On the Existence of Global Bisections of Lie Groupoids

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Abstract

We show that every source connected Lie groupoid always has global bisections through any given point. This bisection can be chosen to be the multiplication of some exponentials as close as possible to a prescribed curve. The existence of bisections through more than one prescribed points is also discussed. We give some interesting applications of these results.

Key words Lie groupoid, bisection, exponential maps.

MSC Primary 17B62, Secondary 17B70.

Funded by CPSF(20060400017).

1 Introduction

The notion of groupoids generalizes that of both groups and the Cartesian product of a set, namely the pair groupoid. It is Ehresmann who first made the concept of groupoid ([6,7]) central to his vision of differential geometry. Lie groupoids, originally called differentiable groupoids, were also introduced by Ehresman (see several papers contained in [5]) and theories of Lie groupoids, especially the relationships with that of Lie algebroids defined by Pradines ([14, 15]) have been investigated by many people and much work has been

done in this field. Readers can find the most basic definitions and examples of (Lie) groupoids in the texts such as Mackenzie's [12], his recent book [13], and [2,11]. The importance of groupoid theories were already shown in the studies on symplectic groupoids and Poisson geometry, for example, illustrated by Weinstein ([17,18]), Coste ([3]), Dazord ([4]), Karasëv ([9]), Zakrzewski ([19]) and many other authors.

By definition, a Lie groupoid is a groupoid where the set Ob of objects and the set Mor of morphisms are both manifolds, the source and target operations are submersions, and all the category operations (source and target, composition, and identity-assigning map) are smooth. In our humble opinion, a Lie groupoid can thus be thought of as a "many-object generalization" of a Lie group, just as a groupoid is a many-object generalization of a group.

On a group G, the left translations $L_g: h \mapsto gh$ form a group which is isomorphic to G itself. In the groupoid world, the concepts of left translations are replaced by a family of elements which is called a bisection of the groupoid [13, I,1.4]. For a groupoid R, we call any section $s \to R$ of the α -fibers for which $\beta \circ s$ is a diffeomorphism a bisection of R (also known as admissible sections in [12]). Bisections may be regarded as generalized elements of the groupoids and similarly, the exponential maps take values in the collections of all bisections. Although in most situations where bisections are used, the question is local and the existence of local bisection are of course correct, it has remained an unsettled problem whether global bisection exist through an arbitrary point. In this paper, we give an affirmative answer to this question.

We will prove that there exists a bisection through a given point of a Lie groupoid which is source connected (Theorem 3.1). Moreover, we show that this bisection is of the form $\exp X_1 \exp X_2 \cdots \exp X_k$, where X_i are some sections of the corresponding Lie algebroid and compact supported. Furthermore, we prove that these X_i can be chosen as close as possible to a prescribed curve (Theorem 3.2).

We also prove that if the groupoid is transitive and the base space is more than 1-dimensional, then for any given points g_1, \dots, g_n , such that their sources and targets are subject to the concordance condition (see (1)), there exists a bisection through all of them (Theorem 4.1).

As stated by Mackenzie in [13]: "Groupoids possess many of the features which give groups their power and importance, but apply in situations which lack the symmetry which is characteristic of group theory and its applications", we shall apply our theorems to varies cases of groupoids and obtain some interesting results.

This paper is organized as follows. Section 2 begins with an account of basic concepts of Lie groupoids, its bisections, Lie algebroids and exponentials (our conventions follow that of [13]). Section 3 gives the main Theorem 3.1 (we also provide a stronger version of this result, Theorem 3.2) and some applications are added in. Section 4 studies the existence problem of bisections through more than one points. Section 5 is devoted to prove Theorem 3.2 which implies Theorem 3.1, and a detailed proof is spilt into several lemmas. To understand

what is going on, the reader is referred to the seven pictures illustrating the geometric images.

2 Lie Groupoids and its Tangent Lie algebroids

Definition 2.1. [13] A groupoid consists of a set R and a subset $M \subset R$, called respectively the groupoid and the base, together with two maps α and β from R to M, called respectively the source and target, and a partial multiplication $(g, h) \mapsto gh$ in R defined on the set of composable pairs:

$$\mathbf{R}^{[2]} \triangleq \{ (g, h) \in \mathbf{R} \times \mathbf{R} | \beta(g) = \alpha(h) \},\$$

all subject to the following conditions:

- i) $\alpha(gh) = \alpha(g)$ and $\beta(gh) = \beta(h)$ for all $(g, h) \in \mathbb{R}^{[2]}$;
- *ii)* f(gh) = (fg)h for all $f, g, h \in \mathbb{R}$ such that $\beta(f) = \alpha(g)$ and $\beta(g) = \alpha(h)$;
- *iii)* $\alpha(x) = \beta(x) = x$ for all $x \in M$;
- iv) $g\beta(g) = \alpha(g)g = g$ for all $g \in \mathbb{R}$;
- v) each $g \in \mathbb{R}$ has a two-sided inverse g^{-1} such that $\alpha(g^{-1}) = \beta(g), \beta(g^{-1}) = \alpha(g)$ and $g^{-1}g = \beta(g), gg^{-1} = \alpha(g)$.

A groupoid R on the base M, with respectively source and target maps α , β , will be denoted by (R $\Rightarrow M$; α, β), or, more briefly, (R, M). We adopt the convention that, whenever we write a multiplication gh, we are assuming that is defined (see Pic. 1))

For $x \in M$, its **orbit**, denoted by O_x , is the set $\beta \circ \alpha^{-1}(x) \subset M$.

Lie groupoid ($\mathbb{R} \Rightarrow M$; α, β) is a groupoid with differential structures and the base space M is an embedded submanifold, the target and source maps are submersions and all the operations are smooth. The orbit O_x is also a submanifold of M. We also notice that, for any α -fiber $P = \alpha^{-1}(x), \beta|_P$: $P \to O_x$ is again a submersion.

We take the tangent Lie algebroid (LieR, ρ) of (R, M) as

$$Lie \mathbf{R} \triangleq \bigcup_{x \in M} T_x \alpha^{-1}(x) = \{ v \in T_x \mathbf{R} | x \in M, \alpha_*(v) = 0 \}.$$

In turn, the bracket of $\Gamma(LieR)$ is determined by the commutator of left invariant vector fields and the anchor map is given by $\rho = \beta_{R*}|_M$.



A bisection of a Lie groupoid (\mathbf{R}, M) is a smooth map $s: M \to \mathbf{R}$ such that

1) $\alpha \circ s = Id_M;$

2) $\beta \circ s$ is a diffeomorphism of M (see Pic. 2).

The collection of all bisections of R is a group, in sense of the following operations.

Identity): The base M serves as the identity, if regarded as a map $M \to \mathbf{R}$;

Multiplication):

Let s and w be two bisections, their multiplication sw is again a bisection defined by (see Pic. 3)

$$sw(x) = s(x)w(\beta \circ s(x)), \qquad \forall x \in M;$$

Inversion):

The inverse of s is defined by (see Pic. 4)

$$s^{-1}(x) = (s \circ (\beta \circ s)^{-1}(x))^{-1}, \qquad \forall x \in M.$$



We recall the result of Kumpera and Spencer.

Lemma 2.2. [10] Let R be any Lie groupoid over base M, and let A = LieRbe its Lie algebroid with anchor ρ . For $X \in \Gamma(A)$, let \overline{X} be the left invariant vector field corresponding to X, and recall that $\rho(X) = \beta_*(\overline{X})$ is its projected vector field on M. Then \overline{X} is complete if and only if $\rho(X)$ is complete. In fact, $\tilde{\phi}_t(g)$ is defined whenever $\phi_t(\beta(g))$ is defined, where $\tilde{\phi}_t(g)$ and $\phi_t(g)$ are the flows generated by \overline{X} and $\rho(X)$, respectively.

As a corollary, let $X \in \Gamma(A)$ be any section which has a **compact support**, we know that X is complete. We denote $\exp(tX) : M \to \mathbb{R}$ $(t \in \mathbb{R})$ the map

$$\exp(tX)(x) \triangleq \phi_t(x), \qquad \forall x \in M,$$

called the exponential of X. One has

$$\phi_t(g) = g \exp(tX)(\beta(g)), \quad \forall g \in \mathbf{R}.$$

Furthermore,

$$\alpha \circ \exp(tX) = Id_M, \qquad \beta \circ \exp(tX) = \phi_t,$$

which shows that $\exp(tX)$ is a bisection of R, for any $t \in \mathbb{R}$.

Another important fact is that

$$(\exp tX)^{-1} = \exp(-tX).$$

In the special case that, the Lie groupoid degenerates to a group, i.e., M = pt is a point, the exponential map becomes the ordinary exponential of Lie groups. We need the following basic results of Lie group theories.

Lemma 2.3. Let G be a Lie group and let $e \in G$ be its unit element. Then there exists an open neighborhood \mathcal{N} of e, and an open neighborhood $\mathcal{O} \subset \mathfrak{g} = T_e G$ near zero, such that

$$\exp:\mathcal{O}\to\mathcal{N}$$

is a diffeomorphism [8, 16].

For a general Lie groupoid (\mathbf{R}, M) and $x \in M$, by $G_x \triangleq \alpha^{-1}(x) \cap \beta^{-1}(x)$ we denote the isotropic group at x. G_x is a Lie group. By $\mathfrak{g}_x \triangleq ker\rho_x \subset \mathbf{A}_x$ we denote the isotropic algebra at x, which is a Lie algebra, in fact, the Lie algebra of G_x .

If $X \in \Gamma(A)$ is a section with compact support and $X_x \in \mathfrak{g}_x$, then $\exp(tX)(x)$ is the usual exponential map $\exp t(X_x)$, regarding $X_x \in \mathfrak{g}_x$.

Definition 2.4. A bisection of (\mathbb{R}, M) is said to be finitely generated if it has the form $\exp X_1 \exp X_2 \cdots \exp X_k$, for some $X_1, \cdots, X_k \in \Gamma(\mathbb{A})$ and every X_i has a compact support. It is said to be finitely generated over \mathcal{U} , an open set $\mathcal{U} \subset M$, if each support of X_i is contained in \mathcal{U} .

3 Main Theorems and their Applications

Theorem 3.1. Let (\mathbb{R}, M) be an α -connected Lie groupoid. Then for any $g \in \mathbb{R}$, there exists a finitely generated bisection s of \mathbb{R} through g, i.e., $s(\alpha(g)) = g$.

An immediate consequence of this theorem is the well known fact that, for a connected Lie group G, every element g can be expressed into

$$g = \exp X_1 \exp X_2 \cdots \exp X_k,$$

for some $X_i \in T_e G$.

In this paper we would like to prove a stronger version of Theorem 3.1 stated as follows. **Theorem 3.2.** Let (\mathbb{R}, M) be Lie groupoid and $x \in M$. Let $g \in \alpha^{-1}(x)$ and let $\widetilde{\mathcal{U}} \subset \alpha^{-1}(x)$ be a connected open set which contains both x and g. Suppose that \mathcal{U} is an open set containing $\beta(\widetilde{\mathcal{U}})$, then there exists a bisection s of \mathbb{R} through g, and s(y) = y for all $y \in \mathcal{U}^c$ (= $M - \mathcal{U}$). Moreover, s is finitely generated over \mathcal{U} .

The proof of this theorem is given in the last section of this paper. As an application of Theorem 3.1 as well as 3.2, we have the following interesting conclusions. We always assume that the reader is familiar with the various kinds of groupoids mentioned below, especially their bisections.

Theorem 3.3. [Homogeneity of manifolds [1].] Let M be a connected smooth manifold. Then, for any two points $x, y \in M$ and any connected open set \mathcal{U} containing x and y, there exists a diffeomorphism $\Phi : M \to M$, such that $\Phi(x) = y$ and $\Phi(m) = m$ for all $m \in \mathcal{U}^c$.

This theorem is of course implied by the following one.

Theorem 3.4. With the same assumptions as in the previous one, for any two points $x, y \in M$, and for an arbitrary open neighbor \mathcal{U} containing x and y, there exist some smooth vector fields X_1, \ldots, X_k compact supported within \mathcal{U} , such that

$$\varphi_k^1 \circ \cdots \circ \varphi_2^1 \circ \varphi_1^1(x) = y,$$

where φ_i^t is the flow of X_i .

Proof of Theorem 3.3 and 3.4. We recommend [13, Example 1.1.7, 1.4.3] for background information on the pair groupoid $M \times M$, for which a bisection is exactly a diffeomorphism of M. The exponential of a vector field which has compact support is its flow. And the multiplication of two bisections, namely diffeomorphism, are exactly their compositions. The conclusion of Theorem 3.4 is exactly the translation of Theorem 3.2 into the pair groupoid case. \Box

In what follows, we consider a vector bundle $E \xrightarrow{q} M$ and we denote by $\Phi(E)$ the *linear frame groupoid* of E, simply called the frame groupoid. Please refer to [12, III] and [13, Example 1.1.12], where it is denoted by $\Pi(E)$), which is the collection of all linear isomorphisms from a fiber of E to some generally different fiber of E, i.e., an element in $\Phi(E)$ is a vector space isomorphisms $\xi : E_x \to E_y$ for $x, y \in M$. The bisections group of $\Phi(E)$ is in fact Aut(E), the group of vector bundle automorphisms of E. It is proved in that $\Phi(E)$ is also a Lie groupoid on M. One may directly draw from Theorem 3.1 the following result.

Theorem 3.5. Let $(E \to M)$ be a vector bundle over a connected smooth manifold M. For any two points $x, y \in M$ and an isomorphism of vector spaces $\phi : E_x \to E_y$, there exists an automorphism $\Phi : E \to E$ of vector bundles such that $\Phi|_{E_x} = \phi$. Of course we can add some structures in E. If (E, [,]) is a Lie algebra bundle, one has the Lie-algebra-bundle frame groupoid $\Phi_{Aut}(E)$ ([13, Example 1.7.12]). If (E, \langle , \rangle) is a Riemannian vector bundle, one gets the orthonormal frame groupoid $\Phi_{\mathcal{O}}(E)$ ([13, Example 1.7.9]). For these two examples, see also Corollary 3.6.11 in [13]. We are then easy to draw the following analogue conclusions.

Theorem 3.6. Let $(E \to M, [,])$ be a Lie algebra bundle over a connected smooth manifold M. For any two points $x, y \in M$ and an isomorphism of Lie algebras $\phi : E_x \to E_y$, there exists an automorphism $\Phi : E \to E$ of Lie algebra bundles such that $\Phi|_{E_x} = \phi$.

Theorem 3.7. Let $(E \to M, \langle , \rangle)$ be a Riemannian vector bundle over a connected smooth manifold M. For any two points $x, y \in M$ and an isomorphism of metric spaces $\phi : E_x \to E_y$, there exists an automorphism $\Phi : E \to E$ of Riemannian vector bundles such that $\Phi|_{E_x} = \phi$.

Finally, we consider the action groupoid $M \triangleleft G$ coming from a right action of a connected Lie group G on a connected manifold M ([13, Example 1.1.9], see also [12]). Here $M \triangleleft G = M \times G$ is a Lie groupoid on M. Recall that a bisection of $M \triangleleft G$ can be identified with a smooth G-valued function $s : M \rightarrow G$ such that the map

$$M \to M, \quad m \mapsto ms(m), \quad \forall m \in M$$

is a diffeomorphism of M. Such kinds of s are called **invertible** functions. So we have the following theorem.

Theorem 3.8. For any prescribed $x \in M$, $g \in G$, one can find an invertible function $s: M \to G$ satisfying s(x) = g.

4 Bisections through Points

Now we consider a more generalized problem: does there exists a bisection through two (or more) given points of a groupoid? The answer is also yes, under some topological conditions.

We recall that for a transitive Lie groupoid (R, M), the map $\beta|_{\alpha^{-1}(x)} : \alpha^{-1}(x) \to M$ is a surjection as well as a submersion, for any $x \in M$.

The following one is the main theorem in this section. We always assume that the natural number $n \ge 2$.

Theorem 4.1. Let (\mathbf{R}, M) be a transitive and α -connected Lie groupoid and suppose that dim $M \ge 2$. For any different n points $g_1, \dots, g_n \in \mathbf{R}$, and let $\alpha(g_i) = x_i, \ \beta(g_i) = y_i, \ i = 1, \dots, n$. Then there exists a bisection s of \mathbf{R} such that $s(x_1) = g_1, \dots, s(x_n) = g_n$ if and only if

$$x_i \neq x_j, \quad and \quad y_i \neq y_j, \quad \forall i \neq j.$$
 (1)

Moreover, this bisection is finitely generated.

In what follows we devote to proving this theorem. We need some preparations and let us introduce some concepts first. For n pairs of points

$$p_i = (x_i, y_i) \in M \times M, \quad i = 1, \cdots, n,$$

they are said to be **concordant** if they are subject to condition (1). If they are concordant and there is a proper arrangement of their indices such that they make a loop, we say they consist a **chain**. That is, for some permutation of the indices $\check{p}_i = p_{\theta(i)} = (\check{x}_i, \check{y}_i) = (x_{\theta(i)}, y_{\theta(i)})$, where $\theta \in \mathcal{S}(n)$ (the permutation group), one has

$$\check{y}_1 = \check{x}_2, \quad \check{y}_2 = \check{x}_3, \quad \cdots, \quad \check{y}_n = \check{x}_1.$$

We shall write

$$\check{p}_1 \frown \check{p}_2 \frown \cdots \frown \check{p}_n$$

to denote such a chain (see Pic. 5).



Definition 4.2. Let $p_i = (x_i, y_i) \in M \times M$, $i = 1, \dots, n$ be some pairs of points in M. p_1, \dots, p_n are said to be **independent**, if they are concordant and there is not any subset of $\{p_i\}$ that can consist a chain.

Remark 4.3. We recall the pair groupoid $M \times M$ of pairs of points p = (x, y). We say an elements $p \in M \times M$ can be expressed by some $p_1, \dots, p_m \in M \times M$, if there are some $m_1, \dots, m_k \in \{1, \dots, m\}$, such that

$$p = q_{m_1} q_{m_2} \cdots q_{m_k},$$

where each q_{m_l} is p_{m_l} or the inverse $p_{m_l}^{-1}$. One can prove that, for concordant n elements p_1, \dots, p_n , they are independent if and only if each $p_i = (x_i, y_i)$ is not possible to be expressed by $p_1, p_2, \dots, \hat{p_i}, \dots, p_n$. This is the reason that we use "independent".

For example, the five pairs in Pic.6 are independent.

Lemma 4.4. Let $p_i = (x_i, y_i) \in M \times M$, $i = 1, \dots, n$ be some pairs of points in M. Then p_1, \dots, p_n are independent if and only if there exists some $\sigma \in S(n)$, such that for $\bar{p}_1 = (\bar{x}_1, \bar{y}_1) = p_{\sigma(1)} = (x_{\sigma(1)}, y_{\sigma(1)}), \dots, \bar{p}_n = (\bar{x}_n, \bar{y}_n) = p_{\sigma(n)} = (x_{\sigma(n)}, y_{\sigma(n)})$, one has

 $\bar{x}_2 \neq \bar{y}_1;$ $\bar{x}_3 \neq \bar{y}_1, \ \bar{x}_3 \neq \bar{y}_2;$ $\dots \dots,$

$$\bar{x}_k \neq \bar{y}_l$$
, for $l = 1, \cdots, k - 1;$
.....,
 $\bar{x}_n \neq \bar{y}_l$, for $l = 1, \cdots, n - 1.$

(In this case and for convenience, we will say \bar{p}_i are well-ordered.)

Proof. " \Leftarrow ": We adopt a negative approach. If there are some elements \bar{p}_{m_1} , \cdots , \bar{p}_{m_k} consist a chain

$$\bar{p}_{m_1} \curvearrowright \bar{p}_{m_2} \curvearrowright \cdots \curvearrowright \bar{p}_{m_k}$$

then from $\bar{y}_{m_1} = \bar{x}_{m_2}$, we conclude $m_1 > m_2$. Similarly, from $\bar{y}_{m_j} = \bar{x}_{m_{j+1}}$, we conclude $m_j > m_{j+1}$ and finally one has $m_1 > m_2 > \cdots > m_k$. On the other hand, $\bar{x}_{m_1} = \bar{y}_{m_k}$ implies $m_k > m_1$: contradiction!

" \Rightarrow ": We give an inductive proof. For n = 2, if two elements $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ are independent, then $p_1 \neq p_2^{-1}$, i.e., either $x_2 \neq y_1$ or $x_1 \neq y_2$ holds. So one can always find $\sigma \in \mathcal{S}(2)$, which is either Id or the flip (1, 2).

Suppose that for $n \ge 2$, the lemma holds. For any independent n+1 elements p_1, \dots, p_{n+1} , we claim that there exist some $k \in \{1, \dots, n+1\}$, such that

$$x_k \neq y_j, \quad \forall j \in \left\{1, \cdots, \widehat{k}, \cdots, n+1\right\}.$$

In fact, if it is not true, then for each x_i , one find some $\psi(i) \in \{1, \dots, \hat{i}, \dots, n+1\}$ and $x_i = y_{\psi(i)}$. Obviously $\psi(i)$ is unique. It is also easy to see that ψ is a permutation of n + 1 numbers. Thus, if we find the smallest number $m \ge 1$ such that $\psi^{m+1}(1) = 1$, then we find a chain:

$$(x_{\psi^m(1)}, y_{\psi^m(1)}) \curvearrowright (x_{\psi^{m-1}(1)}, y_{\psi^{m-1}(1)}) \curvearrowright \cdots \curvearrowright (x_1, y_1),$$

which contradicts with the assumption that p_1, \dots, p_{n+1} are independent.

By this claim we pick $\sigma_0 = (k, n + 1) \in \mathcal{S}(n + 1)$, which is the flip of k and n + 1 and now for

$$\tilde{p}_i = (\tilde{x}_i, \tilde{y}_i) = p_{\sigma_0(i)} = (x_{\sigma_0(i)}, y_{\sigma_0(i)}), i = 1, \cdots, n+1,$$

one has

$$\tilde{x}_{n+1} = x_k \neq \tilde{y}_j = y_{\sigma_0(j)}, \quad \forall j = 1, \cdots, n.$$
(2)

By the inductive assumption, we are able to find $\sigma_1 \in \mathcal{S}(n)$, such that

$$\bar{p}_i = \tilde{p}_{\sigma_1(i)}, \quad i = 1, \cdots, n,$$

satisfies the well-ordered condition. Of course if we write $\bar{p}_{n+1} = \tilde{p}_{n+1}$, then (2) shows that $\bar{p}_1, \dots, \bar{p}_{n+1}$ are also well-ordered.

Lemma 4.5. Let (\mathbb{R}, M) be a transitive and α -connected Lie groupoid. Suppose that dim $M \ge 2$ and let x_1, \dots, x_k be some points of M. Then for any $g \in \mathbb{R}$, if $x = \alpha(g)$ and $y = \beta(g)$ are not contained in the set $\bigcup_{i=1}^{k} \{x_i\}$, then there exists a bisection s such that s(x) = g and $s(x_i) = x_i$, for all $i = 1, \dots, k$.

Proof. Consider the open set $\widetilde{\mathcal{U}} = (\beta^{-1}(x_1) \cup \cdots \cup \beta^{-1}(x_k))^c \cap \alpha^{-1}(x)$. Obviously x and g are contained in $\widetilde{\mathcal{U}}$. Since $\beta|_{\alpha^{-1}(x)} : \alpha^{-1}(x) \to M$ is a surjective submersion, we know that

$$dim\alpha^{-1}(x) - dim(\beta^{-1}(x_i) \cap \alpha^{-1}(x)) = dimM \ge 2.$$

And hence by $\alpha^{-1}(x)$ being connected, $\widetilde{\mathcal{U}}$ is also connected. Let $\mathcal{U} = \beta(\widetilde{\mathcal{U}}) = (\bigcup_{i=1}^{k} \{x_i\})^c$ be the corresponding open set in M. So Theorem 3.2 claims that there exists a bisection s with s(x) = g and $s(x_i) = x_i$, $i = 1, \dots, k$.

Proposition 4.6. Let (\mathbb{R}, M) be a transitive and α -connected Lie groupoid and suppose that dim $M \ge 2$. For n points $g_1, \dots, g_n \in \mathbb{R}$, and let $\alpha(g_i) = x_i$, $\beta(g_i) = y_i, i = 1, \dots, n, if$

$$p_i = (x_i, y_i), \quad i = 1, \cdots, n,$$

are independent, then there exists a bisection s of R such that $s(x_1) = g_1, \dots, s(x_n) = g_n$.

Proof. By Lemma 4.4, it suffices to assume that those p_i are already wellordered. I.e., for each i,

$$x_i \text{ and } y_i \notin \bigcup_{j=1}^{i-1} \{y_j\} \cup \bigcup_{k=i+1}^n \{x_k\}.$$

Thus, Lemma 4.5 tells us that there exists a bisection s_i such that

$$s_i(x_i) = g_i$$
, $s_i(y_j) = y_j$, $\forall j < i$, and $s_i(x_k) = x_k$, $\forall k > i$.

Now, let $w_1 = s_1, w_2 = s_1 s_2, \dots, w_n = s_1 s_2 \dots s_n$ and we claim $s = w_n$ is the bisection we are looking for. We show this fact inductively. Of course we have

$$w_1(x_1) = g_1, \quad w_1(x_j) = x_j, \ j = 2, \cdots, n.$$

Suppose that

$$w_k(x_j) = g_j, \ \forall j \leq k, \quad w_k(x_l) = x_l, \ \forall l \geq k+1$$

is already proved, then for j < k+1,

$$w_{k+1}(x_j) = (w_k s_{k+1})(x_j)$$

= $w_k(x_j) s_{k+1}(\beta \circ w_k(x_j)) = g_j s_{k+1}(y_j) = g_j y_j = g_j$.

For k+1,

$$w_{k+1}(x_{k+1}) = (w_k s_{k+1})(x_{k+1})$$

= $w_k(x_{k+1})s_{k+1}(\beta \circ w_k(x_{k+1})) = x_{k+1}s_{k+1}(x_{k+1}) = g_{k+1}.$

And for l > k+1,

$$w_{k+1}(x_l) = (w_k s_{k+1})(x_l)$$

= $w_k(x_l) s_{k+1}(\beta \circ w_k(x_l)) = x_l s_{k+1}(x_l) = x_l$

This completes the proof.

Proof of Theorem 4.1. " \Rightarrow ": If there exists a bisection s of R such that $s(x_1) = g_1, \dots, s(x_n) = g_n$ and these g_1, \dots, g_n are different points in R, then

$$g_i = s(x_i), \quad g_i^{-1} = s^{-1}(y_i).$$

Since both s and s^{-1} are both embeddings of M into R and $g_i \neq g_j$ $(i \neq j)$, $x_i \neq x_j, y_i \neq y_j$, this shows that these $p_i = (x_i, y_i)$ are concordant n pairs of points in M.

" \Leftarrow ": Suppose that these p_i are concordant. We find all subsets of $\{p_1, \dots, p_n\}$ which consist chains. It is easy to see that any two such subsets are disjoint. Let k be the number of these chains.

We induct with k, which is obviously less than $\frac{1}{2}(n+1)$. If k = 0, i.e., these p_i are independent, Proposition 4.6 already assures the existence of such a bisection through all g_i . We assume that for $k \leq m$ the conclusion is right. Now suppose that one has all the chains S_1, \dots, S_{m+1} , it suffices to assume S_{m+1} is the chain

$$p_1 \curvearrowright p_2 \curvearrowright \cdots \curvearrowright p_r,$$

where $r = \sharp S_{m+1} \leq n$. Choose an arbitrary $y_0 \in M$ such that

$$y_0 \notin \bigcup_{j=1}^n \{x_j\} \cup \bigcup_{j=1}^n \{y_j\}.$$

Since R is transitive, one can find some $h_0 \in \mathbb{R}$ such that $\alpha(h_0) = y_0$, $\beta(h_0) = x_1$. Let $g_0 = g_r h_0^{-1}$. Hence $\alpha(g_0) = x_r$, $\beta(g_0) = y_0$. It is easy to see that g_0 , $g_1, g_2, \dots, \hat{g_r}, \dots, g_n$ are different points in R, and the corresponding

$$p_0 = (x_r, y_0), p_1 = (x_1, y_1), \cdots, \hat{p_r}, \cdots, p_n = (x_n, y_n) \in M \times M$$

has only *m* chains S_1, \dots, S_m . And we are able to determine a bisection \tilde{s} such that

$$\tilde{s}(x_r) = g_0, \quad \tilde{s}(x_i) = g_i, \quad \forall i \in \{1, \cdots, \hat{r}, \cdots, n\}$$

By the mean time, Lemma 4.5 gives a bisection \breve{s} such that

$$\breve{s}(y_0) = h_0, \quad \breve{s}(y_i) = y_i, \quad \forall i \in \{1, \cdots, \widehat{r}, \cdots, n\}.$$

Then it is a direct check that the bisection $\tilde{s}\tilde{s}$ satisfies

$$\begin{split} \tilde{s}\tilde{s}(x_i) & (i\neq r) \\ = & \tilde{s}(x_i)\tilde{s}(\beta\circ\tilde{s}(x_i)) = g_iy_i = g_i, \end{split}$$

and

$$\tilde{s}\tilde{s}(x_r) = \tilde{s}(x_r)\tilde{s}(\beta \circ \tilde{s}(x_r)) = g_0\tilde{s}(x_0) = g_0h_0 = g_r.$$

This shows that $\tilde{s}\tilde{s}$ is just what we need. The preceding process of construction of such a bisection also shows that $\tilde{s}\tilde{s}$ is finitely generated. This completes the proof.

Remark 4.7. From the proof, one is able to see that, in Theorem 4.1, the condition "R is transitive and $dim M \ge 2$ " can be replaced by "each orbit of the base space M is more than 1-dimensional".

In what follows we present some applications of Theorem 4.1 in certain kind of groupoids. They are respectively similar to Theorem 3.3, 3.5, 3.6, 3.7 and 3.8 but concerning several pairs of points, and the proofs are omitted.

Theorem 4.8. Let M be a connected smooth manifold. Then, for any prescribed points $x_i, y_i \in M$ such that $(x_i, y_i), i = 1, \dots, n$ are concordant, there exists a diffeomorphism $\Phi : M \to M$, such that $\Phi(x_i) = y_i, i = 1, \dots, n$.

Theorem 4.9. Let $(E \to M)$ be a vector (resp. Lie algebra, Riemannian) bundle over a connected smooth manifold M. For any prescribed points x_i , $y_i \in M$ and isomorphisms of vector (resp. Lie algebra, Riemannian) spaces $\phi_i : E_{x_i} \to E_{y_i}$, such that (x_i, y_i) , $i = 1, \dots, n$ are concordant, there exists an automorphism $\Phi : E \to E$ of vector (resp. Lie algebra, Riemannian) bundles such that $\Phi|_{E_{x_i}} = \phi_i$.

Theorem 4.10. Let G be a connected Lie group which acts on a manifold M and suppose that dim $M \ge 2$. If the action is transitive, then for any prescribed $x_i \in M, g_i \in G$ such that $(x_i, x_i g_i), i = 1, \dots, n$ are concordant, then one can find an invertible function $s: M \to G$ satisfying $s(x_i) = g_i, i = 1, \dots, n$.

5 Proof of Theorem 3.2

We split the proof of Theorem 3.2 into several steps. In this section, we fix a Lie groupoid (\mathbf{R}, M) which is α -connected. We also assume that the base space M is connected. Let $\mathbf{A} = Lie\mathbf{R}$.

Lemma 5.1. Let $x \in M$, $g \in \alpha^{-1}(x)$ and $c : [0,1] \to \alpha^{-1}(x)$ be a smooth curve such that c(0) = x, c(1) = g. If the base curve $\bar{c} = \beta \circ c : [0,1] \to M$ is an injection and suppose that \bar{c} is contained in some open set $\mathcal{U} \subset M$, then there exists some $X \in \Gamma(A)$ with compact support in \mathcal{U} , such that

$$\exp tX(x) = c(t), \quad \forall t \in [0, 1].$$

In particular, $\exp X$ is a bisection through g and $\exp X|_{\mathcal{U}^c}$ is the identity map.

Proof. For each $t \in [0, 1]$, we define $X_t \in A_{\bar{c}(t)}$ to be

$$X_t = l_{c(t)^{-1}*}c'(t).$$

Since \bar{c} does not intersect with itself, one is able to extend this *t*-function into a well defined section $X \in \Gamma(A)$ which is compact supported in \mathcal{U} . It is clearly that $\exp tX(x) = c(t)$.

Lemma 5.2. For each $x \in M$ and each open neighborhood \mathcal{U} near x, there exists an open set $\mathcal{W} \subset \alpha^{-1}(x)$ containing x, such that for each element $g \in \mathcal{W}$, there exists a bisection $s : M \to \mathbb{R}$ through g and s(y) = y for all $y \in \mathcal{U}^c$. Moreover, s has the form $s = \exp X$, for some $X \in \Gamma(A)$ and X has a compact support contained in \mathcal{U} .

Proof. The target map $\beta : \mathbb{R} \to M$ is a submersion. For the α -fiber $P_x = \alpha^{-1}(x), \beta|_{P_x} : P_x \to O_x$ is also a submersion. Let $\dim O_x = m, \dim P_x = m+n$. Hence, there exist two local coordinate systems $(\mathcal{S}; x_1, \cdots, x_m, y_1, \cdots, y_n)$ of P_x near x and $(\mathcal{T}; x_1, \cdots, x_m)$ of O_x near x and they are subject to the following requirements:

- 1) $\mathcal{S} \cong \mathbb{R}^{m+n}$, x is the origin point;
- 2) $\mathcal{T} \cong \mathbb{R}^m$, x is also the origin point;
- 3) $\beta|_{\mathcal{S}}$ is given canonically by

$$\beta: (x_1, \cdots, x_m, y_1, \cdots, y_n) \mapsto (x_1, \cdots, x_m).$$

4) \mathcal{T} is contained in \mathcal{U} and $\overline{\mathcal{T}} \subset \mathcal{U}$ is compact.

So, for two different points $g, h \in S$, if $\beta(g) \neq \beta(h)$, one is able to find a curve c connecting g and h. Moreover, this curve can be chosen so that it lies entirely in S, such that $\bar{c} = \beta \circ c$, which lies in \mathcal{T} , is a curve without self-intersections. Let $G_x = \alpha^{-1}(x) \cap \beta^{-1}(x)$ be the isotropic group at x. Find the two open sets

Let $G_x = \alpha^{-1}(x) \cap \beta^{-1}(x)$ be the isotropic group at x. Find the two open sets \mathcal{N} and \mathcal{O} as claimed by Lemma 2.3. Of course we can assume that they are both simply connected.

Write $G_0 = G_x \cap S$, which is a closed subset of S. Write $\mathcal{N}_0 = \mathcal{N} \cap S$, which is an open set of G_0 , where G_0 has the relative topology coming from S. Therefore, for each point $h \in \mathcal{N}_0$, one is able to find an open ball B(h) of S with h at the center, such that

$$B(h) \cap \mathcal{N}_0 \subset \mathcal{N}_0$$
.

Now, let

$$\mathcal{W} \triangleq \bigcup_{h \in \mathcal{N}_0} B(h) \subset \mathcal{S}.$$

We claim that this \mathcal{W} is just what we need. In fact, for each $g \in \mathcal{W}$, there are possibly two cases:

Case 1) $\beta(g) = x$, i.e., $g \in G_x \cap \mathcal{W} = \mathcal{N}_0$. Then by Lemma 2.3, we find an $X_x \in \mathcal{O} \subset A_x$, such that $\exp(X_x) = g$. Then extend X_x arbitrarily to a section $X \in \Gamma(A)$ with compact support contained in \mathcal{U} and we get a bisection $\exp X$ through g.

Case 2) $\beta(g) \neq x$. In this case, one can of course find a smooth curve c: [0,1] $\rightarrow \mathcal{W}$, such that c(0) = x, c(1) = g, and more importantly, the curve $\overline{c} = \beta \circ c$ which connects x with $\beta(g)$, is a curve who does not intersect with itself. Then by Lemma 5.1, we also obtain an $X \in \Gamma(A)$ with compact support contained in \mathcal{U} and we get a bisection exp X through g.

Both of the two kinds of X we constructed vanish outside of \mathcal{U} . Hence the bisection exp X maps every $y \in \mathcal{U}^c$ to itself. \Box

With these preparations, we are able to prove the strong version main theorem.

Proof of Theorem 3.2. We choose a smooth curve $c : [0,1] \to \alpha^{-1}(x)$ which lies in $\widetilde{\mathcal{U}}$ such that $c(0) = \alpha(g), c(1) = g$, and hence the open set \mathcal{U} covers the base curve $\overline{c} = \beta \circ c$ which connects $\alpha(g)$ and $\beta(g)$.

For each $t \in [0, 1]$, consider the point $\bar{c}(t) \in M$. Set $\mathcal{U} = M$, Lemma 5.2 says that there is neighborhood $\mathcal{W}_t \subset \alpha^{-1}(\bar{c}(t))$ near $\bar{c}(t)$, such that for each $h \in \mathcal{W}_t$, there is a bisection through h.

Now, these $l_{c(t)}\mathcal{W}_t$ become an open coverage of the curve c. Since the interval [0, 1] is compact, we find the following data (see Pic. 7):

1) a partition of [0, 1]:

$$0 = t_0 < t_1 < t_2 < \dots < t_{k-1} < t_k = 1;$$

- 2) finitely some open sets $\mathcal{W}_i = \mathcal{W}_{t_i}$, $i = 0, 1, \dots, k$, such that $\bar{c}(t_i) \in \mathcal{W}_i$; and hence the collection of $l_{c(t_i)}\mathcal{W}_i$ $(i = 0, 1, \dots, k)$ finitely covers the curve c;
- 3) some points

$$a_1 \in (t_0, t_1), \quad a_2 \in (t_1, t_2), \quad \cdots, \quad a_k \in (t_{k-1}, t_k),$$

such that

$$c(a_1) \in \mathcal{W}_0 \cap l_{c(t_1)} \mathcal{W}_1, \quad c(a_2) \in l_{c(t_1)} \mathcal{W}_1 \cap l_{c(t_2)} \mathcal{W}_2,$$

$$\cdots, \quad c(a_k) \in l_{c(t_{k-1})} \mathcal{W}_{k-1} \cap l_{c(t_k)} \mathcal{W}_k.$$



By the last condition, we know that there is a bisection s_1 through $c(a_1)$. And since $l_{c(t_1)^{-1}}c(a_1) \in \mathcal{W}_1$, there exists a bisection w_1 through $l_{c(t_1)^{-1}}c(a_1)$. Similarly, since $l_{c(t_{i-1})^{-1}}c(a_i) \in \mathcal{W}_{i-1}$, there is a bisection s_i through $l_{c(t_{i-1})^{-1}}c(a_i)$. And since $l_{c(t_i)^{-1}}c(a_i) \in \mathcal{W}_i$, there exists a bisection w_i through $l_{c(t_i)^{-1}}c(a_i)$. \cdots We can require these bisections s_1, \cdots, s_k , and w_1, \cdots, w_k are the identity maps on \mathcal{U}^c and they are all of the forms $s_i = \exp X_i, w_i = \exp Y_i$, for some $X_i, Y_i \in \Gamma(A)$ with compact supports in \mathcal{U} .

We now show that the section $s_1w_1^{-1}s_2w_2^{-1}\cdots s_kw_k^{-1}$ is a bisection through g. We notice the following inductive formulas, hold for all $i = 1, \cdots, k$:

- 1) $\beta \circ s_i(\bar{c}(t_{i-1})) = \bar{c}(a_i);$
- 2) $\beta \circ w_i(\bar{c}(t_i)) = \bar{c}(a_i).$

Using these, one is able to get

3)
$$w_i^{-1}(\bar{c}(a_i)) = w_i(\bar{c}(t_i))^{-1} = c(a_i)^{-1}c(t_i);$$

4) $(s_i w_i^{-1})(\bar{c}(t_{i-1})) = l_{c(t_{i-1})^{-1}}c(t_i) = c(t_{i-1})^{-1}c(t_i).$

And hence we obtain

$$s_1 w_1^{-1} s_2 w_2^{-1} \cdots s_k w_k^{-1}(x)$$

$$= s_1 w_1^{-1}(\bar{c}(t_0)) s_2 w_2^{-1}(\bar{c}(t_1)) \cdots s_i w_i^{-1}(\bar{c}(t_i-1)) \cdots s_k w_k^{-1}(\bar{c}(t_k-1))$$

$$= c(t_1) c(t_1)^{-1} c(t_2) \cdots c(t_{i-1})^{-1} c(t_i) \cdot c(t_{k-1})^{-1} c(t_k)$$

$$= c(t_k) = g.$$

It is also easy to check that s(y) = y for all $y \in \mathcal{U}^c$. Since $w_k^{-1} = \exp(-Y_i)$, we know that $s_1 w_1^{-1} s_2 w_2^{-1} \cdots s_k w_k^{-1}$ is just the bisection we need. This completes the proof.

Acknowledgements

We are grateful to the organizers of the Summer School and Conference on Poisson Geometry (2005) and the International Center of Theoretical Physics in Trieste of Italy, for their hospitality while part of the work was being done.

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