Product and anti-Hermitian structures on the tangent space

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Abstract

Noting that the complete lift of a Rimannian metric g defined on a differentiable manifold M is not 0-homogeneous on the fibers of the tangent bundle TM. In this paper we introduce a new lift $\tilde{g}_2$ which is 0-homogeneous. It determines on slit tangent bundle a pseudo-Riemannian metric, which depends only on the metric g. We study some of the geometrical properties of this pseudo-Riemannian space and define the natural almost complex structure and natural almost product structure which preserve the property of homogeneity and find some new results.

Keywords: Almost complex structure, almost anti-Hermitian structure, almost product structure, complete lift metric, 0-homogeneous lift.

1. Introduction.

The importance of the complete lift g_2 of a Riemannian metric g is well known in Riemannian geometry, Finsler geometry and Physics, and has many applications in Biology too (see [1]). The tensor field g_2 determines a pseudo-Riemannian structure on slit tangent bundle $\widetilde{TM} = TM \setminus \{0\}$, but g_2 is not 0-homogeneous on the fibers of the tangent bundle TM. Therefore, we cannot study some global properties of the pseudo-Riemannian space $(\widetilde{TM}, g_2)$. For instance we can not prove a theorem of Gauss-Bonnet type for this space (see [4]).

In this paper, we define a new kind of lift $\tilde{g}_2$ to TM of the Riemannian metric g. Thus $\tilde{g}_2$ determines on $\widetilde{TM}$ a pseudo-Riemannian structure, which is 0-homogeneous on the fibers of TM and depends only on g. Some geometrical properties of $\tilde{g}_2$ such as the Levi-Civita connection are studied.

Almost complex and almost product structures are among the most important geometrical structures which can be considered on a manifold. Geometric properties of this structures have been studied in (see [2] to [7], [11], [12], [15], [16]). We introduce the natural almost complex and product structures $\widetilde{J}$ and $\widetilde{Q}$ which depend only on g and preserve the property of homogeneity. Then we get almost anti-Hermitian structure $(\tilde{g}_2, \widetilde{J})$ and almost product structure $(\tilde{g}_2, \widetilde{Q})$. By considering twin tensor of $\tilde{g}_2$, we construct almost para-Hermitian and Hermitian structures on $\widetilde{TM}$.

Let M be a smooth manifold, TM its tangent bundle and $\chi(M)$ the algebra of vector fields on M. A K-structure on M is a fields of endomorphisms K on TM such that $K^2 = \varepsilon I$, where $\varepsilon = \pm 1$. Thus $\varepsilon = 1$ corresponds to an *almost product structure*, while $\varepsilon = -1$ provides an *almost complex structure*.

A K-structure is *integrable* if and only if there exists an a linear torsionless connection on M such that $\nabla K = 0$, or equivalently the Nijenhuis tensor N_K vanishes. In this case, ∇ is called *almost complex (product) connection* if K be an almost complex (product) structure.

If g is a metric on M such that $g(KX, KY) = \sigma g(X, Y)$, $\sigma = \pm 1$, for arbitrary vector fields X and Y on M, then we shall say that the metric g is K-metric.

The definition above unifies the following four cases:

The case $\varepsilon = 1, \sigma = 1$ corresponds to the (pseudo-) Riemannian *almost product manifold* (M, g, K), the case $\varepsilon = 1, \sigma = -1$ provides the *almost para-Hermitian manifold* (M, g, K), the case $\varepsilon = -1, \sigma = 1$ is known as the *almost Hermitian manifold* (M, g, K), and finally the case $\varepsilon = -1, \sigma = -1$ corresponds to the *almost anti-Hermitian* manifold (M, g, K).

Let us introduce a $(0,2)$ tensor field h, the twin of g, by $h(X,Y) = g(KX, Y)$.

Then

$$h(X,Y) = \varepsilon\sigma h(Y,X), h(KX,KY) = \sigma h(X,Y).$$

Notice that for $\varepsilon\sigma = 1$, the twin tensor is a metric, while for $\varepsilon\sigma = -1$ the twin tensor is a 2-form.

Let ψ be a (0,3) tensor fields defined by the formula

$$\psi(X,Y,Z) = g((\nabla_X K)Y, Z) \equiv (\nabla_X h)(Y,Z) \tag{1.1}$$

Obviously, if the tensor fields ψ vanishes then $\nabla K = 0$ for a torsionless (Levi-Civita) connection and the Nijenhuis tensor N_K is forced to vanish, too ([2]).

2. The Complete Lift

Let $\Gamma_{ij}{}^{k}$ be the coefficients of the Riemannian connection of M, then $N_j{}^h = \Gamma_{0j}{}^h = y^a \Gamma_{aj}{}^h(x)$ can be regarded as coefficients of the canonical nonlinear connection N of TM, where (x^h, y^h) are the induced coordinates in TM.

N determines a horizontal distribution on $\widetilde{TM}$, which is supplementary to the vertical distribution V, such that, we have:

$$T_u\widetilde{TM} = N_u \oplus V_u, \quad \forall u \in \widetilde{TM}. \tag{2.1}$$

The adapted basis to N and V is given by $\{X_h, X_{\bar{h}}\}$ where

$$X_h = \frac{\partial}{\partial x^h} - y^a \Gamma_{ah}{}^m \frac{\partial}{\partial y^m}, \qquad X_{\bar{h}} = \frac{\partial}{\partial y^h} \tag{2.2}$$

and its dual basis is $\{dx^i, \delta y^i\}$ where

$$\delta y^i = dy^i + y^a \Gamma_{aj}{}^i dx^j. \tag{2.3}$$

The indices $a, b, ..., \bar{a}, \bar{b}, ...,$ run over the range $\{1, 2, ..., n\}$. The summation convention will be used in relation to this system of indices. By straightforward calculations, we have the following lemma.

Lemma 1. The Lie bracket of the adapted frame of TM satisfies the following:

1) $[X_i, X_j] = y^a K_{jia}{}^m X_{\bar{m}}$,

2) $[X_i, X_{\bar{j}}] = \Gamma_{ji}{}^m$,

3) $[X_{\bar{i}}, X_{\bar{j}}] = 0$,

where $K_{jia}{}^{m}$ denote the components of the curvature tensor of M.
Let (M,g) be a Riemannian space, M being a real n-dimensional manifold and (TM,π,M) its tangent bundle. On a domain $U\subset M$ of a local chart, g has the components $g_{ij}(x)$, $(i,j,...=1,...,n)$. Then on the domain of chart $\pi^{-1}(U)\subset TM$ we consider the functions $g_{ij}(x,y)=g_{ij}(x),\forall(x,y)\in\pi^{-1}(U)$ and put

$$\| y \|=\sqrt{g_{ij}(x)y^{i}y^{j}}. \tag{2.4}$$

Then, $\| y\|$ is globally defined on TM, differentiable on $\widetilde{TM}$ and continuous on the null section.

The complete lift of g to TM is defined by

$$g_{2}(x,y)=2g_{ij}(x)dx^{i}\delta y^{j},\ \forall(x,y)\in\widetilde{TM}. \tag{2.5}$$

Then, g_2 is not 0-homogeneous on the fibers of TM.
Namely, for the homothety $h_t:(x,y)\rightarrow(x,ty)$ for all $t\in R^{+}$ we get

$$(g_2\circ h_t)(x,y)=2tg_{ij}(x)dx^{i}\delta y^{j}=tg_2(x,y)\neq g_2(x,y).$$

On $\widetilde{TM}$ we define an almost complex structure J by

$$J(X_i)=-X_{\bar{i}},\quad J(X_{\bar{i}})=X_i,\quad i=1,...,n. \tag{2.6}$$

It is known that $(\widetilde{TM},J,g_2)$ is an almost anti-Hermitian manifold. Moreover, the integrability of the almost complex structure J implies that (M,g) is locally flat. (see [7])
Also, we define almost product structure Q on $\widetilde{TM}$ by

$$Q(X_i)=X_{\bar{i}},\quad Q(X_{\bar{i}})=X_i,\quad i=1,...,n. \tag{2.7}$$

Then, $(\widetilde{TM},Q,g_2)$ is an almost product manifold. Also, the integrability of the almost product structure Q implies that (M,g) is locally flat.
The previous space, called "the geometrical model on TM of the Riemannian space (M,g)", is important in the study of the geometry of initial Riamannian space (M,g) ([6], [7]).

3. The 0-homogeneous lift of the Riemannian metric g

We can eliminate the inconvenience of the complete lift, introducing a new kind of lift to TM of the Riemannian metric g. Then we obtain the Levi-Civita connection for this metric.
Definition . Let $\tilde{g}_2$ be a the tensor field on $\widetilde{TM}$ defined by

$$\tilde{g}_2(x,y)=\frac{2}{\| y\|}g_{ij}(x)dx^{i}\delta y^{i} \tag{3.1}$$

where $\| y\|$ was defined in (2.4).Then $\tilde{g}_2$ is called the 0-homogeneous lift of the Riemannian metric g to $\widetilde{TM}$.
We get, evidently:
Theorem 2. *The following properties hold:*
1. *The pair* $(\widetilde{TM},\tilde{g}_2)$ *is a pseudo-Riemannian space, depending only on the metric* g.

2. *$\tilde{g}_2$ is 0-homogeneous on the fibers of the tangent bundle TM.*

In order to study the geometry of the pseudo-Riemannian space $(\widetilde{TM}, \tilde{g}_2)$ we can apply the theory of the (h,v)-Riemannian metric on TM given in the books [6], [7] and [9]. Looking at the relation (2.5) and (3.1) we can assert:

Theorem 3. *The lifts g_2 and $\tilde{g}_2$ coincide on the hyper unit tangent sphere $g_{ij}(x_0)y^i y^j = 1$, for every point $x_0 \in M$.*

Let $\overline{\nabla}$ be the Riemannian connection of TM with coefficient $\overline{\Gamma}_{BC}{}^A$, that is:

$$\begin{aligned}
&\overline{\nabla}_{X_i} X_j = \overline{\Gamma}_{ji}{}^m X_m + \overline{\Gamma}_{ji}{}^{\bar{m}} X_{\bar{m}}, \qquad \overline{\nabla}_{X_i} X_{\bar{j}} = \overline{\Gamma}_{\bar{j}i}{}^m X_m + \overline{\Gamma}_{\bar{j}i}{}^{\bar{m}} X_{\bar{m}}, \\
&\overline{\nabla}_{X_{\bar{i}}} X_j = \overline{\Gamma}_{j\bar{i}}{}^m X_m + \overline{\Gamma}_{j\bar{i}}{}^{\bar{m}} X_{\bar{m}}, \qquad \overline{\nabla}_{X_{\bar{i}}} X_{\bar{j}} = \overline{\Gamma}_{\bar{j}\bar{i}}{}^m X_m + \overline{\Gamma}_{\bar{j}\bar{i}}{}^{\bar{m}} X_{\bar{m}}
\end{aligned} \tag{3.2}$$

Then, we have

$$\begin{aligned}
&\overline{\nabla}_{X_i} dx^h = -\overline{\Gamma}_{mi}{}^h dx^m - \overline{\Gamma}_{\bar{m}i}{}^h \delta y^m, \\
&\overline{\nabla}_{X_i} \delta y^h = -\overline{\Gamma}_{mi}{}^{\bar{h}} dx^m - \overline{\Gamma}_{\bar{m}i}{}^{\bar{h}} \delta y^m, \\
&\overline{\nabla}_{X_{\bar{i}}} dx^h = -\overline{\Gamma}_{m\bar{i}}{}^h dx^m - \overline{\Gamma}_{\bar{m}\bar{i}}{}^h \delta y^m, \\
&\overline{\nabla}_{X_{\bar{i}}} \delta y^h = -\overline{\Gamma}_{m\bar{i}}{}^{\bar{h}} dx^m - \overline{\Gamma}_{\bar{m}\bar{i}}{}^{\bar{h}} \delta y^m,
\end{aligned} \tag{3.3}$$

Since the torsion tensor $T(X,Y)$ of $\overline{\nabla}$ defined by $T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y]$ vanishes, we have the following relations by means of Lemma 1 and (3.2).

$$\begin{array}{ll}
(1)\ \overline{\Gamma}_{ji}{}^h = \overline{\Gamma}_{ij}{}^h & (2)\ \overline{\Gamma}_{ji}^{\bar{h}} = \overline{\Gamma}_{ij}^{\bar{h}} + y^a K_{jia}^h \\
(3)\ \overline{\Gamma}_{\bar{j}i}{}^h = \overline{\Gamma}_{i\bar{j}}{}^h & (4)\ \overline{\Gamma}_{\bar{j}i}{}^{\bar{h}} = \overline{\Gamma}_{i\bar{j}}{}^{\bar{h}} + \Gamma_{ji}{}^h \\
(5)\ \overline{\Gamma}_{\bar{j}\bar{i}}{}^h = \overline{\Gamma}_{\bar{i}\bar{j}}{}^h & (6)\ \overline{\Gamma}_{\bar{j}\bar{i}}{}^{\bar{h}} = \overline{\Gamma}_{\bar{i}\bar{j}}{}^{\bar{h}}
\end{array} \tag{3.4}$$

Furthermore, we have the following lemma.

Lemma 4. *The connection coefficients $\overline{\Gamma}_{BC}{}^A$ of $\overline{\nabla}$ of the complete metric $\tilde{g}_2$ satisfy the following relations:*

$$\begin{array}{ll}
(1)\ \overline{\Gamma}_{ji}{}^h = \Gamma_{ji}{}^h & (2)\ \overline{\Gamma}_{ji}{}^{\bar{h}} = y^a K_{aij}{}^h \\
(3)\ \overline{\Gamma}_{\bar{j}i}{}^h = \dfrac{1}{2\|y\|^2}(g_{ij}y^h - \delta_i^h y_j) & (4)\ \overline{\Gamma}_{j\bar{i}}{}^h = \dfrac{1}{2\|y\|^2}(g_{ij}y^h - \delta_j^h y_i) \\
(5)\ \overline{\Gamma}_{\bar{j}i}{}^{\bar{h}} = \Gamma_{ji}{}^h & (6)\ \overline{\Gamma}_{j\bar{i}}{}^{\bar{h}} = 0 \\
(7)\ \overline{\Gamma}_{\bar{j}\bar{i}}{}^h = 0 & (8)\ \overline{\Gamma}_{\bar{j}\bar{i}}{}^{\bar{h}} = -\dfrac{1}{2\|y\|^2}(\delta_i^h y_j + \delta_j^h y_i)
\end{array}$$

Proof. The condition compatibility $\overline{\nabla}$ is equivalent with following equations:

$$g_{ir}\overline{\Gamma}_{jm}{}^{\bar{r}} + g_{jr}\overline{\Gamma}_{im}{}^{\bar{r}} = 0 \tag{3.5}$$

$$g_{ir}(\overline{\Gamma}_{jm}{}^r - \overline{\Gamma}_{\bar{j}m}{}^{\bar{r}}) + g_{jr}(\Gamma_{im}{}^r - \overline{\Gamma}_{im}{}^r) = 0 \tag{3.6}$$

$$g_{ir}\overline{\Gamma}_{\bar{j}m}{}^r + g_{jr}\overline{\Gamma}_{\bar{i}m}{}^r = 0 \tag{3.7}$$

$$g_{ir}\overline{\Gamma}_{j\bar{m}}{}^{\bar{r}} + g_{jr}\overline{\Gamma}_{i\bar{m}}{}^{\bar{r}} = 0 \tag{3.8}$$

$$g_{ir}\overline{\Gamma}_{\bar{j}\bar{m}}{}^{\bar{r}} + g_{jr}\overline{\Gamma}_{i\bar{m}}{}^{r} + \frac{1}{\|y\|^2} g_{ij} y_m = 0 \tag{3.9}$$

$$g_{ir}\overline{\Gamma}_{\bar{j}\bar{m}}{}^{r} + g_{jr}\overline{\Gamma}_{\bar{i}\bar{m}}{}^{r} = 0 \tag{3.10}$$

From (3.10) we have $\overline{\Gamma}_{\bar{j}\bar{i}}{}^{h} = 0$, thus we get (7). From (3.4), (3.9) and (3.7), we have

$$g_{ir}\overline{\Gamma}_{\bar{j}m}{}^{r} = -g_{ir}\overline{\Gamma}^{r}_{\bar{i}m} = -g_{ir}\overline{\Gamma}^{r}_{m\bar{i}} = g_{mr}\overline{\Gamma}^{\bar{r}}_{\bar{i}\bar{j}} + \frac{g_{mj}y_i}{\|y\|^2}$$

$$= -g_{ir}\overline{\Gamma}^{r}_{m\bar{j}} - \frac{g_{im}y_j}{\|y\|^2} + \frac{g_{mj}y_i}{\|y\|^2}$$

$$= -g_{ir}\overline{\Gamma}^{r}_{\bar{j}m} - \frac{g_{im}y_j}{\|y\|^2} + \frac{g_{mj}y_i}{\|y\|^2}$$

Thus we get (3). From (3) and (3.4), we have (4).
From (3.9), (4) and (3.4), we have

$$g_{ir}\overline{\Gamma}_{\bar{j}\bar{m}}{}^{\bar{r}} + \frac{1}{2\|y\|^2} g_{im}y_j - \frac{1}{2\|y\|^2} g_{ji}y_m + \frac{g_{ij}y_m}{\|y\|^2} = 0,$$

then we obtain (8).
From (3.4) and (3.5) we have

$$g_{ir}\overline{\Gamma}_{jm}{}^{\bar{r}} = -g_{jr}\overline{\Gamma}_{im}{}^{\bar{r}} = -g_{jr}(\overline{\Gamma}_{mi}{}^{\bar{r}} + y^a K_{ima}{}^{r}) = g_{mr}\overline{\Gamma}_{ji}{}^{\bar{r}} - y^a K_{imaj} = g_{mr}\overline{\Gamma}_{ij}{}^{\bar{r}} + y^a(K_{jiam} - K_{imaj})$$
$$= -g_{ir}\overline{\Gamma}_{mj}{}^{\bar{r}} + y^a(K_{jiam} - K_{imaj}) = -g_{ir}(\overline{\Gamma}_{jm}{}^{\bar{r}} + y^a K_{mja}{}^{r}) + y^a(K_{jiam} - K_{imaj}),$$

thus we get (2).
From (3.4), (3.6) and (3.8), we have

$$g_{ir}\overline{\Gamma}_{j\bar{m}}{}^{\bar{r}} = -g_{jr}\overline{\Gamma}_{i\bar{m}}{}^{\bar{r}} = -g_{jr}(\overline{\Gamma}_{\bar{m}i}{}^{\bar{r}} - \Gamma_{mi}{}^{r}) = g_{jr}(\Gamma_{mi}{}^{r} - \overline{\Gamma}_{\bar{m}i}{}^{\bar{r}})$$
$$= -g_{mr}(\Gamma_{ji}{}^{r} - \overline{\Gamma}_{ji}{}^{r}) = -g_{mr}(\Gamma_{ij}{}^{r} - \overline{\Gamma}_{ij}{}^{r}) = g_{ir}(\Gamma_{mj}{}^{r} - \overline{\Gamma}_{\bar{m}j}{}^{\bar{r}})$$
$$= g_{ir}\Gamma_{mj}{}^{r} - g_{ir}\overline{\Gamma}_{\bar{m}j}{}^{\bar{r}} = g_{ir}\Gamma_{mj}{}^{r} - g_{ir}(\overline{\Gamma}_{j\bar{m}}{}^{\bar{r}} + \Gamma_{mj}{}^{r})$$

thus we obtain (5) and (6). From (3.6) and (5), we have (1).

4. The almost anti-Hermitian structure $(\tilde{g}_2, \tilde{J})$

The almost complex structure J defined in (2.6) has not the property of homogeneity. The $F(\widetilde{TM})$-linear mapping $J : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$, applies the 1-homogeneous vector fields X_i into 0-homogeneous vector fields $X_{\bar{i}}$ $(i = 1, ..., n)$. Therefore, we consider the $F(\widetilde{TM})$-linear mapping $\tilde{J} : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$, given on the adapted basis by

$$\tilde{J}(X_i) = -\|y\| X_{\bar{i}}, \qquad \tilde{J}(X_{\bar{i}}) = \frac{1}{\|y\|} X_i, \ (i = 1, ..., n). \tag{4.1}$$

Obviously, $\tilde{J}$ is a tensor field of type $(1,1)$ on $\widetilde{TM}$, that is homogeneous on the fibers of TM.

Theorem 5. $(\widetilde{TM}, \tilde{g}_2, \tilde{J})$ *is an almost anti-Hermitian manifold.*

Proof. It follows easily that

$$\tilde{g}_2(JX_i, JX_j) = -\tilde{g}_2(X_i, X_j), \quad \tilde{g}_2(JX_{\bar{i}}, JX_{\bar{j}}) = -\tilde{g}_2(X_{\bar{i}}, X_{\bar{j}})$$

$$\tilde{g}_2(JX_{\bar{i}}, JX_j) = -\tilde{g}_2(X_{\bar{i}}, X_j).$$

Hence

$$\tilde{g}_2(\tilde{J}X, \tilde{J}Y) = -\tilde{g}_2(X, Y), \ \forall X, Y \in \chi(\widetilde{TM}).$$

Proposition 6. *The Nijenhuis tensor field of the almost complex structure* $\tilde{J}$ *on* $\widetilde{TM}$ *is give by*

$$\begin{cases} N_{\tilde{J}}(X_i, X_j) = (y_i\delta_j^s - y_j\delta_i^s - y^a K_{jia}{}^s)X_{\bar{s}}, \\ N_{\tilde{J}}(X_i, X_{\bar{j}}) = \dfrac{1}{\|y\|^2}(y_i\delta_j^s - y_j\delta_i^s - y^a K_{jia}{}^s)X_s, \\ N_{\tilde{J}}(X_{\bar{i}}, X_{\bar{j}}) = \dfrac{1}{\|y\|^2}(y_j\delta_i^s - y_i\delta_j^s + y^a K_{jia}{}^s)X_{\bar{s}}. \end{cases}$$

Proof. Recall that the Nijenhuis tensor field $N_{\tilde{J}}$ defined by $\tilde{J}$ is given by

$$N_{\tilde{J}}(X, Y) = [\tilde{J}X, \tilde{J}Y] - \tilde{J}[\tilde{J}X, Y] - \tilde{J}[X, \tilde{J}Y] - [X, Y], \qquad \forall X, Y \in \chi(\widetilde{TM}).$$

Replacing the basis $(X_i, X_{\bar{i}})$ in the above formula and using following relation:

$$X_i(\|y\|) = 0, \quad X_{\bar{i}}(\|y\|) = \frac{y_i}{\|y\|}$$

We get the proof.

Theorem 7. *The almost complex structure* $\tilde{J}$ *is a complex structure on* $\widetilde{TM}$ *if and only if the Riemannian space* (M, g) *is of constant sectional curvature 1.*

Proof. From the condition $N_{\tilde{J}} = 0$, one obtains:

$$\{K_{jia}{}^s - (g_{ia}\delta_j^s - g_{ja}\delta_i^s)\}y^a = 0.$$

Differentiating with respect to y^h, taking $y^a = 0 \quad \forall a \in \{1, \ldots, n\}$, it follows that the curvature tensor field of ∇ has the expression

$$K_{jih}{}^s = g_{ih}\delta_j^s - g_{jh}\delta_i^s \tag{4.2}$$

Using by the Schur theorem (in the case where M is connected and $\dim M \geq 3$) it follows that (M, g) has the constant sectional curvature 1.

Corollary 8. $(\widetilde{TM}, \tilde{g}_2, \tilde{J})$ *is an anti-Hermitian manifold if and only if the space* (M, g) *is of constant sectional curvature 1.*

From (4.2) we have

$$R_{ij} = (n-1)g_{ij}, \qquad (n > 1) \tag{4.3}$$

where R_{rk} is the Ricci tensor and

$$S = n(n-1).$$

where S is the scalar tensor.

Corollary 9. *If the structure* $(\tilde{g}_2, \widetilde{J})$ *is a Hermitian structure on* $\widetilde{TM}$ *then* (M, g) *is an Einstein space with positive scalar curvature.*

Since $R_{ij} = R_{ji}$ then from (4.3) we get:

Corollary 10. *If the almost complex structure* $\widetilde{J}$ *is a complex structure then* $(M, R_{ij}(x))$ *is a Riemannian space.*

5. The almost product structure $(\tilde{g}_2, \widetilde{Q})$

The almost product structure Q defined in (2.7) has not the property of homogeneity. The $F(\widetilde{TM})$-linear mapping $Q : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$, applies the 1-homogeneous vector fields X_i into 0-homogeneous vector fields $X_{\bar{i}}\, (i = 1, ..., n)$. Therefore, we consider the $F(\widetilde{TM})$-linear mapping $\widetilde{Q} : \chi(\widetilde{TM}) \to \chi(\widetilde{TM})$, given on the adapted basis by

$$\widetilde{Q}(X_i) = \| y \| X_{\bar{i}}, \quad \widetilde{Q}(X_{\bar{i}}) = \frac{1}{\| y \|} X_i, \ (i = 1, ..., n). \tag{5.1}$$

Obviously, $\tilde{Q}$ is a tensor field of type $(1,1)$ on $\widetilde{TM}$, that is homogeneous on the fibers of TM. It is not difficult to prove:

Theorem 11. $(\widetilde{TM}, \tilde{g}_2, \tilde{Q})$ *is an almost product manifold.*

In order to find conditions that $\widetilde{Q}$ be a product structure, we have to put zero for the Nijenhuis tensor field of $\widetilde{Q}$,

$$N_{\tilde{Q}}(X, Y) = [\tilde{Q}X, \tilde{Q}Y] - \tilde{Q}[\tilde{Q}X, Y] - \tilde{Q}[X, \tilde{Q}Y] + [X, Y], \qquad \forall X, Y \in \chi(\widetilde{TM}).$$

Theorem 12. $(\widetilde{TM}, \tilde{g}_2, \tilde{Q})$ *is a product manifold if and only if the space* (M, g) *is of constant sectional curvature -1.*

Proof. Similar to proposition 6 and theorem 7, by putting $N_{\tilde{Q}} = 0$ we get

$$K_{jia}{}^{s} = -(g_{ia}\delta^s_j - g_{ja}\delta^s_i). \tag{5.2}$$

Therefore, using by the Schur theorem, it follows that (M, g) has the constant sectional curvature -1.

Theorem 13. *If the structure* $(\tilde{g}_2, \widetilde{Q})$ *is a product structure on* $\widetilde{TM}$ *then* (M, g) *is an Einstein space with negative scalar curvature.*

Proof. From (5.2) we have $R_{ij} = (1-n)g_{ij}, S = n(1-n)$ for $n > 1$.

Since $R_{ij} = R_{ji}$ then we get:

Corollary 14. *If the almost product structure* $\widetilde{Q}$ *is a product structure then* $(M, R_{ij}(x))$ *is a Riemannian space.*

6. Almost Hermitian and para-Hermitian structures on $\widetilde{TM}$

In this section, we get twin tensor of metric $\tilde{g}_2$ and by using it introduce Almost Hermitian and para-Hermitian structures on $\widetilde{TM}$. Then we show that these structures are not Kahlerian or para-Kahlerian.

Lemma15. *The twin tensor of structure* $(\tilde{g}_2, \tilde{J})$ *is a metric that is given by*

$$h_{\tilde{J}} = -2g_{ij}dx^i dx^j + \frac{2g_{ij}}{\|y\|^2}\delta y^i \delta y^j$$

Proof. From relation $h_{\tilde{J}}(X,Y) = \tilde{g}_2(\tilde{J}X,Y)$ we have:

$$h_{\tilde{J}}(X_i, X_j) = -\|y\|\tilde{g}_2(X_{\bar{i}}, X_j) = -2g_{ij},\ h_{\tilde{J}}(X_{\bar{i}}, X_{\bar{j}}) = \frac{1}{\|y\|}\tilde{g}_2(X_i, X_{\bar{j}}) = \frac{2}{\|y\|^2}g_{ij}$$

$$h_{\tilde{J}}(X_i, X_{\bar{j}}) = -\|y\|\tilde{g}_2(X_{\bar{i}}, X_{\bar{j}}) = 0.$$

Theorem 16. $(\widetilde{TM}, h_{\tilde{J}}, \tilde{Q})$ *is an almost para-Hermitian manifold.*

Proof. Straightforward computations, we obtain

$$h_{\tilde{J}}(\tilde{Q}X_i, \tilde{Q}X_j) = 2g_{ij} = -h_{\tilde{J}}(X_i, X_j),\ h_{\tilde{J}}(\tilde{Q}X_{\bar{i}}, \tilde{Q}X_{\bar{j}}) = -\frac{2}{\|y\|^2}g_{ij} = -h_{\tilde{J}}(X_{\bar{i}}, X_{\bar{j}})$$

$$h_{\tilde{J}}(\tilde{Q}X_{\bar{i}}, \tilde{Q}X_j) = 0 = -h_{\tilde{J}}(X_{\bar{i}}, X_j)$$

Therefore

$$h_{\tilde{J}}(\tilde{Q}X, \tilde{Q}Y) = -h_{\tilde{J}}(X,Y)$$

By definition $\Omega_{\tilde{Q}}^{h_{\tilde{J}}}(X,Y) = h_{\tilde{J}}(\tilde{Q}X,Y)$, the associated almost simplectic structure $\Omega^{h_{\tilde{J}}}$ is given in adapted basis by

$$\Omega^{h_{\tilde{J}}} = \frac{4}{\|y\|}g_{ij}(x)dx^i \wedge \delta y^j.$$

Theorem17. *The space* $(\widetilde{TM}, h_{\tilde{J}}, \tilde{Q})$ *cannot be an almost para- Kählerian manifold.*

Proof. since, $d(\frac{1}{\|y\|}) = -\frac{1}{\|y\|^2}d\|y\|$ and $d(g_{ij}dx^i \wedge \delta y^j) = 0$ then, the exterior differential of $\Omega^{h_{\tilde{J}}}$ satisfies the equation:

$$d\Omega^{h_{\tilde{J}}} = -\frac{4}{\|y\|}d\|y\| \wedge \Omega^{h_{\tilde{J}}}.$$

It follows, easily that $d\Omega_{\tilde{J}}^{h_{\tilde{Q}}} \neq 0$ on $\tilde{T}M$, i.e., $\Omega_{\tilde{J}}^{h_{\tilde{Q}}}$ is not closed.

From theorem 12,16, we have:

Theorem18. $(\widetilde{TM}, h_{\tilde{J}}, \tilde{Q})$ *is a para-Hermitian manifold if and only if the space* (M,g) *is of constant sectional curvature -1.*

Lemma19. *The Levi-Civita connection coefficients* $\bar{\bar{\nabla}}^{h_{\tilde{J}}}$ *of* $h_{\tilde{J}}$ *satisfy the following relations:*

$$(1)\ \bar{\Gamma}_{ji}{}^{h}=\Gamma_{ji}{}^{h},\qquad (2)\ \bar{\Gamma}_{ji}{}^{\bar{h}}=\frac{1}{2}y^{a}K_{jia}{}^{h},$$

$$(3)\ \bar{\Gamma}_{\bar{j}i}{}^{h}=-\frac{1}{2\|y\|^{2}}y^{a}K_{aji}{}^{h},\quad (4)\ \bar{\Gamma}_{j\bar{i}}{}^{h}=-\frac{1}{2\|y\|^{2}}y^{a}K_{aij}{}^{h},$$

$$(5)\ \bar{\Gamma}_{\bar{j}i}{}^{\bar{h}}=\Gamma_{ji}{}^{h},\qquad (6)\ \bar{\Gamma}_{j\bar{i}}{}^{\bar{h}}=0,$$

$$(7)\ \bar{\Gamma}_{\bar{j}\bar{i}}{}^{h}=0,\qquad (8)\ \bar{\Gamma}_{\bar{j}\bar{i}}{}^{\bar{h}}=\frac{1}{\|y\|^{2}}(g_{ji}y^{h}-\delta_{j}^{h}y_{i}-\delta_{i}^{h}y_{j}).$$

Theorem 20. $\bar{\nabla}^{h_{\tilde{J}}}$ *is an almost complex connection.*

Proof. From (1.1) we have

$$\tilde{g}_{2}((\bar{\nabla}_{X}^{h_{\tilde{J}}}\tilde{J})Y,Z)\equiv(\bar{\nabla}_{X}^{h_{\tilde{J}}}h_{\tilde{J}})(Y,Z)$$

Since $\bar{\nabla}^{h_{\tilde{J}}}$ is Levi-Civita connection for $h_{\tilde{J}}$ then

$$\tilde{g}_{2}((\bar{\nabla}_{X}^{h_{\tilde{J}}}\tilde{J})Y,Z)=0$$

i.e. $\bar{\nabla}_{X}\tilde{J}=0.$

Similarly previous case, the twin tensor of structure $(\tilde{g}_{2},\tilde{Q})$ is a metric that is

$$h_{\tilde{Q}}=2g_{ij}dx^{i}dx^{j}+\frac{2g_{ij}}{\|y\|^{2}}\delta y^{i}\delta y^{j}$$

Obviously, $h_{\tilde{Q}}$ is 0-homogeneous on the fibers of TM.

Theorem 21.

1. $(\widetilde{TM},h_{\tilde{Q}},\tilde{J})$ *is an almost Hermitian structure on* $\widetilde{TM}$ *.*

2. The associated almost simplectic structure $\Omega_{\tilde{J}}^{h_{\tilde{Q}}}$ *is given in adapted basis by*

$$\Omega_{\tilde{J}}^{h_{\tilde{Q}}}=\frac{4}{\|y\|}g_{ij}(x)\delta y^{i}\wedge dx^{j}$$

Theorem 22. *The space* $(\widetilde{TM},h_{\tilde{Q}},\tilde{J})$ *cannot be an almost Kählerian manifold.*

From corollary 8 and theorem 21 we obtain following theorem.

Theorem 23. $(\widetilde{TM},h_{\tilde{Q}},\tilde{J})$ *is a Hermitian manifold if and only if the space* (M,g) *is of constant sectional curvature 1.*

Lemma 24. *The Levi-Civita connection coefficients* $\bar{\nabla}^{h_{\tilde{Q}}}$ *of* $h_{\tilde{Q}}$ *satisfy the following relations:*

$$(1)\ \bar{\Gamma}_{ji}{}^{h}=\Gamma_{ji}{}^{h},\qquad (2)\ \bar{\Gamma}_{ji}{}^{\bar{h}}=\frac{1}{2}y^{a}K_{jia}{}^{h},$$

$$(3)\ \bar{\Gamma}_{\bar{j}i}{}^{h}=\frac{1}{2\|y\|^{2}}y^{a}K_{aji}{}^{h},\quad (4)\ \bar{\Gamma}_{j\bar{i}}{}^{h}=\frac{1}{2\|y\|^{2}}y^{a}K_{aij}{}^{h},$$

$$(5)\ \bar{\Gamma}_{\bar{j}i}{}^{\bar{h}}=\Gamma_{ji}{}^{h},\qquad (6)\ \bar{\Gamma}_{j\bar{i}}{}^{\bar{h}}=0,$$

$$(7)\ \bar{\Gamma}_{\bar{j}\bar{i}}{}^{h}=0,\qquad (8)\ \bar{\Gamma}_{\bar{j}\bar{i}}{}^{\bar{h}}=\frac{1}{\|y\|^{2}}(g_{ji}y^{h}-\delta_{j}^{h}y_{i}-\delta_{i}^{h}y_{j}).$$

Theorem 25. $\bar{\nabla}^{h_{\tilde{Q}}}$ *is an almost product connection.*

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