

COHOMOGENEITY ONE MANIFOLDS AND SELFMAPS OF NONTRIVIAL DEGREE

THOMAS PÜTTMANN

ABSTRACT. We construct natural selfmaps of compact cohomogeneity one manifolds with finite Weyl group and compute their degrees and Lefschetz numbers. On manifolds with simple cohomology rings this yields in certain cases relations between the order of the Weyl group and the Euler characteristic of a principal orbit. We apply our construction to the compact Lie group $SU(3)$ where we extend identity and transposition to an infinite family of selfmaps of every odd degree. The compositions of these selfmaps with the power maps realize all possible degrees of selfmaps of $SU(3)$.

1. INTRODUCTION

A natural problem in topology is the following: Given a compact oriented manifold M^n , which integers can occur as the degree of maps $M \rightarrow M$? In the fundamental case where M is a sphere S^n all integers can easily be realized since S^n is a $(n-1)$ -fold suspension of S^1 . For other manifolds the problem is usually difficult (see [DuWa] for references and a detailed study in the case of $(n-1)$ -connected $2n$ -manifolds).

In order to construct natural maps of degree $\neq -1, 0, 1$ one might first impose actions of compact Lie groups G on M and then try to find equivariant maps. In the most symmetric case where the action $G \times M \rightarrow M$ is transitive, it is well-known and easy to see that every equivariant map is a diffeomorphism and hence has degree ± 1 . We deal with the case where the action is of cohomogeneity one, i.e., the principal orbits G/H are of codimension 1 or, equivalently, the orbit space M/G is 1-dimensional and hence a closed interval or a circle. We focus on the much more interesting case where the orbit space M/G is a closed interval.

We suppose that the cohomogeneity one manifold M is equipped with an invariant Riemannian metric such that the Weyl group is finite or, equivalently, such that the normal geodesics are closed. Such metrics exist on many cohomogeneity one manifolds M . For example, all invariant metrics with positive sectional curvature necessarily have finite Weyl groups [GWZ]. There might, however, be infinitely many invariant Riemannian metrics with mutually distinct Weyl groups on the same compact manifold M with a fixed action $G \times M \rightarrow M$. The elementary but new construction of this paper highly depends on the order of the Weyl group: We k -fold the closed normal geodesics of M starting from one of the non-principal orbits. This leads to well-defined maps $\psi_k : M \rightarrow M$, $k = j|W|/2 + 1$, for even integers j , and even for all integers j provided a certain condition on the isotropy groups is satisfied. In the following theorem we only deal with the general case. For the exceptional case we refer to section 3.

2000 *Mathematics Subject Classification.* Primary 57S15, Secondary 55M25.
supported by a DFG Heisenberg scholarship and the DFG priority program SPP 1154.

Theorem 1. *A compact oriented Riemannian cohomogeneity one manifold M whose orbit space M/G is a closed interval and whose Weyl group W is finite admits equivariant selfmaps $\psi_k : M \rightarrow M$ for all $k = j|W|/2 + 1$, $j \in 2\mathbb{Z}$ with degree*

$$\deg \psi_k = \begin{cases} k & \text{if the codimensions of both non-principal orbits are odd,} \\ +1 & \text{otherwise,} \end{cases}$$

and Lefschetz number

$$L(\psi_k) = \begin{cases} -j\chi(G/H)/2 & \text{if the codimensions of both non-principal orbits are odd,} \\ \chi(M) & \text{otherwise.} \end{cases}$$

Here, $\chi(G/H)$ denotes the Euler characteristic of a principal orbit.

Degree and Lefschetz number of selfmaps are of course not independent. In the case where M^n is a rational homology sphere they are coupled by the simple equation $L(\psi) = 1 + \deg \psi$ for even n or $L(\psi) = 1 - \deg \psi$ for odd n . Hence, we deduce from Theorem 1 that $\text{rank } G = \text{rank } H$ and $\chi(G/H) = |W|$ if the codimensions of both singular orbits are odd¹. On manifolds M with somewhat more complicated cohomology rings one can still use Theorem 1 to derive restrictions for cohomogeneity one actions $G \times M \rightarrow M$ with $\text{rank } G = \text{rank } H$, i.e., $\chi(G/H) > 0$. For such actions the dimensions of all orbits are necessarily even (hence the dimension of M is necessarily odd), and the Weyl groups are necessarily finite. Using Dirichlet's theorem on prime numbers in arithmetic progressions one obtains, for example, the following consequence of Theorem 1:

Corollary 2. *Let $G \times M \rightarrow M$ be a cohomogeneity one action with $\text{rank } G = \text{rank } H$. If M has the rational cohomology ring of a product*

$$\mathbb{S}^{l_1} \times \mathbb{S}^{l_2} \times \dots \times \mathbb{S}^{l_m}, \quad l_1 < l_2 < \dots < l_m,$$

then $\chi(G/H) = 2^{m-1}|W|$.

In order to illustrate Theorem 1 by examples other than spheres one might scan the classifications [Neu], [Pa], [Hoe] for simply connected cohomogeneity one manifolds in dimensions ≤ 7 where the codimensions of both singular orbits are odd. This scan reveals two nontrivial \mathbb{S}^3 -bundles $M_1^7 \rightarrow \mathbb{S}^4$ and $M_2^7 \rightarrow \mathbb{C}\mathbb{P}^2$ with sections. We briefly discuss these examples in section 5 and deduce from the refined version of Theorem 1 that M_1^7 and M_2^7 admit equivariant selfmaps ψ_k for all $k \in \mathbb{Z}$ with degree k and Lefschetz number $2(1 - k)$ and $3(1 - k)$, respectively. The space M_1^7 provides an example for Corollary 2.

Our main example appears in dimension 8: the compact Lie group $\text{SU}(3)$. Simple polynomial selfmaps of $\text{SU}(3)$ are already given by the power maps $\rho_k : A \mapsto A^k$. These maps are equivariant with respect to conjugation and have degree $\deg \rho_k = k^2$ by a well-known result of Hopf [Ho]. Our construction yields additional selfmaps ψ_k of $\text{SU}(3)$ of every odd degree k and Lefschetz number 0 that extend identity and transposition to an infinite family. These maps are equivariant with respect to the cohomogeneity one action

$$\text{SU}(3) \times \text{SU}(3) \rightarrow \text{SU}(3), \quad (A, B) \mapsto ABA^T.$$

¹Note that $\chi(G/H) = |W|$ implies $W = N(H)/H$. For the linear cohomogeneity one actions on spheres with $\text{rank } G = \text{rank } H$ the identity $W = N(H)/H$ was already observed in [GWZ].

Explicit formulas are given in section 4. Combining the maps ρ_k with the maps ψ_k and a simple argument involving Steenrod squares we prove

Theorem 3. *For any $m \in \mathbb{N}$ and $\ell \in \mathbb{Z}$ there is a selfmap of $SU(3)$ with degree $4^m(2\ell + 1)$. For each of these selfmaps the entries of the target matrix are real polynomials in the complex entries of the argument matrix. Other integers do not appear as degrees of selfmaps of $SU(3)$.*

The author would like to thank A. Rigas for some useful discussions.

2. CONSTRUCTION OF THE EQUIVARIANT SELFMAPS

Before we describe our construction we first summarize the necessary facts about cohomogeneity one manifolds. We refer to the standard sources [Mo], [Bd1], [AA] and the recent paper [GWZ].

Let M be a compact manifold on which a compact Lie group G acts with cohomogeneity one. Then M/G is a circle or a closed interval. In the former case all orbits are principal and M is a G/H -bundle over \mathbb{S}^1 with structure group $N(H)/H$. In particular, M has infinite fundamental group. We only consider the case where M/G is a closed interval. The regular part M_0 of M (i.e., the union of principal orbits) projects to the interior of the interval and the two end points of the interval correspond to non-principal orbits N_0 and N_1 . The manifold M is obtained by gluing the normal disk bundles over N_0 and N_1 along their common boundary. In particular, $\chi(M) = \chi(N_0) + \chi(N_1) - \chi(G/H)$.

Suppose that M is equipped with an invariant Riemannian metric $\langle \cdot, \cdot \rangle$. After rescaling we can assume that M/G is isometric to the unit interval $[0, 1]$. A normal geodesic is an (unparametrized) geodesic that passes through all orbits perpendicularly. Through any point $p \in M$ there is at least one normal geodesic and precisely one if p is contained in a principal orbit. The group G acts transitively on the set of normal geodesics. We fix a normal geodesic segment $\gamma : [0, 1] \rightarrow M$ such that $\gamma(]0, 1[)$ projects to the interior of M/G and $p_0 = \gamma(0) \in N_0$ and $p_1 = \gamma(1) \in N_1$. This segment is a shortest curve from N_0 to N_1 . It extends to a parametrized normal geodesic $\gamma : \mathbb{R} \rightarrow M$. The isotropy groups $G_{\gamma(t)}$ are constant along γ except at the points $\gamma(t)$ with $t \in \mathbb{Z}$, i.e., at points where γ intersects the non-principal orbits. We denote the generic isotropy group along γ by H .

The Weyl group W of $(M, \langle \cdot, \cdot \rangle)$ is by definition the subgroup of elements of G that leave γ invariant modulo the subgroup of elements that fix γ pointwise. It is a dihedral subgroup of $N(H)/H$. The geodesic γ is periodic if and only if W is finite. More precisely, the order $|W|$ of W equals the number of times that γ passes through a fixed principal orbit before it closes. To give an example of what might happen on a standard space consider the diagonal $SO(3)$ -action on $\mathbb{S}^2 \times \mathbb{S}^2(\sqrt{\alpha})$. The normal geodesics are closed if and only if α is rational. In this case, the Weyl group is the dihedral group $D_{|p-q|}$ of order $2|p - q|$ where $\alpha = p/q$ for positive integers p and q with $\gcd(p, q) = 1$. Otherwise, the Weyl group is the infinite dihedral group D_∞ .

In the following we assume that the Weyl group of $(M, \langle \cdot, \cdot \rangle)$ is finite, i.e., the fixed unit speed normal geodesic γ and all their translated copies $g \cdot \gamma$ are closed with period $|W|$.

Our construction is based on the following elementary fact: Let

$$\psi_k : \mathbb{S}^1 \rightarrow \mathbb{S}^1, \quad \lambda \mapsto \lambda^k$$

be the k -th power of \mathbb{S}^1 . Except in the trivial case where $k = 1$ the fixed points of ψ_k are precisely the $|k - 1|$ -roots of 1 and ψ_k has the property $\psi_k(\lambda \cdot \lambda_0) = \psi_k(\lambda) \cdot \lambda_0$ for λ_0 with $\lambda_0^{k-1} = 1$. In other words, if we have a circle of any length with a fixed base point p_0 then k -folding the intrinsic distance to p_0 is a well-defined map ψ_k of the circle with $|k - 1|$ fixed points and k -folding the distance to any of these fixed points leads to the same map ψ_k .

Lemma 2.1. *The assignment $g \cdot \gamma(t) \mapsto g \cdot \gamma(kt)$ leads to a well-defined smooth map $\psi_k : M \rightarrow M$ with $k = j|W|/2 + 1$ for all $j \in 2\mathbb{Z}$, and even for all $j \in \mathbb{Z}$ if the isotropy groups at $\gamma(t_0)$ and $\gamma(t_0 + |W|/2)$ for one and hence all odd integers t_0 are not just conjugate but equal.*

Proof. We first check that the assignment $\psi_k : g \cdot \gamma(t) \mapsto g \cdot \gamma(kt)$ yields a well-defined map on the unparametrized closed normal geodesic $\gamma(\mathbb{R})$. Suppose that $g \cdot \gamma(\mathbb{R}) = \gamma(\mathbb{R})$. Then $g \cdot \gamma(t) = \gamma(2t_0 \pm 2t)$ with $t_0 \in \mathbb{Z}$. A simple computation shows $\psi_k(g \cdot \gamma(t)) = g \cdot \psi_k(\gamma(t))$. The geometric reasons for this equivariance of ψ_k under the Weyl group are that all the points $\gamma(2t_0)$ are fixed points of $\gamma(t) \mapsto \gamma(kt)$ (here the special form of k is essential) and that the k -folding of a circle is the same from every fixed point. Now ψ_k is well-defined on $\gamma(\mathbb{R})$ and hence on the regular part M_0 of M since through any point in M_0 there is only one (unparametrized) normal geodesic. On N_0 the assignment $g \cdot \gamma(t) \mapsto g \cdot \gamma(kt)$ obviously leads to the identity. On N_1 we also get the identity provided $j \in 2\mathbb{Z}$. If the isotropy groups at $\gamma(t_0)$ and $\gamma(t_0 + |W|/2)$ are equal for one odd integer t_0 then by the action of the Weyl group this is true for all odd integers. If j is an odd integer and $k = j|W|/2 + 1$ then

$$g \cdot \gamma(t_0) \mapsto g \cdot \gamma(kt_0) = g \cdot \gamma(t_0 + |W|/2).$$

This actually defines an equivariant diffeomorphism $N_1 \rightarrow N_1$ or $N_1 \rightarrow N_0$ depending on whether $|W|/2$ is even or odd. All in all we have shown that there is a commutative diagram

$$\begin{array}{ccc} \nu N_0 & \xrightarrow{\times k} & \nu N_0 \\ \exp \downarrow & & \downarrow \exp \\ M & \xrightarrow{\psi_k} & M \end{array}$$

where νN_0 denotes the normal bundle of N_0 . This implies the smoothness of ψ_k . \square

3. DEGREE AND LEFSCHETZ NUMBER

Let M be a compact Riemannian cohomogeneity one manifold with finite Weyl group W such that the orbit space M/G is isometric to the interval $[0, 1]$. The manifold M is orientable if and only if the principal orbits are orientable and none of the non-principal orbits is exceptional and orientable (see [Mo]). In this section we assume that M is orientable and equipped with a fixed orientation.

Theorem 3.1. *The Lefschetz number of $\psi_k : M \rightarrow M$, $k = j|W|/2 + 1$, is*

$$L(\psi_k) = \begin{cases} -j\chi(G/H)/2 & \text{if } \text{codim } N_0 \text{ and } \text{codim } N_1 \text{ are both odd,} \\ \chi(M) & \text{otherwise,} \end{cases}$$

if j is even and

$$L(\psi_k) = \begin{cases} -j\chi(G/H)/2 & \text{if } \text{rank } G = \text{rank } H, \\ \chi(N_0) & \text{otherwise,} \end{cases}$$

if j is odd provided that ψ_k is well-defined in this case.

Proof. We perturb the map $\psi_k : M \rightarrow M$ to a map with only finitely many fixed points and compute the Lefschetz number as the sum of fixed point indices. For this perturbation we use the map $l_g : M \rightarrow M$, $p \mapsto g \cdot p$ where $g \in G$ is a general element, i.e., the closure of $\{g^m \mid m \in \mathbb{Z}\}$ is a maximal torus of G . By a classical theorem of Hopf and Samelson [HS], the restriction of l_g to any of the orbits $G \cdot p$ has precisely $\chi(G/G_p)$ fixed points and each of these fixed points has fixed point index one, i.e., $\det(\mathbb{1} - A) > 0$ where A is the derivative of l_g at the fixed point (note that our sign convention is different from that in [HS]). We now consider the composition $l_g \circ \psi_k$ for a general element $g \in G$ sufficiently close to $1 \in G$. The equivariant map ψ_k maps orbits to orbits. It is clear from the construction of ψ_k that only finitely many orbits V_1, \dots, V_m are mapped to themselves, i.e., $\psi_k(V_i) = V_i$. On each of these orbits, ψ_k is an equivariant diffeomorphism. Hence, the restriction of ψ_k to V_i is either the identity or does not have fixed points. The latter property is preserved under small perturbations of ψ_k . Therefore, the map $l_g \circ \psi_k$ can only have fixed points in the orbits V_i on which ψ_k restricts to the identity, and the number of fixed points in V_i is $\chi(V_i)$. Let p be one of these fixed points in the orbit V . In order to compute the fixed point index at p we need the derivative of l_g and ψ_k at p . Both derivatives clearly preserve both tangent and normal space to V at p . The derivative of ψ_k is multiplication with k on the normal space and the identity on the tangent space. The derivative of l_g is close to the identity on the normal space and given by a matrix A with $\det(\mathbb{1} - A) > 0$ on the tangent space by the above mentioned result of Hopf. Hence, if B denotes the derivative of $l_g \circ \psi_k$ at p then the fixed point index at p is given by

$$\det(\mathbb{1} - B) = (\text{sgn}(1 - k))^{\text{codim } V} = (-\text{sgn } j)^{\text{codim } V}.$$

The final step is now to sum these fixed point indices. We first consider the case where j is even. Then ψ_k restricts to the identity on both N_0 and N_1 . Since ψ_k has $|k - 1|$ fixed points on $\gamma([0, |W|])$, we have $|k - 1|/|W| - 1 = |j|/2 - 1$ fixed points on $\gamma(]0, 1[)$. These fixed points correspond precisely to the orbits on which ψ_k restricts to the identity. Hence we get

$$\begin{aligned} L(\psi_k) = L(l_g \circ \psi_k) &= (-\text{sgn } j)^{\text{codim } N_0} \chi(N_0) + (-\text{sgn } j)^{\text{codim } N_0} \chi(N_0) \\ &\quad + (|j|/2 - 1)(-\text{sgn } j) \chi(G/H). \end{aligned}$$

Now suppose j is odd. Then ψ_k restricts to the identity on N_0 but not on N_1 . Since ψ_k has $|k - 1|$ fixed points on $\gamma([0, |W|])$, we have $|k - 1|/|W| - 1/2 = (|j| - 1)/2$ fixed points on $\gamma(]0, 1[)$. Hence,

$$L(\psi_k) = L(l_g \circ \psi_k) = (-\text{sgn } j)^{\text{codim } N_0} \chi(N_0) + \frac{|j| - 1}{2} (-\text{sgn } j) \chi(G/H).$$

It is straightforward to simplify these two formulas for $L(\psi_k)$ using the facts that the Euler characteristic of an odd dimensional manifold vanishes, that $\chi(M) = \chi(N_0) + \chi(N_1) - \chi(G/H)$, and that $\chi(G/H) > 0$ implies that $\text{rank } G = \text{rank } H$ and hence that all orbits have even dimension. In the case where the codimensions of both singular orbits are odd one also has to use the fact that $\chi(N_0) = \chi(N_1)$ since both fibrations $G/H \rightarrow N_i$ have fibers that are even dimensional spheres. \square

The degree of ψ_k is the sum of the oriented preimages of a regular value. Since the two non-principal orbits are mapped to themselves or one to the other we just need to consider the regular part M_0 of M . In order to determine the correct orientations we transfer the problem from M_0 to $G/H \times (\mathbb{R} \setminus \mathbb{Z})$ by the map

$$\phi : G/H \times \mathbb{R} \rightarrow M, \quad (gH, t) \mapsto g \cdot \gamma(t).$$

The restriction of ϕ to each $G/H \times]\ell, \ell + 1[$ with $\ell \in \mathbb{Z}$ is a diffeomorphism. We equip \mathbb{R} with the standard orientation and G/H with an orientation such the restriction of $G/H \times]0, 1[\rightarrow M_0$ of ϕ is orientation preserving. This defines a standard product orientation on $G/H \times \mathbb{R}$. For the rest of the paper, however, we will equip each $G/H \times]\ell, \ell + 1[$ with the orientation inherited from M_0 by ϕ .

Lemma 3.2. *If $\text{codim } N_0$ and $\text{codim } N_1$ are both odd then all $G/H \times]\ell, \ell + 1[$ inherit the standard product orientation from M_0 via ϕ .*

If $\text{codim } N_0$ and $\text{codim } N_1$ are both even, only the pieces $G/H \times]2\ell, 2\ell + 1[$ inherit the standard product orientation from M_0 via ϕ .

If $\text{codim } N_0$ is odd and $\text{codim } N_1$ is even then only the pieces $G/H \times]4\ell, 4\ell + 1[$ and $G/H \times]4\ell - 1, 4\ell[$ inherit the standard product orientation from M_0 via ϕ .

If $\text{codim } N_0$ is even and $\text{codim } N_1$ is odd then only the pieces $G/H \times]4\ell, 4\ell + 1[$ and $G/H \times]4\ell + 1, 4\ell + 2[$ inherit the standard product orientation from M_0 via ϕ .

Proof. Let σ_0 denote the geodesic reflection along N_0 , i.e., σ_0 maps each point $g \cdot \gamma(t)$ in $M \setminus N_1$ to $g \cdot \gamma(-t)$. This equivariant diffeomorphism of $M \setminus N_1$ corresponds to $-\text{id}$ in the normal bundle of the orbit N_0 and hence preserves the orientation of $M \setminus N_1$ if and only if $\text{codim } N_1$ is even. We have the equivariant commutative diagram

$$\begin{array}{ccc} G/H \times]0, 1[& \xrightarrow{(gH, t) \mapsto (gH, -t)} & G/H \times]-1, 0[\\ \phi \downarrow & & \phi \downarrow \\ M_0 & \xrightarrow{\sigma_0} & M_0 \end{array}$$

Note that the map $]0, 1[\rightarrow]-1, 0[, t \mapsto -t$ reverses the orientation of \mathbb{R} . Hence, $G/H \times]-1, 0[$ inherits the same orientation from M_0 as $G/H \times]0, 1[$ if and only if σ_0 reverses orientation, i.e., if and only if $\text{codim } N_0$ is odd. \square

Corollary 3.3. *If M is orientable and $\text{codim } N_0$ and $\text{codim } N_1$ do not have the same parity, then the order $|W|$ of the Weyl group W of M is divisible by 4.*

Theorem 3.4. *The map $\psi_k : M \rightarrow M$, $k = j|W|/2 + 1$ has degree*

$$\deg \psi_k = \begin{cases} k & \text{if } \text{codim } N_0 \text{ and } \text{codim } N_1 \text{ are both odd,} \\ +1 & \text{otherwise,} \end{cases}$$

if j is even, and degree

$$\deg \psi_k = \begin{cases} k & \text{if } \text{codim } N_0 \text{ and } \text{codim } N_1 \text{ are both odd,} \\ 0 & \text{if } \text{codim } N_0 \text{ and } \text{codim } N_1 \text{ are both even, } |W| \notin 4\mathbb{Z}, \\ -1 & \text{if } \text{codim } N_0 \text{ is even, } \text{codim } N_1 \text{ is odd, and } |W| \notin 8\mathbb{Z}, \\ +1 & \text{otherwise,} \end{cases}$$

if j is odd provided that ψ_k is well-defined in this case (see Lemma 2.1).

Proof. The point $\gamma(\tau)$ with $\tau = 1/2$ is a regular value of the map ψ_k whose $|k|$ preimages are given by $\gamma(t_m)$ with

$$t_m = \frac{m|W| + \tau}{k}, \quad m \in \mathbb{Z}.$$

In order to compute whether the differential of ψ_k at $\gamma(t_m)$ preserves or reverses orientation we lift $\psi_k : M_0 \rightarrow M_0$ to the map

$$\tilde{\psi}_k : G/H \times \mathbb{R} \rightarrow G/H \times \mathbb{R}, \quad (gH, t) \rightarrow (gH, kt)$$

and answer the same question for the differential of $\tilde{\psi}_j$ at (eH, t_m) by applying Lemma 3.2 (note again that we use the orientation on $G/H \times (\mathbb{R} \setminus \mathbb{Z})$ induced from M_0 by ϕ). If $\text{codim } N_0$ and $\text{codim } N_1$ are both odd then all pieces $G/H \times]\ell, \ell + 1[$ inherit the standard product orientation. Hence, $\deg \psi_k = k$ and we are done.

Thus assume that $\text{codim } N_0$ or $\text{codim } N_1$ are even. For each preimage p of $\gamma(\tau)$ there is precisely one $m \in \mathbb{Z}$ such that $p = \gamma(t_m)$ and $t_m \in]0, |W|[$. We now have to count how many t_m are contained in each of the intervals $]\ell, \ell + 1[$ for $\ell \in \{0, 1, \dots, |W| - 1\}$. A simple computation shows that $t_m \in]\ell, \ell + 1[$ is equivalent to

$$\frac{j}{2}\ell + \frac{\ell - \tau}{|W|} < m < \frac{j}{2}(\ell + 1) + \frac{\ell + 1 - \tau}{|W|}$$

if $j \geq 0$, and equivalent to

$$\frac{j}{2}(\ell + 1) + \frac{\ell + 1 - \tau}{|W|} < m < \frac{j}{2}\ell + \frac{\ell - \tau}{|W|}$$

if $j < 0$. Suppose first that j is even. For $\ell > 0$ we have $\frac{\ell - \tau}{|W|} \in]0, 1[$ and $\frac{\ell + 1 - \tau}{|W|} \in]0, 1[$. Hence there are precisely $|j|$ of the t_m in each interval $]\ell, \ell + 1[$ and $|j + 1|/2$ in the interval $]0, 1[$. When counted with orientation using Lemma 3.2 the sum of preimages of $\gamma(\tau)$ is hence $+1$ since there are $|W|$ intervals $]\ell, \ell + 1[$ and $|W|$ is divisible by 2 if the codimensions of N_0 and N_1 are both even and $|W|$ is divisible by 4 if the codimensions of N_0 and N_1 have different parities. Note that in the case $j \geq 0$ there is one point more in the interval $]0, 1[$ than in the other intervals and this point has positive orientation, while in the case $j < 0$ there is one point less but the orientation of the real line has changed. Suppose now that j is odd. Then $\frac{\ell - \tau}{|W|} \in]0, \frac{1}{2}[$ if and only if $0 < \ell \leq |W|/2$ and $\frac{\ell + 1 - \tau}{|W|} \in]0, \frac{1}{2}[$ if and only if $0 \leq \ell < |W|/2$. Using these facts it is straightforward to show that there are $|j + 1|/2$ of the t_m in each of the intervals $]\ell, \ell + 1[$ with $\ell = 0$, $\ell = |W|/2$, odd $\ell < |W|/2$, or even $\ell > |W|/2$, while there are $|j - 1|/2$ of the t_m in each of the intervals $]\ell, \ell + 1[$ with even $0 < \ell < |W|/2$ or odd $\ell > |W|/2$. The claimed degrees follow now by applying Lemma 3.2. \square

Proof of Corollary 2. Suppose that M has the rational cohomology ring of

$$\mathbb{S}^{l_1} \times \mathbb{S}^{l_2} \times \cdots \times \mathbb{S}^{l_m}, \quad l_1 < l_2 < \cdots < l_m,$$

and that $G \times M \rightarrow M$ is a cohomogeneity one action with $\text{rank } G = \text{rank } H$, i.e., $\chi(G/H) > 0$ where H is a principal isotropy group. By Lemma 2.1, M admits equivariant selfmaps ψ_k , $k = j|W|/2 + 1$, with degree k and Lefschetz number $L(\psi_k) = -j\chi(G/H)$ for every even integer j . By Dirichlet's theorem there are infinitely many prime numbers in the arithmetic progression $k = j|W|/2 + 1$. Hence, we can assume that k is a prime number. We now inspect the induced map ψ_k^* on the cohomology ring. Let x_1, \dots, x_m be generators of $H^*(M)$ with $x_i \in H^{l_i}(M)$. The equation $\deg \psi_k = k$ means $\psi_k^*(x_1 x_2 \cdots x_m) = k x_1 x_2 \cdots x_m$. Since k is a prime number and $l_1 < l_2 < \cdots < l_m$ it is clear that

$$\psi_k^*(x_j) = k\sigma_j x_j + \text{products of lower dimensional generators}$$

with $\sigma_j = \pm 1$ for precisely one $j \in \{1, \dots, m\}$ and

$$\psi_k^*(x_i) = \sigma_i x_i + \text{products of lower dimensional generators}$$

with $\sigma_i = \pm 1$ for $i \in \{1, \dots, m\}$, $i \neq j$. The total number of σ_i with $\sigma_i = -1$ for $i \in \{1, \dots, m\}$ must be even. The Lefschetz number $L(\psi_k)$ is equal to $-j\chi(G/H)/2$ by Theorem 3.1. On the other hand, $L(\psi_k)$ is the alternating trace of ψ_k^* in cohomology. The simple cohomology ring structure implies

$$-j\chi(G/H)/2 = L(\psi_k) = (1 + (-1)^{l_j} k \sigma_j) \prod_{i \neq j} (1 + (-1)^{l_i} \sigma_i).$$

Since we can assume $j > 2$ and since $\chi(G/H) > 0$, we have $\sigma_i = (-1)^{l_i}$ for all $i \neq j$ and $\sigma_j = -(-1)^{l_j}$. Hence, $\chi(G/H) = 2^{m-1}|W|$ and Corollary 2 is proved. \square

4. THE MAIN EXAMPLE $\text{SU}(3)$

The natural selfmaps of compact Lie groups G are the power maps $\rho_k : A \mapsto A^k$. These maps are equivariant with respect to conjugation and hence determined by their restriction to a maximal torus $T^r \subset G$. Using this fact it is easy to see that the degree of ρ_k is k^r and thus rather few integers can be realized as degrees of power maps. For $\text{SU}(3)$ with rank $r = 2$ these are all squares k^2 .

Conjugation on $\text{SU}(3)$ is an action of cohomogeneity two. We use the cohomogeneity one action

$$\text{SU}(3) \times \text{SU}(3) \rightarrow \text{SU}(3), \quad (A, B) \mapsto ABA^T$$

to construct polynomial selfmaps of $\text{SU}(3)$ whose degrees cover all odd integers and that extend identity and transposition to an infinite family. Polynomial here means that the entries of the target matrix are real polynomials in the complex entries of the argument matrix.

Lemma 4.1. *$\text{SU}(3)$ admits selfmaps ψ_k with degree k for all odd integers k . These maps are equivariant with respect to the cohomogeneity one action above.*

Proof. A straightforward computation shows that

$$\gamma(t) = \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a geodesic that passes perpendicularly through all orbits. This geodesic is closed with period 2π . At times $t = 0$ and $t = \pi$ it passes through the singular orbit $SU(3)/SO(3)$ that consists of the symmetric matrices in $SU(3)$. At $t = \frac{\pi}{2} + \pi\mathbb{Z}$ it passes through the other singular orbit $SU(3)/SU(2)$ where $SU(2)$ is embedded in the upper left corner of $SU(3)$. The isotropy group of $\gamma(t)$ for all other times t is $SO(2)$ embedded in the upper left corner of $SU(3)$ in the standard way. Note that the singular isotropy groups at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$ are not just conjugate but equal. The claim follows from Lemma 2.1 and Theorem 3.4. \square

We now derive an explicit formula for these maps. Let

$$\begin{aligned} f_j(x) &= \sum_{i=0}^j (-1)^i \binom{2j+1}{2i} x^i (1-x)^{j-i}, \\ g_j(x) &= \sum_{i=0}^j (-1)^i \binom{2j+1}{2i+1} x^i (1-x)^{j-i}, \\ h_j(x) &= \frac{1-f_j(x)}{x} = \sum_{i=0}^{j-1} (-1)^i \left(\binom{j}{i+1} x^i + \binom{2j+1}{2i+2} x^i (1-x)^{j-1-i} \right). \end{aligned}$$

Theorem 4.2. *The selfmaps $\psi_k : SU(3) \rightarrow SU(3)$ of odd degree $k = 2j + 1$ are given by the following formula:*

$$\psi_{2j+1}(B) = \frac{1}{2} f_{|j|}(x)(B + B^T) \pm \frac{1}{2} g_{|j|}(x)(B - B^T) + \frac{1}{4} h_{|j|}(x) \begin{pmatrix} \bar{b}_{23} - \bar{b}_{32} \\ \bar{b}_{31} - \bar{b}_{13} \\ \bar{b}_{12} - \bar{b}_{21} \end{pmatrix} \begin{pmatrix} \bar{b}_{23} - \bar{b}_{32} \\ \bar{b}_{31} - \bar{b}_{13} \\ \bar{b}_{12} - \bar{b}_{21} \end{pmatrix}^T$$

where $x = 1 - \frac{1}{4}(\text{trace } B - 1)^2$. The $+$ -sign has to be taken if $j \geq 0$ and the $-$ -sign has to be taken if $j < 0$.

Proof. Let $B = (b_{ij})$ be a fixed matrix in $SU(3)$. We want to solve the equation $B = A\gamma(t)A^T$ for $t \in [0, \frac{\pi}{2}]$ and $A \in SU(3)$ then multiply t by an odd integer k and determine $B' = A\gamma(kt)A^T$. Symbolically, we illustrate this as follows

$$B \rightsquigarrow (A, t) \rightsquigarrow (A, kt) \rightsquigarrow B'.$$

We will see that it is not necessary to determine the entries of A completely with all their ambiguity. We stop at an intermediate level and then return.

Let $A = (z, w, v)$ with $z, w, v \in \mathbb{C}^3$. Then $B = A\gamma(t)A^T$ becomes

$$\begin{aligned} b_{ii} &= (z_i^2 + w_i^2) \cos t + v_i^2, \\ b_{i,i+1} &= (z_i z_{i+1} + w_i w_{i+1}) \cos t + v_i v_{i+1} - \bar{v}_{i-1} \sin t, \\ b_{i,i-1} &= (z_i z_{i-1} + w_i w_{i-1}) \cos t + v_i v_{i-1} + \bar{v}_{i+1} \sin t \end{aligned}$$

with indices cyclic modulo 3. It is straightforward to see that

$$\begin{aligned}
\cos t &= \frac{1}{2}(\text{trace } B - 1), \\
2\bar{v}_1 \sin t &= b_{32} - b_{23}, \\
2\bar{v}_2 \sin t &= b_{13} - b_{31}, \\
2\bar{v}_3 \sin t &= b_{21} - b_{12}, \\
(z_i^2 + w_i^2) \cos t + v_i^2 &= b_{ii}, \\
(z_2 z_3 + w_2 w_3) \cos t + v_2 v_3 &= \frac{1}{2}(b_{23} + b_{32}), \\
(z_3 z_1 + w_3 w_1) \cos t + v_3 v_1 &= \frac{1}{2}(b_{31} + b_{13}), \\
(z_1 z_2 + w_1 w_2) \cos t + v_1 v_2 &= \frac{1}{2}(b_{12} + b_{21}).
\end{aligned}$$

Now, applying the binomial formula to $(\cos t + i \sin t)^k$ leads the following formulas for $\cos kt$ and $\sin kt$ with k odd:

$$\cos(2j+1)t = f_j(\sin^2 t) \cos t, \quad \sin(2j+1)t = g_j(\sin^2 t) \sin t.$$

We obtain

$$\begin{aligned}
b'_{11} &= (z_1^2 + w_1^2) \cos(2j+1)t + v_1^2 \\
&= (z_1^2 + w_1^2) f_j(\sin^2 t) \cos t + v_1^2 f_j(\sin^2 t) + v_1^2 h_j(\sin^2 t) \sin^2 t \\
&= b_{11} f_j(\sin^2 t) + \frac{1}{4}(\bar{b}_{23} - \bar{b}_{32})^2 h_j(\sin^2 t)
\end{aligned}$$

and analogous expressions for b_{22} and b_{33} . Note that these are polynomials in the matrix entries of B since $\sin^2 t = 1 - \frac{1}{4}(\text{trace } B - 1)^2$ is a polynomial in $\text{trace } B$ and, e. g., $\bar{b}_{23} = b_{31}b_{12} - b_{11}b_{32}$, since $B \in \text{SU}(3)$. Similarly, we obtain polynomial expressions for the off-diagonal terms of $B' = \psi_{2j+1}(B)$. All in all this leads to the claimed formula. \square

Corollary 4.3. *For every $m \in \mathbb{N}$ and every odd integer $(2\ell+1)$ there is a polynomial selfmap $\text{SU}(3) \rightarrow \text{SU}(3)$ of degree $4^m(2\ell+1)$.*

Proof. The maps ψ_k are defined for odd k and have degree k . The power maps $\rho_k : A \mapsto A^k$ have degree k^2 . \square

Lemma 4.4. *Let $\phi : \text{SU}(3) \rightarrow \text{SU}(3)$ be any map. Then $\deg \phi = 4^m \cdot (\text{odd number})$ for some $m \in \mathbb{N}$.*

Proof. The cohomology ring $H^*(\text{SU}(3); \mathbb{Z})$ is isomorphic to the cohomology ring $H^*(\mathbb{S}^3 \times \mathbb{S}^5; \mathbb{Z})$, i.e., to $\Lambda(x, y)$ with generators x in dimension 3 and y in dimension 5. If $\phi : \text{SU}(3) \rightarrow \text{SU}(3)$ is any map, we have the commutative diagram

$$\begin{array}{ccc}
H^3(\text{SU}(3); \mathbb{Z}_2) & \xrightarrow{\phi^*} & H^3(\text{SU}(3); \mathbb{Z}_2) \\
\text{Sq}^2 \downarrow & & \text{Sq}^2 \downarrow \\
H^5(\text{SU}(3); \mathbb{Z}_2) & \xrightarrow{\phi^*} & H^5(\text{SU}(3); \mathbb{Z}_2)
\end{array}$$

Vertically, the second Steenrod square Sq^2 yields an isomorphism (see [Bd2], p. 469). The diagram thus says that $\phi^*(y)$ is an even multiple of y if and only if $\phi^*(x)$ is an even multiple of x . Hence, $\phi^*(xy)$ is in $4^m(2\ell+1) \cdot xy$ for some $m \in \mathbb{N}$ and $\ell \in \mathbb{Z}$. \square

Lemma 4.5. *We have $\rho_k^*(x) = kx$ and $\rho_k^*(y) = ky$ while $\psi_k^*(x) = kx$ and $\psi_k^*(y) = y$. Hence, none of the ρ_k is homotopic to any of the ψ_k .*

Proof. For ρ_k we refer to [Ho]. It is easy to see that the subgroup $SU(2)$ embedded in the upper left corner of $SU(3)$ is invariant under ψ_k and ψ_k restricts to a selfmap of $SU(2)$ with degree k . The commutative diagram

$$\begin{array}{ccc} H^3(SU(3); \mathbb{Z}) & \xrightarrow{\psi_k} & H^3(SU(3); \mathbb{Z}) \\ \downarrow & & \downarrow \\ H^3(SU(2); \mathbb{Z}) & \xrightarrow{\times k} & H^3(SU(2); \mathbb{Z}) \end{array}$$

shows $\psi_k^*(x) = kx$. Since $\deg \psi_k = k$, the generator xy of $H^8(SU(3); \mathbb{Z})$ is mapped to kxy . Hence, $\psi_k^*(y) = y$. \square

5. FURTHER EXAMPLES

In this section we supply examples for the various cases in Theorem 3.1 and Theorem 3.4. Classical examples of selfmaps are the k -powers of $\mathbb{S}^n \subset \mathbb{R} \times \mathbb{R}^n$,

$$(\cos t, v \sin t) \mapsto (\cos kt, v \sin kt).$$

These well-defined analytic selfmaps of \mathbb{S}^n are equivariant with respect to the standard $O(n)$ -action on \mathbb{R}^n . If n is odd we have $\text{rank } O(n) = \text{rank } O(n-1)$. From Theorem 3.4 we recover the known fact that the degree of the k -power of \mathbb{S}^n is k if n is odd, and 0 or 1 if n is even, depending on whether k is even or odd. From Theorem 3.1 we recover the Lefschetz number $(1-k)$ if n is odd, and 1 or 2 if n is even, depending on whether k is even or odd.

An example where the codimensions of the singular orbits have different parity and there are equivariant maps with degree -1 is given by the standard action of $SO(1+m) \subset SU(1+m)$ on $\mathbb{C}\mathbb{P}^m$ for odd m . A normal geodesic is given by

$$\gamma(t) = [\cos t : i \sin t : 0 : \dots : 0].$$

The singular orbit at $t = 0$ is $\mathbb{R}\mathbb{P}^m = SO(1+m)/O(m)$ and the singular isotropy groups at $t = \pi/4$ and $t = -\pi/4$ are both equal to $SO(2)SO(m-1)$. The normal geodesic γ is closed with period π . Hence, the Weyl group is the dihedral group of order 4 and $(2j+1)$ -folding the distance to $N_0 = \mathbb{R}\mathbb{P}^m$ yields a well-defined map with degree 1 and Lefschetz number $m+1$ if j is even and degree -1 and Lefschetz number 0 if j is odd. Note that for even m selfmaps of $\mathbb{C}\mathbb{P}^m$ with degree -1 cannot exist and this is perfectly matched by the fact that both singular orbits have even codimensions in this case.

We finally discuss the spaces $M_1^7 = Sp(2) \times_{Sp(1)^2} Sp(1)$ and $M_2^7 = SU(3) \times_{U(2)} SU(2)$ where $U(2)$ acts on $SU(2)$ by conjugation and one of the factors of $Sp(1)^2$ acts trivially on $Sp(1)$ and the other by conjugation. The space M_1^7 is an \mathbb{S}^3 -bundle over \mathbb{S}^4 . Such bundles are classified by their characteristic homomorphism $\mathbb{S}^3 \rightarrow SO(4)$. Using two Cartan embeddings of \mathbb{S}^4 into $Sp(2)$ it is not difficult to see that for M_1^7 this homomorphism is given by $q \mapsto C_q$ where q is a unit quaternion and $C_q : \mathbb{H} \rightarrow \mathbb{H}$ is conjugation by q . In particular, the bundle $M_1^7 \rightarrow \mathbb{S}^4$ has a section but is nontrivial. It is known that M_1^7 is not homotopy equivalent to $\mathbb{S}^3 \times \mathbb{S}^4$ but has the same cohomology and homotopy groups as $\mathbb{S}^3 \times \mathbb{S}^4$ [JW]. The group $Sp(2)$ acts from the left on M_1^7 by cohomogeneity one. The principal orbits are diffeomorphic to $\mathbb{C}\mathbb{P}^3 = Sp(2)/Sp(1)SO(2)$ with Euler characteristic 4 and the singular orbits are both diffeomorphic to \mathbb{S}^4 . We have $\text{rank } G = \text{rank } H$. The Weyl

group is isomorphic to \mathbb{Z}_2 . Hence, by Lemma 2.1 there exist equivariant selfmaps ψ_k with degree $\psi_k = k$ and Lefschetz number $L(\psi_k) = 2(1 - k)$ for all integers k . This example thus illustrates Corollary 2, since $4 = \chi(G/H) = 2 \cdot |W|$.

The space M_2^7 is an \mathbb{S}^3 -bundle over $\mathbb{C}\mathbb{P}^2$. The group $\mathrm{SU}(3)$ acts from the left on M_2^7 by cohomogeneity one. The principal orbits are diffeomorphic to $\mathrm{SU}(3)/\mathrm{T}^2$ with Euler characteristic 6 and the singular orbits are both diffeomorphic to $\mathbb{C}\mathbb{P}^2$. Since the bundle $M_2^7 \rightarrow \mathbb{C}\mathbb{P}^2$ has a section, M_2^7 has the same cohomology and homotopy groups as $\mathbb{S}^3 \times \mathbb{C}\mathbb{P}^2$. Moreover, the corresponding principal $\mathrm{SO}(3)$ -bundle over $\mathbb{C}\mathbb{P}^2$ is the Aloff-Wallach space $W_{1,1}^7$. The first Ponrjagin class of this principal bundle is $-3 \in H^4(\mathbb{C}\mathbb{P}^2) \approx \mathbb{Z}[\mathrm{Zi}]$. Hence, the \mathbb{S}^3 -bundle $M_2^7 \rightarrow \mathbb{C}\mathbb{P}^2$ has $w_2 = 0$ and $p_1 = -3$. By the Dold-Whitney classification [DoWh] it is nontrivial. The Weyl group of the cohomogeneity one $\mathrm{SU}(3)$ action is isomorphic to \mathbb{Z}_2 . Hence, there exist equivariant selfmaps ψ_k with degree $\psi_k = k$ and Lefschetz number $L(\psi_k) = 3(1 - k)$ for all integers k in agreement with the homology of M_2^7 .

REFERENCES

- [AA] A. V. Alekseevsky, D. V. Alekseevsky, *Riemannian G-manifolds with one-dimensional orbit space*, Ann. Global Anal. Geom. **11** (1993), 197–211.
- [Bd1] G. E. Bredon, *Introduction to compact transformation groups*, Pure and Applied Mathematics, Vol. 46, Academic Press, New York, 1972.
- [Bd2] G. E. Bredon, *Topology and Geometry*, GTM 139, Springer, New York 1993.
- [DoWh] A. Dold, H. Whitney, *Classification of oriented sphere bundles over a 4-complex*, Ann. Math. **69** (1959), 667–677.
- [DuWa] H. Duan, S. Wang, *The degree of maps between manifolds*, Math. Z. **244** (2003), 67–89.
- [GWZ] K. Grove, B. Wilking, W. Ziller, *Positively curved cohomogeneity one manifolds and 3-Sasakian geometry*, J. Different. Geom., to appear.
- [Hoe] C. Hoelscher, *Classification of cohomogeneity one manifolds in low dimensions*, Ph. D. thesis, University of Pennsylvania, Philadelphia 2007.
- [Ho] H. Hopf, *Über den Rang geschlossener Liescher Gruppen*, Comment. Math. Helv. **13** (1941), 119–143.
- [HS] H. Hopf, H. Samelson, *Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen*, Comment. Math. Helv. **13** (1941), 240–251.
- [JW] I. M. James, J. H. C. Whitehead, *The homotopy theory of sphere bundles over spheres. I.*, Proc. Lond. Math. Soc. **4** (1954), 196–218.
- [Mo] P. S. Mostert, *On a compact Lie group acting on a manifold*, Ann. of Math. **65** (1957), 447–455; Errata *ibid.* **66**, 589.
- [Neu] W. D. Neumann, “3-dimensional G -manifolds with 2-dimensional orbits”, pp. 220–222 in *Proc. Conf. on Transformation Groups* (New Orleans, La., 1967), Springer, New York, 1968
- [Pa] J. Parker, *4-dimensional G -manifolds with 3-dimensional orbits*, Pac. J. Math. **125** (1986), 187–204.
- [Zi] W. Ziller, *Fatness revisited*, unpublished lecture notes, University of Pennsylvania, 2000.

UNIVERSITÄT BONN, MATHEMATISCHES INSTITUT, BERINGSTR. 1, 53115 BONN, GERMANY
E-mail address: puttmann@math.uni-bonn.de