

ON 4-DIMENSIONAL GRADIENT SHRINKING SOLITONS

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Abstract

In this paper we classify the four dimensional gradient shrinking solitons under certain curvature conditions satisfied by all solitons arising from finite time singularities of Ricci flow on compact four manifolds with positive isotropic curvature. As a corollary we generalize a result of Perelman on three dimensional gradient shrinking solitons to dimension four.

1. Introduction

The goal of this paper is to generalize a result of Perelman on three dimensional gradient shrinking solitons to dimension four. In his surgery paper Perelman proved the following statement **[P2]**:

Theorem 1.1. *Any κ -non-collapsed (for some $\kappa > 0$) complete gradient shrinking soliton M^3 with bounded positive sectional curvature must be compact.*

Combining with Hamilton's convergence (or curvature pinching) result **[H1]** (see also **[I]**) one can conclude that M^3 must be isometric to a quotient of \mathbb{S}^3 . The reader can find more detailed proof of this result in **[CaZ, KL, MT]** and Theorem 9.79 of **[CLN]**. We refer to **[NW]** for the discussion on the uses of such a result in the study of Ricci flow, an alternate proof to the above result, basic frame work for the high dimensional cases and a related result in high dimensions.

For four manifolds, in **[H2]**, Hamilton proved that for any compact Riemannian manifold with positive curvature operator, the Ricci flow deforms it into a metric of constant curvature. Such a result has been generalized by H. Chen **[Ch]** to manifolds whose curvature operator is 2-positive. (Recently, in a foundational work Böhm and Wilking **[BW]** have generalized this result to all dimensions.) However it is still unknown if there exists any four dimensional complete gradient shrinking solitons with positive curvature operator which is not compact.

In **[H4]**, Hamilton initiated another important direction, Ricci flow with surgery, and used the method to study the topology of four manifolds with positive isotropic curvature.

Recall from **[H2]** that there is a natural splitting of $\wedge^2(\mathbb{R}^4)$ into self-dual and anti-self-dual parts and one can write the curvature operator \mathbf{R} as

$$\mathbf{R} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}$$

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according to the decomposition $\wedge^2(\mathbb{R}^4) = \wedge_+ \oplus \wedge_-$. We may choose the basis for \wedge_+ and \wedge_- as

$$\begin{aligned}\varphi_1 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4), & \psi_1 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_2 - e_3 \wedge e_4), \\ \varphi_2 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 + e_4 \wedge e_2), & \psi_2 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_3 - e_4 \wedge e_2), \\ \varphi_3 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 + e_2 \wedge e_3), & \psi_3 &= \frac{1}{\sqrt{2}}(e_1 \wedge e_4 - e_2 \wedge e_3),\end{aligned}$$

where $\{e_1, e_2, e_3, e_4\}$ is a positively oriented basis. The first Bianchi identity implies that $\text{tr}(A) = \text{tr}(C) = \frac{S}{4}$ where S is the scalar curvature. Note that A and C are symmetric. Let $A_1 \leq A_2 \leq A_3$ and $C_1 \leq C_2 \leq C_3$ be eigenvalues of A and C respectively. Then \mathbf{R} has positive isotropic curvature amounts to that $A_1 + A_2 > 0$ and $C_1 + C_2 > 0$.

In [H4], it was shown that on the blow-up limit of any finite time singularity of Ricci flow on a compact 4-manifold initially with positive isotropic curvature, there exists $\delta > 0$ depending only on the initial manifold such that the following pinching estimates hold

$$(1.1) \quad A_1 \geq \delta A_3, \quad C_1 \geq \delta C_3, \quad A_1 C_1 \geq B_3^2$$

where $0 \leq B_1 \leq B_2 \leq B_3$ are singular values of B . We say that \mathbf{R} has *uniformly positive isotropic curvature* if (1.1) holds with $A_1 C_1 > 0$. Note that this implies $\mathbf{R} \geq 0$. In view of the work [H4] (see also related work [ChZ]) for the study of the Ricci flow on four manifolds with positive isotropic curvature it is useful to have a classification of gradient shrinking solitons with uniformly positive isotropic curvature in the sense of (1.1).

On the other hand, in general on a gradient shrinking solitons with positive isotropic curvature, it is not clear to the authors whether or not (1.1) always holds. We say that a Riemannian four manifold M has *weakly uniformly positive isotropic curvature* if there exists $\Sigma > 0$ such that

$$(1.2) \quad \left(\frac{B_3^2}{(A_1 + A_2)(C_1 + C_2)} \right) (x) \leq \Sigma.$$

By Theorem B2.1 of [H4], it is easy to infer that a gradient shrinking soliton with bounded curvature satisfying (1.2) must satisfy

$$(1.3) \quad \left(\frac{B_3^2}{(A_1 + A_2)(C_1 + C_2)} \right) (x) \leq \frac{1}{4}.$$

The main purpose of this article is to show a classification result on the gradient shrinking solitons satisfying a rather weak pinching condition:

$$(1.4) \quad \left(\frac{B_3^2}{(A_1 + A_2)(C_1 + C_2)} \right) (x) \leq \exp(a(r(x) + 1))$$

for some $a > 0$, where $r(x)$ is the distance function to a fixed point on the manifold. As in [NW] we also assume that the curvature tensor satisfies

$$(1.5) \quad |R_{ijkl}|(x) \leq \exp(b(r(x) + 1))$$

for some $b > 0$.

Theorem 1.2. *Any four dimensional complete gradient shrinking soliton with nonnegative curvature operator and positive isotropic curvature satisfying (1.4) and (1.5) is either a quotient of \mathbb{S}^4 or a quotient of $\mathbb{S}^3 \times \mathbb{R}$.*

We should remark that in view of the examples [Ko, Co, FIK] some conditions on the curvature operator are essential to obtain a classification result as above. As a corollary of Theorem 1.2 we have the following four dimensional analogue of Theorem 1.1.

Corollary 1.3. *Any four dimensional gradient shrinking soliton with positive curvature operator satisfying (1.4) and (1.5) must be compact.*

Note that there exists a general compactness result [NWu] under a certain pinching condition on the curvature operator, provided that the curvature operator is bounded. But the condition (1.4) is a much weaker one since the curvature operator pinching of [NWu] implies that there exists $\epsilon > 0$ with $A_1 \geq \epsilon A_3$, $C_1 \geq \epsilon C_3$, which further implies $(A_1 + A_2)(C_1 + C_2) \geq \epsilon' S^2 \geq \epsilon' \delta$ for some positive ϵ' and δ (by Proposition 1.1 of [N]). From the last estimate and the boundedness of curvature one can deduce (1.2).

Combining the pinching result Theorem B1.1 of [P1], and Proposition 11.2 of [P1] (see also [N]), one can conclude that the *asymptotic soliton*, borrowing the terminology from [P2], arising from the singularity of Ricci flow on a four manifold with positive isotropic curvature, has nonnegative curvature operator, satisfies (1.1) and has at most quadratic curvature growth. Hence one can apply Theorem 1.2 to obtain a classification on such *asymptotic solitons*.

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2. A preliminary result

From [H2] we know that

$$\mathbf{R}^\# = 2 \begin{pmatrix} A^\# & B^\# \\ (B^t)^\# & C^\# \end{pmatrix},$$

the traceless part of A and C are W_+ and W_- , the self-dual part and the anti-self-dual part of Weyl curvature, and B is the traceless Ricci curvature. It is easy to see that $\text{tr}(A) = \text{tr}(C) = \frac{S}{4}$. Here S is the scalar curvature. Notice that $A^\#, B^\#, C^\#$ are computed as transformations of $\wedge^2(\mathbb{R}^3)$. For example $A^\# = \det(A)(A^t)^{-1}$, while $B^\# = -\det(B)(B^t)^{-1}$.

Let $\sigma^2 = |\text{Ric}|^2$ and $\tilde{\sigma}^2 = |\text{Ric}_0|^2$, where Ric_0 is the traceless part of Ricci tensor. Also let λ_i be the eigenvalue of Ric_0 . First we shall determine how $\tilde{\sigma}^2$ is related to B . Direct computation shows that

$$B = \frac{1}{2} \begin{pmatrix} R_{1212} - R_{3434} & R_{23} - R_{14} & R_{24} + R_{13} \\ R_{23} + R_{14} & R_{1313} - R_{2424} & R_{34} - R_{12} \\ R_{24} - R_{13} & R_{34} + R_{12} & R_{1414} - R_{2323} \end{pmatrix}.$$

Here R_{ij} are the Ricci tensor components. From this we have the following expression of Ric_0 in terms of B :

$$(2.1) \quad \text{Ric}_0 = \begin{pmatrix} B_{11} + B_{22} + B_{33} & B_{32} - B_{23} & B_{13} - B_{31} & B_{21} - B_{12} \\ B_{32} - B_{23} & B_{11} - B_{22} - B_{33} & B_{21} + B_{12} & B_{13} + B_{31} \\ B_{13} - B_{31} & B_{21} + B_{12} & B_{22} - B_{11} - B_{33} & B_{23} + B_{32} \\ B_{21} - B_{12} & B_{13} + B_{31} & B_{23} + B_{32} & B_{33} - B_{11} - B_{22} \end{pmatrix}.$$

Direct computation shows that

$$\tilde{\sigma}^2 = 4|B|^2 \quad \text{and} \quad \sum_1^4 \lambda_j^3 = -8 \text{tr}(B^\# B^t).$$

By [NW], the classification result follows from the non-positivity of $2 \operatorname{tri}(\mathbf{R})S - \sigma^2 |R_{ijkl}|^2$. Here $\operatorname{tri}(\mathbf{R}) = 2\langle \mathbf{R}^2 + \mathbf{R}^\#, \mathbf{R} \rangle$. We now compute it in terms of A, B, C . First it is easy to see that

$$(2.2) \quad P \doteq 2 \operatorname{tri}(\mathbf{R})S - \sigma^2 |R_{ijkl}|^2 = 4\langle S(\mathbf{R}^2 + \mathbf{R}^\#) - (\frac{S^2}{n} + \tilde{\sigma}^2) \mathbf{R}, \mathbf{R} \rangle.$$

For the case $\dim(M) = 4$ we have that

$$\begin{aligned} \langle \mathbf{R}^2 + \mathbf{R}^\#, \mathbf{R} \rangle &= \operatorname{tr}(A^3) + \operatorname{tr}(C^3) + 2 \operatorname{tr}(A^\# A) + 2 \operatorname{tr}((B^t)^\# B) + 2 \operatorname{tr}(B^\# B^t) \\ &\quad + 2 \operatorname{tr}(C^\# C) + 3 \operatorname{tr}(ABB^t) + 3 \operatorname{tr}(CB^t B) \end{aligned}$$

and

$$\langle \mathbf{R}, \mathbf{R} \rangle = \operatorname{tr}(A^2) + \operatorname{tr}(C^2) + 2|B|^2.$$

Hence

$$\begin{aligned} \frac{1}{4}P &= S \left(\operatorname{tr}(A^3) + \operatorname{tr}(C^3) + 2 \operatorname{tr}(A^\# A + C^\# C) + 2 \operatorname{tr}((B^t)^\# B) \right. \\ &\quad \left. + 2 \operatorname{tr}(B^\# B^t) + 3 \operatorname{tr}(ABB^t) + 3 \operatorname{tr}(CB^t B) \right) \\ &\quad - \left(\frac{S^2}{4} + 4|B|^2 \right) (\operatorname{tr}(A^2) + \operatorname{tr}(C^2) + 2|B|^2). \end{aligned}$$

Let $\overset{\circ}{A}$ be the traceless part of A . Similarly we have $\overset{\circ}{C}$. By choosing suitable basis we may diagonalize $\overset{\circ}{A}$ and $\overset{\circ}{C}$ such that we can assume that

$$A = \begin{pmatrix} \frac{S}{12} + a_1 & 0 & 0 \\ 0 & \frac{S}{12} + a_2 & 0 \\ 0 & 0 & \frac{S}{12} + a_3 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{S}{12} + c_1 & 0 & 0 \\ 0 & \frac{S}{12} + c_2 & 0 \\ 0 & 0 & \frac{S}{12} + c_3 \end{pmatrix}.$$

Now we can write

$$\begin{aligned} \frac{1}{4}P &= -S^2 \left(\frac{1}{6} \sum_1^4 \lambda_i^2 + \sum_1^3 a_i^2 + \sum_1^3 c_i^2 \right) \\ (2.3) \quad &+ 4S \left(\sum_1^3 (a_i^3 + c_i^3) + 6a_1 a_2 a_3 + 6c_1 c_2 c_3 - \frac{1}{2} \sum_1^4 \lambda_i^3 \right) \\ &+ 12S \left(a_1 b_1^2 + a_2 b_2^2 + a_3 b_3^2 + c_1 \tilde{b}_1^2 + c_2 \tilde{b}_2^2 + c_3 \tilde{b}_3^2 \right) \\ &- 2 \left(\sum_1^4 \lambda_i^2 \right)^2 - 4 \left(\sum_{i=1}^4 \lambda_i^2 \right) \left(\sum_{i=1}^3 (a_i^2 + c_i^2) \right). \end{aligned}$$

Here $\sum_1^3 a_i = \sum_1^3 c_i = \sum_1^4 \lambda_j = 0$, $b_i^2 = \sum_{j=1}^3 B_{ij}^2$ and $\tilde{b}_i^2 = \sum_{j=1}^3 B_{ji}^2$. Hence $\sum_1^3 b_i^2 = \sum_1^3 \tilde{b}_i^2 = \frac{1}{4} \sum_1^4 \lambda_j^2$.

Check with some examples. After a scaling, we have that

$$\mathbf{R}_{\mathbb{S}^4} = \begin{pmatrix} \operatorname{id} & 0 \\ 0 & \operatorname{id} \end{pmatrix}, \quad \mathbf{R}_{\mathbb{S}^3 \times \mathbb{R}} = \begin{pmatrix} \operatorname{id} & F \\ F^t & \operatorname{id} \end{pmatrix}, \quad \mathbf{R}_{\mathbb{S}^2 \times \mathbb{S}^2} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}, \quad \mathbf{R}_{\mathbb{S}^2 \times \mathbb{R}^2} = \begin{pmatrix} E & E \\ E & E \end{pmatrix}$$

where

$$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check that $P = 0$ on the above examples. On the complex projective space $\mathbf{R}_{\mathbb{P}^2} = \begin{pmatrix} \operatorname{id} & 0 \\ 0 & 3E \end{pmatrix}$. $P = 0$ in this case too!

With suitable choices of the orthonormal basis for Λ_+ and Λ_- we can assume that $A_i = \frac{S}{12} + a_i$, $C_i = \frac{S}{12} + c_i$. It is easy to see that $\max\{\tilde{b}_i^2, b_i^2\} \leq B_3^2$ for any $1 \leq i \leq 3$.

The main result of this section is to prove a special case of Theorem 1.2.

Proposition 2.1. *Suppose that $BB^t = b^2 \text{id}$ for some b , A and C are positive semi-definite. Then $P \leq 0$ and the universal cover of M is either \mathbb{S}^4 or $\mathbb{S}^3 \times \mathbb{R}$.*

Proof. Observing that

$$\sum a_i^3 + 6a_1a_2a_3 = 3 \sum a_i^3$$

in order to show that $2 \text{tri}(\mathbf{R})S - \sigma^2 |R_{ijkl}|^2 \leq 0$ it suffices to show that

$$-S^2 \sum a_i^2 + 12S \sum a_i^3 - 48b^2 \sum a_i^2 \leq 0.$$

Here we have used that $\sum \lambda_i^2 = 12b^2$. Note that we have the constraints that $\frac{S}{12} + a_i \geq 0$ and $\sum a_i = 0$. Using the fact that $\sum a_i^3 \leq \frac{1}{\sqrt{6}}$ under the constraints $\sum a_i = 0$ and $\sum a_i^2 = 1$, which can be obtained by Proposition 4.1 of [NW], we conclude that

$$\frac{\sum a_i^3}{\sum a_i^2} \leq \frac{1}{\sqrt{6}}$$

where $a^2 = \sum a_i^2$. On the other hand, under the constraints $\frac{S}{12} + a_i \geq 0$ and $\sum a_i = 0$, the maximum of $\sum a_i^2$ is $\frac{S^2}{24}$, which can be better seen by expressing everything in terms of $A_i = \frac{S}{12} + a_i \geq 0$. This shows that $-S^2 \sum a_i^2 + 12S \sum a_i^3 \leq 0$ in view of $S > 0$. We can handle the terms with c_i 's similarly. Furthermore, $-S^2 \sum a_i^2 + 12S \sum a_i^3 - 48b^2 \sum a_i^2 = S^2 \sum c_i^2 + 12S \sum c_i^3 - 48b^2 \sum c_i^2 = 0$ implies either $b = 0$ and $a_3 = c_3 = \frac{S}{4}$, $a_1 = a_2 = c_1 = c_2 = -\frac{S}{12}$, or $a_i = c_i = 0$, which is locally conformally flat. The first case is excluded by the positivity of the isotropic curvature. The second case was reduced to the previous result of authors in [NW]. Evoking the proof of Corollary 4.2 of [NW] we obtain a complete classification for this special case. Note that we have used that A and C are semi-positive definite to ensure that $\max\{\sum a_i^2, \sum c_i^2\} \leq \frac{S^2}{24}$. q.e.d.

In the next section we shall reduce the proof of Theorem 1.2 to this special case.

REMARK 2.2. It was pointed out to us by Christoph Böhm that the method of this section alone is not enough to obtain the classification result for gradient shrinking solitons with positive curvature operator, unlike the three dimensional cases treated in [NW].

3. The proof

First we observe that some of the ordinary differential inequalities in [H2] also hold as partial differential inequalities. We list the ones needed below.

Proposition 3.1. *Let $(M, g(t))$ be a solution to Ricci flow. Let A_i , B_i and C_i be the components of curvature operator as defined in the first section. Then with respect to the time dependent moving frame,*

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right)(A_1 + A_2) &\geq A_1^2 + A_2^2 + 2(A_1 + A_2)A_3 + B_1^2 + B_2^2, \\ \left(\frac{\partial}{\partial t} - \Delta\right)(C_1 + C_2) &\geq C_1^2 + C_2^2 + 2(C_1 + C_2)C_3 + B_1^2 + B_2^2, \\ \left(\frac{\partial}{\partial t} - \Delta\right)B_3 &\leq A_3B_3 + C_3B_3 + 2B_1B_2. \end{aligned}$$

The differential inequality can be understood in the sense of distributions.

Proof. The proof is essentially a repeat of the methods in [H4]. Using a time dependent moving frame we have that

$$\left(\frac{\partial}{\partial t} - \Delta\right) \mathbf{R} = \mathbf{R}^2 + \mathbf{R}^\#.$$

Fix a point (x_0, t_0) , choose a local frame so that A and C are diagonal at x_0 . Notice that $\sum_{i,j=1}^2 A_{ij} g^{ij} \geq A_1 + A_2$ and equality holds at (x_0, t_0) . Hence at (x_0, t_0) we have that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \left(\sum_{i,j=1}^2 A_{ij} g^{ij}\right) &= \sum_{i,j=1}^2 g^{ij} \left(A^2 + BB^t + 2A^\#\right)_{ij} \\ &\geq A_1^2 + A_2^2 + 2(A_1 + A_2)A_3 + B_1^2 + B_2^2. \end{aligned}$$

In the last line one uses the same line of argument as in [H2]. This shows the partial differential inequality in the sense of barrier. It then follows from the PDE theory, in viewing of the concavity of $A_1 + A_2$, that the inequality also holds in the sense of distribution. The other two inequalities can be shown similarly. q.e.d.

Now we let $\psi_1 = A_1 + A_2$, $\psi_2 = C_1 + C_2$, $\varphi = B_3$. Our assumption on M has positive isotropic curvature implies that $\psi_1 > 0, \psi_2 > 0$. In the computations below we also assume $B_3 > 0$. But it will be clear later on that this is not necessary. Proposition 3.1 implies

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \log\left(\frac{\varphi^2}{\psi_1 \psi_2}\right) &\leq 2|\nabla \log \varphi|^2 - |\nabla \log \psi_1|^2 - |\nabla \log \psi_2|^2 - \frac{2B_1(B_3 - B_2)}{B_3} \\ &\quad - \frac{(A_1 - B_1)^2 + (A_2 - B_2)^2 + 2A_2(B_2 - B_1)}{A_1 + A_2} \\ &\quad - \frac{(C_1 - B_1)^2 + (C_2 - B_2)^2 + 2C_2(B_2 - B_1)}{C_1 + C_2}. \end{aligned}$$

We denote the last three expressions as $-E$. It is clear that $-E \leq 0$ with equality holds only if $A_1 = C_1 = B_1 = B_2 = A_2 = C_2 = B_3$. In particular we have that $B_1 = B_2 = B_3$, namely $BB^t = b^2 \text{id}$. Using the above partial differential inequality we have that

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \left(\frac{\varphi^2}{\psi_1 \psi_2}\right)^2 &\leq \left(\frac{\varphi^2}{\psi_1 \psi_2}\right)^2 (4|\nabla \log \varphi|^2 - 2|\nabla \log \psi_1|^2 - 2|\nabla \log \psi_2|^2 - 2E) \\ &\quad - 4\left(\frac{\varphi^2}{\psi_1 \psi_2}\right)^2 |2\nabla \log \varphi - \nabla \log \psi_1 - \nabla \log \psi_2|^2. \end{aligned}$$

Now we compute the gradient terms.

$$\begin{aligned} &4|\nabla \log \varphi|^2 - 2|\nabla \log \psi_1|^2 - 2|\nabla \log \psi_2|^2 - 4|2\nabla \log \varphi - \nabla \log \psi_1 - \nabla \log \psi_2|^2 \\ &= -2|2\nabla \log \varphi - \nabla \log \psi_1 - \nabla \log \psi_2|^2 + 2\langle \nabla \log \frac{\varphi}{\psi_1}, \nabla \log(\varphi \psi_1) \rangle \\ &\quad + 2\langle \nabla \log \frac{\varphi}{\psi_2}, \nabla \log(\varphi \psi_2) \rangle - 2\langle \nabla \log \frac{\varphi}{\psi_1}, \nabla \log \frac{\varphi}{\psi_1} \rangle \\ &\quad - 2\langle \nabla \log \frac{\varphi}{\psi_2}, \nabla \log \frac{\varphi}{\psi_2} \rangle - 4\langle \nabla \log \frac{\varphi}{\psi_1}, \nabla \log \frac{\varphi}{\psi_2} \rangle \\ &= -2|\nabla \log \frac{\varphi}{\psi_1} + \nabla \log \frac{\varphi}{\psi_2}|^2 + 8\langle \nabla \log \varphi, \nabla \log \psi_1 \rangle + 8\langle \nabla \log \varphi, \nabla \log \psi_2 \rangle \\ &\quad - 4|\nabla \log \psi_1|^2 - 4|\nabla \log \psi_2|^2 - 4|\nabla \log \varphi|^2 - 4\langle \nabla \log \psi_1, \nabla \log \psi_2 \rangle \\ &= -2|\nabla \log \frac{\varphi}{\psi_1} + \nabla \log \frac{\varphi}{\psi_2}|^2 - 2|\nabla \log \frac{\varphi}{\psi_1}|^2 - 2|\nabla \log \frac{\varphi}{\psi_2}|^2 \\ &\quad + 2\langle \nabla \log \frac{\varphi^2}{\psi_1 \psi_2}, \nabla(\log \psi_1 \psi_2) \rangle. \end{aligned}$$

Putting together we have that

$$(3.1) \quad \begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 &\leq -2 \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 E + \langle \nabla \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2, \nabla(\log \psi_1\psi_2) \rangle \\ &\quad -2 \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 \left(\left| \nabla \log \frac{\varphi}{\psi_1} + \nabla \log \frac{\varphi}{\psi_2} \right|^2 \right. \\ &\quad \left. + \left| \nabla \log \frac{\varphi}{\psi_1} \right|^2 + \left| \nabla \log \frac{\varphi}{\psi_2} \right|^2 \right). \end{aligned}$$

It is clear that the right hand side of the above inequality can be rewritten so that $\varphi > 0$ is not really required. Since (M, g) is a gradient shrinking soliton, letting f be the potential function, the computation in the Section 1 of [NW] implies that

$$\frac{\partial}{\partial t} \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 = \langle \nabla f, \nabla \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 \rangle.$$

Now multiply both sides of (3.1) by $e^{-f+\log(\psi_1+\psi_2)}$ and integrate over the manifold:

$$\begin{aligned} &\int_M \langle \nabla f, \nabla \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 \rangle e^{-f+\log(\psi_1\psi_2)} - \int_M \left(\Delta \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2\right) e^{-f+\log(\psi_1\psi_2)} \\ &\leq -2 \int_M \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 E e^{-f+\log(\psi_1\psi_2)} + \int_M \langle \nabla \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2, \nabla(\log \psi_1\psi_2) \rangle e^{-f+\log(\psi_1\psi_2)} \\ &\quad -2 \int_M \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 \left(\left| \nabla \log \frac{\varphi}{\psi_1} + \nabla \log \frac{\varphi}{\psi_2} \right|^2 \right) e^{-f+\log(\psi_1\psi_2)} \\ &\quad -2 \int_M \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 \left(\left| \nabla \log \frac{\varphi}{\psi_1} \right|^2 + \left| \nabla \log \frac{\varphi}{\psi_2} \right|^2 \right) e^{-f+\log(\psi_1\psi_2)}. \end{aligned}$$

All the integrals are finite by the derivative estimates of Shi [Sh], the assumption (1.4) and (1.5), and Lemma 1.3 of [NW] asserting that $f(x) \geq \frac{1}{8}r^2(x) - C$ (with $C > 0$ and $r(x)$ being distance function to a fixed point). As in [NW], integration by parts can be performed on the term involving the Laplacian operator in the left hand side of the above inequality. After the integration by parts and some cancelations we have that

$$\begin{aligned} 0 &\leq - \int_M \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 E e^{-f+\log(\psi_1\psi_2)} \\ &\quad -2 \int_M \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 \left(\left| \nabla \log \frac{\varphi}{\psi_1} + \nabla \log \frac{\varphi}{\psi_2} \right|^2 \right) e^{-f+\log(\psi_1\psi_2)} \\ &\quad -2 \int_M \left(\frac{\varphi^2}{\psi_1\psi_2}\right)^2 \left(\left| \nabla \log \frac{\varphi}{\psi_1} \right|^2 + \left| \nabla \log \frac{\varphi}{\psi_2} \right|^2 \right) e^{-f+\log(\psi_1\psi_2)}, \end{aligned}$$

which implies that

$$E = \left| \nabla \log \frac{\varphi}{\psi_1} + \nabla \log \frac{\varphi}{\psi_2} \right| = \left| \nabla \log \frac{\varphi}{\psi_1} \right| = \left| \nabla \log \frac{\varphi}{\psi_2} \right| = 0.$$

In particular, we conclude that $BB^t = b^2 \text{id}$.

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