

A note on mean volume and surface densities for a class of birth-and-growth stochastic processes

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Abstract

Many real phenomena may be modelled as locally finite unions of d -dimensional time dependent random closed sets in \mathbb{R}^d , described by birth-and-growth stochastic processes, so that their mean volume and surface densities, as well as the so called mean *extended* volume and surface densities, may be studied in terms of relevant quantities characterizing the process. We extend here known results in the Poissonian case to a wider class of birth-and-growth stochastic processes, proving in particular the absolute continuity of the random time of capture of a point $x \in \mathbb{R}^d$ by processes of this class.

Keywords: Stochastic birth-and-growth processes; random closed sets; mean densities

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1 Problem and main results

A great variety of real phenomena in material science and in biomedicine, such as crystallization processes (see [8], and references therein; see also [20] for the crystallization processes on sea shells), tumor growth [3, 9], spread of fires in the woods, etc., can be described by space-time structured stochastic birth-and-growth processes (see, e.g., [10]). Roughly speaking, a *birth-and-growth (stochastic) process* is a dynamic germ-grain model [19, 14], used to model situations in which *nuclei* are born in time and are located in space randomly, and each nucleus generates a *grain* (a random closed set) evolving in time. So it can be described by a marked point process $N = \{(T_j, X_j)\}_{j \in \mathbb{N}}$ modelling births at random times $T_j \in \mathbb{R}_+$ and related random spatial locations (*nuclei*) $X_j \in \mathbb{R}^d$ ($d \geq 2$), and by a growth model according to which each nucleus generates a grain $\Theta_{T_j}^t(X_j)$ evolving in time. Under regularity assumptions on the birth and growth model, the union set Θ^t of such grains at time t is then a locally finite union of random closed sets and the mean volume and surface densities associated to the birth-and-growth process $\{\Theta^t\}_t$ can be defined. Sometimes it is of interest to consider the so-called mean *extended* densities of Θ^t , defined as the mean densities of the union of the grains $\Theta_{T_j}^t(X_j)$ ignoring overlapping; for instance the mean density of the $d-1$ -dimensional measure of the union of the topological boundaries $\partial\Theta_{T_j}^t(X_j)$ might be studied whenever the process $\{\Theta^t\}_t$ is given by the union of $(d-1)$ -dimensional grains free to grow in space. A natural question is whether any relationship exists between these densities and, in particular, if it is possible to describe them in terms of relevant quantities associated with the process, like the intensity measure of the nucleation process N and the growth rate. In current literature, the particular case in which N is given by a marked Poisson process has

been studied extensively, and great importance has been given to the concept of *causal cone* and its relationship with the mean (extended) volume density (e.g., see [6, 7, 8, 16]). In particular a relationship between the measure of the causal cone with respect to the intensity measure of the nucleation process and the mean extended volume density has been proven in [6], where the property of independence of the grains, due to the Poisson assumption, plays a fundamental role. Since such quantities are well defined also for more general birth-and-growth processes, aim of the present paper is to extend known results in the Poissonian case to a wider family of processes. To this end we introduce here a class of birth-and-growth processes, denoted by \mathcal{G} , satisfying quite general assumptions, and we show that the quoted result on the mean extended volume density (see Proposition 3.4) and, in particular, an equation for the mean extended surface density (see Proposition 3.5) hold for any process in \mathcal{G} . In particular, in order to do this, we prove that the so called time of capture $T(x)$ of a point $x \in \mathbb{R}^d$ associated with a process $\{\Theta^t\}_t \in \mathcal{G}$ is a continuous random variable with density (see Theorem 3.3). Examples of non-Poissonian birth-and-growth processes in \mathcal{G} are also provided.

2 Preliminaries and notations

We recall that a *random closed set* Θ in \mathbb{R}^d is a measurable map $\Theta : (\Omega, \mathfrak{F}, \mathbb{P}) \rightarrow (\mathbb{F}, \sigma_{\mathbb{F}})$, where \mathbb{F} denotes the class of the closed subsets in \mathbb{R}^d , and $\sigma_{\mathbb{F}}$ is the σ -algebra generated by the so called *hit-or-miss topology* (see [18]). Denoted by T_j the \mathbb{R}_+ -valued random variable representing the *time of birth* of the j -th nucleus, and by X_j the \mathbb{R}^d -valued random variable representing the *spatial location* of the nucleus born at time T_j , defined on the same probability space, let $\Theta_{T_j}^t(X_j)$ be the random closed set obtained as the evolution up to time $t \geq T_j$ of the nucleus born at time T_j in X_j . The family $\{\Theta^t\}_t$ of random closed sets given by

$$\Theta^t = \bigcup_{T_j \leq t} \Theta_{T_j}^t(X_j), \quad t \in \mathbb{R}_+,$$

is called *birth-and-growth (stochastic) process*. The *nucleation process* $\{(T_j, X_j)\}$ is usually described by a marked point process (MPP) N in \mathbb{R}_+ with marks in \mathbb{R}^d . (For basic definitions and results about MPPs we refer to [13, 17, 19]). Thus, it is defined as a random measure given by

$$N = \sum_{j=1}^{\infty} \delta_{T_j, X_j},$$

where $\delta_{t,x}$ denotes here the Dirac measure on $\mathbb{R}_+ \times \mathbb{R}^d$ concentrated at (t, x) ; so, for any $B \times A \in \mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}^d}$ ($\mathcal{B}_{\mathbb{R}^d}$ is the Borel σ -algebra of \mathbb{R}^d), $N(B \times A)$ is the random number of nuclei born in the region A , during time B . We recall that the *marginal process* $\tilde{N}(\cdot) := N(\cdot \times \mathbb{R}^d)$ is itself a point process. Throughout the paper we denote by Λ and $\tilde{\Lambda}$ the *intensity measure* of N and \tilde{N} , respectively, so defined

$$\Lambda(B \times A) := \mathbb{E}[N(B \times A)], \quad \tilde{\Lambda}(B) := \mathbb{E}[\tilde{N}(B)], \quad A \in \mathcal{B}_{\mathbb{R}^d}, B \in \mathcal{B}_{\mathbb{R}};$$

the measure $\tilde{\Lambda}$ is usually assumed to be locally finite, and it is well known the following decomposition of the intensity measure (see, e.g., [17]):

$$\Lambda(dt \times dx) = \tilde{\Lambda}(dt)Q(t, dx), \tag{1}$$

where, $\forall t \in \mathbb{R}_+$, $Q(t, \cdot)$ is a probability measure on \mathbb{R}^d , called the *mark distribution* at time t .

Models of volume growth have been studied extensively, since the pioneering work by Kolmogorov [16] (see also [6]). We denote by \mathcal{H}^n the n -dimensional Hausdorff measure and recall that $\mathcal{H}^d(B)$ coincides with the usual d -dimensional Lebesgue measure of B for any Borel set $B \subset \mathbb{R}^d$, while for $1 \leq n < d$ integer $\mathcal{H}^n(B)$ coincides with the classical n -dimensional measure of B if B is contained in a C^1 n -dimensional manifold embedded in \mathbb{R}^d . Throughout the paper we assume $d \geq 2$ and the *normal growth model* (see, e.g., [7]), according to which at \mathcal{H}^{d-1} -almost every point of the actual grain surface at time t (i.e. at \mathcal{H}^{d-1} -a.e. $x \in \partial\Theta_{T_j}^t(X_j)$) growth occurs with a given strictly positive normal velocity

$$v(t, x) = G(t, x)n(t, x), \quad (2)$$

where $G(t, x)$ is a given deterministic *growth field*, and $n(t, x)$ is the unit outer normal at point $x \in \partial\Theta_{T_0}^t(X_0)$. We assume that

$$0 < g_0 \leq G(t, x) \leq G_0 < \infty \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d,$$

for some $g_0, G_0 \in \mathbb{R}$, and $G(t, x)$ is sufficient regular such that the evolution problem given by (2) for the growth front $\partial\Theta_{t_0}^t(x)$ is well posed. It follows that for any fixed $t \in \mathbb{R}_+$, the topological boundary of each grain is a random closed set with locally finite \mathcal{H}^{d-1} -measure \mathbb{P} -almost surely (see also [5]). Furthermore, for the birth-and-growth model defined above, the so-called *causal cone* associated with a point $x \in \mathbb{R}^d$ and a time $t \in \mathbb{R}_+$ is well defined (see e.g. [7] for its analytical properties).

Definition 2.1 (Causal cone) *The causal cone $\mathcal{C}(t, x)$ of a point x at time t is the space-time region in which at least one nucleation has to take place so that the point x is covered by grains by time t :*

$$\mathcal{C}(t, x) := \{(s, y) \in [0, t] \times \mathbb{R}^d : x \in \Theta_s^t(y)\}.$$

To any point $x \in \mathbb{R}^d$ it is also associated a random variable $T(x)$, said the *time of capture of point x* , defined by

$$T(x) := \min\{t > 0 : x \in \Theta^t\}. \quad (3)$$

We know that any random closed set Θ in \mathbb{R}^d with locally finite \mathcal{H}^n measure \mathbb{P} -a.s., induces a random measure $\mu_\Theta(\cdot) := \mathcal{H}^n(\Theta \cap \cdot)$ on \mathbb{R}^d (for a discussion of the measurability of the random variables $\mathcal{H}^n(\Theta \cap \cdot)$, we refer to [4, 21]), and it is clear that $\mu_{\Theta(\omega)}$ is singular with respect to \mathcal{H}^d for \mathbb{P} -a.e. $\omega \in \Omega$ if $n < d$. On the other hand, the expected measure $\mathbb{E}[\mu_\Theta](\cdot) := \mathbb{E}[\mathcal{H}^n(\Theta \cap \cdot)]$ may be absolutely continuous with respect to \mathcal{H}^d , in dependence of the probability law of Θ ; in such case the random closed set Θ is said to be *absolutely continuous in mean* (see [11]).

For any fixed $t \in \mathbb{R}_+$ the following measures on \mathbb{R}^d associated to a birth-and-growth process $\{\Theta^t\}_t$ as above, and their respective densities (provided that the topological boundary $\partial\Theta_{T_j}^t(X_j)$ of each grain $\Theta_{T_j}^t(X_j)$ is absolutely continuous in mean), can be introduced (see [6, 9]):

Definition 2.2 (Mean volume and surface measures and densities) *For any $t \in \mathbb{R}_+$*

- *the measure $\mathbb{E}[\mu_{\Theta^t}](\cdot) := \mathbb{E}[\mathcal{H}^d(\Theta^t \cap \cdot)]$ on \mathbb{R}^d is said mean volume measure at time t , while the quantity $V_V(t, x)$ such that, for any $A \in \mathcal{B}_{\mathbb{R}^d}$,*

$$\mathbb{E}[\mu_{\Theta^t}](A) = \int_A V_V(t, x) dx,$$

is called mean volume density (or crystallinity) at point x and time t (dx stands for $\mathcal{H}^d(dx)$);

- the measure $\mathbb{E}[\mu_{\Theta^t}^{\text{ex}}](\cdot) := \mathbb{E}[\sum_{j:T_j \leq t} \mathcal{H}^d(\Theta_{T_j}^t(X_j) \cap \cdot)]$ on \mathbb{R}^d is said mean extended volume measure at time t , while the quantity $V_{\text{ex}}(t, x)$ such that, for any $A \in \mathcal{B}_{\mathbb{R}^d}$,

$$\mathbb{E}[\mu_{\Theta^t}^{\text{ex}}](A) = \int_A V_{\text{ex}}(t, x) dx,$$

is called mean extended volume density at point x and time t ;

- the measure $\mathbb{E}[\mu_{\partial\Theta^t}](\cdot) := \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta \cap \cdot)]$ on \mathbb{R}^d is said mean surface measure at time t , while the quantity $S_V(t, x)$ such that, for any $A \in \mathcal{B}_{\mathbb{R}^d}$,

$$\mathbb{E}[\mu_{\partial\Theta^t}](A) = \int_A S_V(t, x) dx,$$

is called mean surface density at point x and time t ;

- the measure $\mathbb{E}[\mu_{\partial\Theta^t}^{\text{ex}}](\cdot) := \mathbb{E}[\sum_{j:T_j \leq t} \mathcal{H}^{d-1}(\partial\Theta_{T_j}^t(X_j) \cap \cdot)]$ on \mathbb{R}^d is said mean extended surface measure at time t , while the quantity $S_{\text{ex}}(t, x)$ such that, for any $A \in \mathcal{B}_{\mathbb{R}^d}$,

$$\mathbb{E}[\mu_{\partial\Theta^t}^{\text{ex}}](A) = \int_B S_{\text{ex}}(t, x) dx,$$

is called mean extended surface density at point x and time t .

In other words, the mean extended volume and surface measures represent the mean of the sum of the volume measures and of the surface measures of the grains which are born and grown until time t , supposed *free to grow*, ignoring overlapping. Note that in the particular case in which Θ^t is stationary, $V_V(t, \cdot)$ and $S_V(t, \cdot)$ are constant and they are said *volume fraction* and *surface density* of Θ^t , respectively (see, e.g., [19], p. 342).

We mentioned that a problem of interest in real applications is to find relationships about the above mean densities, being relevant quantities describing the geometric process $\{\Theta^t\}_t$. Recent results in this direction show that, if $G(t, x)$ is such that the topological boundary of the grains satisfies a certain regularity condition (related to rectifiability properties), then an evolution equation for the mean densities can be proved; namely (see Proposition 25 in [12]),

Proposition 2.3 *Let $\{\Theta^t\}_t$ be a birth-and-growth process with growth model as above such that:*

- the marginal process \tilde{N} is such that $\mathbb{E}[\tilde{N}([t, t + \Delta t]) \mathbf{1}_{\tilde{N}([t, t + \Delta t]) \geq 2}] = o(\Delta t) \forall t > 0$;
- $\forall t > 0$, denoted by Θ_r^t the closed r -neighborhood of Θ^t (i.e. $\Theta_r^t := \{x \in \mathbb{R}^d : \exists y \in \Theta^t \text{ with } |x - y| \leq r\}$), the following limit holds for any bounded Borel set A with $\mathcal{H}^d(\partial A) = 0$

$$\lim_{r \downarrow 0} \frac{\mathbb{E}[\mathcal{H}^d((\Theta_r^t \setminus \Theta^t) \cap A)]}{r} = \mathbb{E}[\mathcal{H}^{d-1}(\partial\Theta^t \cap A)]; \quad (4)$$

- the time of capture $T(x)$ is a continuous random variable with density.

Then the following evolution equation holds in weak form

$$\frac{\partial}{\partial t} V_V(t, x) = G(t, x) S_V(t, x). \quad (5)$$

(For a discussion about equation (4) see [1, 12].)

Note that, by linearity arguments, an analogous relationship for the mean extended densities follows:

$$\frac{\partial}{\partial t} V_{\text{ex}}(t, x) = G(t, x) S_{\text{ex}}(t, x), \quad (6)$$

to be taken, as usual, in weak form.

While it is easily seen that $V_V(t, x) = \mathbb{P}(x \in \Theta^t)$ for \mathcal{H}^d -a.e. $x \in \mathbb{R}^d$ (see Section 3.3), an analogous result about V_{ex} , and so about S_{ex} by (6), is known in current literature only in the case of Poisson type nucleation processes. Namely, it has been shown in [6], Theorem 1, that if N is a marked Poisson process with intensity measure $\Lambda(dt \times dx) = \alpha(t, x) dt dx$ with α such that $\tilde{\Lambda}([0, t]) < \infty$ and $\alpha(t, \cdot) \in L^1(\mathbb{R}^d) \forall t \in \mathbb{R}_+$, then

$$V_{\text{ex}}(t, x) = \Lambda(\mathcal{C}(t, x)) \quad (7)$$

for $\mathcal{H}^1 \times \mathcal{H}^d$ -a.e. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

By Definition 2.1 it follows that $V_V(t, x) = \mathbb{P}(N(\mathcal{C}(t, x)) > 0)$, therefore whenever the nucleation process N is given by a marked Poisson process with intensity measure Λ as above we have that

$$\frac{\partial}{\partial t} V_V(t, x) = (1 - V_V(t, x)) \frac{\partial}{\partial t} V_{\text{ex}}(t, x). \quad (8)$$

In the next section we will show that, while Eq. (8) is true only in the Poissonian case thanks to the property of independence of increments which characterizes Poisson processes, Eq. (7), and consequently a formula for the mean extended surface density S_{ex} by (6), holds for a wider class of birth-and-growth processes, which can be taken as model in various real applications.

3 Extensions to the non-Poissonian case

3.1 A class of birth-and-growth stochastic processes

Definition 3.1 (The class \mathcal{G}) Let \mathcal{G} be the family of all birth-and-growth processes $\{\Theta^t\}_t$ with growth model as above such that Θ^t satisfies equation (4) for any $t \in \mathbb{R}_+$ and the following assumptions on the nucleation process N are fulfilled:

(A1) $\mathbb{E}[\tilde{N}([t, t + \Delta t]) \mathbf{1}_{\tilde{N}([t, t + \Delta t]) \geq 2}] = o(\Delta t)$ for all $t > 0$;

(A2) with respect to the decomposition of Λ in (1), $\tilde{\Lambda}$ is locally finite with density λ , and for all $t > 0$ the mark distribution $Q(t, \cdot)$ admits density $q(t, \cdot)$ on \mathbb{R}^d .

A few comments about the assumptions defining the class \mathcal{G} :

- Condition (A1) is closely related to the notion of *simple* point process (see, e.g., [15]) and it is used in the proof of the evolution equation (5). Besides, observing that

$$\mathbb{P}(\tilde{N}([t, t + \Delta t]) \geq 2) \leq \sum_{n=2}^{\infty} n \mathbb{P}(\tilde{N}([t, t + \Delta t]) = n) \stackrel{(A1)}{=} o(\Delta t),$$

it guarantees that for any infinitesimal time interval Δt at most one nucleation can occur, i.e.

$$\mathbb{P}(\tilde{N}([t, t + \Delta t]) > 0) = \mathbb{P}(\tilde{N}([t, t + \Delta t]) = 1) + o(\Delta t), \quad (9)$$

which is usually assumed in modelling many real situations.

- Condition (A2) implies that $\tilde{\Lambda}([0, t]) < \infty$, which is a common assumption in the theory of point processes, and, in particular, that the intensity measure Λ is absolutely continuous with respect to $\mathcal{H}^1 \times \mathcal{H}^d$, by (1). This, together with the growth model assumptions, guarantees that the boundary of each grain is absolutely continuous in mean, so that the mean surface density S_V and the mean extended surface density S_{ex} are well defined. Further, denoted by $\mathcal{S}_x(s, t) := \{y \in \mathbb{R}^d : (s, y) \in \mathcal{C}(t, x)\}$ the section of the causal cone $\mathcal{C}(t, x)$ at time $s < t$ and $S(y; x, t) := \sup\{s \geq 0 \mid (y, s) \in \mathcal{C}(t, x)\}$ if $(y, 0) \in \mathcal{C}(t, x)$, Proposition 4.1 in [7] ensures that, if Λ is absolutely continuous with respect to $\mathcal{H}^1 \times \mathcal{H}^d$ with density $\alpha(t, x)$, then $\Lambda(\mathcal{C}(t, x))$ is continuously differentiable with respect to t and, in particular,

$$\frac{\partial}{\partial t} \Lambda(\mathcal{C}(t, x)) = G(t, x) \int_0^t \int_{\partial \mathcal{S}_x(s, t)} \alpha(s, y) dK_{x, t, s}(y) ds, \quad (10)$$

with the measure

$$dK_{x, t, s}(y) = \frac{|\nabla_x S|(y; x, t)}{|\nabla_y S|(y; x, t)} d\mathcal{H}^{d-1}(y). \quad (11)$$

So, by condition (A2), we have that (10) holds for any process in \mathcal{G} with $\alpha(s, y) = \lambda(s)q(s, y)$.

Now we show by simple examples that the class \mathcal{G} is not trivial and strictly contains the birth-and-growth processes with Poissonian nucleation process.

Proposition 3.2 *Let $\{\Theta^t\}_t$ be a birth-and-growth process with $G(t, x)$ sufficiently regular as in previous assumptions and N marked Poisson process with intensity measure Λ satisfying condition (A2). Then $\{\Theta^t\}_t \in \mathcal{G}$.*

Proof. By the well known definition of marked Poisson point process we have that the marginal process \tilde{N} is a Poisson process with intensity measure $\tilde{\Lambda}(dt) = \lambda(t)dt$, by assumption (A2). So we have to prove only condition (A1). Let $t \in \mathbb{R}_+$ be fixed. Recalling the Poisson property $\mathbb{P}(\tilde{N}([t, t + \Delta t]) \geq 1) = \tilde{\Lambda}([t, t + \Delta t]) + o(\tilde{\Lambda}([t, t + \Delta t]))$, we have that

$$\begin{aligned} \mathbb{E}[\tilde{N}([t, t + \Delta t]) \mathbf{1}_{\tilde{N}([t, t + \Delta t]) \geq 2}] &= \sum_{n=2}^{\infty} n \frac{\tilde{\Lambda}([t, t + \Delta t])^n}{n!} e^{-\tilde{\Lambda}([t, t + \Delta t])} \\ &= \tilde{\Lambda}([t, t + \Delta t]) \mathbb{P}(\tilde{N}([t, t + \Delta t]) \geq 1) = o(\Delta t). \end{aligned}$$

□

Example 1 Let $\{\Theta^t\}_t$ be a birth-and-growth process with $G(t, x)$ sufficiently regular as in previous assumptions and nucleation process $N^{(1)}$ given by the birth of only one nucleus (T, X) with $T \geq 0$ continuous random variable with density and X random point in \mathbb{R}^d with distribution $Q \ll \mathcal{H}^d$. Clearly $\{\Theta^t\}_t \in \mathcal{G}$, and for any $t \in \mathbb{R}_+$, $V_{\text{ex}}(t, x) = V_V(t, x)$ \mathcal{H}^d -a.e. $x \in \mathbb{R}^d$, since $\Theta^t = \Theta_T^t(X)$ (with $\Theta_T^t(X) = \emptyset$ if $T > t$).

In the next example we provide a non-trivial (i.e. like $N^{(1)}$) birth-and-growth process belonging to the class \mathcal{G} , with nucleation process N which is not given by a marked Poisson process.

Example 2 Let $G(t, x)$ be sufficiently regular as in previous assumptions, and let $T_1 \geq 0$ be a continuous random variable with probability density function f . We assume that the first nucleus is born at the random time T_1 and that a new nucleation occurs at times $T_1 + 1, T_1 + 2, \dots$

(i.e. $T_j = T_1 + j - 1$). Let the spatial locations X_1, X_2, \dots of the nuclei be IID and independent of T_1 , with distribution $Q \ll \mathcal{H}^d$. It follows that

$$\begin{aligned}\mathbb{P}(\tilde{N}([0, t]) = 0) &= \mathbb{P}(T_1 > t), \\ \mathbb{P}(\tilde{N}([0, t]) = n) &= \mathbb{P}(t - n < T_1 \leq t - n + 1), \quad \text{for } n = 1, 2, \dots, [t] + 1, \\ \mathbb{P}(\tilde{N}([0, t]) = n) &= 0, \quad \text{for } n > [t] + 1,\end{aligned}$$

where $[t]$ is the integer part of t . As a consequence, $\tilde{N}([0, t]) \leq [t] + 1$ \mathbb{P} -a.s. and

$$\tilde{\Lambda}([0, t]) = \sum_{n=1}^{[t]+1} n \mathbb{P}(t - n < T_1 \leq t - n + 1) = \sum_{j=0}^{[t]} \mathbb{P}(T_1 \leq t - j) = \sum_{j=0}^{[t]} \int_0^{t-j} f(t) dt;$$

thus conditions (A1) and (A2) are satisfied.

3.2 Absolute continuity of the time of capture $T(x)$

Proposition 2.3 gives sufficient conditions on the birth-and-growth process for the existence of an evolution equation for its mean densities. The condition of absolute continuity of the time of capture $T(x)$ (defined in (3)) of a given point $x \in \mathbb{R}^d$ is not trivial to check, in general. In [10] it is shown that if the mark distribution $Q(t, \cdot)$ admits density on \mathbb{R}^d for all $t > 0$, then $T(x)$ is a continuous random variable, i.e. $\mathbb{P}(T(x) = t) = 0$ for all $t \in \mathbb{R}$; results about the absolute continuity of $T(x)$ for general birth-and-growth processes are not available in current literature yet. In the following theorem we prove that for any birth-and-growth process in \mathcal{G} , $T(x)$ is an absolutely continuous random variable, i.e it admits a probability density function.

Theorem 3.3 *For any birth-and-growth process in the class \mathcal{G} , the random variable $T(x)$ admits probability density function for all $x \in \mathbb{R}^d$.*

Proof. By Besicovitch derivation theorem (see Theorem 2.22 in [2]) we know that every positive Radon measure η on \mathbb{R}^d can be represented in the form $\eta = \eta_{\ll} + \eta_{\perp}$, where η_{\ll} and η_{\perp} are the absolutely continuous part of η with respect to \mathcal{H}^d and the singular part of η , respectively, and that η_{\perp} is given by the restriction of η to the \mathcal{H}^d -negligible set

$$E = \left\{ y \in \mathbb{R}^d : \lim_{r \downarrow 0} \frac{\eta(B_r(y))}{\mathcal{H}^d(B_r(y))} = \infty \right\}, \quad (12)$$

where $B_r(y)$ is the ball of radius r centered in y .

Let P^x be the probability measure on \mathbb{R} of $T(x)$, i.e. $P^x(A) := \mathbb{P}(T(x) \in A)$ for all Borel sets $A \subset \mathbb{R}$, and observe that for all $t > 0$

$$\begin{aligned}P^x(B_{\Delta t}(t)) &= \mathbb{P}(T(x) \in [t - \Delta t, t + \Delta t]) \\ &= \mathbb{P}(t < T(x) \leq t + \Delta t) + \mathbb{P}(t - \Delta t < T(x) \leq t) \\ &= \mathbb{P}(\{N(\mathcal{C}(t + \Delta t, x)) > 0\} \cap \{N(\mathcal{C}(t, x)) = 0\}) \\ &\quad + \mathbb{P}(\{N(\mathcal{C}(t, x)) > 0\} \cap \{N(\mathcal{C}(t - \Delta t, x)) = 0\}) \\ &= \mathbb{P}(\{N(\mathcal{C}(t + \Delta t, x) \setminus \mathcal{C}(t, x)) > 0\} \cap \{N(\mathcal{C}(t, x)) = 0\}) \\ &\quad + \mathbb{P}(\{N(\mathcal{C}(t, x) \setminus \mathcal{C}(t - \Delta t, x)) > 0\} \cap \{N(\mathcal{C}(t - \Delta t, x)) = 0\}) \\ &\leq \mathbb{P}(N(\mathcal{C}(t + \Delta t, x) \setminus \mathcal{C}(t, x)) > 0) + \mathbb{P}(N(\mathcal{C}(t, x) \setminus \mathcal{C}(t - \Delta t, x)) > 0) \\ &\leq \mathbb{E}(N(\mathcal{C}(t + \Delta t, x) \setminus \mathcal{C}(t, x))) + \mathbb{E}(N(\mathcal{C}(t, x) \setminus \mathcal{C}(t - \Delta t, x))) \\ &= \Lambda(\mathcal{C}(t + \Delta t, x) \setminus \mathcal{C}(t, x)) + \Lambda(\mathcal{C}(t, x) \setminus \mathcal{C}(t - \Delta t, x)).\end{aligned} \quad (13)$$

Note that $\mathcal{C}(s_1, x) \subset \mathcal{C}(s_2, x)$ for any s_1, s_2 with $s_1 < s_2$, and, by assumption (A2), we know that $\Lambda(\mathcal{C}(t, x))$ is continuously differentiable with respect to t with partial derivative given by equation (10) (with $\alpha(s, y) = \lambda(s)q(s, y)$). Then, being $\mathcal{H}^1(B_{\Delta t}(t)) = 2\Delta t$, we get that for all $t > 0$

$$\begin{aligned} \limsup_{\Delta t \downarrow 0} \frac{P^x(B_{\Delta t}(t))}{2\Delta t} &\stackrel{(13)}{\leq} \limsup_{\Delta t \downarrow 0} \frac{\Lambda(\mathcal{C}(t + \Delta t, x)) - \Lambda(\mathcal{C}(t, x))}{2\Delta t} + \limsup_{\Delta t \downarrow 0} \frac{\Lambda(\mathcal{C}(t, x)) - \Lambda(\mathcal{C}(t - \Delta t, x))}{2\Delta t} \\ &= \frac{1}{2} \left(\frac{\partial^+}{\partial t} \Lambda(\mathcal{C}(t, x)) + \frac{\partial^-}{\partial t} \Lambda(\mathcal{C}(t, x)) \right) \\ &= \frac{\partial}{\partial t} \Lambda(\mathcal{C}(t, x)) \stackrel{(10)}{<} \infty. \end{aligned}$$

Thus we conclude that the set E in(12) is empty, and so $P_{\perp} \equiv 0$, i.e. $T(x)$ is an absolutely continuous random variable. \square

3.3 Mean extended volume and surface densities and causal cone

Let us observe that for any d -dimensional random closed set Ξ , by applying Fubini's theorem (in $\Omega \times \mathbb{R}^d$, with the product measure $\mathbb{P} \times \mathcal{H}^d$), we have

$$\mathbb{E}[\mathcal{H}^d(\Xi \cap A)] = \int_A \mathbb{P}(x \in \Xi) dx \quad \forall A \in \mathcal{B}_{\mathbb{R}^d},$$

and so, considering the birth-and-growth process $\{\Theta^t\}_t$, we have that for any $t \in \mathbb{R}_+$

$$V_V(t, x) = \mathbb{P}(x \in \Theta^t), \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d. \quad (14)$$

Then it is clear that Eq. (7) may be true for non-Poisson birth-and-growth process as well. For instance consider the process $\{\Theta^t\}_t$ in Example 1; we know that in this case $V_{\text{ex}}(t, x) = V_V(t, x)$, so for any $t \in \mathbb{R}_+$ the following chain of equality holds for \mathcal{H}^d -a.e. $x \in \mathbb{R}^d$:

$$V_{\text{ex}}(t, x) = \mathbb{P}(x \in \Theta^t) = \mathbb{P}(N(\mathcal{C}(t, x)) > 0) = \mathbb{E}[N(\mathcal{C}(t, x))] = \Lambda(\mathcal{C}(t, x)).$$

Such relationship between V_{ex} and the causal cone is proved in [6] in the Poissonian case, using the fact that, since nuclei are assumed to be born accordingly with a marked Poisson process, for any fixed $t \in \mathbb{R}_+$ the associated grains are independently and identically distributed as a typical grain. We show here that Eq. (7) holds for any birth-and-growth process in \mathcal{G} , and so reobtaining the Poissonian case as special case by Proposition 3.2.

Proposition 3.4 *Let $\{\Theta^t\}_t \in \mathcal{G}$. Then, for all $t \in \mathbb{R}_+$,*

$$V_{\text{ex}}(t, x) = \Lambda(\mathcal{C}(t, x)) \quad \text{for } \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d.$$

Proof. Since $\Theta_{T_j}^t(X_j) = \emptyset$ if $T_j > t$, by the definition of the mean extended volume measure in Definition 2.2 we have that, for any fixed $t > 0$,

$$\mathbb{E}[\mu_{\Theta^t}^{\text{ex}}(\cdot)] = \sum_j \mathbb{E}[\mathcal{H}^d(\Theta_{T_j}^t(X_j) \cap \cdot)],$$

and so its mean density $V_{\text{ex}}(t, \cdot)$ is given by the sum of the mean volume densities of each individual grain. Hence we get that for \mathcal{H}^d -a.e. $x \in \mathbb{R}^d$

$$\begin{aligned} V_{\text{ex}}(t, x) &= \sum_j \mathbb{P}(x \in \Theta_{T_j}^t(X_j)) = \sum_j \mathbb{E}[\mathbf{1}_{x \in \Theta_{T_j}^t(X_j)}] = \sum_j \mathbb{E}[\mathbf{1}_{(T_j, X_j) \in \mathcal{C}(t, x)}] \\ &= \mathbb{E}\left[\sum_j \mathbf{1}_{(T_j, X_j) \in \mathcal{C}(t, x)}\right] = \mathbb{E}[N(\mathcal{C}(t, x))] = \Lambda(\mathcal{C}(t, x)). \end{aligned}$$

□

Now we are ready to state the main result of this section, which follows as a corollary of Theorem 3.3 and Proposition 3.4.

Proposition 3.5 *For any birth-and-growth process $\{\Theta^t\}_t \in \mathcal{G}$ the following equality for the mean extended surface density holds for all $t \in \mathbb{R}_+$:*

$$S_{\text{ex}}(t, x) = \int_0^t \int_{\partial \mathcal{S}_x(s, t)} \lambda(s) q(s, y) dK_{x, t, s}(y) ds, \quad \mathcal{H}^d\text{-a.e. } x \in \mathbb{R}^d,$$

with $dK_{x, t, s}(y)$ defined as in (11).

Proof. By the definition of the class \mathcal{G} and Theorem 3.3, Proposition 2.3 applies and so Eq. (6) holds. Then the assertion directly follows by Proposition 3.4 and Eq. (10). □

4 Final remarks

From the previous sections we know that

$$\frac{\partial}{\partial t} V_V(t, x) = \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(x \in \Theta^{t+\Delta t} \setminus \Theta^t)}{\Delta t} = \lim_{\Delta t \downarrow 0} \frac{\mathbb{P}(N(\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t, x)) \geq 1 \cap N(\mathcal{C}(t, x)) = 0)}{\Delta t}, \quad (15)$$

while

$$\frac{\partial}{\partial t} V_{\text{ex}}(t, x) = \lim_{\Delta t \downarrow 0} \frac{\Lambda(\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t, x))}{\Delta t};$$

so, in general, V_V cannot be written in terms of V_{ex} only, except in the trivial case in which only one nucleation may occur (see Example 1), and in the particular case of Poissonian nucleation process. Indeed, if N is a marked Poisson process, then it is well known that it is a Poisson point process on the product space $\mathbb{R} \times \mathbb{R}^d$, and so the events $\{N(\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t, x)) \geq 1\}$ and $\{N(\mathcal{C}(t, x)) = 0\}$ are independent because $[\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t, x)] \cap \mathcal{C}(t, x) = \emptyset$; therefore by (15) and observing that $\mathbb{P}(N(\mathcal{C}(t, x)) = 0) = 1 - V_V(t, x)$ and

$$\mathbb{P}(N(\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t, x)) \geq 1) = \Lambda(\mathcal{C}(t+\Delta t, x) \setminus \mathcal{C}(t, x)) + o(\Delta t),$$

we reobtain Eq. (8).

Furthermore, we mention that three different kinds of nucleation can be considered in order to model various real situations.

1. *Free nucleation.* Nuclei are allowed to be born also in an already crystallized region; i.e. if at a random time T_j a new nucleation occurs, the probability law associated to X_j does not depend on the space occupied by Θ^{T_j} , so that X_j may belong to Θ^{T_j} .

2. *Thinned nucleation.* Nuclei which are born in an already crystallized region are removed; i.e. if nucleation occurs according to a *free* process N_0 , then the considered nucleation process N can be described as a “thinning” (e.g., see [19]) of the MPP N_0 . Namely, in accordance to the previous notations, we have that if $N_0 = \sum_j \delta_{T_j, X_j}$, then

$$N = \sum_j \delta_{T_j, X_j} (1 - \mathbf{1}_{\bigcup_{i=1}^{j-1} \Theta_{T_i}^{T_j^-}(X_i)}(X_j)).$$

We may notice that if N_0 is a marked Poisson point process with intensity measure Λ_0 , then the thinned process N is not Poissonian any longer, having intensity measure $\Lambda(dt \times dx) = \Lambda_0(dt \times dx)(1 - \mathbb{P}(x \in \Theta^{t-}))$.

3. *New nuclei are forced to be born in the free space $\mathbb{R}^d \setminus \Theta$.* Similarly to the thinned process described above, in modelling real applications sometimes it is assumed that new nuclei can be born only in the free space; for instance consider the case in which every new nucleation occurs in the free space uniformly in a bounded region $A \subset \mathbb{R}^d$, i.e. if a nucleation X_j occurs at time T_j , then X_j is a random point uniformly distributed in $A \setminus \Theta^{T_j^-}$. It is clear that, in this case, the probability distribution of every mark X_j associated to T_j depends on the whole history of the process, and in particular on the crystallized region at time T_j .

Throughout the paper we have considered a general nucleation process N , so that our results apply making no distinction between the three types of nucleation described above. Note that if $\{\Theta^t\}_t$ is a birth-and-growth process in \mathcal{G} with free nucleation process N_0 , then the birth-and-growth process with thinned nucleation N associated to N_0 belongs to \mathcal{G} as well. This might be useful whenever the free nucleation process is simpler to handle.

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