mathrm{pr}_1^* D(X'/S') = \mathrm{pr}_2^* D(X'/S') = D(X''/S'')\,.$$

Pulling back to X_2, we have

$$\mathrm{pr}_1^* D(X_1/S_1) = \mathrm{pr}_2^* D(X_1/S_1)\,.$$

By [3, VIII, 1.9], there exists a closed subscheme C of X_0 such that $D(X_1/S_1) = C \times_{X_0} X_1$. Let $C_1, C_2, \ldots, C_{n'}$ be all connected components of X_0. By Lemma 3.5, $n' = n$, and by rearranging the order of $C_1, C_2, \ldots, C_{n'}$, we may assume that $C_i \times_{X_0} X_1 = D_i$ for all $i \in [1, n]$. By [7, §(8.6)] and by replacing B with a suitably large finitely generated A_0-subalgebra of A_1, we may assume that $C_i \times_{X_0} X' = D'_i$ for all $i \in [1, n]$. Then these data $X' \to S'$, $D'_1, D'_2, \ldots, D'_n$, $a_1, a_2, \ldots, a_n$ satisfy conditions in the begin of §5. Let $\mathscr{P}'$, $\mathscr{Q}'$, θ', ϑ' and $\mathfrak{d}'$ be the notations for f' defined in Remark 3.20; and let $(\mathscr{M}', \mathscr{N}', \sigma', \tau', \varphi')$ be a semistable log structure of f'. Let $\rho'\colon \mathbb{N}^n_{S'} \to \mathcal{O}_{S'}$ be the homomorphism of monoids defined by $\rho'(\varepsilon_i) = a_i$, where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is a basis of $\mathbb{N}^n$. Then there is a commutative diagram



where $\gamma\colon \mathbb{N}^n_{S'} \to \mathscr{Q}$ is the canonical morphism. As $\gamma_{\bar{s}'}$ is an isomorphism, by Lemma 5.1 there exists an affine étale neighborhood N of $\bar{s}'$ such that $\gamma|_N$ lifts to a chat $\mathbb{N}^n_N \to \mathscr{N}'|_N$. Note that A_1 is a strictly Henselian local ring. By [7, (18.8.1)], $S_1 \to S'$ factors through N. So by replacing S' with N, we obtains that γ lifts to a chat $\rho\colon \mathbb{N}^n_{S'} \to \mathscr{N}'$ and the composite morphism

$$\mathbb{N}^n_{S'} \xrightarrow{\rho} \mathscr{N}' \to \mathcal{O}_{S'}$$

is equal to ρ'. By Theorem 7.10, there is an isomorphism $\varphi\colon \mathrm{pr}_1^*\mathscr{N}' \xrightarrow{\sim} \mathrm{pr}_2^*\mathscr{N}'$ of log structures on X'' such that $\mathrm{pr}_{23}^*(\varphi) \circ \mathrm{pr}_{12}^*(\varphi) = \mathrm{pr}_{13}(\varphi)$ on X'''. By Lemma 3.21, both $\mathrm{pr}_1^*(\rho)$ and $\mathrm{pr}_2^*(\rho)$ are lifts of the canonical morphism $\mathbb{N}^n_{S''} \to \mathscr{Q}''$, where $\mathscr{Q}''$ is defined in Remark 3.20. So there exists an element

$$u = (u_1, u_2, \ldots, u_n) \in (\mathcal{O}^*_{S''})^n(S'')$$

such that $\varphi \circ \mathrm{pr}_1^*(\rho) = u \cdot \mathrm{pr}_2^*(\rho)$ and $\mathrm{pr}_{23}^*(u_i) \circ \mathrm{pr}_{12}^*(u_i) = \mathrm{pr}_{13}(u_i)$ in $\mathcal{O}^*_{S'''}(S''')$ for all $i \in [1, n]$. We have $a_i \otimes 1 = u_i \cdot (1 \otimes a_i)$ in A''. Let v_i denote the image of u_i in A_2^*. Then $a_i \otimes 1 = v_i \cdot (1 \otimes a_i)$ in A_2 and v_i defines an isomorphism $\psi_i\colon \mathcal{O}_{S_2} \xrightarrow{\sim} \mathcal{O}_{S_2}$ of $\mathcal{O}_{S_2}$-modules such that $\mathrm{pr}_{23}^*(\psi_i) \circ \mathrm{pr}_{12}^*(\psi_i) = \mathrm{pr}_{13}(\psi_i)$ on S_3. By flat descent of quasi-coherent sheaf, there exists an invertible $\mathcal{O}_S$-module $\mathscr{L}_i$ and an isomorphism $\phi_i\colon q^*\mathscr{L}_i \xrightarrow{\sim} \mathcal{O}_{S_1}$ of $\mathcal{O}_{S_1}$-modules such that

$$\begin{array}{ccc} \mathrm{pr}_1^* q^*\mathscr{L} & \xrightarrow{\mathrm{pr}_1^*(\phi_i)} & \mathcal{O}_{S_2} \\ \| & & \downarrow{\scriptstyle \psi_i} \\ \mathrm{pr}_2^* q^*\mathscr{L} & \xrightarrow[\mathrm{pr}_2^*(\phi_i)]{} & \mathcal{O}_{S_2} \end{array}$$

is commutative, where $q\colon S_1 \to S_0$ is the canonical morphism. Since A_0 is a noetherian local ring, $\mathscr{L}_i \cong \mathcal{O}_{S_0}$. So ϕ_i defines an element $w_i \in A_1^*$ such that $v_i = w_i^{-1} \otimes w_i$. Put $b_i \coloneqq w_i a_i$. Then $b_i \otimes 1 = 1 \otimes b_i$ in A_2. By [13, I, 2.18], $b_i \in A_0$. Since A_1 is flat over $\widehat{A}_0$, by Lemma 3.1, $b_i = w'_i a_i$ for some $w'_i \in \widehat{A}_0^*$. Now replacing T_1 with $(w'_i)^{-1}T_1$ in (8.4), we obtain

$$\widehat{\mathcal{O}}_{X_0,x'_i} \cong \widehat{A}_0[[T_1,T_2]]/(T_1T_2 - b_i)\,. \tag{8.5}$$

As $A_0 = (R_{\mathfrak{p}})^{\mathrm{h}}$, there exists a finitely generated étale R-algebra B, a prime ideal $\mathfrak{q}$ of B, elements $c_1, c_2, \ldots, c_n \in \mathfrak{q}$, and an isomorphism $\nu\colon (B_{\mathfrak{q}})^{\mathrm{h}} \xrightarrow{\sim} A_0$ of R-algebras such that $\nu(c_i) = b_i$ for all $i \in [1,n]$. Put $L \coloneqq \operatorname{Spec} B$ and $Z \coloneqq Y \times_{\operatorname{Spec} R} L$. Let $l \in L$ be the point defined by $\mathfrak{q}$. As $\kappa(l) = k'$, we may regard that $Z_l = (X_0)_{s_0}$. By (8.5) and Lemma 8.7, we have

$$\widehat{\mathcal{O}}_{Z,x'_i} \cong \widehat{\mathcal{O}_{L,l}}[[T_1,T_2]]/(T_1T_2 - c_i)\,.$$

As B is finitely generated over $\mathbb{Z}$, by Lemma 8.8 there exists a local chart of $Z \to L$ at x'_i. By base extension $S_0 \to L$, we obtains a local chart of $X_0 \to S_0$ at x'_i. So $X_0 \to S_0$ satisfies (N_1). By Lemma 8.5 $X_0 \to S_0$ also satisfies (N_3). By base extension $S \to S_0$, we know that $f\colon X \to S$ satisfies (N_3) and is a semistable curve over S. □

From above lemma and Corollary 7.12, we obtains that

Theorem 8.11. *Any semistable curve over a locally noetherian scheme satisfies* (N_3)*, thus has a canonical semistable log structure.*

Theorem 8.12. *Let S be a noetherian scheme and $f\colon X \to S$ be a proper and faithfully flat morphism. Then X is a semistable curve over S if and only if for every closed point $y \in S$, $X \times_S \operatorname{Spec}\kappa(y)_s \to \operatorname{Spec}\kappa(y)_s$ is a semistable curve.*

Theorem 8.13. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & S' \\ {\scriptstyle p}\downarrow & \square & \downarrow{\scriptstyle q} \\ X & \xrightarrow[f]{} & S \end{array}$$

be a Cartesian square of schemes such that

(1) *f' is a semistable curve,*
(2) *q is faithfully flat,*
(3) *f is proper and of finite presentation.*

Then f is also a semistable curve.

Proof. Obviously f is also faithfully flat. So by Definition 8.3, we may assume that $S = \operatorname{Spec} k$ where k is a separably closed field, and S' is affine. By Theorem 7.11, we may further assume that S' is of finite type over S. Then the theorem is by Theorem 7.18. □

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