

Singular Yamabe metrics and initial data with *exactly* Kottler–Schwarzschild–de Sitter ends

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Abstract

We construct large families of initial data sets for the vacuum Einstein equations with positive cosmological constant which contain *exactly Delaunay ends*; these are non-trivial initial data sets which coincide with those for the Kottler–Schwarzschild–de Sitter metrics in regions of infinite extent. From the purely Riemannian geometric point of view, this produces complete, constant positive scalar curvature metrics with exact Delaunay ends which are not globally Delaunay. The ends can be used to construct new compact initial data sets via gluing constructions. The construction provided applies to more general situations where the asymptotic geometry may have *non-spherical* cross-sections consisting of Einstein metrics with positive scalar curvature.

1 Introduction

There exists very strong evidence suggesting that we live in a world with strictly positive cosmological constant Λ [38, 42]. This leads to a need for a better understanding of the space of solutions of Einstein equations with $\Lambda > 0$. The most general method available for constructing such solutions proceeds by solving a Cauchy problem [2, 10, 21]. In view of the general relativistic constraint equations this, subsequently, requires understanding the corresponding collection of initial data sets. In particular one is led to the question of boundary conditions satisfied by the fields. When Λ vanishes a natural set of boundary conditions arises from the obvious model solution – the Minkowski space-time. A tempting further restriction is then the requirement of a well defined and finite total mass, leading to a well understood set of asymptotic boundary conditions [1, 5, 11, 34]. When $\Lambda > 0$ the question of asymptotic conditions seems to be much less clear cut. One wants to consider a class of space-times which includes all solutions of physical interest. Until there is overwhelming evidence to the contrary, “physical interest” should carry a notion of “non-singular”. The simplest possibility, widely adopted, is to assume that the Cauchy surface \mathcal{S} is a compact manifold without boundary. However, an appealing more general way of

ensuring regularity of the initial data is to suppose that (\mathcal{S}, g) is a complete Riemannian manifold. One would then like to understand the space of solutions of those general relativistic constraint equations with (\mathcal{S}, g) – complete.

An interesting class of asymptotic models for such initial data has already been explored in the mathematical literature, the time symmetric initial data provided by the *Delaunay*¹ metrics [19, 20, 41]. These describe the family of complete rotationally symmetric, conformally flat metrics with constant positive scalar curvature, and are in fact well known to general relativists as the time-symmetric slices of the Kottler–Schwarzschild–de Sitter solutions [22, 25] (however the connection between these two subjects has apparently not been previously noted. In the Riemannian geometric context, the Delaunay metrics form the local asymptotic model for isolated singularities of locally conformally flat constant positive scalar curvature metrics [8, 9, 24, 36] (in dimensions $n \leq 5$ this also holds in the non-conformally flat setting [26]). The known results concerning the existence of complete constant positive scalar curvature metrics with *asymptotically Delaunay* ends [7, 24, 27, 30–32, 36, 37, 40] may thus be reinterpreted, via their space-time development, as the existence of space-times satisfying the Einstein field equations with a positive cosmological constant which have asymptotically Kottler–Schwarzschild–de Sitter ends.

The object of this work is to point out that every constant positive scalar curvature (CPSC) asymptotically Delaunay metric is naturally accompanied by a CPSC metric with an *exactly Delaunay* end, and moreover these metrics may be chosen to coincide away from the end in question. Such metrics are of interest in general relativity for at least four reasons:

1. They provide, via their maximal development, a large class of space-times satisfying the Einstein field equations with a positive cosmological constant with exactly controlled geometry in the asymptotic regions; in fact the space-time development is explicitly known in the domain of dependence of the Delaunay regions.
2. They demonstrate that the special horizon behavior, with alternating cosmological and event horizons, which is exhibited by the Kottler–Schwarzschild–de Sitter space-time, occurs in large classes of non-stationary solutions.
3. Any two metrics which carry exactly Delaunay ends with identical mass (Delaunay) parameters may be glued together using obvious identifications on the ends. (A more difficult *end-to-end* asymptotic gluing theorem of this sort was established by Ratzkin [37], however with exactly Delaunay ends this construction is effortless.) Thus Delaunay ends can easily be used as bridges to create wormholes, or to make connected sums of initial data sets. Wormhole constructions are already known to be possible by completely different techniques [16] in the setting of a non-positive cosmological constant. Here we provide such a construction for positive cosmological constants, with the added bonus of an explicit knowledge of the space-time development in the domain of dependence of the middle part of the connecting neck, which may be of arbitrary (quantized by multiples of the period of the exact Delaunay metric) length.
4. The asymptotically Delaunay metrics are uniquely characterised by a simple geometric criterion [24, 26], see Section 2.3 below.

A natural setting for our considerations is provided by the *generalised Kottler metrics* and *generalised Delaunay metrics*, as described in Sections 2.2 and 2.4 below. Our gluing construction applies in this more general setting.

¹These are also often called *Fowler* solutions; see §2.3 for further remarks on the history and choice of terminology used here.

In an accompanying paper [14], by one of us (PTC) and Erwann Delay, analogous constructions are carried out with a negative cosmological constant. With hindsight, within the family of Kottler metrics with $\Lambda \in \mathbb{R}$, the gluing in the current setting is the easiest, while that in [14] is the most difficult. This is due to the fact that for $\Lambda > 0$, as considered here, one deals with one linearised operator with a one-dimensional kernel; in the case $\Lambda = 0$ the kernel is $(n + 1)$ -dimensional; while for $\Lambda < 0$ one needs to deal with a one-parameter family of operators with $(n + 1)$ -dimensional kernels.

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2 Kottler–Schwarzschild–de Sitter space and metrics of constant positive scalar curvature with asymptotically Delaunay ends

In this section we review some results concerning the Kottler–Schwarzschild–de Sitter space and CPSC metrics which are asymptotically Delaunay. In order to fix notations and conventions we start with some standard facts.

Recall that initial data for the Einstein field equations with a cosmological constant Λ on an n -dimensional manifold M consist of a pair (g, K) consisting of a Riemannian metric g on M and a symmetric 2-tensor K satisfying the vacuum constraint equations

$$R(g) - (2\Lambda + |K|_g^2 - (\text{tr}_g K)^2) = 0 \quad (2.1)$$

$$D_i(K^{ij} - \text{tr}_g K g^{ij}) = 0 \quad (2.2)$$

where $R(g)$ is the scalar curvature (Ricci scalar) of the metric g . If one considers time-symmetric initial data, for which $K \equiv 0$, then these equations reduce to the requirement that g has constant scalar curvature $R(g) = 2\Lambda$. Here we restrict to the case where Λ is positive, and note that the normalization $\Lambda = \frac{n(n-1)}{2}$ corresponds to $R(g) = n(n-1)$, the scalar curvature of the standard sphere of radius one in \mathbb{R}^{n+1} .

2.1 Kottler–Schwarzschild–de Sitter metrics

The Kottler–Schwarzschild–de Sitter space-time [25] metric in $n + 1$ dimensions, with cosmological constant $\Lambda > 0$ and mass $m \in \mathbb{R}$ may be written as

$$ds^2 = -V dt^2 + V^{-1} dr^2 + r^2 \overset{\circ}{h}, \quad \text{where } V = V(r) = 1 - \frac{2m}{r^{n-2}} - \frac{r^2}{\ell^2}, \quad (2.3)$$

where $\ell > 0$ is related to the cosmological constant Λ by the formula $2\Lambda = n(n-1)/\ell^2$, while $\overset{\circ}{h}$ denotes the standard metric on the unit $(n-1)$ -sphere in \mathbb{R}^n . To avoid a singularity lying at finite distance on the level sets of t we will assume $m > 0$. Equation (2.3) provides then a spacetime metric satisfying the Einstein equations with cosmological constant $\Lambda > 0$ and with well behaved spacelike hypersurfaces when one restricts the coordinate r to an interval (r_b, r_c) on which $V(r)$ is positive; such an interval exists if and only if

$$\left(\frac{2}{(n-1)(n-2)} \right)^{n-2} \Lambda^{n-2} m^2 n^2 < 1. \quad (2.4)$$

When $n = 3$ this corresponds to the condition that $9m^2\Lambda < 1$, and the case of equality is referred to as the extreme Kottler–Schwarzschild–de Sitter space-time (for which the coordinate expression (2.3) is no longer valid). In the limit where Λ tends to zero with m held constant, the space-time metric approaches the Schwarzschild metric with mass m , and in the limit where m goes to zero with Λ held constant the metric tends to that of the de Sitter space-time with cosmological constant Λ .

The breakdown of the coordinate description above at the horizons $r = r_b$ and $r = r_c$ can be handled by taking extensions [4, 22]: In fact, the Kottler–Schwarzschild–de Sitter metric admits an analytic extension (analogous to the Kruskal extension of the Schwarzschild metric) as an r -periodic metric on $(t, r, \theta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{S}^{n-1}$. This is most easily seen via the associated conformal Carter–Penrose diagrams [22]. The time-symmetric slice $t = 0$ of the (extended) Kottler–Schwarzschild–de Sitter metrics are thus a one-parameter family (parameterized by their mass m) of periodic, spherically symmetric, metrics on $\mathbb{R} \times \mathbb{S}^{n-1}$ with constant positive scalar curvature $R = 2\Lambda$.

Finally note that, due to the spherical symmetry, each of these metrics is conformally flat.

2.1.1 Extreme limit

It is of some interest to enquire what happens when $m \rightarrow \mathring{m}$, where \mathring{m} denotes the values at which equality is achieved in (2.4). In this limit r_b and r_c coalesce to a single value which we will denote by \mathring{r} . From the space-time point of view the situation is the following: recall that the Carter–Penrose diagram for the maximally extended KSdS space-times with $0 < m < \mathring{m}$ is built out of diamond shaped regions corresponding to $r_b < r < r_c$, where the Killing vector ∂_t is time-like, and of triangle shaped (either upright, or upside-down) regions where ∂_t is spacelike [22]. After passing to the limit $m \rightarrow \mathring{m}$ the diamond-shaped regions disappear, and the resulting diagram consists of a string of triangles. The Killing vector ∂_t is then spacelike everywhere, except on the degenerate horizons $\mathring{r} = r_b = r_c$.

On the level sets of t a rather different analysis applies, this is discussed in Section 2.3.1.

2.2 Generalised Kottler metrics

All the results discussed in Section 2.1 remain valid if $m \neq 0$ and if the metric \mathring{h} in (2.3) is an *Einstein metric on an $(n - 1)$ -dimensional manifold N^{n-1} with scalar curvature equal to $(n - 1)(n - 2)$* [6]. We will refer to such metrics as *generalised Kottler metrics*. Note that $m = 0$ requires (N^{n-1}, \mathring{h}) to be the unit round metric if one does not want $r = 0$ to be a singularity at finite distance along the level sets of t .

2.3 Delaunay metrics

The Delaunay metrics, in dimension $n \geq 3$, may be defined as the (two parameter) family of metrics

$$g = u^{4/(n-2)}(dy^2 + \mathring{h}), \quad (2.5)$$

where \mathring{h} is the unit round metric on \mathbb{S}^{n-1} , which are spherically symmetric and have constant scalar curvature $R(g) = n(n - 1)$. Thus the functions $u = u(y) > 0$ must satisfy the ODE

$$u'' - \frac{(n-2)^2}{4}u + \frac{n(n-2)}{4}u^{\frac{n+2}{n-2}} = 0. \quad (2.6)$$

The two parameters correspond respectively to a minimum value ε for u , with

$$0 \leq \varepsilon \leq \bar{\varepsilon} = \left(\frac{n-2}{n}\right)^{\frac{n-2}{4}} \quad (2.7)$$

(ε is called the Delaunay parameter or neck size) and a translation parameter along the cylinder. A straightforward ODE analysis (see [32]) shows that all the positive solutions are periodic. The degenerate solution with $\varepsilon = 0$ corresponds to the round metric on a sphere from which two antipodal points have been removed. The solution with $\varepsilon = \bar{\varepsilon}$ corresponds to the rescaling of the cylindrical metric so that the scalar curvature has the desired value.

Note that the Delaunay ODE was first studied by Fowler [19, 20], however the name used here and elsewhere in the literature is inspired from the analogy with the Delaunay surfaces: the complete, periodic CMC surfaces of revolution in \mathbb{R}^3 [18]. As is well known, the analogy between the “conformally flat metrics of constant positive scalar curvature” and “complete embedded CMC surfaces of in \mathbb{R}^3 ” goes far beyond this correspondence (see, e.g., [30]).

Regarding the Delaunay metrics as singular solutions of the Yamabe equation on (\mathbb{S}^n, g_0) one has a number of uniqueness results. Among these are the facts that no solution with a single singular point exists, and that any solution with exactly two isolated singular points must be conformally equivalent to a Delaunay metric. These results can be proved by a generalization of the classical Alexandrov reflection argument (the method of moving planes), see [23]. The first general existence result for complete conformally flat metrics of constant scalar curvature with asymptotically Delaunay ends is due to Schoen [40].

Of immediate interest to us is the fact that conformally flat metrics, with constant positive scalar curvature, and with an *isolated singularity of the conformal factor* are necessarily asymptotic to a Delaunay metric [24]; in fact, in dimensions $n = 3, 4, 5$ the conformal flatness condition is not needed [26]. Specifically, with respect to spherical coordinates about an isolated singularity of the conformal factor, there is a half-Delaunay metric which g converges to, exponentially fast in r , along with all of its derivatives. This fact is used in [27, 31, 32, 36, 37] where complete, constant scalar curvature metrics, conformal to the round metric on $\mathbb{S} \setminus \{p_1, \dots, p_k\}$ were studied and constructed. (This is one instance of the more general “singular Yamabe problem”.)

By the uniqueness of solutions to ODEs, or otherwise, we have:

PROPOSITION 2.1 *The time symmetric initial data sets for Kottler–Schwarzschild–de Sitter space in spatial dimension n with $\Lambda = \frac{n(n-1)}{2}$, are precisely the Delaunay metrics with constant positive scalar curvature $R = n(n-1)$.*

This correspondence continues to hold for any choice of positive cosmological constant Λ provided that one homothetically rescales the Delaunay metrics so that $R = 2\Lambda$.

Comparing (2.3) and (2.5) we find

$$r = u^{\frac{2}{n-2}}, \quad r \frac{dy}{dr} = V^{-1/2}, \quad (2.8)$$

which allows us to determine y as a function of r on any interval of r 's on which V has no zeros.

2.3.1 Extreme limit

Let \hat{m} and \hat{r} be as in Section 2.1.1 and suppose that $0 < m < \hat{m}$, denote by $r_* \in (r_b, r_c)$ the value at which the maximum value V_* of V is attained, shifting y by a

constant we can assume that the corresponding value $y_* = y(r_*)$ of the y coordinate in (2.6) is zero. We have $r_* \rightarrow \mathring{r}$ and $V_* \rightarrow 0$ as $m \rightarrow \mathring{m}$, and it clearly follows from (2.8) that the correspondence $y \leftrightarrow r$ breaks down in the limit. This singular behavior with respect to the r coordinate is of course resolved by the coordinate y of (2.5). A somewhat more explicit way of seeing this is to replace r by a new coordinate w through the formula

$$r = r_* + \sqrt{V_*}w$$

which scales up the interval $r \in (r_b, r_c)$ to $w \in ((r_b - r_*)/\sqrt{V_*}, (r_c - r_*)/\sqrt{V_*})$. Equations (2.8) become

$$r_* + \sqrt{V_*}w = u^{\frac{2}{n-2}}, \quad \frac{dy}{dw} = \frac{\sqrt{V_*}}{(r_* + \sqrt{V_*}w)\sqrt{V}}, \quad (2.9)$$

which are regular in the limit $m \rightarrow \mathring{m}$. In the new coordinates we have

$$\frac{1}{V}dr^2 + r^2\mathring{h} = \frac{V_*}{V}dw^2 + (r_* + \sqrt{V_*}w)^2\mathring{h} \longrightarrow dw^2 + r_*^2\mathring{h} \quad \text{as} \quad m \rightarrow \mathring{m},$$

with the limit being uniform over compact sets of the w coordinate. This shows in which sense the space sections of the KSdS metrics approach a cylindrical geometry in the extreme limit. It should, however, be borne in mind that the space-time picture of Section 2.1.1 is rather different.

2.4 Generalised Delaunay metrics

Similarly to Section 2.2, the analysis presented at the beginning of Section 2.3 remains valid when the parameter ϵ is positive and if the metric \mathring{h} in (2.5) is an *Einstein metric on an $(n-1)$ -dimensional manifold N^{n-1} with scalar curvature equal to $(n-1)(n-2)$* . We refer to the resulting metrics as *generalised Delaunay metrics*. As before, $\epsilon = 0$ requires (N^{n-1}, \mathring{h}) to be the unit round sphere if one wants to avoid a singularity at the set $u(y) = 0$.

2.5 Complete metrics with constant positive scalar curvature and asymptotically Delaunay ends

Conformal gluing constructions for constant scalar curvature metrics $\tilde{g} = u^{4/(n-2)}g$ with $R(\tilde{g}) = R(g) = n(n-1)$ have given rise to a wide variety of such metrics with asymptotically Delaunay ends. The linearisation of this equation about a solution leads to the operator $L_g = \Delta_g + n$. The key *nondegeneracy* assumption of any conformal gluing construction is that L_g is surjective when acting on appropriately defined function spaces. The Delaunay metrics themselves are nondegenerate in this sense [32], moreover the solutions constructed by Mazzeo-Pacard on $\mathbb{S} \setminus \{p_1, \dots, p_k\}$ [27] are non-degenerate. On the other hand, the standard metric on the n -sphere, (\mathbb{S}, g_0) , is degenerate due to the fact that the restrictions of the linear functions in \mathbb{R}^{n+1} span an $(n+1)$ -dimensional co-kernel of L_{g_0} . In addition to the original construction of Schoen [40], the constructions of [31] and [37] use non-degenerate solutions as building blocks to produce new non-degenerate solutions.

All of the constructions alluded to above are in the setting where the metrics are locally conformally flat everywhere. This is clearly not necessary. A general conformal gluing theorem was established by Byde [7]:

THEOREM 2.2 (Byde [7]) *Let (M, g) be a compact Riemannian manifold, possibly with boundary, of constant scalar curvature $n(n-1)$, which is non-degenerate in the*

sense described above, and let $x_0 \in \text{int}(M)$ be a point in a neighborhood of which g is conformally flat. Then there is a constant ρ_0 and a one parameter family of complete metrics g_ρ on $M \setminus \{x_0\}$ defined for $\rho \in (0, \rho_0)$, conformal to g , with constant scalar curvature $n(n-1)$. Moreover, each g_ρ is asymptotically Delaunay and $g_\rho \rightarrow g$ uniformly on compact sets in $M \setminus \{x_0\}$ as $\rho \rightarrow 0$.

This result is exactly analogous to results for constant mean curvature surfaces established in [28, 29]. Byde goes further and shows how one can also glue asymptotically Delaunay ends onto non-compact, non-degenerate solutions (though without the uniform convergence to the original metric away from the gluing locus).

Note that, in light of Proposition 2.1, all of these results, and others, on the existence of CPSC metrics with asymptotically Delaunay ends have an immediate reinterpretation, after considering the maximal development of the initial data set, as statements regarding the existence of space-times with asymptotically Kottler–Schwarzschild–de Sitter ends.

3 Perturbation to exactly Delaunay, or generalised Delaunay ends

The gluing construction of Corvino-Schoen [17, 39] (compare [13]) generalises to the positive cosmological constant setting as follows:

THEOREM 3.1 *Let N^{n-1} be compact, let (M, g) satisfy $R(g) = n(n-1)$, and suppose that M contains an end $E \approx [0, \infty) \times N^{n-1}$ on which g is asymptotic to a generalised Delaunay metric $\mathring{g} = \mathring{g}_\varepsilon$, with $0 < \varepsilon < (\frac{n-2}{n})^{\frac{n-2}{4}}$, together with derivatives up to order four. Then for every $\delta > 0$ there is an ε' satisfying $|\varepsilon - \varepsilon'| < \delta$ and a metric g' with $R(g') = n(n-1)$, which differs from g only far away on E , and which is a generalised Delaunay metric with Delaunay parameter ε' on the complement E' of a compact subset of E .*

REMARK 3.2 By taking the maximal, globally hyperbolic, space-time development of the time-symmetric initial data set (M, g') , we obtain a solution of the vacuum Einstein equations with cosmological constant $\Lambda = n(n-1)/2$ such that the metric on the domain of dependence of the end E' is isometric to a subset of the Kottler–Schwarzschild–de Sitter space-time.

PROOF: Let g asymptote to a generalised Delaunay metric \mathring{g}_ε on $\mathbb{R} \times N^{n-1}$. We can write \mathring{g}_ε as

$$\mathring{g}_\varepsilon = dx^2 + e^{2f(x)} \mathring{h}, \quad (3.1)$$

where \mathring{h} is an Einstein metric on N^{n-1} , normalised as described above.

Consider a connected component of the set on which $V > 0$, where V is the function appearing in (2.3) for the metric \mathring{g}_ε . It follows from (2.3) that \sqrt{V} is the normal component of the Killing vector ∂_t on the level sets of t . From the general results in [33] it follows that any such function, for a static space-time, solves equation (3.2) below. It further follows from the analysis in [12] that \sqrt{V} can be smoothly continued to a real-analytic function on M , which we call \mathring{N} , by changing signs across the zero level sets of V . Furthermore, both \mathring{N} and $V = \pm \mathring{N}^2$ are functions of x only in the representation (3.1) of \mathring{g} .

Let $T = T(\varepsilon)$ be the period of f , and let $\Omega_i = [iT + \sigma, iT + T + \sigma] \times N^{n-1}$, where σ will be chosen below.

Let $(\mathbb{R} \times N^{n-1}, \mathring{g}_{\varepsilon'})$ be a generalised Delaunay metric with parameter ε' near ε , $|\varepsilon - \varepsilon'| < \delta$.

Let $g_{\varepsilon'}$ be a metric on Ω_i obtained by interpolating between g and $\mathring{g}_{\varepsilon'}$ using any i -independent cut-off function smoothly varying from zero to one. The cut-off should be supported away from the end-points of the interval $[i, i + T]$.

To achieve constant scalar curvature we will, first, correct the metric $g_{\varepsilon'}$ to a new metric $\mathring{g}_{\varepsilon'}$ using the operator L of [13], as restricted to time-symmetric data, so that $Y \equiv 0$ there. The correction will be of the order of the perturbation introduced, namely $O(|\varepsilon - \varepsilon'|)$. This will, however, not quite solve the problem because the operator L at $g = \mathring{g}_{\varepsilon}$ has a cokernel, which consists of functions solving the ‘‘static KIDs equation’’:

$$D_i D_j N = N R_{ij} + \Delta_g N g_{ij} . \quad (3.2)$$

We thus have to understand the space of solutions of (3.2):

LEMMA 3.3 *Let $g = \mathring{g}$ be a generalised Delaunay metric as in (3.1).*

1. *If (N^{n-1}, \mathring{h}) is the round sphere and if $m = 0$, then the space of solutions of (3.2) is $(n + 1)$ -dimensional, spanned by \mathring{N} together with functions of the form $e^f \alpha^i$, where f is as in (3.1) and α^i is the restriction of the Euclidean coordinate x^i to \mathbb{S}^{n-1} under the standard embedding $\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$.*
2. *Otherwise all solutions of (3.2) are proportional to \mathring{N} .*

PROOF: Let v^A be coordinates on the level sets of x ; we have $\Gamma_{AB}^x = -f' \mathring{h}_{AB}$, $\Gamma_{xj}^x = 0$, $\Gamma_{xB}^A = f' \delta_B^A$, and $\Gamma_{BC}^A = \mathring{\Gamma}_{BC}^A$, where the $\mathring{\Gamma}_{BC}^A$'s are the Christoffel symbols of \mathring{h} . Since \mathring{N} depends only upon x , and satisfies (3.2), we immediately find

$$R_{xA} = 0$$

away from the zero-set of V ; by continuity this holds everywhere (this conclusion could also have been reached directly from the warped product structure of \mathring{g}). But then (3.2) gives

$$0 = D_x D_A N = e^f \partial_x (e^{-f} \partial_A N) ,$$

hence

$$N(x, v^A) = e^{f(x)} \mathring{P}(v^A) + \mathring{M}(x) , \quad (3.3)$$

for some functions $\mathring{P} = \mathring{P}(v^A)$, $\mathring{M} = \mathring{M}(x)$.

Set

$$\alpha := N / \mathring{N} ,$$

then α is smooth away from the zero-level sets of \mathring{N} . Since both N and \mathring{N} satisfy (3.2) one finds that α is a solution of the equation

$$\mathring{N} D_i D_j \alpha + D_i \alpha D_j \mathring{N} + D_j \alpha D_i \mathring{N} = 0 . \quad (3.4)$$

From $D_A \mathring{N} = 0$ we obtain

$$D_A D_B \alpha = 0 . \quad (3.5)$$

Let $\lambda_{AB} = -\Gamma_{AB}^x = f' \mathring{h}_{AB}$ be the second fundamental form of the level sets of x , (3.5) can be rewritten as

$$\mathring{\mathcal{D}}_A \mathring{\mathcal{D}}_B \alpha + \alpha_x \lambda_{AB} \equiv \mathring{\mathcal{D}}_A \mathring{\mathcal{D}}_B \alpha + \underbrace{\alpha_x f'}_{=: \varphi} \mathring{h}_{AB} = 0 , \quad (3.6)$$

where $\overset{\circ}{\mathcal{D}}$ is the covariant derivative operator of the metric $\overset{\circ}{h}$, and $\alpha_x = \partial_x \alpha$. Applying $\overset{\circ}{\mathcal{D}}^B$ to (3.6) and commuting derivatives one obtains (recall that the Ricci tensor of $\overset{\circ}{h}$ equals $(n-2)\overset{\circ}{h}$)

$$\overset{\circ}{\mathcal{D}}_A \left(\overset{\circ}{\mathcal{D}}_B \overset{\circ}{\mathcal{D}}^B \alpha + (n-2)\alpha - \varphi \right) = 0 .$$

Contracting A with B in (3.6) we find $\varphi = \overset{\circ}{\mathcal{D}}_B \overset{\circ}{\mathcal{D}}^B \alpha / (n-1)$, which allows us to conclude that there exists a constant C such that

$$\overset{\circ}{\mathcal{D}}_B \overset{\circ}{\mathcal{D}}^B (\alpha + C) = -(n-1)(\alpha + C) . \quad (3.7)$$

Suppose, first, that $(N^{n-1}, \overset{\circ}{h})$ is the unit round sphere. Equation (3.7) shows that solutions of (3.6) are linear combinations of the constant function $\alpha^0 = 1$ and of the functions $\alpha^i = x^i|_{\mathbb{S}^{n-1}}$, where x^i is a canonical coordinate in \mathbb{R}^n , with \mathbb{S}^{n-1} being embedded in \mathbb{R}^n in the obvious way. Hence, there exist functions $\lambda_\mu(x)$, $\mu = 0, \dots, n$, such that

$$\alpha(x, v^A) = \lambda_\mu(x) \alpha^\mu(v^A) . \quad (3.8)$$

But then (3.3) implies $\lambda_i(x) = e^f \overset{\circ}{N}^{-1} \overset{\circ}{\lambda}_i$ for some constants $\overset{\circ}{\lambda}_i$, without however imposing any constraints on $\lambda_0(x)$ which remains undetermined so far.

On the other-hand, if $(N^{n-1}, \overset{\circ}{h})$ is *not* the unit round sphere, then by a theorem of Obata [35] the function α does not depend upon v^A , so that (3.8) again holds with $\alpha^i \equiv 0$ and $\alpha^0 = 1$.

Inserting (3.8) into (3.4) with $ij = xx$ one finds

$$\partial_x \left\{ \overset{\circ}{N}^2(x) \partial_x \left[\lambda_0(x) + \overset{\circ}{N}^{-1}(x) e^{f(x)} \overset{\circ}{\lambda}_i \alpha^i(v^A) \right] \right\} = 0 . \quad (3.9)$$

In order to analyse this equation, it is useful to compare (2.3) with (3.1) to conclude that

$$\frac{dr}{dx} = \pm \frac{1}{\overset{\circ}{N}} = \pm \frac{1}{\sqrt{1 - \frac{2m}{r^{n-2}} - \frac{r^2}{\ell^2}}} . \quad (3.10)$$

We then have $\overset{\circ}{N}^2 \partial_x = \pm \overset{\circ}{N}^3 \partial_r$ and

$$\begin{aligned} \pm \overset{\circ}{N}^2 \partial_x \left[\lambda_0 + \overset{\circ}{N}^{-1} e^f \overset{\circ}{\lambda}_i \alpha^i(v^A) \right] &= \overset{\circ}{N}^3 \partial_r \left[\lambda_0 + \frac{r}{\sqrt{1 - \frac{r^2}{\ell^2} - \frac{2m}{r^{n-2}}}} \overset{\circ}{\lambda}_i \alpha^i(v^A) \right] \\ &= \overset{\circ}{N}^3 \partial_r \lambda_0 + 1 - \frac{mn}{r^{n-2}} \overset{\circ}{\lambda}_i \alpha^i(v^A) . \end{aligned}$$

So (3.9) will hold if and only if this is a function which depends at most upon v^A . Hence λ_0 is a constant and $\overset{\circ}{\lambda}_i = 0$ unless $m = 0$, in which case the $\overset{\circ}{\lambda}_i$'s are arbitrary, as desired. \square

Returning to the proof of Theorem 3.1, the metric $\tilde{g}_{\varepsilon'}$ is obtained by solving the equation

$$\left(R(\tilde{g}_{\varepsilon'}) - n(n-1) \right) \Big|_{\Omega_i} \in (\text{Im } L)^\perp$$

using the implicit function theorem, compare [13, Theorem 5.9]; this can be done on Ω_i for all i large enough. As already mentioned, the perturbation introduced is $O(|\varepsilon - \varepsilon'|)$. In view of Lemma 3.3, the obstruction to solving the problem is thus the vanishing of

$$\int_{\Omega_i} \overset{\circ}{N} \left(R(\tilde{g}_{\varepsilon'}) - R(\overset{\circ}{g}_{\varepsilon'}) \right) d\mu_{\tilde{g}} , \quad (3.11)$$

where \mathring{N} is the static KID associated with the generalised Delaunay metric \mathring{g}_ε . We need the following identity, from [15]:

$$\sqrt{\det g} \mathring{N}(R_g - R_b) = \partial_i \left(\mathbb{U}^i(\mathring{N}) \right) + \sqrt{\det g} (\rho + Q), \quad (3.12)$$

where

$$\mathbb{U}^i(\mathring{N}) := 2\sqrt{\det g} \left(\mathring{N}g^{i[k}g^{j]l}\mathring{\mathcal{D}}_j g_{kl} + D^{[i}\mathring{N}g^{j]k}e_{jk} \right), \quad (3.13)$$

$$\rho := (-\mathring{N}\text{Ric}(b)_{ij} + \mathring{\mathcal{D}}_i\mathring{\mathcal{D}}_j\mathring{N} - \Delta_b\mathring{N}b_{ij})g^{ik}g^{j\ell}e_{k\ell}, \quad (3.14)$$

$$Q := \mathring{N}(g^{ij} - b^{ij} + g^{ik}g^{j\ell}e_{k\ell})\text{Ric}(b)_{ij} + Q'. \quad (3.15)$$

Brackets over a symbol denote anti-symmetrisation, with an appropriate numerical factor (1/2 in the case of two indices). Here Q' denotes an expression which is bilinear in

$$e \equiv e_{ij}dx^i dx^j := (g_{ij} - b_{ij})dx^i dx^j,$$

and in $\mathring{\mathcal{D}}_k e_{ij}$, where $\mathring{\mathcal{D}}$ denotes now the covariant derivative operator of the metric b , linear in \mathring{N} , $d\mathring{N}$ and $\text{Hess}\mathring{N}$, with coefficients which are constants in an ON frame for b . The idea behind this calculation is to collect all terms in R_g that contain second derivatives of the metric in $\partial_i \mathbb{U}^i$; in what remains one collects in ρ the terms which are linear in e_{ij} , while the remaining terms are collected in Q ; one should note that the first term at the right-hand-side of (3.15) does indeed not contain any terms linear in e_{ij} when Taylor expanded at $g_{ij} = b_{ij}$. Note that ρ vanishes when $b = \mathring{g}_\varepsilon$ by choice of \mathring{N} . So the integrand is quadratic in e_{ij} , up to terms $O(|\varepsilon - \varepsilon'|)e_{ij}$, and up to the divergence which produces a boundary term

$$\int_{\partial\Omega_i} \mathbb{U}^i dS_i.$$

For our next lemma it is convenient to write two generalised Delaunay metrics g and b as

$$g = \frac{dr^2}{N^2} + r^2 \mathring{h}, \quad b = \frac{dr^2}{\mathring{N}^2} + r^2 \mathring{h}. \quad (3.16)$$

We claim:

LEMMA 3.4 *Let g and b be two generalised Delaunay metrics with mass parameters m and m_0 . Let r be such that $\mathring{N}(r) \neq 0$ and $N(r) \neq 0$. If $\{r\} \times N^{n-1}$ is positively oriented, then*

$$\int_{\{r\} \times N^{n-1}} \mathbb{U}^i dS_i = 2\omega_{n-1}(n-1)\mathring{N}N^{-1}(m - m_0). \quad (3.17)$$

where ω_{n-1} is the volume of N^{n-1} .

PROOF: Let us denote by Γ_{jk}^i the Christoffel symbols of the metric b . We have $\Gamma_{rr}^r = -\partial_r \mathring{N}/\mathring{N}$, $\Gamma_{AB}^r = -r\mathring{N}^2 \mathring{h}_{AB}^{\circ}$, $\Gamma_{BC}^A = \mathring{\Gamma}_{BC}^A$ (where, as before, the $\mathring{\Gamma}_{BC}^A$'s are the Christoffel symbols of the metric \mathring{h}), $\Gamma_{rB}^A = r^{-1}\delta_B^A$, while the remaining Γ 's vanish. It holds that $e = (N^{-2} - \mathring{N}^{-2})dr^2$, from which one easily finds

$$\mathbb{U}^r = 2(n-1)\mathring{N}N^{-1}(m - m_0)\sqrt{\det \mathring{h}}, \quad (3.18)$$

and the result follows by integration. \square

We are ready to show that one can choose ε' — equivalently m' — so that the obstruction vanishes. So we consider the integral (3.11). We wish to use (3.12) with $g = \tilde{g}_{\varepsilon'}$. Note that the integration in (3.11) is taken with respect to the measure $d\mu_{\tilde{g}}$, while (3.12) involves $d\mu_g$. The difference between the two volume integrals comes thus with a prefactor $O(|\varepsilon - \varepsilon'|)$, and produces an error term which is $O((\varepsilon - \varepsilon')^2)$:

$$\begin{aligned} & \int_{\Omega_i} \mathring{N} \left(R(\tilde{g}_{\varepsilon'}) - R(\mathring{g}_{\varepsilon'}) \right) d\mu_{\tilde{g}} \\ &= \int_{\Omega_i} \mathring{N} \left(R(\tilde{g}_{\varepsilon'}) - R(\mathring{g}_{\varepsilon'}) \right) d\mu_g + O((\varepsilon - \varepsilon')^2) \\ &= \int_{\{i+T\} \times N^{n-1}} \mathbb{U}^i dS_i - \int_{\{i\} \times N^{n-1}} \mathbb{U}^i dS_i + O((\varepsilon - \varepsilon')^2). \end{aligned} \quad (3.19)$$

We now choose σ in the definition of Ω_i so that the number

$$\lambda := 4\omega_{n-1}(n-1)\mathring{N}|_{x=\sigma}$$

does not vanish; note that λ equals, up to $O(|\varepsilon - \varepsilon'|)$, the number in front of $(m - m_0)$ in (3.17). Since g approaches \mathring{g} together with its first derivatives as i goes to infinity, the first integral in the last line of (3.19) is $o(1)$, where $o(1)$ tends to zero as i tends to infinity. By Lemma 3.4 the second integral in the last line of (3.19) equals $\lambda(m' - \mathring{m}) = O(|\varepsilon - \varepsilon'|)$, where m' is the mass parameter of $\gamma_{\varepsilon'}$ while \mathring{m} is that of \mathring{g} . We infer that

$$\int_{\Omega_i} \mathring{N} \left(R(\tilde{g}_{\varepsilon'}) - R(\mathring{g}_{\varepsilon'}) \right) d\mu_{\tilde{g}} = \lambda(m' - \mathring{m}) + o(1) + O((\varepsilon - \varepsilon')^2).$$

Clearly this can be made positive or negative when i is large enough by choosing ε' appropriately; by continuity there exists an ε' which makes the integral vanish, and the result is proved. \square

4 Concluding remarks

Our work leads naturally to the following questions:

1. In light of the results in [26] one should expect that, at least for $n = 3, 4, 5$, a construction similar to Byde's [7] could be carried out without any assumption of conformal flatness in a neighborhood of the omitted point (which forms the end of the resulting complete metric). Moreover, with or without the conformally flat condition, it should be straightforward to iterate Byde's construction to produce any number of asymptotically Delaunay ends. If one could add asymptotically Delaunay ends at any chosen set of points in any positive constant scalar curvature manifold, then our analysis here could then be used to replace them with exactly Delaunay ends. (Furthermore, all of this should be doable via a *local* deformation of the metric near points *without static KIDs*, using the techniques of [3, 13, 16].) Alternatively, can one generically deform a constant positive scalar curvature metric, keeping the scalar curvature fixed, to a metric which is conformally flat near a set of prescribed points? Proposition 4.1 of [41] could perhaps be used as an intermediate step here.
2. In Byde's construction, or in a variation thereof as just suggested, can one ensure that the range of masses of the resulting Delaunay ends covers an interval of the form $(0, \epsilon)$, for some $\epsilon > 0$? This is a natural condition which has appeared

elsewhere as a necessary hypothesis (see e.g. [37]). Now, it is clear that the masses of our exactly Delaunay ends are continuous functions of the initial mass. Given any two points with the associated families of exactly Delaunay ends, one could then always adjust the masses to be the same, ensuring that the ends can be glued together.

3. We did not carry out the gluing in situations when the metric g approaches a cylindrical metric $dy^2 + \dot{h}$ along the asymptotic end; such metrics arise in black-hole space-times with degenerate horizons. This deserves further attention.
4. It would be of interest to extend the current gluings to general relativistic initial data with non-vanishing extrinsic curvature.

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