Classical and new log log-theorems

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Abstract

We present a unified approach to celebrated log log-theorems of Carleman, Wolf, Levinson, Sjöberg, Matsaev on majorants of analytic functions. Moreover, we obtain stronger results by replacing original pointwise bounds with integral ones. The main ingredient is a complete description for radial projections of harmonic measures of strictly star-shaped domains in the plane, which, in particular, explains where the log log-conditions come from.

1 Introduction. Statement of results

Our starting point is classical theorems due to Carleman, Wolf, Levinson, and Sjöberg, on majorants of analytic functions.

Definition 1 *A nonnegative measurable function M on a segment $[a,b]\subset\mathbb{R}$ belongs to the class $\mathcal{L}^{++}[a,b]$ if*

$$\int_a^b \log^+\log^+ M(t)\,dt<\infty.$$

(For any real-valued function h, we write $h^+=\max\{h,0\}$, $h^-=h^+-h$.)

Carleman was the first who remarked a special role of functions of the class $\mathcal{L}^{++}$ in complex analysis, by proving the following variant of the Liouville theorem.

Theorem A (T. Carleman [3]) *If an entire function f in the complex plane $\mathbb{C}$ has the bound*

$$|f(re^{i\theta})|\le M(\theta)\quad \forall\theta\in[0,2\pi],\ \forall r\ge r_0, \tag{1}$$

with $M\in\mathcal{L}^{++}[0,2\pi]$, then $f\equiv const.$

This phenomenon appears also in the Phragmén–Lindelöf setting.

Theorem B (F. Wolf [22]) *If a holomorphic function f in the upper half-plane $\mathbb{C}_+=\{z\in\mathbb{C}:\ \operatorname{Im} z>0\}$ satisfies the condition*

$$\limsup_{z\to x_0}|f(z)|\le 1\quad \forall x_0\in\mathbb{R}$$

and for any $\epsilon > 0$ *and all* $r > R(\epsilon)$, $\theta \in (0, \pi)$, *one has*

$$|f(re^{i\theta})| \leq [M(\theta)]^{\epsilon r}$$

with $M \in \mathcal{L}^{++}[0, \pi]$, *then* $|f(z)| \leq 1$ *on* $\mathbb{C}_+$.

The most famous statement of this type is the following local result known as the Levinson–Sjöberg theorem.

Theorem C (N. Levinson [13], N. Sjöberg [21], F. Wolf [23]) *If a holomorphic function* f *in the domain* $Q = \{x + iy : |x| < 1, |y| < 1\}$ *has the bound*

$$|f(x + iy)| \leq M(y) \quad \forall x + iy \in Q,$$

with $M \in \mathcal{L}^{++}[-1, 1]$, *then for any compact subset* K *of* Q *there is a constant* C_K, *independent of the function* f, *such that* $|f(z)| \leq C_K$ *in* K.

For further developments of Theorem C, including higher dimensional variants, see [4], [5], [7], [8], [9]. Theorems A and B were extended to subharmonic functions in higher dimensions in [24].

A similar feature of majorants from the class $\mathcal{L}^{++}$ was discovered by Beurling in a problem of extension of analytic functions [2]. It also appears in relation to holomorphic functions from the MacLane class in the unit disk [10], [14], and in a description of non-quasi-analytic Carleman classes [6].

The next result, due to Matsaev, does not look like a $\log\log$-theorem, however (as will be seen from our considerations) it is also about the class $\mathcal{L}^{++}$; further results in this direction can be found in [16].

Theorem D (V.I. Matsaev [15]) *If an entire function* f *satisfies the relation*

$$\log|f(re^{i\theta})| \geq -Cr^{\alpha}|\sin\theta|^{-k} \quad \forall \theta \in (0, \pi), \ \forall r > 0,$$

with some $C > 0$, $\alpha > 1$, *and* $k \geq 0$, *then it has at most normal type with respect to the order* α, *that is,* $\log|f(re^{i\theta})| \leq Ar^{\alpha} + B$.

All these theorems can be formulated in terms of subharmonic functions (by taking $u(z) = \log|f(z)|$ as a pattern), however our main goal is to replace the pointwise bounds like (1) with some integral conditions. A model situation is the following form of the Phragmén–Lindelöf theorem.

Theorem E (Ahlfors [1]) *If a subharmonic function* u *in* $\mathbb{C}_+$ *with nonpositive boundary values on* $\mathbb{R}$ *satisfies*

$$\lim_{r \to \infty} r^{-1} \int_0^{\pi} u^+(re^{i\theta}) \sin\theta \, d\theta = 0,$$

then $u \leq 0$ *in* $\mathbb{C}_+$.

We will show that all the above theorems are particular cases of results on the class $\mathcal{A}$ defined below and that the $\log\log$-conditions appear as conditions for continuity of certain logarithmic potentials.

Definition 2 *Let ν be a probability measure on a segment $[a,b]$; we will identify it occasionally with its distribution function $\nu(t) = \nu([a,t])$. Suppose $\nu(t)$ is strictly increasing and continuous on $[a,b]$, and denote by μ its inverse function extended to the whole real axis as $\mu(t) = a$ for $t < 0$ and $\mu(t) = b$ for $t > 1$. We will say that such a measure ν belongs to the class $\mathcal{A}[a,b]$ if*

$$\lim_{\delta\to 0}\sup_x \int_0^\delta \frac{\mu(x+t)-\mu(x-t)}{t}\,dt = 0. \qquad (2)$$

Note that this class is completely different from MacLane's class $\mathcal{A}$ [14] that consists of holomorphic functions in the unit disk with asymptotic values at a dense subset of the circle. MacLane's class is however described by the condition $|f(re^{i\theta})| \le M(r)$, $M \in \mathcal{L}^{++}[0,1]$.

Our results extending Theorems A–C and E are as follows.

Theorem 1 *Let a subharmonic function u in the complex plane satisfy*

$$\int_0^{2\pi} u^+(te^{i\theta})\,d\nu(\theta) \le V(t) \quad \forall t \ge t_0, \qquad (3)$$

with $\nu \in \mathcal{A}[0,2\pi]$ and a nondecreasing function V on $\mathbb{R}_+$. Then there exist constants $c > 0$ and $A \ge 1$, independent of u, such that

$$u(te^{i\theta}) \le cV(At) \quad \forall t \ge t_0. \qquad (4)$$

Theorem 2 *If a subharmonic function u in the upper half-plane $\mathbb{C}_+$ satisfies the conditions*

$$\limsup_{z\to x_0} u(z) \le 0 \quad \forall x_0 \in \mathbb{R}$$

and

$$\lim_{t\to\infty} t^{-1}\int_0^\pi u^+(te^{i\theta})\,d\nu(\theta) = 0$$

with $\nu \in \mathcal{A}[0,\pi]$, then $u(z) \le 0$ $\forall z \in \mathbb{C}_+$.

Theorem 3 *Let a subharmonic function u in $Q = \{x+iy : |x| < 1,\ |y| < 1\}$ satisfy*

$$\int_{-1}^1 u^+(x+iy)\,d\nu(y) \le 1 \quad \forall x \in (-1,1) \qquad (5)$$

with $\nu \in \mathcal{A}[-1,1]$. Then for each compact set $K \subset Q$ there is a constant C_K, independent of the function u, such that $u(z) \le C_K$ on K.

Relation of these results to the log log-theorems becomes clear by means of the following statement.

Definition 3 *Denote by* $\mathcal{L}^-[a,b]$ *the class of all nonnegative integrable functions* g *on the segment* $[a,b]$*, such that*

$$\int_a^b \log^- g(s)\, ds < \infty. \tag{6}$$

Proposition 1 *If the density* ν' *of an absolutely continuous increasing function* ν *belongs to the class* $\mathcal{L}^-[a,b]$*, then* $\nu \in \mathcal{A}[a,b]$*. Consequently, if a holomorphic function* f *has a majorant* $M \in \mathcal{L}^{++}$*, then* $\log|f|$ *has the corresponding integral bound with the weight* $\nu \in \mathcal{A}$ *with the density* $\nu'(t) = \min\{1, 1/M(t)\}$*.*

We recall that positive measures ν on the unit circle with $\nu' \in \mathcal{L}^-[0,2\pi]$ are called *Szegö measures*. Proposition 1 states, in particular, that absolutely continuous Szegö measures belong to the class $\mathcal{A}[0,2\pi]$.

An integral version of Theorem D has the following form.

Theorem 4 *Let a function* u*, subharmonic in* $\mathbb{C}$ *and harmonic in* $\mathbb{C}\setminus\mathbb{R}$*, satisfy the inequality*

$$\int_{-\pi}^{\pi} u^-(re^{i\theta})\Phi(|\sin\theta|)\, d\theta \le V(r) \quad \forall r \ge r_0, \tag{7}$$

where $\Phi \in \mathcal{L}^-[0,1]$ *is nondecreasing and the function* V *is such that* $r^{-1-\delta}V(r)$ *is increasing in* r *for some* $\delta > 0$*. Then there are constants* $c > 0$ *and* $A \ge 1$*, independent of* u*, such that*

$$u(re^{i\theta}) \le cV(Ar) \quad \forall r \ge r_1 = r_1(u).$$

Our proofs of Theorems 1–4 rest on a presentation of measures of the class $\mathcal{A}[0,2\pi]$ as radial projections of harmonic measures of star-shaped domains. Let Ω be a bounded Jordan domain containing the origin. Given a set $E \subset \partial\Omega$, $\omega(z,E,\Omega)$ will denote the harmonic measure of E at $z \in \Omega$, i.e., the solution of the Dirichlet problem in Ω with the boundary data 1 on E and 0 on $\partial\Omega \setminus E$. The measure $\omega(0,E,\Omega)$ generates a measure on the unit circle $\mathbb{T}$ by means of the radial projection $\zeta \mapsto \zeta/|\zeta|$. It is convenient for us to consider it as a measure on the segment $[0,2\pi]$, so we put

$$\widehat{\omega}_\Omega(F) = \omega(0, \{\zeta \in \partial\Omega : \arg\zeta \in F\}, \Omega) \tag{8}$$

for each Borel set $F \subset [0,2\pi]$.

The inverse problem is as follows. *Given a probability measure on the unit circle* $\mathbb{T}$*, is it the radial projection of the harmonic measure of any domain* Ω*?*

For our purposes we specify Ω to be *strictly star-shaped*, i.e., of the form

$$\Omega = \{re^{i\theta} :\ r < r_\Omega(\theta),\ 0 \le \theta \le 2\pi\} \tag{9}$$

with r_Ω a positive continuous function on $[0,2\pi]$, $r_\Omega(0) = r_\Omega(2\pi)$.

Theorem 5 *A continuous probability measure ν on $[0, 2\pi]$ is the radial projection of the harmonic measure of a strictly star-shaped domain if and only if $\nu \in \mathcal{A}[0, 2\pi]$.*

Corollary 4 *Every absolutely continuous measure from the Szegö class on the unit circle is the radial projection of the harmonic measure of some strictly star-shaped domain.*

Theorem 5 is proved by a method originated by B.Ya. Levin in theory of majorants in classes of subharmonic functions [11].

Theorems 1–3 and 5 (some of them in a slightly weaker form) were announced in [18] and proved in [19] and [20]. The main objective of the present paper, Theorem 4, is new. Since its proof rests heavily on Theorem 5, we present a proof of the latter as well, having in mind that the papers [19] and [20] are not easily accessible. Moreover, we include the proofs of Theorems 1–3, too, motivated by the same accessability reason as well as by the idea of showing the whole picture.

2 Radial projections of harmonic measures (Proofs of Theorem 5 and Proposition 1)

Measures from the class $\mathcal{A}$ have a simple characterization as follows.

Proposition 2 *Let μ and ν be as in Definition 2. Then the function*

$$N(x) = \int_0^1 \log|x - t|\, d\mu(t)$$

is continuous on $[0, 1]$ if and only if $\nu \in \mathcal{A}[a, b]$.

Proof. The function $N(x)$ is continuous on $[0, 1]$ if and only if for any $\epsilon > 0$ one can choose $\delta \in (0, 1)$ such that

$$I_x(\delta) = \int_{|t-x|<\delta} \log|x - t|\, d\mu(t) > -\epsilon$$

for all $x \in [0, 1]$. Integrating I_x by parts, we get

$$|I_x(\delta)| = \int_0^\delta \frac{r_x(t)}{t}\, dt + r_x(\delta)|\log\delta|,$$

where $r_x(t) = \mu(x+t) - \mu(x-t)$. Therefore, continuity of $N(x)$ implies (2). On the other hand, since $r_x(t)$ increases in t, we have

$$r_x(\delta)|\log\delta| = 2r_x(\delta)\int_\delta^{\sqrt{\delta}} \frac{dt}{t} \le 2\int_\delta^{\sqrt{\delta}} \frac{r_x(t)}{t}\, dt,$$

which gives the reverse implication. □

In the proof of Theorem 5, we will use this property in the following form.

Proposition 3 *Let μ and ν be as in Definition 2 for the class $\mathcal{A}[0,2\pi]$. Then the function*

$$h(z) = \int_0^{2\pi} \log|e^{i\theta} - z|\, d\mu(\theta/2\pi)$$

is continuous on $\mathbb{T}$ if and only if $\nu \in \mathcal{A}[0,2\pi]$.

Proof of Theorem 5. 1) First we prove the sufficiency: *every $\nu \in \mathcal{A}[0,2\pi]$ has the form $\nu = \widehat{\omega}_\Omega$ (8) for some strictly star-shaped domain Ω. In particular, for any compact set $K \in \Omega$ there is a constant $C(K)$ such that*

$$\omega(z, E, \Omega) \le C(K)\,\nu(\arg E) \quad \forall z \in E \tag{10}$$

for every Borel set $E \subset \partial\Omega$, where $\arg E = \{\arg\zeta :\ \zeta \in E\}$.

Let

$$u(z) = \frac{1}{\pi}\int_0^{2\pi} \log|e^{i\theta} - z|\, d\mu(\theta/2\pi)$$

with μ the inverse function to $\nu \in \mathcal{A}[0,2\pi]$. The function u is subharmonic in $\mathbb{C}$ and harmonic outside the unit circle $\mathbb{T}$. By Proposition 3, it is continuous on $\mathbb{T}$ and thus, by Evans' theorem, in the whole plane. Let v be a harmonic conjugate to u in the unit disk $\mathbb{D}$, which is determined uniquely up to a constant. Since $u \in C(\overline{\mathbb{D}})$, radial limits $v^*(e^{i\psi})$ of v exist a.e. on $\mathbb{T}$. Let us fix such a point $e^{i\psi_0}$ and choose the constant in the definition of v in such a way that $v^*(e^{i\psi_0}) = \psi_0$.

Consider then the function $w(z) = z\exp\{-u(z) - iv(z)\}$, $z \in \mathbb{D}$. By the Cauchy-Riemann condition, $\partial v/\partial\phi = r\partial u/\partial r$, which implies

$$\begin{aligned}
\arg w(re^{i\psi}) &= \psi - v(re^{i\psi_0}) - \int_{\psi_0}^{\psi} \frac{\partial v(re^{i\phi})}{\partial\phi}\, d\phi = \psi_0 - v(re^{i\psi_0}) \\
&+ \frac{1}{2\pi}\int_{\psi_0}^{\psi}\int_0^{2\pi} \left[1 - \frac{2r^2 - 2r\cos(\theta-\phi)}{|r - e^{i(\theta-\phi)}|^2}\right] d\mu(\theta/2\pi)\, d\phi \\
&= \psi_0 - v(re^{i\psi_0}) + \frac{1}{2\pi}\int_{\psi_0}^{\psi}\int_0^{2\pi} \frac{1-r^2}{|r - e^{i(\theta-\phi)}|^2}\, d\mu(\theta/2\pi)\, d\phi.
\end{aligned}$$

By changing the integration order and passing to the limit as $r \to 1$, we derive that for each $\psi \in [0,2\pi]$ there exists the limit

$$\lim_{r\to 1} \arg w(re^{i\psi}) = \mu(\psi/2\pi) - \mu(\psi_0/2\pi).$$

Therefore the function $\arg w$ is continuous up to the boundary of the disk; in particular, we can take $\psi_0 = 0$. Since $|w|$ is continuous in $\overline{\mathbb{D}}$ as well, so is w.

By the boundary correspondence principle, w gives a conformal map of $\mathbb{D}$ onto the domain

$$\Omega = \{re^{i\theta} :\ r < \exp\{-u(\exp\{2\pi i\nu(\theta)\})\},\ 0 \le \theta \le 2\pi\}. \tag{11}$$

It is easy to see that the domain Ω is what we sought. Let f be the conformal map of Ω to $\mathbb{D}$, inverse to w. For $z \in \Omega$ and $E \subset \partial\Omega$, we have

$$\begin{aligned}\omega(z, E, \Omega) &= \omega(f(z), f(E), U) = \frac{1}{2\pi}\int_{\arg f(E)} \frac{1-|f(z)|^2}{|f(z)-e^{it}|^2}\,dt \\ &= (1-|f(z)|^2)\int_{\arg E} \frac{d\nu(s)}{|f(z)-e^{2\pi i\nu(s)}|^2},\end{aligned}$$

which proves the claim.

2) Now we prove the necessity: *if ω is of the form (9), then $\widehat{\omega}_\Omega \in \mathcal{A}[0, 2\pi]$.*

We use an idea from the proof of [11, Theorem 2.4]. Let w be a conformal map of $\mathbb{D}$ to Ω, $w(0) = 0$. Since Ω is a Jordan domain, w extends to a continuous map from $\overline{\mathbb{D}}$ to $\overline{\Omega}$, and we can specify it to have $\arg w(1) = 0$. Define

$$f(z) = u(z) + iv(z) = \log\frac{w(z)}{z} \text{ for } |z| \le 1, \quad f(z) = f(|z|^{-2}z) \text{ for } |z| > 1.$$

It is analytic in $\mathbb{D}$ and continuous in $\mathbb{C}$. Define then the function

$$\lambda(z) = u(z) + \frac{1}{\pi}\int_0^{2\pi} \log|e^{i\psi} - z|\,dv(e^{i\psi}), \tag{12}$$

δ-subharmonic in $\mathbb{C}$ and harmonic in $\mathbb{C}\setminus\mathbb{T}$. Let us show that it as actually harmonic (and, hence, continuous) everywhere. To this end, take any function $\alpha \in C(\mathbb{T})$ and a number $r < 1$, and apply Green's formula for $u(z)$ and $A(z) = |z|\alpha(z/|z|)$ in the domain $D_r = \{r < |z| < r^{-1}\}$:

$$\int_{D_r} (A\Delta u - u\Delta A) = \left[\frac{\rho}{2\pi}\int_0^{2\pi}\left(\rho\alpha(e^{i\psi})\frac{\partial u(\rho e^{i\psi})}{\partial\rho} - u(\rho e^{i\psi})\alpha(e^{i\psi})\right)d\psi\right]_{\rho=r}^{\rho=R}. \tag{13}$$

Using the definition of the function f outside $\mathbb{D}$ and the Cauchy-Riemann equations $\partial v/\partial\phi = \rho\partial u/\partial\rho$ if $\rho < 1$ and $\partial v/\partial\phi = -\rho\partial u/\partial\rho$ if $\rho > 1$ (which follows from the definition of f), we can write the right hand side of (13) as

$$-\frac{r + r^{-1}}{2\pi}\int_0^{2\pi} \alpha(e^{i\psi})\,d_\psi v(re^{i\psi}) + \frac{r - r^{-1}}{2\pi}\int_0^{2\pi} u(re^{i\psi})\alpha(e^{i\psi})\,d\psi.$$

When $r \to 1$, (13) takes the form

$$\int_{\mathbb{T}} \alpha\Delta u = -\frac{1}{\pi}\int_0^{2\pi} \alpha(e^{i\psi})\,dv(e^{i\psi}),$$

which implies the harmonicity of the function $\lambda(z)$ (12) in the whole plane.

Now we recall that $v(e^{i\psi}) = \arg w(e^{i\psi}) - \psi$. Since the harmonic measure of the w-image of the arc $\{e^{i\theta} : 0 < \theta < \psi\}$ equals $\psi/2\pi$, we have

$$\widehat{\omega}_\Omega(\arg w(e^{i\psi})) = \psi/2\pi$$

and thus $\arg w(e^{i\psi}) = \mu(\psi/2\pi)$ with μ the inverse function to $\widehat{\omega}_\Omega(\psi)$. Therefore, $v(e^{i\psi}) = \mu(\psi/2\pi) - \psi$.

Consider, finally, the function

$$\gamma(z) = \frac{1}{\pi}\int_0^{2\pi} \log|e^{i\psi} - z|\, d\mu(\psi/2\pi) = \lambda(z) - u(z) + \frac{1}{\pi}\int_0^{2\pi} \log|e^{i\psi} - z|\, d\psi.$$

Since it is continuous on $\mathbb{T}$, Proposition 3 implies $\widehat{\omega}_\Omega \in \mathcal{A}[0, 2\pi]$, and the theorem is proved. □

Note that all the dilations $t\Omega$ of Ω ($t > 0$) represent the same measure from $\mathcal{A}[0, 2\pi]$, and Ω with a given projection $\widehat{\omega}_\Omega$ is unique up to the dilations.

Now we prove Proposition 1 that presents a wide subclass of $\mathcal{A}$ with a more explicit description.

Proof of Proposition 1. Let $\nu : [0, 1] \to [0, 1]$ be an absolutely continuous, strictly increasing function, $\nu' \in \mathcal{L}^-[0, 1]$. Since $\text{mes}\,\{t : \nu'(t) = 0\} = 0$, its inverse function μ is absolutely continuous ([17], p. 297), so

$$\mu(t) = \int_0^t g(s)\, ds$$

with g a nonnegative function on $[0, 1]$. We have

$$\infty > \int_0^1 \log^- \nu'(t)\, dt = \int_0^1 \log^- \frac{1}{\mu'(t)}\, d\mu(t) = \int_0^1 g(t) \log^+ g(t)\, dt,$$

so g belongs to the Zygmund class $\mathbf{L}\log\mathbf{L}$.

Let $\Delta(t)$ denote the modulus of continuity of the function μ. Note that it can be expressed in the form

$$\Delta(t) = \int_0^t h(s)\, ds$$

where h is the nonincreasing equimeasurable rearrangement of g. Then

$$\begin{aligned}\int_0^1 \frac{\Delta(t)}{t}\, dt &= \int_0^1 t^{-1} \int_0^1 h(s)\, ds\, dt = \int_0^1 h(s) \log s^{-1}\, ds \\ &= \int_{E_1 \cup E_2} h(s) \log s^{-1}\, ds,\end{aligned}$$

where $E_1 = \{s \in (0, 1) : h(s) > s^{-1/2}\}$, $E_2 = (0, 1) \setminus E_1$. Since $h \in \mathbf{L}\log\mathbf{L}\,[0, 1]$,

$$\int_{E_1} h(s) \log s^{-1}\, ds \le 2 \int_{E_1} h(s) \log h(s)\, ds < \infty.$$

Besides,

$$\int_{E_2} h(s) \log s^{-1} \, ds \le \int_{E_2} s^{-1/2} \log s^{-1} \, ds < \infty.$$

Therefore,

$$\int_0^1 \frac{\Delta(t)}{t} \, dt < \infty$$

and thus

$$\lim_{\delta \to 0} \int_0^\delta \frac{\Delta(t)}{t} \, dt = 0,$$

which gives (2). □

Corollary 4 follows directly from the definition of the Szegö class, Theorem 5 and Proposition 1.

3 Proofs of Theorems 1 and 2

Here we show how the integral variants of Carleman's and Wolf's theorems can be derived from Theorem 5.

We will need an elementary

Lemma 5 *Let $r(\theta) \in C[0, 2\pi]$, $1 < r_1 \le r(\theta) \le r_2$, let ν be a positive measure on $[0, 2\pi]$ and $V(t)$ be a nonnegative function on $[0, \infty]$. If a nonnegative function $v(te^{i\theta})$ satisfies*

$$\int_0^{2\pi} v(te^{i\theta}) \, d\nu(\theta) \le V(t) \quad \forall t \ge t_0,$$

then for any $R_2 > R_1 \ge t_0$,

$$\int_{R_1}^{R_2} \int_0^{2\pi} v(t\, r(\theta) e^{i\theta}) \, d\nu(\theta) \, dt \le r_1^{-1} \int_{r_1 R_1}^{r_2 R_2} V(t) \, dt.$$

Proof of Lemma 5 is straightforward:

$$\int_{R_1}^{R_2} \int_0^{2\pi} v(t\, r(\theta) e^{i\theta}) \, d\nu(\theta) \, dt = \int_0^{2\pi} \int_{R_1 r(\theta)}^{R_2 r(\theta)} v(te^{i\theta}) \, dt \, \frac{d\nu(\theta)}{r(\theta)}$$

$$\le r_1^{-1} \int_0^{2\pi} \int_{R_1 r_1}^{R_2 r_2} v(te^{i\theta}) \, dt \, d\nu(\theta) \le r_1^{-1} \int_{R_1 r_1}^{R_2 r_2} V(t) \, dt.$$

□

Proof of Theorem 1. By Theorem 5, there exists a domain Ω of the form (9) that contains $\overline{\mathbb{D}}$ such that

$$\omega(z, E, \Omega) \le c_1 \nu(\arg E), \quad \forall z \in \overline{\mathbb{D}}, \ E \subset \partial\Omega, \tag{14}$$

with a constant $c_1 > 0$, see (10). Let $r_1 = \min r(\theta)$. $r_2 = \max r(\theta)$. By the Poisson–Jensen formula applied to the function $v_t(z) = u^+(tz)$ ($t > 0$) in the domain $s\Omega$ ($s > 1$) we have, due to (14),

$$\begin{aligned} v_t(z) &\leq \int_{\partial s\Omega} v_t(\zeta)\,\omega(z, d\zeta, s\Omega) = \int_{\partial\Omega} v_t(s\zeta)\,\omega(s^{-1}z, d\zeta, \Omega) \\ &\leq c_1 \int_0^{2\pi} v_t(s\,r(\theta)e^{i\theta})\,d\nu(\theta), \quad z \in \overline{\mathbb{D}}. \end{aligned}$$

The integration of this relation over $s \in [1, R]$ ($R > 1$) gives, by Lemma 5,

$$(R-1)v_t(z) \leq c_1 \int_1^R \int_0^{2\pi} v_t(s\,r(\theta)e^{i\theta})\,d\nu(\theta)\,ds \leq c_2 t^{-1} r_1^{-1} \int_{tr_1}^{tr_2 R} V(s)\,ds$$

for each $t \geq t_0$. So,

$$u(te^{i\theta}) \leq c(R)V(t\,r_2 R), \quad t \geq t_0,$$

which proves the theorem. □

Remarks. 1. It is easy to see that the constant A in (4) can be chosen arbitrarily close to $r_2/r_1 \geq 1$.

2. Note that we have used inequality (3) in the integrated form only, so the following statement is actually true: *If a subharmonic function u on $\mathbb{C}$ satisfies*

$$\int_{t_0}^t \int_0^{2\pi} u^+(se^{i\theta})\,d\nu(\theta)\,ds \leq W(t) \quad \forall t \geq t_0 \tag{15}$$

with $\nu \in \mathcal{A}[0, 2\pi]$ and a nondecreasing function W, then there are constants $c > 0$ and $A \geq 1$, independent of u, such that $u(te^{i\theta}) \leq ct^{-1}W(At)$ for all $t \geq t_0$.

Now we prove Theorem 2 as a consequence of Theorem 1.

Proof of Theorem 2. The function v equal to u^+ in $\mathbb{C}_+$ and 0 in $\mathbb{C} \setminus \mathbb{C}_+$ is a subharmonic function in $\mathbb{C}$ satisfying the condition

$$\int_0^{2\pi} v^+(te^{i\theta})\,d\nu(\theta) \leq V_1(t)$$

with $\nu \in \mathcal{A}[0, 2\pi]$ and $V_1(t) = o(t)$, $t \to \infty$. Therefore, it satisfies the conditions of Theorem 1 with the majorant $V(t) = \sup\{V_1(s) : s \leq t\}$. So, $\sup_\theta u^+(te^{i\theta}) = o(t)$ as $t \to \infty$, and the conclusion holds by the standard Phragmén–Lindelöf theorem. □

4 Proof of Theorem 3

The integral version of the Levinson–Sjöberg theorem will be proved along the same lines as Theorem 1, however the local situation needs a more refined adaptation.

We start with two elementary statements close to Lemma 5.

Lemma 6 *Let a nonnegative integrable function v in the square $Q = \{|x|, |y| < 1\}$ satisfy (5) with a continuous strictly increasing function ν. Then for any $d \in (0,1)$ there exists a constant $M_1(d)$, independent of u, such that for each $y_0 \in (-1,1)$ one can find a point $y_1 \in (-1,1) \cap (y_0 - d, y_0 + d)$ with*

$$\int_{-1}^{1} v(x + iy_1)\, dx < M_1(d).$$

Proof. Assume $y_0 \geq 0$, then

$$\int_{y_0-d}^{y_0} \int_{-1}^{1} v(x+iy)\, dx\, d\nu(y) = \int_{-1}^{1} \int_{y_0-d}^{y_0} v(x+iy)\, d\nu(y)\, dx \leq 2.$$

Therefore for some $y_1 \in (y_0 - d, y_0)$,

$$\int_{-1}^{1} v(x+iy_1)\, dx \leq 2[\nu(y_0) - \nu(y_0-d)]^{-1} \leq 2[\Delta_*(\nu, d)]^{-1}$$

with $\Delta_*(\nu, d) = \inf\{\nu(t) - \nu(t-d) : t \in (0,1)\} > 0$. □

Lemma 7 *Let a function v satisfy the conditions of Lemma 6, a function r be continuous on a segment $[a,b] \subset [-1,1]$, $0 < r_1 = \min r(y) \leq \max r(y) = r_2 < 1$, and $\delta \in (0, 1-r_2)$. Then there exists $t \in (0,\delta)$ such that*

$$\int_a^b v(t + r(y) + iy)\, d\nu(y) < M_2(\delta)$$

with $M_2(\delta)$ independent of v.

Proof. We have

$$\begin{aligned} \int_0^{\delta} \int_a^b v(t + r(y) + iy)\, d\nu(y) &= \int_a^b \int_{r(y)}^{\delta + r(y)} v(s+iy)\, ds\, d\nu(y) \\ \leq \int_{r_1}^{\delta + r_2} \int_a^b v(s+iy)\, d\nu(y)\, ds &\leq \delta + r_2 - r_1. \end{aligned}$$

Thus one can find some $t \in (0,\delta)$ such that

$$\int_a^b v(t + r(y) + iy)\, d\nu(y) < \delta^{-1}(\delta + r_2 - r_1).$$

□

Proof of Theorem 3. Consider the measure ν_1 on $[-i, i]$ defined as

$$\nu_1(E) = \nu(-iE), \quad E \subset [-i, i].$$

The conformal map $f(z) = \exp\{z\pi/2\}$ of the strip $\{|\mathrm{Im}\, z| < 1\}$ to the right half-plane $\mathbb{C}_r$ pushes the measure ν_1 forward to the measure $f^*\nu$ on the semicircle $\{e^{i\theta} : -\pi/2 \le \theta \le \pi/2\}$, producing a measure of the class $\mathcal{A}[-\pi/2, \pi/2]$; we extend it to some measure $\nu_2 \in \mathcal{A}[-\pi.\pi]$. By Theorem 5, there is a strictly star-shaped domain $\Omega \supset \overline{\mathbb{D}}$ such that the radial projection of its harmonic measure at 0 is the normalization $\nu_2/\nu_2([-\pi, \pi])$ of ν_2.

Let $\Omega_1 = \Omega \cap \mathbb{C}_r$, then for every Borel set $E \subset \Gamma = \partial\Omega_1 \cap \mathbb{C}_r$ and any compact set $K \subset \Omega_1$,

$$\omega(w, E, \Omega_1) \le C_1(K)\, \nu_2(\arg E) \quad \forall w \in K.$$

The pre-image $\Omega_2 = f^{[-1]}(\Omega_1)$ of Ω_1 has the form

$$\Omega_2 = \{z = x + iy :\ x < \varphi(y),\ y \in (0, 1)\}$$

with some function $\varphi \in C[-1, 1]$. Let

$$\Gamma_2 = \{x + iy :\ x = \varphi(y),\ y \in (0, 1)\},$$

then for every Borel $E \subset \Gamma_2$ and any compact subset K of Ω_2,

$$\omega(z, E, \Omega_2) \le C_2(K)\, \nu(\mathrm{Im}\, E) \quad \forall z \in K. \tag{16}$$

For the domain

$$\Omega_3 = \{z = x + iy :\ x > -\varphi(y),\ y \in (0, 1)\}$$

we have, similarly, the relation

$$\omega(z, E, \Omega_3) \le C_3(K)\, \nu(\mathrm{Im}\, E) \quad \forall z \in K \tag{17}$$

for each $E \subset \Gamma_3 = \{x + iy :\ x = -\varphi(y),\ y \in (0, 1)\}$ and compact set $K \subset \Omega_3$.

Let now K be an arbitrary compact subset of the square Q. We would be almost done if we were able to find some reals $h_2(K)$ and $h_3(K)$ such that

$$K \subset \{\Omega_2 + h_2(K)\} \cap \{\Omega_3 + h_3(K)\} \subset \overline{\{\Omega_2 + h_2(K)\} \cap \{\Omega_3 + h_3(K)\}} \subset Q.$$

However this is not the case for any K unless $\varphi \equiv \mathit{const}$. That is why we need partition.

Given K compactly supported in Q, choose a positive $\lambda < (4\,\mathrm{dist}\,(K, \partial Q))^{-1}$ and then $\tau \in (0, \lambda)$ such that the modulus of continuity of φ at 4τ is less than λ. Take a finite covering of K by disks $B_j = \{z : |z - z_j| < \tau\}$, $z_j \in K$, $1 \le j \le n$. To prove the theorem, it suffices to estimate the function u on each B_j.

Let $Q_j = \{z \in Q : |\mathrm{Im}\,(z - z_j)| < 2\tau\}$, then $B_j \subset Q_j$ and $\mathrm{dist}\,(B_j, \partial Q_j) = \tau$. Take also

$$\Omega_2^{(j)} = \Omega_2 \cap Q_j, \quad \Gamma_2^{(j)} = \Gamma_2 \cap \overline{\Omega}_2^{(j)} = \{x + iy :\ x = \varphi(y),\ a_j \le y \le b_j\}.$$

Now we can find reals $h_2^{(j)}$ and $h_3^{(j)}$ such that

$$\Gamma_2^{(j)} + h_2^{(j)} = \{x+iy : x = r_2^{(j)}(y)\} \subset Q_j \cap \{x+iy : 1-4\lambda < x < 1 < 2\lambda\}$$

and

$$\Gamma_3^{(j)} + h_3^{(j)} = \{x+iy : x = r_3^{(j)}(y)\} \subset Q_j \cap \{x+iy : -1+2\lambda < x < -1+4\lambda\}.$$

Furthermore, by Lemma 7, there exist $t_2^{(j)} \in (0,\lambda)$ and $t_3^{(j)} \in (-\lambda, 0)$ such that

$$\int_{a_j}^{b_j} u^+(t_k^{(j)} + r_k^{(j)}(y) + iy)\, d\nu(y) < M_2(\lambda), \quad k = 2,3. \tag{18}$$

Finally we can find, due to Lemma 6, $y_1^{(j)} \in (a_j, a_j+\tau)$ and $y_2^{(j)} \in (b_j-\tau, b_j)$ such that

$$\int_{-1}^{1} u^+(x+iy_m)\, dx < M_1(\tau), \quad m = 1,2. \tag{19}$$

Denote

$$\Omega^{(j)} = \{x+iy : r_3^{(j)}(y) + t_3^{(j)} < x < r_2^{(j)}(y) + t_2^{(j)},\ y_1^{(j)} \le y \le y_2^{(j)}\}.$$

Since $\overline{B_j} \subset \Omega^{(j)}$, relations (16) and (17) imply

$$\omega(z, E, \Omega^{(j)}) \le C(B_j)\nu(\operatorname{Im} E) \quad \forall z \in B_j \tag{20}$$

for all E in the vertical parts of $\partial\Omega^{(j)}$. For E in the horizontal parts of $\partial\Omega^{(j)}$, we have, evidently,

$$\omega(z, E, \Omega^{(j)}) \le C(B_j) \operatorname{mes} E \quad \forall z \in B_j. \tag{21}$$

Now we can estimate $u(z)$ for $z \in B_j$. By (18)–(21),

$$\begin{aligned} u(z) &\le \int_{\partial\Omega^{(j)}} u^+(\zeta)\omega(z, d\zeta, \Omega^{(j)}) \\ &\le C(B_j)\sum_{k=2}^{3}\int_{a_j}^{b_j} u^+(t_k^{(j)} + r^{(j)}(y) + iy)\, d\nu(y) \\ &+ C(B_j)\sum_{m=1}^{2}\int_{-1}^{1} u^+(x+iy_m)\, dx \\ &\le 2C(B_j)(M_1(\tau) + M_2(\lambda)), \end{aligned}$$

which completes the proof. □

5 Proof of Theorem 4

By Theorem 1 and Proposition 1, it suffices to prove

Proposition 4 *If a function u satisfies the conditions of Theorem 4, then there exists a function $f \in \mathcal{L}^-[-\pi,\pi]$ and a constant $c_1 > 0$, the both independent of u, such that*

$$\int_{-\pi}^{\pi} u^+(re^{i\theta}) f(\theta)\, d\theta \le c_1 V(r) \quad \forall r > r_0. \tag{22}$$

Proof. What we will do is a refinement of the arguments from the proof of the original Matsaev's theorem (see [15], [12]). Let

$$D_{r,R,a} = \{z \in \mathbb{C}:\ r < |z| < R,\ |\arg z - \pi/2| < \pi(1/2 - a)\}, \quad 0 < a < 1/4,$$

$b = (1-2a)^{-1}$, $S(\theta,a) = \sin b(\theta - a\pi)$. Carleman's formula for the function u harmonic in $D_{r,R,a}$ has the form

$$2bR^{-b}\int_{\pi a}^{\pi-\pi a} u(Re^{i\theta})S(\theta,a)\,d\theta - b(r^{-b} + r^b R^{-2b})\int_{\pi a}^{\pi-\pi a} u(re^{i\theta})S(\theta,a)\,d\theta$$
$$-(r^{-b+1} - r^{b+1}R^{-2b})\int_{-\pi a}^{\pi a} u_r'(re^{i\theta})S(\theta,a)\,d\theta$$
$$+b\int_r^R \left[u(xe^{i\pi a}) + u(xe^{i\pi(1-a)})\right](x^{-b-1} - x^{b-1}R^{-2b})\,dx = 0.$$

It implies the inequality

$$\int_{\pi a}^{\pi-\pi a} u^+(Re^{i\theta})S(\theta,a)\,d\theta \le c(r,u)R^b + \int_{\pi a}^{\pi-\pi a} u^-(Re^{i\theta})S(\theta,a)\,d\theta$$
$$+ R^b\int_r^R \left[u^-(xe^{i\pi a}) + u^-(xe^{i\pi(1-a)})\right](x^{-b-1} - x^{b-1}R^{-2b})\,dx. \tag{23}$$

Fix some $\tau \in (0, 1/4)$ such that

$$\beta := (1-2\tau)^{-1} < 1 + \delta \tag{24}$$

with δ as in the statement of Theorem 4. Inequality (23) gives us the relation

$$\begin{aligned}
I_0 &:= \int_0^\tau \Phi(\sin\pi a)\int_{\pi a}^{\pi-\pi a} u^+(Re^{i\theta})S(\theta,a)\,d\theta\,da \\
&\le c(r,u)\int_0^\tau R^b\Phi(\sin\pi a)\,da + \int_0^\tau \Phi(\sin\pi a)\int_{\pi a}^{\pi-\pi a} u^-(Re^{i\theta})S(\theta,a)\,d\theta\,da \\
&+ \int_0^\tau \Phi(\sin\pi a)\int_r^R \left[u^-(xe^{i\pi a}) + u^-(xe^{i\pi(1-a)})\right] R^b x^{-b-1}\,dx\,da \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{25}$$

We can represent I_0 as

$$I_0 = \int_0^\pi u^+(Re^{i\theta})\Psi(\theta)\,d\theta$$

with

$$\Psi(\theta) = \int_0^{\lambda(\theta)} S(\theta,a)\Phi(\sin\pi a)\,da \tag{26}$$

and

$$\lambda(\theta) = \min\{\theta/\pi, 1-\theta/\pi, \tau\}. \tag{27}$$

Note that $S(\theta,a) \geq 0$ when $a \leq \lambda(\theta)$, and $S'_a(\theta,a) \leq 0$ for all $a < 1/4$. Since $\Phi(t)$ is nondecreasing, this implies the bound

$$\Psi(\theta) \geq \int_{\lambda(\theta)/2}^{\lambda(\theta)} S(\theta,a)\Phi(\sin\pi a)\,da \geq f(\theta) = \lambda^2(\theta)\,\Phi\left(\sin\frac{\pi\lambda(\theta)}{2}\right)$$

and thus,

$$I_0 \geq \int_0^\pi u^+(Re^{i\theta})f(\theta)\,d\theta \tag{28}$$

with $f \in \mathcal{L}^-[0,\pi]$.

Let us now estimate the right hand side of (25). We have

$$I_1 \leq c(r,u)R^\beta \int_0^\tau \Phi(\sin\pi a)\,da \leq c_1(r,\tau,u)R^\beta; \tag{29}$$

$$I_2 = \int_0^\pi u^-(Re^{i\theta})\Psi(\theta)\,d\theta \leq \int_0^\pi u^-(Re^{i\theta})\Phi(\sin\theta)\,d\theta; \tag{30}$$

$$\begin{aligned} I_3 &\leq \int_0^\tau \int_r^R \Phi(\sin\pi a)\left[u^-(xe^{i\pi a}) + u^-(xe^{i\pi(1-a)})\right]\left(\frac{R}{x}\right)^\beta x^{-1}\,dx\,da \\ &= R^\beta \int_r^R x^{-\beta-1}\left[\int_0^{\pi\tau} + \int_{\pi(1-\tau)}^\pi\right] u^-(xe^{i\theta})\Phi(\sin\theta)\,d\theta\,dx \\ &\leq R^\beta \int_r^R x^{-\beta-1}\int_0^\pi u^-(xe^{i\theta})\Phi(\sin\theta)\,d\theta\,dx. \end{aligned} \tag{31}$$

We insert (28)–(31) into (25):

$$\begin{aligned} \int_0^\pi u^+(Re^{i\theta})f(\theta)\,d\theta &\leq c_1(r,\tau,u)R^\beta + \int_0^\pi u^-(Re^{i\theta})\Phi(\sin\theta)\,d\theta \\ &+ R^\beta \int_r^R x^{-\beta-1}\int_0^\pi u^-(xe^{i\theta})\Phi(\sin\theta)\,d\theta\,dx \\ &= J_1(R) + J_2(R) + J_3(R). \end{aligned} \tag{32}$$

By the choice of β (24), $J_1(R) = o(V(R))$ as $R \to \infty$. Condition (7) implies $J_2(R) \le V(R)$, $R > r_0$. As to the term J_3, take any $\epsilon \in (0, 1+\delta-\beta)$, then

$$\begin{aligned} J_3(R) &\le R^\beta \int_r^R x^{-\beta-1} V(x)\, dx = R^\beta \int_r^R x^{-\beta-\epsilon} V(x) x^{\epsilon-1}\, dx \\ &\le R^\beta R^{-\beta-\epsilon} V(R) \int_r^R x^{\epsilon-1}\, dx \le \epsilon^{-1} V(R). \end{aligned}$$

These bounds give us

$$\int_0^\pi u^+(Re^{i\theta}) f(\theta)\, d\theta \le c_2 V(R) \quad \forall R > r_1(u).$$

Absolutely the same way, we get a similar inequality in the lower half-plane and, as a result, relation (22). □

Remark. We do not know if condition (7) can be replaced by a more general one in terms of the class $\mathcal{A}$.

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