

Conformal changes of generalized complex structures

by

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Dedicated to Acad. Prof. Radu Miron on the occasion of his 80-eth birthday

ABSTRACT. A conformal change of $TM \oplus T^*M$ is a morphism of the form $(X, \alpha) \mapsto (X, e^\tau \alpha)$ ($X \in TM, \alpha \in T^*M, \tau \in C^\infty(M)$). We characterize the generalized almost complex and almost Hermitian structures that are locally conformal to integrable and to generalized Kähler structures, respectively, and give examples of such structures.

1 Introduction

In the last few years, the generalized complex and Kähler structures became an important subject of theoretical quantum field theory, where they provide new sigma models (e.g., [17]) and allow to express certain supersymmetries (e.g., [9]). This also led to an extensive, purely mathematical research of the subject (e.g., [7]). In this note we discuss a mathematical question, that of characterizing generalized almost complex and almost Hermitian structures which become integrable, respectively, Kähler after local conformal changes. The corresponding classical cases of locally conformal symplectic and locally conformal Kähler structures were studied intensively (e.g., [4]). Like in the

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classical case, the characterization includes a closed 1-form ϖ called the Lee form, which defines the local conformal changes. We construct locally conformal generalized complex structures and locally conformal generalized Kähler structures, which are not globally conformal, on the Hopf manifolds and on a product $M \times S^1$ where M is a generalized Sasakian manifold [15]. Finally, we discuss the induced structure on hypersurfaces where the pullback of ϖ vanishes.

2 Preliminaries

Throughout the paper we use the following notation: M is an m -dimensional manifold, X, Y, \dots are either contravariant vectors or vector fields, α, β, \dots are either covariant vectors or 1-forms, $\mathcal{X}, \mathcal{Y}, \dots$ are pairs $(X, \alpha), (Y, \beta), \dots$, $\chi^k(M)$ is the space of k -vector fields, $\Omega^k(M)$ is the space of differential k -forms, Γ are spaces of global cross sections of vector bundles, d is the exterior differential and L is the Lie derivative. All the manifolds and mappings are assumed smooth.

Generalized geometric structures in the sense of Hitchin [8] are similar to classical structures but defined on the *big tangent bundle* $T^{big}M = TM \oplus T^*M$ with the neutral metric

$$g((X, \alpha), (Y, \beta)) = \frac{1}{2}(\alpha(Y) + \beta(X))$$

and the Courant bracket [2]

$$[(X, \alpha), (Y, \beta)] = ([X, Y], L_X\beta - L_Y\alpha + \frac{1}{2}d(\alpha(Y) - \beta(X))).$$

A maximal g -isotropic subbundle E of $T^{big}M$ (or of the complexification $T_c^{big}M = T^{big}M \otimes_{\mathbb{R}} \mathbb{C}$) is an *almost Dirac structure* and if ΓE is closed by the Courant bracket E is a Dirac structure.

A generalized almost complex structure is a vector bundle endomorphism $\Phi \in \text{End}(T^{big}M)$ that satisfies the following conditions

$$\Phi^2 = -Id, \quad g(\mathcal{X}, \Phi\mathcal{Y}) + g(\Phi\mathcal{X}, \mathcal{Y}) = 0.$$

Furthermore, if the *Nijenhuis torsion* of Φ vanishes, i.e.,

$$(2.1) \quad \mathcal{N}_\Phi(\mathcal{X}, \mathcal{Y}) = [\Phi\mathcal{X}, \Phi\mathcal{Y}] - \Phi[\mathcal{X}, \Phi\mathcal{Y}] - \Phi[\Phi\mathcal{X}, \mathcal{Y}] + \Phi^2[\mathcal{X}, \mathcal{Y}] = 0,$$

where the brackets are Courant brackets, Φ is an *integrable* or a *generalized complex structure*.

The generalized, almost complex structure Φ is equivalent with the pair (E, \bar{E}) of its $(\pm\sqrt{-1})$ -eigenbundles (the bar denotes complex conjugation), which are complex almost Dirac structures such that $E \cap \bar{E} = 0$, hence, Φ may be defined by E . Φ is integrable iff E is Dirac.

The structure Φ has the following representation by *classical tensor fields*

$$\Phi \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} A & \sharp_\pi \\ \flat_\sigma & -{}^t A \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix},$$

where

$$\pi \in \chi^2(M), \sigma \in \Omega^2(M), A \in \text{End}(TM), \sharp_\pi \alpha = i(\alpha)\pi, \flat_\sigma X = i(X)\sigma,$$

t denotes transposition, and the following conditions hold

$$(2.2) \quad \pi(\alpha \circ A, \beta) = \pi(\alpha, \beta \circ A), \sigma(AX, Y) = \sigma(X, AY), A^2 = -Id - \sharp_\pi \flat_\sigma.$$

In this classical representation the integrability conditions of Φ are [3, 13]:

- i) the bivector field π defines a Poisson structure on M , i.e., $[\pi, \pi] = 0$ where the bracket is the Schouten-Nijenhuis bracket with the sign convention of [12];
- ii) the *Schouten concomitant* of the pair (π, A) vanishes, i.e.,

$$R_{(\pi, A)}(\alpha, X) = \sharp_\pi(L_X(\alpha \circ A) - L_{AX}\alpha) - (L_{\sharp_\pi \alpha}A)(X) = 0;$$

- iii) the Nijenhuis tensor of A (defined by (2.1) with Lie brackets) satisfies the condition

$$\mathcal{N}_A(X, Y) = \sharp_\pi[i(X \wedge Y)d\sigma];$$

- iv) the *associated 2-form* $\sigma_A(X, Y) = \sigma(AX, Y)$ satisfies the condition

$$d\sigma_A(X, Y, Z) = \sum_{\text{Cycl}(X, Y, Z)} d\sigma(AX, Y, Z).$$

A *generalized, Riemannian metric* is a Euclidean (positive definite) metric G on the bundle $T^{\text{big}}M$, which is compatible with the neutral metric g of $T^{\text{big}}M$ in the sense that the musical isomorphism

$$\sharp_G : T^{\text{big}}M = TM \oplus T^*M \rightarrow T^*M \oplus TM \approx T^{\text{big}}M,$$

where \approx is defined by $(\alpha, X) \Leftrightarrow (X, \alpha)$ and

$$2g(\sharp_G(X, \alpha), (Y, \beta)) = G((X, \alpha), (Y, \beta)),$$

satisfies the conditions [7]

$$\sharp_G^2 = Id, \quad g(\sharp_G(X, \alpha), \sharp_G(Y, \beta)) = g((X, \alpha), (Y, \beta)).$$

It turns out that a generalized, Riemannian metric is equivalent with a pair (γ, ψ) where γ is a classical Riemannian metric on M and $\psi \in \Omega^2(M)$. More exactly,

$$(\gamma, \psi) \Leftrightarrow \sharp_G \begin{pmatrix} X \\ \alpha \end{pmatrix} = \begin{pmatrix} \varphi & \sharp_\gamma \\ b_\beta & {}^t\varphi \end{pmatrix} \begin{pmatrix} X \\ \alpha \end{pmatrix}$$

where $\varphi = -\sharp_\gamma \circ b_\psi$, $b_\beta = b_\gamma \circ (Id - \varphi^2)$ [7].

A *generalized almost Hermitian structure* is a pair (Φ, G) , where Φ is a generalized almost complex structure and G is a generalized Riemannian metric, such that the following skew-symmetry condition holds

$$G(\Phi\mathcal{X}, \mathcal{Y}) + G(\mathcal{X}, \Phi\mathcal{Y}) = 0 \quad (\mathcal{X}, \mathcal{Y} \in \Gamma T^{big} M).$$

Using the g -skew-symmetry of Φ we see that the previous condition is equivalent with the commutation condition $\sharp_G \circ \Phi = \Phi \circ \sharp_G$, which implies that the pair $(\Phi^c = \sharp_G \circ \Phi, G)$ (c comes from *complementary*) is a second generalized almost Hermitian structure that commutes with Φ . A commuting pair (Φ, Φ^c) defines G by $\sharp_G = -\Phi \circ \Phi^c$.

Theorem 2.1. [7] *A generalized almost Hermitian structure (G, Φ) is equivalent with a quadruple (γ, ψ, J_+, J_-) , where γ is a classical, Riemannian metric on M , ψ is a 2-form, and (γ, J_\pm) are two classical almost Hermitian structures of M defined as follows by the matrix of Φ :*

$$(2.3) \quad J_\pm = A + \sharp_\pi \circ b_{\psi \pm \gamma}.$$

The generalized, almost Hermitian manifold (M, G, Φ) is *generalized, Hermitian* if the structure Φ is integrable and *generalized, almost Kähler* if the complementary structure Φ^c is integrable. If both Φ and Φ^c are integrable (M, G, Φ) is a *generalized, Kähler* manifold. The classical structures with the same names yield the simplest examples.

Theorem 2.2. *The structure (G, Φ) is generalized Kähler iff one of the following hypotheses holds: 1) J_{\pm} are integrable and one has the equalities*

$$(2.4) \quad d_+^C \omega_+ = -d\psi, \quad d_-^C \omega_- = d\psi,$$

where $\omega_{\pm}(X, Y) = \gamma(J_{\pm}X, Y)$ are the Kähler forms of the Hermitian structures (γ, J_{\pm}) and the operators d_{\pm}^C are defined by the structures J_{\pm} via the formulas

$$d^C = C^{-1}dC = \sqrt{-1}(\bar{\partial} - \partial) \quad (C\lambda = (\sqrt{-1})^{p-q}\lambda, \lambda \in \Omega^{p,q}(M));$$

2) J_{\pm} are integrable and one has the equalities

$$(2.5) \quad (\nabla_X J_{\pm})(Y) = \mp \frac{1}{2} \sharp_{\gamma} [(i(X \wedge Y)d\psi) \circ J_{\pm} + i(X \wedge (J_{\pm}Y))d\psi],$$

where ∇ is the Levi-Civita connection of the metric γ ; 3) the $(3, 0)$ and $(0, 3)$ type components of $d\psi$ are zero and the connections

$$(2.6) \quad \nabla_X^{\pm} Y = \nabla_X Y \pm \frac{1}{2} \sharp_{\gamma} [i(X \wedge Y)d\psi]$$

satisfy the condition $\nabla^{\pm} J_{\pm} = 0$, respectively.

Characterizations 1) and 3) were proven by Gualtieri [7], where it is also shown that (2.4) is equivalent with

$$(2.7) \quad d\omega_+(J_+X, J_+Y, J_+Z) = -d\omega_-(J_-X, J_-Y, J_-Z) = d\psi(X, Y, Z).$$

The connections (2.6) are called the Bismut connections and they are the unique metric connections with covariant torsion $d\psi$. Characterization 2) follows by replacing F_{\pm} by J_{\pm} in Proposition 4.6 of [15].

3 Locally conformal integrable structures

Consider the automorphism $\mathcal{C}_{\tau} : T^{big}M \rightarrow T^{big}M$ defined by [6, 14, 16]

$$\mathcal{C}_{\tau}(X, \alpha) = (X, e^{\tau}\alpha), \quad \tau \in C^{\infty}(M).$$

We call it a *conformal change* of $T^{big}M$ because it produces a conformal change of the metric g :

$$g(\mathcal{C}_{\tau}(X, \alpha), \mathcal{C}_{\tau}(Y, \beta)) = e^{\tau}g((X, \alpha), (Y, \beta)).$$

Furthermore, if τ is locally constant the change will be called a *homothety*.

The natural way to apply a conformal change to any $\Phi \in \text{End}(T^{\text{big}}M)$ is by conjugation. In particular, the generalized almost complex structure Φ and the generalized Riemannian metric operator \sharp_G will change as follows

$$\Phi \mapsto \Phi' = \mathcal{C}_{-\tau} \circ \Phi \circ \mathcal{C}_\tau, \quad \sharp_G \mapsto \sharp_{G'} = \mathcal{C}_{-\tau} \circ \sharp_G \circ \mathcal{C}_\tau.$$

Accordingly, one gets

$$(3.1) \quad \begin{pmatrix} A & \sharp_\pi \\ \flat_\sigma & -{}^t A \end{pmatrix} \mapsto \begin{pmatrix} A & \sharp_{e^\tau \pi} \\ \flat_{e^{-\tau} \sigma} & -{}^t A \end{pmatrix}, \quad \begin{pmatrix} \varphi & \sharp_\gamma \\ \flat_\beta & {}^t \varphi \end{pmatrix} \mapsto \begin{pmatrix} \varphi & \sharp_{e^{-\tau} \gamma} \\ \flat_{e^{-\tau} \beta} & {}^t \varphi \end{pmatrix}$$

(the minus sign in $e^{-\tau} \gamma$ is because we look at γ as the covariant tensor of the metric). It follows that if $G \Leftrightarrow (\gamma, \psi)$ then $G' \Leftrightarrow (e^{-\tau} \gamma, e^{-\tau} \psi)$.

If (G, Φ) is a generalized almost Hermitian structure (G', Φ') is a generalized almost Hermitian structure too. Moreover, formulas (2.3) and (3.1) show that the corresponding pair of classical Hermitian structures does not change, i.e., $J'_\pm = J_\pm$.

Definition 3.1. A generalized almost complex structure Φ is *globally conformal integrable* if there exists a conformal change \mathcal{C}_τ such that Φ' is integrable. If such changes \mathcal{C}_τ exist locally (i.e., in a neighborhood of each point), Φ is a *locally conformal integrable structure*. Similarly, one has notions of *(locally) generalized Hermitian, almost Kähler and Kähler structures*.

We obtain the conditions of conformal integrability by applying conditions i)-iv) of Section 2 to the tensor fields $(A, e^\tau \pi, e^{-\tau} \sigma)$. The result is

Proposition 3.1. *The generalized almost complex structure Φ is globally conformal integrable if there exists a function $\tau \in C^\infty(M)$ such that $\varpi = d\tau$ satisfies the conditions*

$$(3.2) \quad [\pi, \pi] = -2(\sharp_\pi \varpi) \wedge \pi,$$

$$(3.3) \quad R_{(\pi, A)}(\alpha, X) = \varpi(AX) \sharp_\pi \alpha - \varpi(X) A \sharp_\pi \alpha,$$

$$(3.4) \quad \begin{aligned} \mathcal{N}_A(X, Y) - \sharp_\pi [i(X \wedge Y) d\sigma] &= -\sigma(X, Y) \sharp_\pi \varpi \\ + \varpi(X) [(Id + A^2)(Y)] - \varpi(Y) [(Id + A^2)(X)], \end{aligned}$$

$$\begin{aligned}
(3.5) \quad d\sigma_A(X, Y, Z) &- \sum_{Cycl(X, Y, Z)} d\sigma(A X, Y, Z) \\
&= -[\varpi \wedge \sigma_A + (\varpi \circ A) \wedge \sigma](X, Y, Z).
\end{aligned}$$

Proof. Condition i) is $[e^\tau \pi, e^\tau \pi] = 0$ and a straightforward calculation shows its equivalence with (3.2). If we use the formula

$$L_{fX}A = fL_XA + (AX) \otimes df - X \otimes (df \circ A) \quad (f \in C^\infty(M))$$

in ii) for Φ' the result is (3.3). Furthermore, a simple calculation gives the following expression of iii) for Φ' :

$$\begin{aligned}
(3.6) \quad \mathcal{N}_A(X, Y) - \sharp_\pi[i(X \wedge Y)d\sigma] &= -\sharp_\pi[i(X \wedge Y)(d\tau \wedge \sigma)] \\
&= -\sigma(X, Y)\sharp_\pi d\tau - (X\tau)(\sharp_\pi \circ b_\sigma)(Y) + (Y\tau)(\sharp_\pi \circ b_\sigma)(X).
\end{aligned}$$

In view of (2.2) this result is equivalent to (3.4). Finally, the new associated 2-form is $e^{-\tau}\sigma_A$ and condition iv) for Φ' becomes (3.5). \square

Proposition 3.2. *Let Φ be a generalized complex structure on M and let Φ' be obtained by a conformal change of Φ . Assume that $\dim M > 2$ and that Φ satisfies one of the following conditions: 1) π is non degenerate, 2) $\forall x \in M$, $A_x^2 \neq -Id$ and A_x has no real eigenvalue, 3) $\text{rank } \pi > 2$ and σ is non degenerate. Then Φ' is integrable iff the conformal change is a homothety.*

Proof. If Φ is integrable, Φ' is integrable too iff the right hand side of the equalities (3.2)-(3.5) vanishes. Condition $(\sharp_\pi \varpi) \wedge \pi = 0$ holds iff either $\text{rank } \pi = 2$ or $\sharp_\pi \varpi = 0$. In case 1) we must have the latter equality, which also implies $\varpi = 0$, and we are done. To discuss case 2), assume that $d_x \tau \neq 0$ and take a vector field X such that $X\tau \neq 0$ on a neighborhood U_x . Then, the annulation of the right hand side of (3.3) yields $A|_{im \sharp_\pi} = fId$ on U_x . If we apply this equality to a 1-form $b_\sigma Y$ where the vector field Y is arbitrary and use (2.2), we see that $A|_{U_x}$ satisfies an equation of the form

$$\mathcal{P}(A) = A^3 - fA^2 + A - fId = (A - fId)(A^2 + Id) = 0.$$

Since $A^2 + Id \neq 0$, the minimal polynomial of A is either $A - fId$ or $\mathcal{P}(A)$ and A must have a real eigenvalue. Thus, the hypothesis of case 2) implies $d\tau = 0$ as required. In case 3), since $\text{rank } \pi > 2$ we have $\sharp_\pi d\tau = 0$ and the annulation of the right hand side of (3.4) reduces to

$$(3.7) \quad (X\tau)\sharp_\pi b_\sigma(Y) - (Y\tau)\sharp_\pi b_\sigma(X) = 0.$$

On the other hand, $\text{rank } \pi > 2$ implies that $\forall X \in \chi^1(M)$ with $\sharp_\pi \flat_\sigma(X) \neq 0$ there exists $\lambda \in \Omega^1(M)$ such that $\sharp_\pi \flat_\sigma X, \sharp_\pi \lambda$ are linearly independent. If σ is non degenerate we may put $\lambda = \flat_\sigma Y$ and (3.7) implies $X\tau = 0$. Furthermore, if $\sharp_\pi \flat_\sigma(X) = 0$, (3.7) reduces to $(X\tau)\sharp_\pi \flat_\sigma(Y) = 0$ for any Y and we get $X\tau = 0$ again. Therefore $d\tau = 0$. \square

Accordingly, we get the following characterization of the locally conformal integrable, generalized, almost complex structures.

Theorem 3.1. *Let (M, Φ) be a generalized almost complex manifold that satisfies the hypotheses of Proposition 3.2. Then Φ is locally conformal integrable iff there exists a closed 1-form $\varpi \in \Omega^1(M)$ such that conditions (3.2)-(3.5) hold. The structure Φ is globally conformal integrable iff ϖ is exact.*

Proof. If ϖ exists we have a covering $M = \cup U_a$ such that $\varpi|_{U_a} = d\tau_a$ for some local functions τ_a and $\mathcal{C}_{-\tau_a} \Phi \mathcal{C}_{\tau_a}$ are integrable. Conversely, if we have a covering U_a of M with functions τ_a such that $\mathcal{C}_{-\tau_a} \Phi \mathcal{C}_{\tau_a}$ are integrable then Proposition 3.2 shows that $d\tau_a = d\tau_b$ on $U_\alpha \cap U_\beta$. Thus, the local forms $d\tau_a$ glue up to the required global closed form ϖ . The last assertion of the theorem is obvious. \square

Like in the classical case [11], we call ϖ the *Lee form*. It is worth noticing that if $\varpi \wedge d\sigma = 0$ the first equality of (3.6) shows that $(\pi, A, d\sigma - \varpi \wedge \sigma)$ is a Poisson quasi-Nijenhuis structure [10].

In order to get the characterization of generalized, locally conformal Kähler structures we go from a generalized almost Hermitian structure (G, Φ) to the equivalent quadruple (γ, ψ, J_\pm) , change it to $(e^{-\tau}\gamma, e^{-\tau}\psi, J_\pm)$ and ask the latter to satisfy Gualtieri's conditions (2.4). The result is

Proposition 3.3. *The generalized almost Hermitian structure (G, Φ) is conformal generalized Kähler iff J_\pm are integrable and there exists $\tau \in C^\infty(M)$ such that the form $\varpi = d\tau$ satisfies the conditions*

$$(3.8) \quad d\psi \pm d^{\mathcal{C}}\omega_\pm = \varpi \wedge \psi \mp (\varpi \circ J) \wedge \omega_\pm.$$

Proof. The requirement for J_\pm is clear. For Φ' , (2.7) becomes

$$(d\psi - d\tau \wedge \psi)(X, Y, Z) = \pm(d\omega_\pm - d\tau \wedge \omega_\pm)(J_\pm X, J_\pm Y, J_\pm Z).$$

If we evaluate the wedge products and take into account that $\omega_\pm(J_\pm X, J_\pm Y) = \omega_\pm(X, Y)$ we get (3.8). \square

Proposition 3.4. *If M is a generalized Kähler manifold of dimension greater than 4 then a conformal change leads to a generalized Kähler structure iff it is a homothety.*

Proof. By (3.8) the required condition is

$$(\varpi) \wedge \psi \mp (\varpi \circ J) \wedge \omega_{\pm} = 0.$$

This implies $\varpi \wedge (\varpi \circ J) \wedge \omega_{\pm} = 0$. Since $\text{rank } \omega_{\pm} > 4$ a well known Cartan lemma tells that this condition holds iff $\varpi = d\tau = 0$. \square

As a consequence we get

Theorem 3.2. *If $\dim M > 4$, the generalized almost Hermitian structure (γ, ψ, J_{\pm}) is a locally conformal, generalized Kähler structure iff J_{\pm} are integrable and there exists a closed 1-form ϖ (the Lee form) that satisfies condition (3.8). The same structure is globally generalized Kähler iff ϖ is exact.*

The proof is the same like for Theorem 3.1.

In order to state some other conditions that are equivalent to (3.8) we recall the Weyl connection defined by a Riemannian metric γ and a closed 1-form ϖ :

$$\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\varpi(X)Y - \frac{1}{2}\varpi(Y)X + \frac{1}{2}\gamma(X, Y)\sharp_{\gamma}\varpi,$$

where ∇ is the Levi-Civita connection of γ . The Weyl connection is the Levi-Civita connection of $e^{-\tau}\gamma$ for the local functions τ that satisfy $d\tau = \varpi$ and it is the unique torsionless connection such that $\tilde{\nabla}_X \gamma = \varpi(X)\gamma$.

Proposition 3.5. *In Theorem 3.2, condition (3.8) may be replaced by each of the following conditions: i) the Weyl connection satisfies the conditions*

$$(3.9) \quad (\tilde{\nabla}_X J_{\pm})(Y) = \mp \frac{1}{2}\sharp_{\gamma}\{[i(X \wedge Y)(d\psi - \varpi \wedge \psi)] \circ J_{\pm} \\ + i[X \wedge (J_{\pm}Y)](d\psi - \varpi \wedge \psi)\},$$

ii) the connections

$$(3.10) \quad \tilde{\nabla}_X^{\pm} Y = \tilde{\nabla}_X Y \pm \frac{1}{2}\sharp_{\gamma}[i(X \wedge Y)(d\psi - \varpi \wedge \psi)]$$

satisfy the condition $\tilde{\nabla}^{\pm} J_{\pm} = 0$, respectively.

Proof. Instead of using (2.7), use (2.5), respectively, (2.6) to express the fact that $(e^{-\tau}\gamma, e^{-\tau}\psi, J_{\pm})$ is a generalized Kähler structure. The integrability of J_{\pm} implies the annulation of the $(3, 0)$, $(0, 3)$ components of $d\psi$ [7]. \square

Condition i) of Proposition 3.5 shows that if $d\psi = \varpi \wedge \psi$ then (γ, J_{\pm}) is a pair of classical, locally conformal Kähler structures with the same metric and the same Lee form. Condition ii) is interesting because, at least in the generalized Kähler case, it may be related to physics [5].

Example 3.1. Take $M = \mathbb{R}^{2n} \setminus \{0\} \approx S^{2n-1} \times \mathbb{R}$ by the diffeomorphism $\kappa(x) = (x/||x||, \ln||x||/\ln\lambda)$ ($x \in \mathbb{R}^{2n} \setminus \{0\}$) defined for any choice of $\lambda \in (0, 1)$. Denote by x^i ($i = 1, \dots, 2n$) the natural coordinates on \mathbb{R}^{2n} and consider the symplectic form $\omega = \sum_{h=1}^n dx^h \wedge dx^{n+h}$ and an arbitrary, constant $(1, 1)$ -tensor field A that satisfies the condition $\omega(AX, Y) = \omega(X, AY)$ (such tensor fields obviously exist). Then ω_A (defined like σ_A) is closed, (ω, A) is a Hitchin pair [3] and it has a corresponding generalized complex structure Φ with the chosen tensor field A , the Poisson bivector field π defined by $\sharp_{\pi} \circ \flat_{\omega} = -Id$ and the 2-form σ defined by $\flat_{\sigma} = \flat_{\omega} \circ A^2 + \flat_{\omega}$. If we apply to Φ the conformal change $\mathcal{C}_{\ln||x||^2}$ we get a conformal integrable, generalized, almost complex structure Φ' with the tensor fields $(A, ||x||^2\pi, (1/||x||^2)\sigma)$. Now, consider the quotient $\mathcal{H}^{2n} = (\mathbb{R}^{2n} \setminus \{0\})/\Delta_{\lambda}$ where Δ_{λ} is the infinite cyclic group generated by the transformation $x \mapsto \lambda x$, which is called the Hopf manifold and where κ induces a diffeomorphism $\mathcal{H}^{2n} \approx S^{2n-1} \times S^1$. It is obvious that Φ' is invariant by Δ_{λ} . Hence, there exists an induced generalized, almost complex structure Ψ on \mathcal{H} and Ψ is locally conformal integrable via the conformal changes $\mathcal{C}_{-\ln||x||^2}$. The conditions (3.2)-(3.5) are satisfied for the closed 1-form

$$\varpi = -\frac{2 \sum_{i=1}^{2n} x^i dx^i}{||x||^2}.$$

Since ϖ is proportional to the length element of S^1 (see the isomorphism κ) ϖ is not exact and Ψ is not globally conformal integrable.

Example 3.2. The Hopf manifold \mathcal{H}^{2n} ($n > 1$) also has locally conformal generalized Kähler structures that are not globally conformal. Indeed, take the flat metric $\gamma_0 = \sum_{i=1}^{2n} (dx^i)^2$ of \mathbb{R}^{2n} , an arbitrary constant 2-form ψ_0 , and two γ_0 -compatible, constant, complex structures J_{\pm} , for instance

$$J_+(\frac{\partial}{\partial x^h}) = \frac{\partial}{\partial x^{n+h}}, \quad J_+(\frac{\partial}{\partial x^{n+h}}) = -\frac{\partial}{\partial x^h},$$

$$J_-\left(\frac{\partial}{\partial x^{2h-1}}\right) = \frac{\partial}{\partial x^{2h}}, \quad J_-\left(\frac{\partial}{\partial x^{2h}}\right) = -\frac{\partial}{\partial x^{2h-1}},$$

where $h = 1, \dots, n$. The quadruple $(\gamma_0, \psi_0, J_\pm)$ defines a generalized Kähler structure (G_0, Φ) on $\mathbb{R}^{2n} \setminus \{0\}$ and $(\gamma_0, \psi_0, J_\pm) \mapsto (\gamma_0/\|x\|^2, \psi_0/\|x\|^2, J_\pm)$ produces a conformal generalized Kähler structure (G'_0, Φ') . The latter projects to a locally conformal generalized Kähler structure of \mathcal{H}^{2n} , which satisfies (3.8) for $\varpi = -2d \ln \|x\|$, and is not globally conformal generalized Kähler because ϖ is not exact. If $J_+ = J_-$ this example reduces to a well known example of a classical locally conformal Kähler structure that is not globally conformal Kähler [11]. It is also known that the manifold \mathcal{H}^4 has no generalized Kähler structures with a constant J_+ [7].

Example 3.3. Recall that a generalized Sasakian structure on a manifold M is equivalent with a pair of classical, normal, almost contact, metric structures $(F_\pm, Z_\pm, \xi_\pm, \gamma)$ complemented by a pair of forms $\psi \in \Omega^2(M), \kappa \in \Omega^1(M)$ such that the quadruple

$$(e^t(\gamma + dt^2), e^t(\psi + \kappa \wedge dt), J_\pm = F_\pm + dt \otimes Z_\pm - \xi_\pm \otimes \frac{\partial}{\partial t})$$

defines a generalized Kähler structure on $M \times \mathbb{R}$ [15]. Thus, the similar quadruple without the factors e^t defines a conformal generalized Kähler structure. The later is invariant by translations along the factor \mathbb{R} and it descends to a locally, not globally, conformal, generalized Kähler structure on $M \times S^1$.

Let $(\gamma, \psi, J_\pm, \varpi)$ be a locally conformal generalized Kähler structure on M . A hypersurface $\iota : N \hookrightarrow M$ that satisfies the condition $\iota^* \varpi = 0$ will be called a *Lee hypersurface* and we will describe the induced structure of an orientable, Lee hypersurface.

It is known that any orientable hypersurface of a Hermitian manifold (M, γ, J) has an induced metric, almost contact structure (F, Z, ξ) such that its $\sqrt{-1}$ -eigenbundle is a CR-structure (e.g., [1]). The induced structure is obtained by taking a normal unit vector field ν of N and by defining

$$(3.11) \quad Z = -J\nu, \quad \xi = \flat_\gamma Z, \quad F|_{TN \cap J(TN)} = J, \quad F(Z) = 0.$$

Following [15] we can say more about the induced structure. Indeed, using the normal bundle $\nu N = \text{span}\{\nu\}$ it follows easily that, if we look at J as a generalized complex structure, the hypersurface N is a CRF-submanifold with the generalized CRF-structure defined by the classical almost contact

structure (3.11) (see Proposition 2.5 and Definition 3.1 of [15]). Thus, the induced structure is not just CR, it is a classical CRF-structure (see Definition 3.2 of [15]). From Proposition 3.1 of [15] it follows that, in our case, the supplementary integrability condition (besides the CR condition) is

$$(3.12) \quad [Z, F^2X + \sqrt{-1}FX] \in H \oplus Q_c,$$

where H is the $\sqrt{-1}$ -eigenbundle of F and $Q = \text{span}\{Z\}$ is the 0-eigenbundle of F . Using the property $F^3 + F = 0$ it follows that (3.12) may be changed to

$$(3.13) \quad F \circ (L_Z F) \circ F = 0,$$

equivalently,

$$(3.14) \quad L_Z F = (\xi \circ L_Z F) \otimes Z.$$

Another fact to be noticed is that if ω is the Kähler form of (γ, J) then $\iota^* \omega = \Xi$ where $\Xi(X, Y) = \gamma(FX, Y)$ ($X, Y \in \chi^1(N)$) is the fundamental form of the structure (F, Z, ξ, γ) . In particular, we get

Proposition 3.6. *Any orientable hypersurface of a classical Kähler manifold has an induced classical CRF-structure with a closed fundamental form.*

Back to our subject, the announced result about Lee hypersurfaces is

Proposition 3.7. *An orientable Lee hypersurface of a locally conformal, generalized, Kähler manifold inherits two metric, almost contact structures $(F_{\pm}, Z_{\pm}, \xi_{\pm}, \gamma)$ with the fundamental forms Ξ_{\pm} that satisfy the condition*

$$(3.15) \quad \varpi(\nu)\Xi_{\pm} = \pm i(Z)\iota^*(d\psi \pm d^C \omega_{\pm}).$$

Proof. Pull back (3.8) by ι , then apply the operator $i(Z)$. □

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