ON A CLASSIFICATION OF THE GRADIENT SHRINKING SOLITONS

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Abstract

The main purpose of this article is to provide an alternate proof to a result of Perelman on gradient shrinking solitons. In dimension three we also generalize the result by removing the κ-non-collapsing assumption. In high dimension this new method allows us to prove a classification result on gradient shrinking solitons with vanishing Weyl curvature tensor, which includes the rotationally symmetric ones.

1. Introduction

In his surgery paper Perelman proved the following statement [**P2**]:

Theorem 1.1. *Any κ-non-collapsed gradient shrinking soliton M^3 with bounded positive sectional curvature must be compact.*

Combining with Hamilton's convergence (or curvature pinching) result [**H1**] (see also [**I**]) one can conclude that M^3 must be isometric to a quotient of $\mathbb{S}^3$. The use of such a result is that it rules out the possible complications caused by the existence of noncompact singularity models and implies a classification of finite time singularities models, which then makes surgery procedure possible in the case of dimension three. More precisely, ancient solutions, which are noncompact in interesting cases, can be obtained as the Cheeger-Gromov limit of the sequence of blow-ups, via the compactness result of Hamilton [**H3**], as we approach to the singular time. The gradient shrinking solitons arise from the non-collapsed ancient solutions as the blow-down limits [**P1**], at least in the case that the ancient solution has nonnegative curvature operator. By ruling out the noncompact shrinking solitons with positive curvature one can conclude that the shrinking soliton arisen from the ancient solutions must be cylinder $\mathbb{S}^2 \times \mathbb{R}$ or its quotient. This provides the phototype for the surgery. This relation of the gradient shrinking solitons with the Ricci flow suggests the importance of studying the noncompact gradient shrinking solitons.

On the other hand, Perelman's proof, of which one can find a detailed exposition in [**CZ**, **KL**, **MT**] (see also pages 377-386 of [**CLN**]), is geometric and relies on detailed analysis of the level sets of the potential function, and more importantly, the Gauss-Bonnet formula for surfaces. The authors could not adapt Perelman's argument to the high dimensions. The main goal of this article is to provide an alternate approach and generalize the above result of Perelman to the dimensions greater than 3. Instead of assuming the uniform bound on curvature, we only need very mild growth control on the curvature. Maybe more importantly we do not assume that the gradient shrinking soliton is κ-non-collapsed, as required by the

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above mentioned result of Perelman. For the high dimensional case, our method gives a classification of gradient shrinking solitons which are locally conformally flat. The following is a consequence of our results.

Theorem 1.2. *Let (M^n, g) be a gradient shrinking soliton whose Ricci curvature is nonnegative. If $n \geq 4$ we assume that (M, g) is locally conformally flat. Assume further that*

$$|R_{ijkl}|(x) \leq \exp(a(r(x)+1)) \tag{1.1}$$

for some $a > 0$, where $r(x)$ is the distance function to a fixed point on the manifold. Then its universal cover is either $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{S}^{n-1}\times\mathbb{R}$. In the case that M is compact, the assumptions that the Ricci curvature is nonnegative and the growth condition (1.1) are not needed.

In particular, if (M^n, g) has positive Ricci curvature it must be compact.

As a corollary we have a more general result than Theorem 1.1.

Corollary 1.3. *Let (M^3, g) be a gradient shrinking soliton whose Ricci curvature is positive and satisfying (1.1). Then M must be compact.*

Some new invariant cones, which bounds the Weyl curvature by the scalar curvature, have been discovered in [**BW2**] very recently. This might be related to our result.

The rotationally symmetric gradient shrinking solitons has been studied in [**K**]. It can be easily checked that the rotational symmetric manifolds have vanishing Weyl curvature. Hence our result gives a self-contained classification on rotationally symmetric gradient shrinking solitons. (The proof in [**K**] appealed the strong result of Böhm-Wilking. However it does not require the curvature growth condition for the noncompact case.)

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2. Preliminaries

Recall that (M, g) is a gradient shrinking soliton if there exists a function f such that its Hessian f_{ij} satisfying

$$R_{ij} + f_{ij} - \frac{1}{2}g_{ij} = 0.$$

As shown in [**CLN**], Theorem 4.1, there exists a family of metrics $g(t)$, a solution to Ricci flow with the property that $g(0) = g$ and a family of diffeomorphisms $\phi(t)$, which is generated by the vector field $X = \frac{1}{\tau}\nabla f$, such that $\phi(0) = \mathrm{id}$ and $g(t) = \tau(t)\phi^*(t)g$ with $\tau(t) = 1 - t$, as well as $f(t) = \phi^*(t)f$. The following can be checked some straight forward computations [**CLN**].

Lemma 2.1. *For $\tau > 0$,*

$$\frac{\partial}{\partial \tau}|\operatorname{Ric}|^2 = -\frac{2}{\tau}|\operatorname{Ric}|^2 - \langle \nabla |\operatorname{Ric}|^2, \nabla f\rangle, \tag{2.1}$$

$$\frac{\partial}{\partial \tau}S^2 = -\frac{2}{\tau}S^2 - \langle \nabla S^2, \nabla f\rangle. \tag{2.2}$$

Here S is the scalar curvature.

This particularly holds at $t = 0$ (namely $\tau = 1$). A direct consequence is that

$$\frac{\partial}{\partial t}\left(\frac{|\operatorname{Ric}|^2}{S^2}\right) = \langle \nabla \left(\frac{|\operatorname{Ric}|^2}{S^2}\right), \nabla f\rangle. \tag{2.3}$$

We shall also need the following results. First we need Proposition 1.1 of [**N**] to bound the scalar curvature from below.

Proposition 2.2. *Assume that (M, g) is a non-flat gradient shrinking soliton. Assume that it has nonnegative Ricci curvature. Then there exists $\delta = \delta(M) > 0$ such that $S \geq \delta$.*

The following result on the bound of f as well as its gradient is implicit in the argument of [**P2**] (see also the proof of Proposition 1.1 of [**N**]).

Lemma 2.3. *Assume the same assumption as in Proposition 2.2. There exist constants $B = B(M, f)$, $C = C(M, f) > 0$ such that*

$$f(x) \geq \frac{1}{8} r^2(x) - C \tag{2.4}$$

and

$$f(x) \leq 2r^2(x), \qquad |\nabla f|(x) \leq 4r(x) \tag{2.5}$$

for $r(x) \geq B$. Here $r(x)$ is the distance function to some fixed point $o \in M$ with respect to $g(0)$ metric.

Under the assumption that (M, g) has nonnegative Ricci curvature, it is also easy, from the soliton equation, to have that (e.g. from the proof of Proposition 1.1 in [**N**])

$$|\nabla S|^2 \leq 4S^2 |\nabla f|^2. \tag{2.6}$$

Using the soliton equation and the assumption that $R_{ij} \geq 0$ we also have that

$$|f_{ij}|^2 \leq \max\{\frac{n}{2}, S^2\}. \tag{2.7}$$

We need these inequalities to justify the finiteness of some integrals. Most importantly recall the following local derivative estimates of Shi (cf. Theorem 13.1 of [**H3**]).

Theorem 2.4. *For any $\alpha > 0$ there exists a constant $C(n, K, r, \alpha)$ such that if $(M, g(t))$ is a solution to Ricci flow with $t \in [0, t_1]$, $0 < t_1 \leq \frac{\alpha}{K}$, $p \in M$ and*

$$|R_{ijkl}|(x, t) \leq K$$

for all $x \in B_{g(0)}(p, r)$, $\quad t \in [0, t_1]$, then

$$|\nabla_s R_{ijkl}|(y, t) \leq \frac{C(n, \sqrt{K} r, \alpha) K}{\sqrt{t}}$$

for all $y \in B_{g(0)}(p, \frac{r}{2})$ and $t \in (0, t_1]$. Moreover, for the above (y, t)

$$|\nabla^m R_{ijkl}|(y, t) \leq \frac{C(n, m, K, r, \alpha)}{t^{\frac{m}{2}}}.$$

3. Three dimensional case

We first give a different proof to Perelman's theorem mentioned in the introduction. In fact what we prove is a more general result since we assume neither that gradient shrinking soliton is κ-noncollapsed nor that the curvature is uniformly bounded. Most argument of the proof can also be used in dimensions $n \geq 4$.

We assume that the Ricci curvature satisfies that for any $\epsilon > 0$, there exists $\beta(\epsilon) > 0$ such that

$$|\operatorname{Ric}|(y, t) \leq \exp(\epsilon r^2(x) + \beta(\epsilon)) \tag{3.1}$$

for all $y \in B_{g(-\frac{1}{2})}(x, \frac{r(x)}{2})$ and $t \in [-\frac{1}{2}, 0]$. Here $r(x)$ is the distance function to some fixed point $o \in M$ with respect to the metric $g(0)$. Notice that (3.1) can be easily verified if we assume that $|\operatorname{Ric}|$ is uniformly bounded at $t=0$. The main purpose of this section is to show the following result.

Theorem 3.1. *Let (M^3, g) be a complete gradient shrinking soliton with the positive sectional curvature and the Ricci curvature satisfies (3.1). Then M must be the quotient of $\mathbb{S}^3$.*

Note that we do not need to assume (M, g) is κ-non-collapsed. The proof also concludes that $M = \mathbb{S}^3/\Gamma$ directly without appealing to Hamilton's result. In the later section we in fact directly obtain a classification of solitons under the assumption that $\operatorname{Ric} \geq 0$.

First we recall a result of Hamilton. In [**H1**], the following result was proved for solutions to Ricci flow on three manifold M.

Proposition 3.2.

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|\operatorname{Ric}|^2}{S^2}\right) = -\frac{2}{S^4}\left|S\nabla_p R_{ij} - \nabla_p S R_{ij}\right|^2 - \frac{P}{S^3} + \langle \nabla\left(\frac{|\operatorname{Ric}|^2}{S^2}\right), \nabla \log S^2\rangle, \tag{3.2}$$

where

$$P = \frac{1}{2}\left((\mu+\nu-\lambda)^2(\mu-\nu)^2 + (\lambda+\nu-\mu)^2(\lambda-\nu)^2 + (\lambda+\mu-\nu)^2(\lambda-\mu)^2\right)$$

with μ, ν and λ are eigenvalues of Ric.

If (M^3, g) is gradient shrinking soliton, combining the discussions above we have that at $t=0$,

$$\begin{aligned} 0 \;=\;& \Delta\left(\frac{|\operatorname{Ric}|^2}{S^2}\right) - \langle \nabla\left(\frac{|\operatorname{Ric}|^2}{S^2}\right), \nabla f\rangle - \frac{2}{S^4}\left|S\nabla_p R_{ij} - \nabla_p S R_{ij}\right|^2 \\ & - \frac{P}{S^3} + \langle \nabla\left(\frac{|\operatorname{Ric}|^2}{S^2}\right), \nabla \log S^2\rangle. \end{aligned} \tag{3.3}$$

Now multiply $|\operatorname{Ric}|^2 e^{-f}$ on the both sides of the above equation then integrate by parts. Here we have assumed that all integrals involved are finite and the integration by parts can be performed, which we justify later.

$$\begin{aligned} 0 \;=\;& \int_M -\langle \nabla\left(\frac{|\operatorname{Ric}|^2}{S^2}\right), \nabla|\operatorname{Ric}|^2\rangle e^{-f} - \frac{2|\operatorname{Ric}|^2}{S^4}\left|S\nabla_p R_{ij} - \nabla_p S R_{ij}\right|^2 e^{-f} \\ & \int_M -\frac{P}{S^3}|\operatorname{Ric}|^2 e^{-f} + \langle \nabla\left(\frac{|\operatorname{Ric}|^2}{S^2}\right), \nabla \log S^2\rangle |\operatorname{Ric}|^2 e^{-f}. \end{aligned}$$

Using that

$$\nabla\left(\frac{|\operatorname{Ric}|^2}{S^2}\right) = \frac{\nabla|\operatorname{Ric}|^2}{S^2} - \frac{\nabla S^2}{S^4}|\operatorname{Ric}|^2$$

we have that

$$\begin{aligned} & \int_M -\langle \nabla\left(\frac{|\operatorname{Ric}|^2}{S^2}\right), \nabla|\operatorname{Ric}|^2\rangle e^{-f} + \langle \nabla\left(\frac{|\operatorname{Ric}|^2}{S^2}\right), \nabla \log S^2\rangle |\operatorname{Ric}|^2 e^{-f} \\ =\;& -\int_M \left|\nabla\left(\frac{|\operatorname{Ric}|^2}{S^2}\right)\right|^2 S^2 e^{-f}. \end{aligned}$$

Hence we have that

$$\begin{aligned} 0 \;=\;& \int_M -\left|\nabla\left(\frac{|\operatorname{Ric}|^2}{S^2}\right)\right|^2 S^2 e^{-f} - \frac{2|\operatorname{Ric}|^2}{S^4}\left|S\nabla_p R_{ij} - \nabla_p S R_{ij}\right|^2 e^{-f} \\ & \int_M -\frac{P}{S^3}|\operatorname{Ric}|^2 e^{-f}. \end{aligned}$$

In particular, $\frac{|\operatorname{Ric}|^2}{S^2}$ is a constant,

$$S\nabla_p R_{ij} - \nabla_p S R_{ij} = 0 \tag{3.4}$$

and $P = 0$. If we choose a orthornormal frame such R_{ij} is diagonal, the equality (3.4) implies that

$$S\nabla_p R_{jj} = \nabla_p S R_{jj} \tag{3.5}$$

$$S\nabla_p R_{ij} = 0, \text{ for } i \neq j \tag{3.6}$$

Using the second Bianchi identity:

$$\frac{1}{2}\nabla_i S = \sum_p \nabla_p R_{ip} = \nabla_i R_{ii}$$

we have that

$$\frac{1}{2} S\nabla_i S = S\nabla_j R_{jj} = \nabla_i S R_{ii}.$$

On the other hand, $P = 0$ implies that $R_{11} = R_{22} = R_{33} = \frac{1}{3}S$. We thus have that

$$\frac{1}{2} S\nabla_i S = (\nabla_i S)\frac{S}{3}$$

which implies that S is a constant. Then (3.5) and (3.6) implies that $\nabla_p R_{ij} = 0$ for any p, i, j. This implies that M is a compact locally symmetric space with positive curvature. The claim then follows from classical known results.

Now with the help of Proposition 2.2 and Lemma 2.3 we now justify the finiteness of the integrals involved and the integration by parts.

First note that if we assume that $\sup_{x\in M} |R_{ijkl}|(x) \leq C$ for some $C > 0$, namely the curvature is bounded, invoking the Bernstein-Bando-Shi type derivative estimates (cf. [**CK**], Theorem 7.1), we have that $|\nabla^m R_{ijkl}|$ are uniformly bounded on M. Hence all the integrals involved are finite which then implies, via cut-off function argument, that the integrations by parts are completely legal, in view of the fast decay of e^{-f} ensured by lemma 2.3 and the lower bound of S provided by Proposition 2.2.

For the general case, notice first that the assumption on $|\operatorname{Ric}|$ is equivalent to the same assumption on $|R_{ijkl}|$ (with some factor of absolute constant). Hence we have that for any $\epsilon > 0$, there exists $\beta(\epsilon) > 0$ such that

$$|R_{ijkl}|(y,t) \leq \exp(\epsilon r^2(x) + \beta(\epsilon)) \tag{3.7}$$

for all $y \in B_{g(-\frac{1}{2})}(x, \frac{r(x)}{2})$ and $t \in [-\frac{1}{2}, 0]$.

Below we estimate $\Delta\left(\frac{|\operatorname{Ric}|^2}{S^2}\right)|\operatorname{Ric}|^2$. The others are similar. Applying the local derivative estimate of Shi (cf. Theorem 13.1 of [**H3**]) we have that

$$\begin{aligned} |\nabla_p R_{ijkl}|(x,0) &\leq C_1 \exp(\frac{3}{2}\epsilon r^2(x) + \beta_1(\epsilon)) \\ |\nabla_p\nabla_q R_{ijkl}|(x,0) &\leq C_2 \exp(\frac{9}{4}\epsilon r^2(x) + \beta_2(\epsilon)). \end{aligned}$$

Direct computation shows that

$$\begin{aligned} \Delta\left(\frac{|\operatorname{Ric}|^2}{S^2}\right) &= \frac{\Delta|\operatorname{Ric}|^2}{S^2} - 2\frac{\langle\nabla|\operatorname{Ric}|^2, \nabla\log S^2\rangle}{S^2} + 2|\operatorname{Ric}|^2\frac{|\nabla\log S^2|^2}{S^2} \\ &\quad -\frac{\Delta S^2}{S^4}|\operatorname{Ric}|^2. \end{aligned}$$

At $t=0$ there exists absolute constants C_i, $i=3,4,5$ and $\beta_3(\epsilon)$ depending only on β_1 and β_2 such that for $r(x) >> 1$,

$$\begin{aligned} I &= \left(\left|\frac{\Delta|\operatorname{Ric}|^2}{S^2}\right| |\operatorname{Ric}|^2\right)(x,0) \le \frac{C_3}{\delta^2}\exp(6\epsilon r^2(x)+\beta_3(\epsilon)), \\ II &= \left(\left|\frac{\langle \nabla|\operatorname{Ric}|^2, \nabla \log S^2\rangle}{S^2}\right| |\operatorname{Ric}|^2 + |\operatorname{Ric}|^4 \frac{|\nabla \log S^2|^2}{S^2}\right)(x,0) \\ &\le \frac{C_4}{\delta^2}\exp(6\epsilon r^2(x)+\beta_3(\epsilon)), \\ III &= \left(\left|\frac{\Delta S^2}{S^4}\right| |\operatorname{Ric}|^4\right)(x,0) \le \frac{C_5}{\delta^2}\exp(6\epsilon r^2(x)+\beta_3(\epsilon)). \end{aligned}$$

In the last one we have used the estimates (2.5), (2.6), (2.7), as well as

$$\nabla_i S = 2R_{ij} f_j$$

which then implies

$$\Delta S = \langle \nabla S, \nabla f\rangle + 2R_{ij} f_{ij} \le |\nabla S||\nabla f| + 2S|\sum_{ij} f_{ij}^2|^{1/2}.$$

Putting the above estimates together with (2.4) we conclude that at $t=0$,

$$\int_M \left|\Delta\left(\frac{|\operatorname{Ric}|^2}{S^2}\right)\right| |\operatorname{Ric}|^2 e^{-f}\, d\mu_0 < \infty.$$

Similarly one can establish the finiteness of other integrals involved. Once we have the the finiteness of the integration, the integrations by parts can be checked by approximation via the cut-off functions. This is somewhat standard we hence omit the details.

Remark 3.3. An argument similar to the one used here was originated by Huisken in his classification of mean convex shrinking solitons of mean curvature flow in $\mathbb{R}^{n+1}$ [**Hu2**].

4. High dimension-preliminaries

Most results in this section are either known (cf. [**Hu1**, **H2**]) or can be derived easily from the known ones in the literature. We include them here for the completeness. We also adapt them into the form needed by us.

Recall the evolution formula of the curvature under the Ricci flow [**H1**]:

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) R_{ijkl} &= 2(\mathbf{R}^2 + \mathbf{R}^{\#})_{ijkl} \\ &\quad - (R_{ip}R_{pjkl} + R_{jp}R_{ipkl} + R_{kp}R_{ijpl} + R_{lp}R_{ijkp}) \end{aligned}$$

where $Q(\mathbf{R}) = \mathbf{R}^2 + \mathbf{R}^{\#}$ is defined via the Lie algebra structure of $\wedge^2(n)$, which can be identified with the Lie algebra of $O(n)$. The below is a brief explanation.

Let (E, g) be a Euclidean space with metric g. We can make the following identifications: $\otimes^2 E$, the tensor space, can be identified with $GL(n,\mathbb{R})$, the linear transformations on E (for any $x\otimes y \in \otimes^2 E$, $x\otimes y(z) = \langle y, z\rangle x$ is the corresponding element of $GL(n,\mathbb{R})$); under this identification, the space symmetric two tensors S^2E corresponds to the symmetric transformations $S^2(E)$; $\wedge^2 E$ can be identified with $so(n)$ ($e_i\wedge e_j = e_i\otimes e_j - e_j \otimes e_i$ is identified with E_{ij} with 1 at (i,j)-th position and -1 at (j,i)-th position. The metric on TM extends naturally to all the related tensor spaces such as $\otimes^2 TM$, S^2TM, $\wedge^2 TM$. With respect to the previous identification, the metric on $so(n)$ is given by $\langle A, B\rangle = -\frac{1}{2}\operatorname{tr}(AB)$ $(= \frac{1}{2}\operatorname{tr}(A^tB))$ such that $\{e_i\wedge e_j\}_{i<j}$ is an orthonormal basis of $\wedge^2 TM$. The identification also equips $\wedge^2 TM$ with a Lie

algebra structure, which is of fundamental importance in the study of evolution of curvature operators under Ricci flow. This was first observed by Hamilton [**H2**]. Let us recall this fact first. For an orthonormal basis ϕ_α of $\wedge^2 TM$ (say $\phi_\alpha = e_i \wedge e_j$, which is identified with E_{ij}), the Lie bracket is given by

$$[\phi_\alpha, \phi_\beta] = c_{\alpha\beta\gamma}\phi_\gamma.$$

It is easy to check, by simple linear algebra, that

$$\langle [\phi, \psi], \omega\rangle = -\langle [\omega, \psi], \phi\rangle.$$

This immediately implies that $c_{\alpha\beta\gamma}$ is anti-symmetric. If $A, B \in S^2(\wedge^2 TM)$ one can define

$$(A\#B)_{\alpha\beta} = \frac{1}{2} c_{\alpha\gamma\eta} c_{\beta\delta\theta} A_{\gamma\delta} B_{\eta\theta}.$$

It is easy to see that $A\#B$ is symmetric too. Also from the anti-symmetry of $c_{\alpha\beta\gamma}$

$$A\#B = B\#A.$$

The easy computation also shows that

$$\langle (A\#B)(\phi), \psi\rangle = \frac{1}{2}\sum_{\alpha\beta} \langle [A(\omega_\alpha), B(\omega_\beta)], \phi\rangle \cdot \langle [\omega_\alpha, \omega_\beta], \psi\rangle$$

if $\{\omega_\alpha\}$ is an orthonormal basis. This particularly implies that $\mathrm{tr}((A\#B)\cdot C)$ is symmetric in A, B, C since

$$\begin{aligned} \mathrm{tr}((A\#B)\cdot C) &= \sum_\gamma \langle (A\#B)\cdot C(\omega_\gamma), \omega_\gamma\rangle \\ &= \frac{1}{2}\sum_{\alpha\beta\gamma} \langle [A(\omega_\alpha), B(\omega_\beta)], C(\omega_\gamma)\rangle \langle [\omega_\alpha, \omega_\beta], \omega_\gamma\rangle. \end{aligned}$$

Now define

$$\mathrm{tri}(A, B, C) = \mathrm{tr}((AB + BA + 2A\#B)C)$$

which is symmetric in all variables. If we write

$$\mathbf{R}(e_i \wedge e_j) = \frac{1}{2}\sum_{k,l} R_{ijkl}\, e_k \wedge e_l$$

we would have that

$$|R_{ijkl}|^2 = 4\langle \mathbf{R}, \mathbf{R}\rangle.$$

We denote $\mathrm{tri}(\mathbf{R}) = \mathrm{tri}(\mathbf{R}, \mathbf{R}, \mathbf{R}) = \langle 2(\mathbf{R}^2 + \mathbf{R}^\#), \mathbf{R}\rangle$ and $Q(\mathbf{R}) = \mathbf{R}^2 + \mathbf{R}^\#$.

The curvature operator $\mathbf{R}$ has an orthogonal splitting, with respect irreducible $O(n)$ representation, into the trace part $\mathbf{R}_{\mathrm{I}} = \frac{S}{n(n-1)}\,\mathrm{I}$, the traceless Ricci part $\mathbf{R}_{\mathrm{Ric}_0} = \frac{2}{n-2}\,\mathrm{Ric}_0 \wedge \mathrm{id}$, where Ric_0 denotes the traceless part of the Ricci curvature, and the Weyl curvature $\mathbf{R}_W$ (cf. [**BW1**]). We denote the three subspaces by $\langle \mathrm{I}\rangle$, $\langle \mathrm{Ric}_0\rangle$ and $\langle W\rangle$ respectively. Equipped with the above notations we have that

Lemma 4.1.

$$\left(\frac{\partial}{\partial t} - \Delta\right)|R_{ijkl}|^2 = 8\,\mathrm{tri}(\mathbf{R}) - 2|\nabla_p R_{ijkl}|^2. \tag{4.1}$$

Direct calculation then yields the following

Proposition 4.2. *Assume that* $S > 0$. *Then*

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|R_{ijkl}|^2}{S^2}\right) &= \frac{4}{S^3}\left(2\,\mathrm{tri}(\mathbf{R})S - \sigma^2|R_{ijkl}|^2\right) \\ &\quad - \frac{2}{S^4}\left|S\nabla_p R_{ijkl} - \nabla_p S R_{ijkl}\right|^2 + \langle \nabla\left(\frac{|R_{ijkl}|^2}{S^2}\right), \nabla \log S^2\rangle, \end{aligned} \tag{4.2}$$

where $\sigma^2 = |\mathrm{Ric}|^2$.

Note that Tachibana [**T**] proved that (see also [**CLN**], pages 267-269), under the assumption that $\mathbf{R}\ge 0$,

$$-2\,\mathrm{tri}(\mathbf{R})+\mathrm{Ric}(\mathbf{R},\mathbf{R})\ge 0$$

where $\mathrm{Ric}(\mathbf{R},\mathbf{R})=R_{ip}R_{ijkl}R_{pjkl}$.

In [**Hu1**], Huisken obtained the following identities.

$$(\mathbf{R}_{\mathrm{I}})_{ijkl}(Q(\mathbf{R}))_{ijkl}=4\langle Q(\mathbf{R}),\mathbf{R}_{\mathrm{I}}\rangle=\frac{2}{n(n-1)}S\sigma^2; \tag{4.3}$$

$$(\mathbf{R}_{\mathrm{Ric}_0})_{ijkl}(Q(\mathbf{R}))_{ijkl}=\frac{4}{n(n-1)}S\tilde\sigma^2-\frac{8}{(n-2)^2}\lambda_i^3+\frac{4}{n-2}(\mathbf{R}_W)_{ijij}\lambda_i\lambda_j; \tag{4.4}$$

$$(\mathbf{R}_W)_{ijkl}(Q(\mathbf{R}))_{ijkl}=2\,\mathrm{tri}(\mathbf{R}_W)+\frac{2}{n-2}(\mathbf{R}_W)_{ijij}\lambda_i\lambda_j; \tag{4.5}$$

where λ_i are the eigenvalues of Ric_0 and $\tilde\sigma^2=\sum\lambda_i^2$. Below we first show these equations via the following lemma, which essentially follows from [**BW1**]. In [**Hu1**], the result was shown by direct but long computations which were omitted. With the help of [**BW1**], the result can be obtained without much computation. We include the derivation for the sake of completeness. First we need to following lemma which has been essentially proved in [**BW1**].

Lemma 4.3.

$$\mathbf{R}+\mathbf{R}\,\#\,\mathrm{I}=\mathrm{Ric}(\mathbf{R})\wedge\mathrm{id}\,. \tag{4.6}$$

Hence for any $\mathbf{R}_1,\mathbf{R}_2\in S_B(\wedge^2(n))$*, let*

$$B(\mathbf{R}_1,\mathbf{R}_2)=\mathbf{R}_1\,\mathbf{R}_2+\mathbf{R}_2\,\mathbf{R}_1+2\,\mathbf{R}_1\,\#\,\mathbf{R}_2\,.$$

Let $\mathbf{R}_I^i\in\langle\mathrm{I}\rangle$, $\mathbf{R}_0\in\langle\mathrm{Ric}_0\rangle$, $W_i,W\in\langle W\rangle$ $(i=1,2)$*. Then the following hold*

$$\begin{aligned}
B(\mathbf{R}_{\mathrm{I}},W)&=0,\\
B(\mathbf{R}_{\mathrm{I}}^1,\mathbf{R}_{\mathrm{I}}^2)&\in\langle\mathrm{I}\rangle\\
B(W_1,W_2)&\in\langle W\rangle\\
B(\mathbf{R}_{\mathrm{I}},\mathbf{R}_0)&\in\langle\mathrm{Ric}_0\rangle\\
B(\mathbf{R}_0,W)&\in\langle\mathrm{Ric}_0\rangle
\end{aligned}$$

$$\frac12 B(\mathbf{R}_0,\mathbf{R}_0)=\frac{1}{n-2}\,\mathrm{Ric}_0\wedge\mathrm{Ric}_0-\frac{2}{(n-2)^2}(\mathrm{Ric}_0^2)_0\wedge\mathrm{id}+\frac{\tilde\sigma^2}{n(n-2)}\,\mathrm{I}\,. \tag{4.7}$$

Moreover

$$\mathrm{Ric}_0\wedge\mathrm{Ric}_0=-\frac{\tilde\sigma^2}{n(n-1)}\,\mathrm{I}-\frac{2}{n-2}(\mathrm{Ric}_0^2)_0\wedge\mathrm{id}+(\mathrm{Ric}_0\wedge\mathrm{Ric}_0)_W\,. \tag{4.8}$$

Equipped with the above lemma we have that

$$\begin{aligned}
\mathrm{tri}(\mathbf{R},\mathbf{R},\mathbf{R}_{\mathrm{I}})&=\mathrm{tri}(\mathbf{R},\mathbf{R}_{\mathrm{I}},\mathbf{R})\\
&=\frac{2S}{n(n-1)}\langle\mathrm{Ric}\wedge\mathrm{id},\mathbf{R}\rangle\\
&=\frac{2S}{n(n-1)}\langle\frac{S}{n}\,\mathrm{id}\wedge\mathrm{id}+\mathrm{Ric}_0\wedge\mathrm{id},\frac{S}{n(n-1)}\,\mathrm{id}\wedge\mathrm{id}+\frac{2}{n-2}\,\mathrm{Ric}_0\wedge\mathrm{id}\rangle
\end{aligned}$$

If we let $\bar\lambda=\frac{S}{n}$, we have that

$$\begin{aligned}
&\langle\frac{S}{n}\,\mathrm{id}\wedge\mathrm{id}+\mathrm{Ric}_0\wedge\mathrm{id},\frac{S}{n(n-1)}\,\mathrm{id}\wedge\mathrm{id}+\frac{2}{n-2}\,\mathrm{Ric}_0\wedge\mathrm{id}\rangle\\
=\;&\langle\bar\lambda\,\mathrm{id}\wedge\mathrm{id}+\mathrm{Ric}_0\wedge\mathrm{id},\frac{\bar\lambda}{n-1}\,\mathrm{id}\wedge\mathrm{id}+\frac{2}{n-2}\,\mathrm{Ric}_0\wedge\mathrm{id}\rangle=\frac{n}{2}\bar\lambda^2+\frac12\sum\lambda_i^2\\
=\;&\frac12\sigma^2.
\end{aligned}$$

This proves (4.3). For (4.4), let $\mathbf{R}_0 = \mathbf{R}_{\mathrm{Ric}_0}$. We need to compute $\mathrm{tri}(\mathbf{R}, \mathbf{R}, \mathbf{R}_0)$. Using the symmetry

$$\begin{aligned}
\mathrm{tri}(\mathbf{R}, \mathbf{R}, \mathbf{R}_0) &= \mathrm{tri}(\mathbf{R}, \mathbf{R}_0, \mathbf{R}) \\
&= \langle B(\mathbf{R}_{\mathrm{I}}, \mathbf{R}_0), \mathbf{R}\rangle + \langle B(\mathbf{R}_0, \mathbf{R}_0), \mathbf{R}\rangle + \langle B(\mathbf{R}_W, \mathbf{R}_0), \mathbf{R}\rangle \\
&= \langle B(\mathbf{R}_{\mathrm{I}}, \mathbf{R}_0), \mathbf{R}_0\rangle + \langle B(\mathbf{R}_0, \mathbf{R}_0), \mathbf{R}_{\mathrm{I}} + \mathbf{R}_0 + \mathbf{R}_W\rangle + \langle B(\mathbf{R}_W, \mathbf{R}_0), \mathbf{R}_0\rangle \\
&= 2\,\mathrm{tri}(\mathbf{R}_0, \mathbf{R}_0, \mathbf{R}_{\mathrm{I}}) + 2\,\mathrm{tri}(\mathbf{R}_0, \mathbf{R}_0, \mathbf{R}_W) + \mathrm{tri}(\mathbf{R}_0, \mathbf{R}_0, \mathbf{R}_0).
\end{aligned}$$

Using (4.7) and (4.8) we have that

$$\mathrm{tri}(\mathbf{R}_0, \mathbf{R}_0, \mathbf{R}_{\mathrm{I}}) = \frac{1}{n(n-1)}\tilde{\sigma}^2 S$$

In a similar way,

$$\mathrm{tri}(\mathbf{R}_0, \mathbf{R}_0, \mathbf{R}_0) = -\frac{4}{(n-2)^2}\sum \lambda_i^3$$

and

$$2\,\mathrm{tri}(\mathbf{R}_0, \mathbf{R}_0, \mathbf{R}_W) = \frac{2}{n-2}(R_W)_{ijij}\lambda_i\lambda_j.$$

The above three give (4.4). For (4.5), notice that

$$\mathrm{tri}(\mathbf{R}_W, \mathbf{R}, \mathbf{R}) = \mathrm{tri}(\mathbf{R}_0, \mathbf{R}_W, \mathbf{R}_0) + \mathrm{tri}(\mathbf{R}_W, \mathbf{R}_W, \mathbf{R}_W).$$

Then the claimed equality follows from the above computation on $\mathrm{tri}(\mathbf{R}_0, \mathbf{R}_0, \mathbf{R}_W)$.

Finally one can arrive at the following formula.

$$\begin{aligned}
2\,\mathrm{tri}(\mathbf{R})S - \sigma^2|R_{ijkl}|^2 &= -4|\,\mathbf{R}_W\,|^2\sigma^2 + 2S\,\mathrm{tri}(\mathbf{R}_W, \mathbf{R}_W, \mathbf{R}_W) \\
&\quad -\frac{4}{n(n-1)(n-2)}S^2\tilde{\sigma}^2 - \frac{4}{n-2}\tilde{\sigma}^4 \\
&\quad -\frac{8}{(n-2)^2}S\sum\lambda_i^3 + \frac{6}{n-2}S(\mathbf{R}_W)_{ijij}\lambda_i\lambda_j.
\end{aligned} \tag{4.9}$$

This follows from (4.3)-(4.5) along with the observation that

$$\begin{aligned}
|\,\mathbf{R}\,|^2 &= |\,\mathbf{R}_{\mathrm{I}}\,|^2 + |\,\mathbf{R}_{\mathrm{Ric}_0}\,|^2 + |\,\mathbf{R}_W\,|^2 \\
&= \frac{S^2}{2n(n-1)} + \frac{1}{n-2}\sum\lambda_j^2 + |\,\mathbf{R}_W\,|^2
\end{aligned}$$

and

$$\sigma^2 = \frac{S^2}{n} + \tilde{\sigma}^2.$$

Hence

$$4\sigma^2|\,\mathbf{R}\,|^2 = \frac{2S^2}{n(n-1)}\sigma^2 + \frac{4S^2\tilde{\sigma}^2}{(n-2)n} + \frac{4}{n-2}\tilde{\sigma}^4 + 4|\,\mathbf{R}_W\,|^2\sigma^2.$$

In the case that $\mathbf{R}_W = 0$, which is automatical if $n = 3$ and amounts to that (M, g) is locally conformally flat if $n \geq 4$, we have that

$$2\,\mathrm{tri}(\mathbf{R})S - \sigma^2|R_{ijkl}|^2 = -\frac{4}{n(n-1)(n-2)}S^2\tilde{\sigma}^2 - \frac{4}{n-2}\tilde{\sigma}^4 - \frac{8}{(n-2)^2}S\sum\lambda_i^3. \tag{4.10}$$

Similarly, using that

$$\left(\frac{\partial}{\partial t} - \Delta\right) R_{ik} = 2R_{ijkl}R_{jl} - 2R_{il}R_{lk}$$

we also have the high dimensional analogue of Proposition 3.2.

Proposition 4.4. *Assume that $S > 0$. Then*

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{\sigma^2}{S^2}\right) = \frac{4}{S^3}\left(SR_{ijkl}R_{jl}R_{ik} - \sigma^4\right) - \frac{2}{S^4}|S\nabla_p R_{ij} - \nabla_p S R_{ij}|^2 + \langle \nabla\left(\frac{\sigma^2}{S^2}\right), \nabla \log S^2\rangle. \tag{4.11}$$

In the case $\dim(M) = 3$ the above recovers Hamilton's computation Proposition 3.2.

5. High dimension-locally conformally flat

We first prove the following algebraic result.

Proposition 5.1. *Assume that (M, g) is locally conformally flat. Let $\sigma, \tilde{\sigma}, \lambda_i$ be as in the last section. Then*

$$2\,\mathrm{tri}(\mathbf{R})S - \sigma^2|R_{ijkl}|^2 = -\frac{4}{n-2}\left(\frac{1}{n(n-1)}S^2\tilde{\sigma}^2 + \tilde{\sigma}^4 + \frac{2}{n-2}S\sum\lambda_i^3\right) \le 0.$$

If the equality holds, then either
(i) $\lambda_i = 0$ for all $1 \le i \le n$, or
(ii) there exists $a > 0$ such that

$$\begin{aligned}\lambda_l &= \frac{1}{\sqrt{n(n-1)}}a, \qquad \text{for } 1 \le l \le n-1;\\ \lambda_n &= -\sqrt{\frac{n-1}{n}}a\end{aligned}$$

and $S = \sqrt{n(n-1)}a$.

Proof. Let

$$f(S, \lambda_1, \cdots, \lambda_n) = \frac{1}{n(n-1)}S^2\sum\lambda_i^2 + \frac{2}{n-2}S\sum\lambda_i^3 + \left(\sum\lambda_i^2\right)^2.$$

The goal is to show that $f \ge 0$ under the constraint that $\sum\lambda_i = 0$ and analyze the equality case. Since it is homogenous we can consider the extremal values of f under the further constraint $\sum\lambda_i^2 = 1$. Viewing f as a quadratic form in S, the result follows, by elementary consideration, if we show that

$$\left(\sum\lambda_i^3\right)^2 \le \frac{(n-2)^2}{n(n-1)}$$

under the constraints $\sum\lambda_i = 0$ and $\sum\lambda_i^2 = 1$. Let $g = \sum_i\lambda_i^3$. By the Lagrangian multipliers methods, at the critical points we have that

$$\begin{aligned}3\lambda_j^2 - \lambda - 2\mu\lambda_j &= 0, \qquad \text{for } 1 \le j \le n,\\ \sum\lambda_i &= 0,\\ \sum\lambda_i^2 &= 1.\end{aligned}$$

This implies that $\lambda = \frac{3}{n}$ and

$$\lambda_j = \frac{\mu + \epsilon_j\sqrt{\mu^2 + \frac{9}{n}}}{3}$$

with $\epsilon_j \in \{-1, 1\}$. We shall compute all possible values of λ_j. We shall divide into two cases.

Case 1 : $n = 2k$. Let $\epsilon = \sum_j\epsilon_j$ which takes value in $\{-2k, -2(k-1), \cdots, -2, 0, 2, \cdots, 2(k-1), 2k\}$. Since $\sum_j\lambda_j = 0$, it is easy to see that ϵ can not take the value $2k$ or

$-2k$. If $\epsilon=0$ we have that $\mu=0$, which then implies that, after permutation of the index $\lambda_j=\frac{1}{\sqrt{n}}$ for $1\le j\le k$ and $\lambda_j=-\frac{1}{\sqrt{n}}$ for $k\le j\le 2k$. In this case $g=0$.

In general assume that $\epsilon=2(k-i)$ for some $1\le i\le k$. We shall consider only the case $1\le i\le k-1$ since the rest is symmetric to it. Without the loss of the generality we may assume that $\epsilon_j=1$ for $1\le j\le 2k-i$ and $\epsilon_j=-1$ for $2k-i\le j\le 2k$. In this case

$$\begin{aligned}\mu &= -\frac{3(k-i)}{\sqrt{(2k-i)2ki}},\\ \lambda_l &= \sqrt{\frac{i}{2k(2k-i)}}, \quad \text{if } 1\le l\le 2k-i,\\ \lambda_l &= -\sqrt{\frac{2k-i}{2ki}}, \quad \text{if } 2k-i<l\le 2k.\end{aligned}$$

This implies that

$$g=-\frac{n-2i}{\sqrt{(n-i)i}\sqrt{n}}.$$

Noticing that $\frac{(n-2i)^2}{(n-i)i}$ is monotone decreasing in i, we can conclude that $g\ge-\frac{n-2}{\sqrt{n(n-1)}}$. Symmetrically, for $\epsilon=-2(k-i)$ we can find $g=\frac{n-2i}{\sqrt{(n-i)i}\sqrt{n}}$. Combining them together we have that

$$-\frac{n-2}{\sqrt{n(n-1)}}\le g\le\frac{n-2}{\sqrt{n(n-1)}}.$$

The minimum is achieved when $i=1$, which implies the second part of the statement in the proposition.

Case 2: $n=2k+1$. Again due to the fact that $\sum\lambda_i=0$, ϵ takes value in $\{-(2k-1),\cdots,-1,1,\cdots,2k-1\}$. Assume that $\epsilon=2(k-i)+1$ for some $1\le i\le 2k$. We shall only consider $1\le i\le k$ since the other half is symmetric to this case. Now we assume that $\epsilon_j=1$ for all $1\le j\le 2k-i+1$, and $\epsilon_j=-1$ for $2k-i+2\le j\le 2k+1$. Now we have that

$$\begin{aligned}\mu &= -\frac{3}{2}\cdot\frac{2(k-i)+1}{\sqrt{2k+1}\sqrt{i}\sqrt{2k-i+1}},\\ \lambda_l &= \frac{\sqrt{j}}{\sqrt{2k-j+1}\sqrt{2k+1}}, \quad \text{for } 1\le 1\le l\le 2k-i+1,\\ \lambda_l &= -\frac{\sqrt{2k-i+1}}{\sqrt{i}\sqrt{2k+1}}, \quad \text{for } 2k-i+2\le l\le 2k+1.\end{aligned}$$

From this we can compute that

$$g=-\frac{n-2i}{\sqrt{n}\sqrt{n-i}\sqrt{i}}.$$

Again by elementary inequality

$$-\frac{n-2i}{\sqrt{n-i}\sqrt{i}}\ge-\frac{n-2}{\sqrt{n-1}\sqrt{n}}.$$

Hence we conclude that $g^2\le\frac{(n-2)^2}{(n-1)n}$. The minimum achieves when $i=1$.

Combining the above two cases, we complete the proof that $f\ge 0$. From the above discussion, it is straight forward to check that the listed cases are the only two when the inequality can achieve the equality. q.e.d.

Corollary 5.2. *Let (M^n,g) ($n\ge 4$) be a locally conformally flat gradient shrinking soliton whose Ricci curvature is nonnegative satisfying (3.7). Then its universal cover is either $\mathbb{R}^n$, $\mathbb{S}^n$ or $\mathbb{S}^{n-1}\times\mathbb{R}$. In the case that M is compact, the assumptions* *that the Ricci curvature is nonnegative and the growth condition (3.7) are not needed. In particular, if (M^n, g) has positive Ricci curvature it must be compact.*

Proof. Notice that S satisfies the equation $\left(\frac{\partial}{\partial t} - \Delta\right) S = 2|\operatorname{Ric}|^2$. By the strong maximum principle we may assume that $S > 0$, otherwise $M = \mathbb{R}^n$.

Now as in Section 3 we have that

$$(5.1) \quad 0 = \Delta\left(\frac{|R_{ijkl}|^2}{S^2}\right) - \langle \nabla\left(\frac{|R_{ijkl}|^2}{S^2}\right), \nabla f\rangle - \frac{2}{S^4}\left|S\nabla_p R_{ijkl} - \nabla_p S R_{ijkl}\right|^2 - \frac{P}{S^3} + \langle \nabla\left(\frac{|R_{ijkl}|^2}{S^2}\right), \nabla \log S^2\rangle.$$

Here

$$P = -4(2\operatorname{tri}(\mathbf{R})S - \sigma^2|R_{ijkl}|^2),$$

which is nonnegative by the lemma. Multiplying $|R_{ijkl}|^2 e^{-f}$ and integrating by parts, which can be justified similarly as in Section 3, we have that

$$0 = \int_M -\left|\nabla\left(\frac{|R_{ijkl}|^2}{S^2}\right)\right|^2 S^2 e^{-f} - \frac{2|R_{ijkl}|^2}{S^4}\left|S\nabla_p R_{ijkl} - \nabla_p S R_{ijkl}\right|^2 e^{-f} \int_M -\frac{P}{S^3}|R_{ijkl}|^2 e^{-f}.$$

By the lemma we have that

$$\nabla_p S R_{ijkl} = S\nabla_p R_{ijkl} \tag{5.2}$$

which implies that

$$\nabla_p S R_{ik} = S\nabla_p R_{ik}.$$

Also the argument of Section 3 implies that

$$2\operatorname{tri}(\mathbf{R})S - \sigma^2|R_{ijkl}|^2 = -2\left(\frac{1}{12}S^2\tilde{\sigma}^2 + \tilde{\sigma}^4 + S\sum \lambda_i^3\right) = 0$$

and $\frac{|R_{ijkl}|^2}{S^2}$ is a constant.

If $\lambda_i = 0$, then $R_{ik} = \frac{S}{n}\delta_{ik}$. By the second Bianchi identity we have that

$$\frac{1}{2}S\nabla_i S = S\nabla_p R_{ip} = \frac{S}{4}\delta_{ip}\nabla_p S.$$

which implies that $\nabla_p S = 0$. Then we have $\nabla_p R_{ijkl} = 0$ by (5.2).

If the second case happens, by the lemma we have that $R_{ij} = \frac{\delta_{ij}}{n-1}S$ for $1 \le i, j \le n-1$ and $R_{nj} = 0$ for $1 \le j \le n$. The same computation as in $n = 3$ shows that $\nabla_p S = 0$, hence $\nabla_p R_{ijkl} = 0$, which means that (M, g) is locally symmetric. The conclusion follows from the fact that (M, g) is either Einstein or its Ricci curvature has constant rank $n - 1$ and with $n - 1$ identical nonzero eigenvalues. q.e.d.

Remark 5.3. (1) The compactness part should be compared with the result in [**NW**], where under certain curvature operator pinching condition, the manifold is shown to be compact.

(2) Whether or not the argument here is sufficient to show that *any shrinking gradient soliton with positive curvature operator must be compact* is an interesting question. The Kähler case has been resolved in [**N**]. We hope to return to the remaining cases in the future study.

Since Proposition 5.1 also holds when $n = 3$, and $\mathbf{R}_W = 0$ automatically we have the following corollary which generalizes Theorem 1.1.

Corollary 5.4. *Let (M^3, g) be a gradient shrinking soliton whose Ricci curvature is nonnegative satisfying (3.1). Then its universal cover is either $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{R}$. In the case that M is compact, the assumptions that the Ricci curvature is nonnegative is not needed. In particular, if (M^3, g) has positive Ricci curvature it must be compact.*

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