The Sectional Curvature Remains Positive when Taking Quotients by Certain Nonfree Actions[*]

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Abstract

We study some cases when the sectional curvature remains positive under the taking of quotients by certain nonfree isometric actions of Lie groups. We consider the actions of the groups S^1 and S^3 such that the quotient space can be endowed with a smooth structure using the fibrations $S^3/S^1 \simeq S^2$ and $S^7/S^3 \simeq S^4$. We prove that the quotient space carries a metric of positive sectional curvature, provided that the original metric has positive sectional curvature on all 2-planes orthogonal to the orbits of the action.

Introduction

In this paper we consider a certain method for constructing closed Riemannian manifolds of positive sectional curvature. There are only few known examples of such manifolds (one can find a list of all known examples, for example, in the introduction to the paper [Ba]).

The main tool for constructing new examples is the taking of quotients by free isometric actions of Lie groups. More precisely, suppose that a Lie group G acts on a closed smooth Riemannian manifold M by isometries; i.e., the metric of M is preserved by the action. Suppose further that this action is free; i.e., for every point $p \in M$ and every element $g \in G$ such that $g \neq 1$ we have $g \cdot p \neq p$. In this case the quotient space M/G has the structure of a smooth manifold such that the projection map $\tau : M \to M/G$ is smooth.

Introduce a metric on M/G as follows. Let X and Y be two vectors tangent to M/G at the point q, and let $q = \tau(p)$ for some $p \in M$. Then there exist unique vectors X' and Y' tangent to M at the point p that are orthogonal to the orbit of G and mapped by τ to X and Y, respectively. Let g be the metric of M. Put $g_*(X, Y) = g(X', Y')$. Since G acts by isometries, this relation defines correctly some metric g_* on M/G, which we call the *quotient metric* of the metric g.

To compute the curvature of the quotient metric, one uses the curvature formulas for Riemannian submersions. A map $f : M \to N$ between manifolds is a *submersion* iff it is surjective together with its differential at each point. Let f be a submersion between Riemannian manifolds (M, g_M) and (N, g_N).

Consider two vectors X' and Y' at a point $p \in M$ that are orthogonal to the submanifold $f^{-1}\big(f(p)\big)$. Suppose that f maps them to the respective vectors X and Y. The map f is a *Riemannian submersion* iff $g_M(X', Y') = g_N(X, Y)$ for all X' and Y'. If f is a Riemannian submersion and R_M and R_N are the curvature tensors of the manifolds M and N, then the relations on the curvature found in [O'Ne] imply that

$$R_M(X', Y', Y', X') \le R_N(X, Y, Y, X).$$

The projection map $\tau : M \to M/G$ is a Riemannian submersion. Therefore, we obtain the following fact: *if a Lie group G acts freely by isometries and the metric of M has positive sectional curvature on all 2-planes orthogonal to the orbits of G, then the quotient metric on M/G has positive sectional curvature.*

In this article we prove a more general version of the above statement:

Theorem. *Let M be a smooth manifold with an action of a group G, the latter being either S^1 or $S^3 = Sp(1)$. Denote by N the fixed point set of the action and by $\tau : M \to M/G$ the projection map. Let $W \subset M$ be a neighborhood of N. Assume that M is endowed with a metric $(\cdot, \cdot)_0$ preserved by the action of G, and the following conditions hold:*

1. *The metric $(\cdot, \cdot)_0$ has positive sectional curvature on all 2-planes orthogonal to the orbits of G.*

2. *For each point $p \in M$, its isotropy group is either trivial or the whole G.*

3. *The space N is a compact submanifold of M of codimension $2 \cdot (1 + \dim G)$.*

Then the quotient space M/G posesses a smooth structure such that τ is smooth at the free points of the action; moreover, this space carries a metric of positive sectional curvature that coincides with the quiotient metric of $(\cdot, \cdot)_0$ outside W.

To construct the smooth structure on M/G, we use infinite cones over the Hopf fibration $S^3/S^1 \simeq S^2$ and the fibration $S^7/S^3 \simeq S^4$. Some problems arise from the fact that the quotient metric for the standard metric on the sphere in both of the fibrations is the standard metric on the sphere of radius $1/2$; therefore, we have a conic singularity in the fixed point set N.

In the first section we give a construction which enables us to smooth the quotient metric in a neighborhood of N. In the second and third sections we construct some particular metric on the normal bundle over N and analyze its curvature. In the fourth section we show that, under certain conditions, one can glue two metrics in a neighborhood of a submanifold with positivity of sectional curvature preserved. Finally, in the fifth section we construct the smooth structure on M/G and the sought metric of positive sectional curvature.

1   Smoothing the Conic Metric

Consider the following metric on $\mathbb{R}^n$:

$$g = dr^2 + g^2(r) d\varphi^2. \tag{1.1}$$

Here $r(x)$ is the length of the vector x, $d\varphi^2$ is the standard metric of sectional curvature 1 on the sphere rS^{n-1}, and $g(r)$ is some positive function in $C^2(0, \infty)$ with the third derivative at zero.

The considered metric is C^2-smooth iff the following conditions hold:

$$g(0) = 0, \quad g'(0) = 1, \quad g''(0) = 0. \tag{1.2}$$

By straightforward computations using the formulas for curvature of Riemannian submersions, one can prove (this fact is, however, not used in this paper directly) that the considered metric has positive sectional curvature iff the following conditions hold:

$$g'''(0) < 0, \quad g''(r) < 0, \quad |g'(r)| < 1 \ \text{ for } \ r > 0. \tag{1.3}$$

Now, we construct auxiliary functions g_ε to be used later. Though the existence of such functions looks quite natural from the geometrical point of view (see the figure below), the formal proof faces certain difficulties; therefore, the proof is described at length.

First, consider the function $g_0(r) = r(1 - r^2)$ for $0 \leq r < 1/2$. It satisfies (1.2), (1.3), and the inequality $g_0'(r) \geq 1/4$. Next, for each $\delta > 0$ consider a *mollifier* $\omega_\delta(x)$, i.e., an even nonnegative smooth function with support $[-\delta, \delta]$ such that its integral over the whole real axis equals 1. Given functions f and g defined on the whole real axis, denote by $f * g$ their convolution, i.e., the integral

$$(f * g)(x) = \int\limits_{-\infty}^{\infty} f(t)g(x - t)\, dt.$$

Lemma 1.1.

1. *If a function f is Lipshitz continuous and has the derivative f' almost everywhere, then $(f * \omega_\delta)' = f' * \omega_\delta$.*

2. *If a function f is concave, then such is the function $f * \omega_\delta$.*

3. *$1 * \omega_\delta = 1$ and $x * \omega_\delta = x$.*

4. *If $f''' = 0$ everywhere, then $f * \omega_\delta = f + c$ for some constant c.*

Proof. 1. It suffices to apply the Lebesgue dominated convergence theorem.

2. The function f is concave iff the following inequality holds for $x, y \in \mathbb{R}$ and $\lambda \in [0, 1]$:
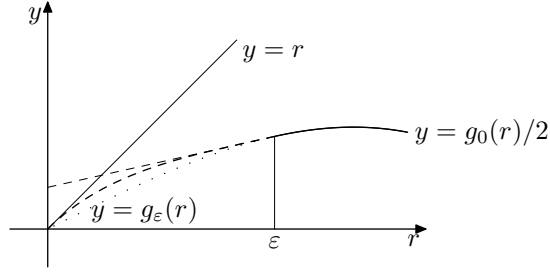$$f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$$

If follows from the definition of convolution that the same inequality holds for the function $f * w_\delta$.

3. The first statement is obvious. The second one follows from the formulas

$$(x * \omega_\delta)(0) = 0 \ \text{ and } \ (x * \omega_\delta)' = x' * \omega_\delta.$$

4. Since $f'(x)$ is a linear function, we have $(f * \omega_\delta)'(x) = f'(x)$ and thus $f * \omega_\delta = f + c$. $\qquad\square$

Lemma 1.2. *For each $\varepsilon > 0$, there exists a function $g_\varepsilon \in C^2(0, 1/2)$ with the following properties:*

1. g_ε satisfies (1.2), $g_\varepsilon''(r) \le -r$, and $g_\varepsilon'(r) \le g_0'(r)$.

2. If $r \ge \varepsilon$, then $g_\varepsilon(r) = g_0(r)/2$.

Proof. We may assume that $\varepsilon < 1/2$. Consider the set $\mathcal{A}$ of all functions of class $C^2[0, \varepsilon]$ that satisfy the relations

$$g(0) = 0, \quad g''(0) = 0, \quad g''(r) \le -r, \tag{1.4}$$
$$g'(r) \le g_0'(r) + g'(0) - 1. \tag{1.5}$$

It is clear that $\mathcal{A}$ is convex. Consider the operator $T_\varepsilon : \mathcal{A} \to \mathbb{R}^4$ defined as follows:

$$T_\varepsilon(g) = \big(g(\varepsilon), g'(\varepsilon), g''(\varepsilon), g'(0) \big).$$

It suffices to prove that the set $T_\varepsilon(\mathcal{A})$ contains the point

$$A = \left(\frac{g_0(\varepsilon)}{2}, \frac{g_0'(\varepsilon)}{2}, \frac{g_0''(\varepsilon)}{2}, 1 \right).$$

Since this set is convex and has finite dimension, it suffices to check that it is dense in a neighborhood of the point A. Consider the auxiliary function $\varphi(r) = (r^3 - 3\varepsilon^2 r + 2\varepsilon^3)/6$. Note that $\varphi''(r) = r$ and $\varphi(\varepsilon) = \varphi'(\varepsilon) = 0$.

Take a point $B = (b_0, b_1, b_2, b_3)$. If B and A are sufficiently close to each other, then the following conditions hold:

$$b_2 + \varphi''(\varepsilon) < 0, \quad \varepsilon b_1 + \varphi(0) < b_0 < (b_3 - 1)\varepsilon + g_0(\varepsilon),$$
$$b_1 < b_3 - 1 + g_0'(\varepsilon). \tag{1.6}$$

In order to prove that the set $T_\varepsilon(\mathcal{A})$ is dense in a neighborhood of the point A, we construct a one-parameter family of functions $g_\delta \in \mathcal{A}$ such that $T_\varepsilon(g_\delta) \to B$ when $\delta \to +0$. Construct a function $g_1 \in C^2[0, \varepsilon]$ with the properties

$$g_1'' \le 0, \quad g_1(\varepsilon) = b_0, \quad g_1'(\varepsilon) = b_1, \quad g_1''(\varepsilon) = b_2 + \varphi''(\varepsilon),$$
$$g_1(0) > \varphi(0), \quad g_1'(r) - \varphi'(r) \le b_3 - 1 + g_0'(r). \tag{1.7}$$

It can be sought in the form

$$g_1(r) = b_0 + b_1(r - \varepsilon) - \int_0^{\varepsilon - r} u(t)\, dt.$$

The function $u \in C^1[0, \varepsilon]$ must satisfy the following conditions:

$$u(0) = 0, \quad u'(0) = -(b_2 + \varphi''(\varepsilon)), \quad u' \geq 0,$$

$$\int_0^\varepsilon u(t)\, dt < b_0 - b_1\varepsilon - \varphi(0), \quad u(\varepsilon - r) \leq \varphi'(r) + g_0'(r) + b_3 - 1 - b_1.$$

Such function u exists if the conditions (1.6) hold.

We have $g_1(\varepsilon) - \varphi(\varepsilon) < (b_3 - 1)\varepsilon + g_0(\varepsilon)$ and at the same time $g_1(0) - \varphi(0) > g_0(0)$. Therefore, there is a point $r_0 \in (0, \varepsilon)$ such that

$$g_1(r_0) - \varphi(r_0) = (b_3 - 1)r_0 + g_0(r_0).$$

Consider the following function g_2 defined on the whole real axis:

$$g_2(r) = \begin{cases} (b_3 - 1)r + g_0(r), & r < r_0, \\ g_1(r) - \varphi(r), & r_0 \leq r < \varepsilon, \\ b_0 + b_1(r - \varepsilon) + \dfrac{b_2}{2}(r - \varepsilon)^2, & r \geq \varepsilon. \end{cases}$$

It is obvious that this function is continuous everywhere and C^2-smooth at all points but r_0. Take the convolution $g_\delta = g_2 * \omega_\delta$. It is well-known that $g_\delta \in C^\infty$, and $T_\varepsilon(g_\delta) \to T_\varepsilon(g_2) = B$ as $\delta \to 0$. It remains to prove that $g_\delta \in \mathcal{A}$ for small δ. The equalities $g_\delta(0) = g_\delta''(0) = 0$ follow from Lemma 1.1 (item 1) and from the fact that g_2 is odd in a neighborhood of zero. The inequality $g_\delta''(r) \leq -r$ holds in a neighborhood of zero due to Lemma 1.1 (items 1 and 3), and outside this neighborhood this inequality follows from Lemma 1.1 (item 2) and the concavity of the function $g_2 + \varphi$ for $r > 0$.

We now show that g_δ satisfies (1.5). In a neighborhood of the segment $[0, \varepsilon]$ we have $g_2' \leq b_3 - 1 + g_0'$, so Lemma 1.1 (items 1 and 4) yields

$$g_\delta' = g_2' * \omega_\delta \leq (b_3 - 1 + g_0') * \omega_\delta = g_0' - 1 + g_\delta'(0).$$

$\square$

Lemma 1.3. *Let $0 < t < 1/3$ and $g_0(t_0) = g_\varepsilon(t)$. Then $g_\varepsilon'(t) \leq g_0'(t_0)$ and $7tg_\varepsilon'(t) \geq t_0 g_0'(t_0) + g_\varepsilon(t)$.*

Proof. We have $g_\varepsilon' \leq g_0'$; therefore, $g_\varepsilon \leq g_0$ and $t_0 \leq t$. Thus $g_\varepsilon'(t) \leq g_0'(t) \leq g_0'(t_0)$.

The functions g_0 and g_ε are concave; moreover, $g_0'(0) = g_\varepsilon'(0) = 1$, $g_0(1/2) = 3/8$, and $g_\varepsilon(1/2) = 3/16$; therefore, for $0 \leq t \leq 1/2$ we have

$$\frac{3}{4}t \leq g_0(t) \leq t, \quad \frac{3}{8}t \leq g_\varepsilon(t) \leq t.$$

Now,

$$t_0 g_0'(t_0) \leq t_0 \leq \frac{4}{3} g_0(t_0) = \frac{4}{3} g_\varepsilon(t) \leq \frac{4}{3}t, \quad tg_\varepsilon'(t) \geq tg_\varepsilon'(1/3) \geq \frac{t}{3}.$$

$\square$

2   Certain Metrics on Vector Bundles

Some results of these section follow from [V, theorem 3.5] and [Be, theorem 9.59]. However, we state them together with proofs for completeness of exposition.

Suppose that N is an n-dimensional compact manifold and M is vector bundle over N of rank $m - n$; i.e.,

$$M = \big\{ (p, v) \mid p \in N, \ v \in V_p \big\}. \tag{2.1}$$

Here V_p is an $(m - n)$-dimensional vector space, which we call the *fiber* at the point p. We assume that each V_p is endowed with an inner product $\langle \cdot, \cdot \rangle$ depending smoothly on p. We can embed N into M as $\{v = 0\}$. Denote by $\pi : M \to N$ the projection map. A line of the form $(p, tv)_{t \geq 0}$ is called a *ray emanating from p in the direction v*.

Define the function $r : M \to \mathbb{R}$ as follows:

$$r(p, v) = \sqrt{\langle v, v \rangle}. \tag{2.2}$$

It is obvious that the function r is continuos and belongs to $C^\infty(M^\circ)$, where $M^\circ = M \setminus N$ is the regular part of M. A *neighborhood of N* is a submanifold $\{r < \rho\} \subset M$ for $\rho > 0$.

Note that if $K = \{r = 1\} \in M$, then there is a standard identification $M^\circ \simeq K \times (0, \infty)$. That makes it possible to identify different *radial fibers*, i.e., the submanifolds of the kind $\{r = t\}$ for $t > 0$. Denote by $\partial/\partial r$ the image of the standard vector field 1 on $(0, \infty)$ under this identification. The images of vector fields on K under this identification are called *radial basic fields*. *Vertical fields* are the radial basic fields that are tangent to the fibers. Note that for each radial basic field A we have

$$\left[\frac{\partial}{\partial r}, A \right] = 0. \tag{2.3}$$

Moreover, the Lie bracket of each two radial basic fields is a radial basic field.

A *trivialization* of the bundle M is a pair (U, Θ), where U is an open subset in N and Θ is a diffeomorphism of $U \times \mathbb{R}^{m-n}$ onto $\pi^{-1}(U)$ such that for each $p \in U$, the restriction of Θ to $p \times \mathbb{R}^{m-n}$ is a linear isomorphism onto V_p that maps the standard inner product on $\mathbb{R}^{m-n}$ to the inner product $\langle \cdot, \cdot \rangle$ on V_p. We assume that the bundle M is *locally trivial*; i.e., for each $p \in N$, there exists a trivialization (U, Θ) such that $p \in U$.

Let (U_1, Θ_1) and (U_2, Θ_2) be two trivializations. Then there is a smooth map $\Psi_{12} : U_1 \cap U_2 \to SO(m - n)$ such that for $p \in U_1 \cap U_2$ and $x \in \mathbb{R}^{m-n}$ we have

$$\Theta_2(p, x) = \Theta_1(p, \Psi_{12}(p) \cdot x). \tag{2.4}$$

Consider a metric of the form $dr^2 + g^2(r)d\varphi^2$ on $\mathbb{R}^{m-n}$ as in (1.1). Note that this metric is invariant under the standard action of the group $SO(m-n)$. We call a vector field V on $\mathbb{R}^{n-m}$ a *turn field* iff it has the form $V(x) = A \cdot x$ for some matrix $A \in \mathfrak{so}(m - n)$. Note that turn fields are Killing fields of the action of $SO(m-n)$, so we have the following relation for vector fields A and B and a turn field C:

$$(\nabla_A C, B) + (\nabla_B C, A) = 0. \tag{2.5}$$

Furthermore, a turn field is mapped by each element of $SO(m-n)$ to another turn field, so (2.4) makes it possible to define turn fields on each fiber V_p. A vector field on M is called a turn field iff it is tangent to the fibers and is a turn field on each fiber.

Consider a trivialization (U, Θ). We call the images of vector fields on U under Θ *locally horizontal*. The images under Θ of vector fields on $\mathbb{R}^{m-n}$ that are constant in the Cartesian coordinates are called *locally vertical* fields on M. For a vector field X on M, denote by X' its orthogonal projection to the fibers with respect to some metric on M.

Lemma 2.1. *Suppose that (U, Θ_1) and (U, Θ_2) are two trivializations and X_1 and X_2 are the locally horizontal fields in the first and second trivializations, respectively, that are produced by the same vector field on U. Then $X_1 - X_2$ is a turn field.*

Proof. Consider the field X on U that produces both X_1 and X_2. Let Ψ_{12} be the map that connects the two trivializations by (2.4). Then

$$(X_2 - X_1)_{\Theta_1(p,x)} = d\Theta_{1(p,x)}\big(0, (X\Psi_{12})(p) \cdot \Psi_{12}(p)^{-1} \cdot x\big).$$

Since $\Psi_{12}(p) \in SO(m-n)$, we have $(X\Psi_{12})(p) \cdot \Psi_{12}(p)^{-1} \in \mathfrak{so}(m-n)$ and the lemma follows. $\square$

This lemma shows that the following definition is correct:

Definition 2.1. *Let g be a function satisfying (1.2). A metric $(\cdot, \cdot)$ on M is called proper or g-proper iff the following conditions hold:*

1. *The restriction of the metric $(\cdot, \cdot)$ to each fiber V_p is equal to $dr^2 + g^2(r)d\varphi^2$.*

2. *The orthogonal projection to the fibers of each vector field that is locally horizontal in some trivialization is a turn field.*

Lemma 2.2. *Let $(\cdot, \cdot)$ be a proper metric. Then:*

1. *At every point $p \in N$, the spaces $T_p N$ and $T_p V_p$ are orthogonal.*

2. *Each fiber V_p is totally geodesic.*

Proof. 1. At the points of N, the orthogonal projection of every locally horizontal field to the fiber is zero.

2. Consider a trivialization (U, Θ). Suppose that A and B are two locally vertical vector fields and X is a locally horizontal field. It suffices to prove that $(\nabla_A B, X - X') = 0$. We have

$$[A, X] = [B, X] = [A, B] = 0, \quad X(A, B) = 0,$$
$$(\nabla_A X', B) + (\nabla_B X', A) = 0,$$
$$(A, X) = (A, X'), \quad (B, X) = (B, X'),$$
$$2(\nabla_A B, X) = A(B, X) + B(A, X) - X(A, B),$$
$$2(\nabla_A B, X') = A(B, X') + B(A, X') - (\nabla_A X', B) - (\nabla_B X', A).$$

The proof is complete. $\square$

Consider a trivialization (U, Θ). Assume that we are given:

1. A linear map $Q_p : T_p N \to \mathfrak{so}(m-n)$ that depends smoothly on the point $p \in N$;

2. A function g satisfying (1.2);

3. A metric g_N on N.

Lemma 2.3. *There exists a unique g-proper metric $(\,\cdot\,,\cdot\,)$ on $\pi^{-1}(U)$ such that*

1. For $p \in U$, $x \in \mathbb{R}^{m-n}$, and a locally horizontal field X we have

$$X'_{\Theta(p,x)} = d\Theta_{(p,x)}\big(0, Q_p(X_p) \cdot x\big). \tag{2.6}$$

2. The map π is a Riemannian submersion onto N endowed with the metric g_N.

Furthermore, the submanifold N is totally geodesic with respect to this metric, and the following relations hold at any point $p \in N$ for all locally vertical fields A, B, and C and locally horizontal fields X and Y:

$$A(B,C) = 0, \quad A(X,Y) = 0, \quad A(B,X) = \big\langle Q_p(X_p) \cdot A, B \big\rangle. \tag{2.7}$$

The fields A and B on the right-hand side of the last equation are identified with the vector fields on $\mathbb{R}^{m-n}$ that produces them.

Moreover, if we identify a radial fiber $\{r = t\}$ with $K = \{r = 1\}$ in a standard way, then the metric $(\,\cdot\,,\cdot\,)$ depends only on the map Q_p, the metric g_N, and the value $g(t)$ (and does not explicitly depend on t).

Proof. The fact that the metric is g-proper determines its form on the fibers V_p. The metric $(\,\cdot\,,\cdot\,)$ is given by formulas

$$(A, X) = (A, X'), \quad (X, Y) = (X', Y') + \pi^* g_N(X, Y). \tag{2.8}$$

Here the field A is locally vertical, the fields X and Y are locally horizontal, and the fields X' and Y' are determined from (2.6). Since Q_p is a linear map from $T_p N$ to $\mathfrak{so}(m-n)$, (2.8) defines a smooth proper Riemannian metric. The equalities (2.7) can be checked directly. The fact that N is totally geodesic is equivalent to the following equation holding at the points of N:

$$(\nabla_X X, A) = 0. \tag{2.9}$$

But at these points we have

$$(\nabla_X X, A) = X(A, X) - (X, \nabla_X A) = -(X, \nabla_A X) = -\frac{1}{2} A(X, X). \tag{2.10}$$

It follows that (2.9) is equivalent to the second equation in (2.7). The last statement of the lemma is verified directly using (2.8) and the form of the metric on the fibers. $\qquad\square$

Let M_0 be a manifold endowed with a Riemannian metric $(\cdot,\cdot)_0$; let N be its compact totally geodesic submanifold. Define g_N as the restriction of the metric $(\cdot,\cdot)_0$ to N. Consider as M the *normal bundle* over N; i.e., let

$$V_p = T_p N^\perp = \{v \in T_p M_0 \mid v \perp T_p N\}.$$

Take the inner product $\langle\cdot,\cdot\rangle$ on the fiber V_p to be the restriction of the metric $(\cdot,\cdot)_0$ to $T_p N^\perp$. Consider the map $\Phi : M \to M_0$ defined as follows:

$$\Phi(p,v) = \exp_p v. \tag{2.11}$$

It is well-known that this map is a diffeomorphism for small r. Using Φ, we pull back the metric $(\cdot,\cdot)_0$ to a neighborhood of N in M.

Lemma 2.4. *Let g be a function satisfying (1.2). Then there exists unique g-proper metric $(\cdot,\cdot)_1$ on M such that the 1-jets of the metrics $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ coincide at the points of N, and π is a Riemannian submersion, if we take the metric g_N on N. Furthermore, if we identify some radial fiber $\{r = t\}$ with K, then the metric $(\cdot,\cdot)_1$ on this fiber depends only on the metric $(\cdot,\cdot)_0$ and the value $g(t)$ (and does not depend explicitly on t). Moreover, for each field X, its orthogonal projection to the fibers X' is independent of the function g.*

Proof. It suffices to prove local existence and uniqueness of the metric $(\cdot,\cdot)_1$. Let (U,Θ) be some trivialization on M. Suppose that the vector fields A, B, and C are locally vertical and X and Y are locally horizontal. It is clear that, if the sought metric exists, then it may be constructed using Lemma 2.3 for some map Q_p. In this case the restrictions of the metrics $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ to N conincide automatically, so in order to make the 1-jets of the two metrics coincide at the points of N, it suffices to obtain the equalities (2.7) together with the following equalities for the metric $(\cdot,\cdot)_0$ at the points of N:

$$(A,X)_0 = 0, \quad (A,B)_0 = \langle A,B\rangle \tag{2.12}$$

It can be seen from the construction of the normal bundle that (2.12) and the first equation in (2.7) hold. The second equation in (2.7) follows from the fact that N is totally geodesic (see (2.9) and (2.10)). The last equation in (2.7) is equivalent to the following one:

$$\langle Q_p(X_p) \cdot A, B\rangle = A(B,X)_0. \tag{2.13}$$

But the expression on the right-hand side depends multilinearly on A, B, and X, so (2.13) defines the unique map Q_p. It remains to prove that the map Q_p defined above takes its values in $\mathfrak{so}(m-n)$, i.e., that the following equality holds at the points of N:

$$A(A,X)_0 = 0.$$

Indeed, $\nabla_A A = 0$ at the points of N, so

$$A(A,X)_0 = (A,\nabla_A X)_0 = (A,\nabla_X A)_0 = \frac{1}{2}X(A,A)_0 = 0.$$

The last two statements of this lemma follow from the last statement of the previous lemma, (2.6) and (2.13). $\qquad\square$

We now establish some properties of proper metrics that will be useful in computing the curvature. Consider a vector field X on N. There exists a unique field $\widehat{X}$ on M which is π related to X and orthogonal to the fibers; we call it a *basic field* (produced by the field X). Note that the vector $\partial/\partial r$ is orthogonal to the radial fibers, so all basic fields are tangent to the radial fibers.

Lemma 2.5. *If $(\cdot\,,\cdot)$ is a proper metric on M, then*

1. *For all radial basic fields A and X such that A is tangent to the fibers, the value $(A, X)/(A, A)$ is constant on every ray.*

2. *If a field X is basic, then it is radial basic.*

Proof. 1. Consider a trivialization (U, Θ). For radial basic fields X which are tangent to the fiber, the statement follows from the form of the metric on the fibers, so we may assume that X is locally horizontal. Then $(A, X) = (A, X')$ and X' is a turn field; therefore, it is tangent to the fibers and is radial basic.

2. Consider the radial basic field Y that coincides with X on the radial fiber $\{r = t\}$ for some t. This is possible since X is tangent to the radial fibers. Now, X and Y are π related to the same field on N, and Y is orthogonal to the fibers due to the first statement of this lemma. Therefore, $X = Y$. $\qquad\square$

Lemma 2.6. *Let $(\cdot\,,\cdot)$ be a proper metric and let A and B be radial basic fields. Denote by II the second fundamental form of the radial fibers. Then the following equalities hold:*

$$\left(\mathrm{II}(A, B), \frac{\partial}{\partial r} \right) = -\frac{1}{2} \frac{\partial}{\partial r}(A, B), \tag{2.14}$$

$$\left(\frac{DA}{\partial r}, B \right) = \frac{1}{2} \frac{\partial}{\partial r}(A, B), \tag{2.15}$$

$$\left(\frac{DA}{\partial r}, \frac{\partial}{\partial r} \right) = 0. \tag{2.16}$$

Proof. The statement of the lemma follows from the equalities

$$\left(\mathrm{II}(A, B), \frac{\partial}{\partial r} \right) = \left(\nabla_A B, \frac{\partial}{\partial r} \right) = \left(\nabla_B A, \frac{\partial}{\partial r} \right),$$

$$\left(\nabla_A B, \frac{\partial}{\partial r} \right) = A\left(B, \frac{\partial}{\partial r} \right) - \left(B, \nabla_A \frac{\partial}{\partial r} \right) = -\left(B, \frac{DA}{\partial r} \right),$$

$$\frac{\partial}{\partial r}(A, B) = \left(A, \frac{DB}{\partial r} \right) + \left(B, \frac{DA}{\partial r} \right),$$

$$\left(\frac{DA}{\partial r}, \frac{\partial}{\partial r} \right) = \left(\nabla_A \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) = \frac{1}{2} A\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right) = 0.$$

$\square$

3   The Curvature of a Certain Family of Proper Metrics

As in Lemma 2.4, suppose that M_0 is a manifold with a given metric $(\cdot\,,\cdot)_0$, N is its totally geodesic submanifold, and M is the normal bundle over N. The

metric $(\cdot\,,\cdot)_0$ is pulled back to M via the map Φ from (2.11). We also assume that g is a given function satisfying (1.2), and $L \geq 0$ is a constant. Using Lemma 2.4, consider the g proper metric $(\cdot\,,\cdot)_1$ on M such that the map π is a Riemannian submersion. Construct the metric $(\cdot\,,\cdot)$ in a neighborhood of N by the formula

$$(X, Y) = (X, Y)_1 - Lr^2 \cdot (\pi^* g_N)(X, Y). \qquad (3.1)$$

Here X and Y are arbitrary vectors. It is clear that the constructed metric is smooth. It is positive definite for $r^2 < 1/L$ and its 1-jets at the points of N coincide with those of the metric $(\cdot\,,\cdot)_0$. Furthermore, on each radial fiber $\{r = t\}$ the metric $(\cdot\,,\cdot)$ is determined by the metric $(\cdot\,,\cdot)_0$, the constant L, and the value $g(t)$. Moreover, for each field X its orthogonal projection X' to the fibers depends only on the original metric $(\cdot\,,\cdot)_0$ (that is, it does not depend on the function g and the constant L). Thus, the metric $(\cdot\,,\cdot)_0$ defines what fields are basic in the metric $(\cdot\,,\cdot)$.

The map π restricted to each radial fiber is a Riemannian submersion with totally geodesic fibers from M endowed with the metric $(\cdot\,,\cdot)$ to N endowed with the metric $(1 - Lr^2)g_N$. Therefore (see [Be, p. 9 C]), for all basic fields X and Y and vertical fields A and B we have

$$[A, X] \in T(V_p), \quad (\nabla_X Y, A) = \frac{1}{2}\big(A, [X, Y]\big),$$
$$(\nabla_A X, B) = 0, \quad (\nabla_X A, B) = \big([X, A], B\big), \qquad (3.2)$$
$$(\nabla_X A, Y) = -\frac{1}{2}\big(A, [X, Y]\big) = (\nabla_A X, Y).$$

Denote the covariant derivative and curvature tensor of the radial fibers by the symbols $\overline{\nabla}$ and $\overline{R}$. It is well-known that if the fields A and B are tangent to the radial fibers, then

$$\nabla_A B = \overline{\nabla}_A B + \mathrm{II}(A, B), \qquad (3.3)$$
$$R(A, B, B, A) = \overline{R}(A, B, B, A)$$
$$-\big(\mathrm{II}(A, A), \mathrm{II}(B, B)\big) + \big(\mathrm{II}(A, B), \mathrm{II}(A, B)\big). \qquad (3.4)$$

Suppose that the fields A and B are vertical, and the fields X and Y are basic. It follows from (2.15) that

$$\left(\frac{DA}{\partial r}, B\right) = \frac{1}{2}\frac{\partial}{\partial r}(A, B) = \frac{g'(r)}{g(r)}(A, B),$$
$$\left(\frac{DX}{\partial r}, A\right) = \left(\frac{DA}{\partial r}, X\right) = \frac{1}{2}\frac{\partial}{\partial r}(X, A) = 0,$$
$$\left(\frac{DX}{\partial r}, Y\right) = \frac{1}{2}\frac{\partial}{\partial r}(X, Y) = -\frac{Lr}{1 - Lr^2}(X, Y),$$

From here and (2.16) we obtain

$$\frac{DA}{\partial r} = \frac{g'(r)}{g(r)}A, \qquad (3.5)$$
$$\frac{DX}{\partial r} = -\frac{Lr}{1 - Lr^2}X. \qquad (3.6)$$

We are now ready to compute the sectional curvature of the metric $(\cdot\,,\cdot)$. To this end, consider two vector fields

$$E = a\frac{\partial}{\partial r} + A + X, \quad F = b\frac{\partial}{\partial r} + B + Y.$$

We assume that the vector fields A and B are vertical, the fields X and Y are basic, and $[A, B] = 0$. We also assume that the vectors E and F form an orthonormal system at the point where the curvature is computed. We introduce the fields

$$P = A + X, \quad Q = B + Y, \quad \widehat{A} = a\frac{\partial}{\partial r} + A, \quad \widehat{B} = b\frac{\partial}{\partial r} + B,$$
$$C = aB - bA, \quad Z = aY - bX, \quad G = C + Z. \tag{3.7}$$

We have

$$R(E, F, F, E) = R(P, Q, Q, P) + R\left(\frac{\partial}{\partial r}, G, G, \frac{\partial}{\partial r}\right) - 2R\left(\frac{\partial}{\partial r}, G, P, Q\right). \tag{3.8}$$

For each radial basic field H we have

$$R\left(\frac{\partial}{\partial r}, H, H, \frac{\partial}{\partial r}\right) = \left(\frac{D}{\partial r}\nabla_H H, \frac{\partial}{\partial r}\right) - \left(\nabla_H \frac{DH}{\partial r}, \frac{\partial}{\partial r}\right)$$
$$= \frac{\partial}{\partial r}\left(\nabla_H H, \frac{\partial}{\partial r}\right) + \left|\frac{DH}{\partial r}\right|^2$$
$$= \left|\frac{DH}{\partial r}\right|^2 - \frac{1}{2}\frac{\partial^2}{\partial r^2}(H, H).$$

From here, (3.5), and (3.6) we obtain

$$R\left(\frac{\partial}{\partial r}, G, G, \frac{\partial}{\partial r}\right) = -\frac{g''(r)}{g(r)}(C, C) + \left(\frac{L}{1 - Lr^2} + \frac{(Lr)^2}{(1 - Lr^2)^2}\right)(Z, Z). \tag{3.9}$$

Furthermore,

$$R\left(A, B, C, \frac{\partial}{\partial r}\right) = A\left(\nabla_B C, \frac{\partial}{\partial r}\right) - B\left(\nabla_A C, \frac{\partial}{\partial r}\right) + \left(\nabla_A C, \frac{DB}{\partial r}\right) - \left(\nabla_B C, \frac{DA}{\partial r}\right)$$
$$= \frac{1}{2}\frac{\partial}{\partial r}\left(B(A, C) - A(B, C)\right) + \left(\nabla_A C, \frac{DB}{\partial r}\right) - \left(\nabla_B C, \frac{DA}{\partial r}\right)$$
$$= \frac{g'(r)}{g(r)}\left(B(A, C) - A(B, C) + (\nabla_A C, B) - (\nabla_B C, A)\right)$$
$$= 0,$$
$$R\left(\frac{\partial}{\partial r}, Z, X, Y\right) = \frac{\partial}{\partial r}(\nabla_Z X, Y) - Z\left(\frac{DX}{\partial r}, Y\right) + \left(\frac{DX}{\partial r}, \nabla_Z Y\right) - \left(\frac{DY}{\partial r}, \nabla_Z X\right)$$
$$= \frac{Lr}{1 - Lr^2}\left(-2(\nabla_Z X, Y) + Z(X, Y) - (\nabla_Z Y, X) + (\nabla_Z X, Y)\right)$$

$$= 0,$$

$$R\left(A, B, Z, \frac{\partial}{\partial r}\right) = \frac{1}{2}\frac{\partial}{\partial r}\left(B(A, Z) - A(B, Z)\right) + \left(\nabla_A Z, \frac{DB}{\partial r}\right) - \left(\nabla_B Z, \frac{DA}{\partial r}\right)$$

$$= \frac{g'(r)}{g(r)}\left((\nabla_A Z, B) - (\nabla_B Z, A)\right) = 0,$$

$$R\left(\frac{\partial}{\partial r}, C, A, Y\right) = \left(\frac{D}{\partial r}\nabla_C A, Y\right) - \left(\nabla_C \frac{DA}{\partial r}, Y\right) = 0,$$

$$R\left(\frac{\partial}{\partial r}, C, X, Y\right) = \frac{\partial}{\partial r}(\nabla_C X, Y) - C\left(\frac{DX}{\partial r}, Y\right) + \left(\frac{DX}{\partial r}, \nabla_C Y\right) - \left(\frac{DY}{\partial r}, \nabla_C X\right)$$

$$= -\left(\frac{Lr}{1 - Lr^2} + \frac{g'(r)}{g(r)}\right)(C, [X, Y]),$$

$$R\left(\frac{\partial}{\partial r}, Z, A, Y\right) = \frac{\partial}{\partial r}(\nabla_Z A, Y) - Z\left(\frac{DA}{\partial r}, Y\right) + \left(\frac{DA}{\partial r}, \nabla_Z Y\right) - \left(\frac{DY}{\partial r}, \nabla_Z A\right)$$

$$= \frac{1}{2}\frac{\partial}{\partial r}(A, [Y, Z]) + \frac{1}{2}\left(\frac{Lr}{1 - Lr^2} - \frac{g'(r)}{g(r)}\right)(A, [Y, Z])$$

$$= \frac{1}{2}\left(\frac{Lr}{1 - Lr^2} + \frac{g'(r)}{g(r)}\right)(A, [Y, Z]).$$

Hence, we find that

$$R\left(\frac{\partial}{\partial r}, G, P, Q\right)$$

$$= \frac{1}{2}\left(\frac{Lr}{1 - Lr^2} + \frac{g'(r)}{g(r)}\right)\left(-2(C, [X, Y]) + (A, [Y, Z]) - (B, [X, Z])\right)$$

$$= -\frac{3}{2}\left(\frac{Lr}{1 - Lr^2} + \frac{g'(r)}{g(r)}\right)(C, [X, Y]). \tag{3.10}$$

Finally, from (2.14) and (3) we conclude that

$$R(P, Q, Q, P)$$

$$= \bar{R}(P, Q, Q, P) - \left(\frac{g'(r)}{g(r)}(A, A) - \frac{Lr}{1 - Lr^2}(X, X)\right)\left(\frac{g'(r)}{g(r)}(B, B) - \frac{Lr}{1 - Lr^2}(Y, Y)\right)$$

$$+ \left(\frac{g'(r)}{g(r)}(A, B) - \frac{Lr}{1 - Lr^2}(X, Y)\right)^2$$

$$= \bar{R}(P, Q, Q, P) - \left(\frac{g'(r)}{g(r)}\right)^2\left((A, A)(B, B) - (A, B)^2\right)$$

$$- \left(\frac{Lr}{1 - Lr^2}\right)^2\left((X, X)(Y, Y) - (X, Y)^2\right)$$

$$+ \frac{g'(r)}{g(r)}\frac{Lr}{1 - Lr^2}\left((A, A)(Y, Y) + (B, B)(X, X) - 2(A, B)(X, Y)\right). \tag{3.11}$$

To obtain the formula for the curvature at the points of M°, we have to insert (3.9)–(3.11) into (3.8).

Lemma 3.1. *Let $p \in N \subset M$. Consider two vectors E and F at the point p such that $E = A + X$ and $F = B + Y$, where the vectors A and B are tangent to the fiber and the vector fields X and Y are basic. In this case the following formula holds at the point p:*

$$
\begin{aligned}
R(E, F, F, E) = R(X, Y, Y, X) &- g'''(0)\big((A, A)(B, B) - (A, B)^2\big) \\
&+ L\big((A, A)(Y, Y) + (B, B)(X, X) - 2(A, B)(X, Y)\big) \\
&+ 3\big(B, \nabla_A[X, Y]\big).
\end{aligned} \tag{3.12}
$$

Proof. Since the expressions on the right- and on the left-hand sides depend on the values of the fields A and B only at the point p, we can assume without loss of generality that the fields A and B are constant in the Cartesian coordinate system of the fiber. If $A = B = 0$, then (3.12) is obvious. Therefore, henceforth we assume that $A \neq 0$. Let $a = \sqrt{(A, A)_p}$ and $b = (A, B)_p / a$. Consider the ray emanating from p in the direction of the vector A and analyze the limits of some expressions on that ray as $r \to 0$. It is clear that $A = a \partial / \partial r$ on the ray. Next, consider the field $\overline{B} = B - b \partial / \partial r$ on the ray. In that case $\overline{B}$ is tangent to the radial fibers. Applying the formulas for curvature at the regular points of the ray obtained before, we get ($C = a\overline{B}$ and $Z = aY - bX$)

$$
\begin{aligned}
R\bigg(a\frac{\partial}{\partial r} + X, b\frac{\partial}{\partial r} &+ \overline{B} + Y, b\frac{\partial}{\partial r} + \overline{B} + Y, a\frac{\partial}{\partial r} + X \bigg) \\
&= R(X, \overline{B}+Y, \overline{B}+Y, X) - \frac{g''(r)}{g(r)}(C, C) + \bigg(\frac{L}{1-Lr^2} + \frac{(Lr)^2}{(1-Lr^2)^2} \bigg)(Z, Z) \\
&\quad + 3\bigg(\frac{Lr}{1-Lr^2} + \frac{g'(r)}{g(r)} \bigg)(C, [X, Y]).
\end{aligned}
$$

But the value of $R(E, F, F, E)$ at the point p is the limit of this expression as $r \to 0$ and equals

$$
R(X, \overline{B} + Y, \overline{B} + Y, X) - a^2 g'''(0)(\overline{B}, \overline{B}) + L(Z, Z) + 3A\big(\overline{B}, [X, Y]\big). \tag{3.13}
$$

Note that $\big(\partial/\partial r, [X, Y]\big) = 0$. Therefore,

$$
A\big(\overline{B}, [X, Y]\big) = A\big(B, [X, Y]\big).
$$

Using the symmetries of the curvature tensor, from (3.13) we obtain the following equality at the point p:

$$
R(X, \overline{B} + Y, \overline{B} + Y, X) = R(X, Y, Y, X) + L(\overline{B}, \overline{B})(X, X). \tag{3.14}
$$

It remains to put (3.14) into (3.13) and use the equality $(\nabla_A B)_p = 0$. $\qquad\square$

Consider vectors A, B, X, and Y in some vector space with an inner product $\langle \cdot, \cdot \rangle$. Define

$$
V(A, B) = \langle A, A \rangle \langle B, B \rangle - \langle A, B \rangle^2, \tag{3.15}
$$

$$
W(A, B; X, Y) = \langle A, A \rangle \langle Y, Y \rangle + \langle B, B \rangle \langle X, X \rangle - 2\langle A, B \rangle \langle X, Y \rangle. \tag{3.16}
$$

Lemma 3.2. *The following statements are true:*

1. $W(A, B; X, Y) \geq 0$.

2. *If $\{A, B\} \perp \{X, Y\}$ and the vectors $A+X$ and $B+Y$ form an orthonormal system, then*

$$W(A, B; X, Y) \geq \frac{1}{2}\big(\langle A, A\rangle + \langle B, B\rangle\big)\big(\langle X, X\rangle + \langle Y, Y\rangle\big). \qquad (3.17)$$

3. *If $\{A, B\} \perp \{X, Y\}$, then*

$$V(A + X, B + Y) = V(A, B) + V(X, Y) + W(A, B; X, Y). \qquad (3.18)$$

Proof. 1. It follows from the Cauchy–Bunyakovskiĭ inequality that

$$\begin{aligned} W(A, B; X, Y) &\geq \langle A, A\rangle\langle Y, Y\rangle + \langle B, B\rangle\langle X, X\rangle \\ &\quad - 2\sqrt{\langle A, A\rangle\langle B, B\rangle\langle X, X\rangle\langle Y, Y\rangle} \\ &= \big(\sqrt{\langle A, A\rangle\langle Y, Y\rangle} - \sqrt{\langle B, B\rangle\langle X, X\rangle}\,\big)^2. \end{aligned}$$

2. Consider the matrix

$$G_{AB} = \begin{pmatrix} \langle A, A\rangle & \langle A, B\rangle \\ \langle A, B\rangle & \langle B, B\rangle \end{pmatrix}.$$

Let λ and μ be its eigenvalues; it is obvious that $\lambda, \mu \in \mathbb{R}$. Furthermore,

$$\begin{aligned} W(A, B; X, Y) &= \langle A, A\rangle\big(1 - \langle B, B\rangle\big) + \langle B, B\rangle\big(1 - \langle A, A\rangle\big) + 2\langle A, B\rangle^2 \\ &= \langle A, A\rangle + \langle B, B\rangle - 2V(A, B) \\ &= \lambda + \mu - 2\lambda\mu, \end{aligned}$$

$$\begin{aligned} \big(\langle A, A\rangle + \langle B, B\rangle\big)\big(\langle X, X\rangle + \langle Y, Y\rangle\big) \\ = (\lambda + \mu)(2 - \lambda - \mu). \end{aligned}$$

Thus, the required inequality is equivalent to the following:

$$\lambda + \mu - 2\lambda\mu \geq \lambda + \mu - \frac{1}{2}(\lambda + \mu)^2.$$

The last inequality is obvious.

3. This statement is verified by straightforward computations. $\qquad \square$

Lemma 3.3. *The following inequality holds for some constant M_1 depending only on the metric $(\cdot, \cdot)_0$:*

$$\big|[X, Y]'\big| \leq M_1 g(r)\frac{|X| \cdot |Y|}{1 - Lr^2}. \qquad (3.19)$$

Here the fields X and Y are basic.

Proof. Fix the fields on N that produce X and Y. Then the expression on the right-hand side is independent of L (since at a point (p, v) it is equal to $M_1 g(r) |X_p| \cdot |Y_p|$). Moreover, the fields X and Y do not depend on L and g; since the orthogonal projection to the fibers of each fixed vector field is also independent of L and g, the field $[X, Y]'$ depends only on the original metric $(\cdot, \cdot)_0$. Therefore, we may assume that $L = 0$ and $g(r) = r$. It remains to choose the constant M_1 such that the inequality (3.19) holds in that case. This is possible since the value $[X, Y]'$ at the point (p, v) depends only on the values X_p and Y_p bilinearly (and does not depend on the values of X and Y at other points); moreover, at the points of N we have $[X, Y]' = 0$. □

Lemma 3.4. *Let the metric g_N have positive sectional curvature and let $g'''(0) < 0$. Then there exists a constant L_1 such that the metric $(\cdot, \cdot)$ has positive sectional curvature at the points of N provided that $L \geq L_1$.*

Proof. Fix a point $p \in N$. We may assume that the fields A and B are constant in the Cartesian coordinate system of the fibers. Then

$$\big(B, \nabla_A [X, Y]\big)_p = A\big(B, [X, Y]\big)_p.$$

Consider the ray emanating from p in the direction of the vector A and use the previous lemma. We have

$$
\begin{aligned}
A\big(B, [X, Y]\big)_p &= |A| \cdot \frac{\partial}{\partial r}\bigg|_{r=0} \big(B, [X, Y]\big) \\
&= |A| \lim_{r \to 0} \frac{\big(B, [X, Y]\big)}{r} \\
&\leq |A| \cdot |B| \cdot M_1 \cdot |X| \cdot |Y| \lim_{r \to 0} \frac{g(r)}{r} \\
&= M_1 |A| \cdot |B| \cdot |X| \cdot |Y|.
\end{aligned}
$$

Therefore, we have the following inequality at the point p:

$$\big|\big(B, \nabla_A [X, Y]\big)\big| \leq \frac{M_1}{4}\big(|A|^2 + |B|^2\big)\big(|X|^2 + |Y|^2\big).$$

Choose $\delta > 0$ such that $-g'''(0) \geq \delta$ and $R(X, Y, Y, X) \geq \delta V(X, Y)$. Using Lemmas 3.1 and 3.2, we obtain the following inequality for an orthonormal system E and F:

$$
\begin{aligned}
R(E, F, F, E) &\geq \delta\big(V(X, Y) + V(A, B)\big) + L \cdot W(A, B; X, Y) \\
&\quad - 2M_1 \cdot W(A, B; X, Y) \\
&= \delta + (L - \delta - 2M_1) \cdot W(A, B; X, Y).
\end{aligned}
$$

It remains to take $L \geq 2M_1 + \delta$. □

Lemma 3.5. *For various $\varepsilon > 0$, consider the metrics $(\cdot, \cdot)$ constructed using (3.1) from the fixed metric $(\cdot, \cdot)_0$ and the function g_ε from Lemma 1.2. Suppose that the metric g_N has positive sectional curvature. Then there exist constants L_0 and $\rho_0 > 0$ independent of ε and such that for $L \geq L_0$ the metric $(\cdot, \cdot)$ has positive sectional curvature on $\{r < \rho_0 / L\}$.*

Proof. The positivity of the sectional curvature at the points of N follows from the previous lemma, so we have only to establish it at the points of M°. For some constant $\lambda \in \mathbb{R}$ consider the metric $(\cdot\,,\cdot)_3$ constructed using (3.1) from the fixed metric $(\cdot\,,\cdot)_0 + \lambda \cdot \pi^* g_N$, the function $g = g_0$ from section 1 and a constant L_3 so big that this metric would have positive curvature at the points of N when $\lambda = 0$. In this case the sectional curvature of the metric $(\cdot\,,\cdot)_3$ in some neighborhood of N is bounded from below by some number $\delta_3 > 0$ when λ lies in some neighborhood of zero.

Consider a radial fiber $\{r = t\}$. If t is sufficiently small, we can find the value t_0 such that $g_0(t_0) = g_\varepsilon(t)$. The metric $(\cdot\,,\cdot)$ on the radial fiber $\{r = t\}$ coincides with the metric $(\cdot\,,\cdot)_3$ on the radial fiber $\{r = t_0\}$ for $\lambda = (1 - Lt^2)/(1 - L_3 t_0^2) - 1$, the latter being close to 0 for sufficiently small t and Lt^2. Then the curvature tensor $\bar{R}$ of the restriction of the metric $(\cdot\,,\cdot)$ to the radial fiber $\{r = t\}$ coincides with the curvature tensor $\bar{R}_3$ of the restriction of the metric $(\cdot\,,\cdot)_3$ to the radial fiber $\{r = t_0\}$; thus, from (3.11) we conclude that

$$R(P,Q,Q,P) \geq \delta_3 V(P,Q) + \frac{g_0'(t_0)^2 - g_\varepsilon'(t)^2}{g_\varepsilon(t)^2} V(A,B)$$
$$+ \left(\left(\frac{L_3 t_0}{1 - L_3 t_0^2} \right)^2 - \left(\frac{Lt}{1 - Lt^2} \right)^2 \right) V(X,Y)$$
$$+ \left(\frac{g_\varepsilon'(t)}{g_\varepsilon(t)} \frac{Lt}{1 - Lt^2} - \frac{g_0'(t_0)}{g_0(t_0)} \frac{L_3 t_0}{1 - L_3 t_0^2} \right) W(A,B;X,Y).$$

Using Lemma 1.3, we obtain the following inequality for large L and small t and Lt:

$$R(P,Q,Q,P) \geq \frac{\delta_3}{2} V(P,Q) + \frac{L}{7} W(A,B;X,Y).$$

It follows from here and Lemma 3.3 that

$$R(E,F,F,E) \geq \frac{\delta_3}{2} V(P,Q) + \frac{L}{7} W(A,B;X,Y)$$
$$+ L(Z,Z) + (C,C) - 4M_1 |C| \cdot |X| \cdot |Y|.$$

But

$$W(A,B;X,Y) + (Z,Z) = W(\widehat{A}, \widehat{B}; X,Y);$$

therefore,

$$R(E,F,F,E) \geq \frac{\delta_3}{2} V(P,Q) + \frac{L}{7} W(\widehat{A}, \widehat{B}; X,Y)$$
$$+ (C,C) - 4M_1 |C| \cdot |X| \cdot |Y|. \tag{3.20}$$

Assume that $W(\widehat{A}, \widehat{B}; X,Y) \leq \delta$ for some $\delta > 0$. We have

$$V(P,Q) + W(\widehat{A}, \widehat{B}; X,Y) + (C,C)$$
$$\geq V(A,B) + (C,C) + V(X,Y) + W(\widehat{A}, \widehat{B}; X,Y) = 1.$$

On the other hand, it follows from Lemma 3.2 (item 2) that the value $|C|\cdot|X|\cdot$ $|Y|$ is infinitesimal as $\delta \to 0$, so for δ sufficiently small we have $R(E, F, F, E) > 0$. If, on the contrary, $W(\widehat{A}, \widehat{B}; X, Y) \geq \delta$, then, recalling that all the values participating in the inequality (3.20) are bounded, we can choose L such that $R(E, F, F, E) > 0$ in this case as well. It remains to choose ρ_0 sufficiently small. $\qquad\square$

4   Gluing Metrics in a Neighborhood of a Submanifold

The proofs of the lemmas in this section base on some ideas taken from [G, statement 2.1].

Lemma 4.1. *Let $\varepsilon > 0$. Then there exists a function $\varphi : (0, +\infty) \to \mathbb{R}$ with the following properties:*

1. *$\varphi \in C^\infty$.*

2. *$0 \leq \varphi(x) \leq 1$ for all x.*

3. *$\varphi(x) = 1$ for $x \leq \delta_1$ and $\varphi(x) = 0$ for $x \geq \delta_2$, where δ_1 and δ_2 are some constants such that $0 < \delta_1 < \delta_2 \leq \varepsilon$.*

4. *$\left| x\varphi'(x) \right| \leq \varepsilon$, $\left| x^2 \varphi''(x) \right| \leq \varepsilon$ for all x.*

Proof. Consider a function $f \in C^\infty(\mathbb{R})$ such that $0 \leq f \leq 1$, $f(x) = 1$ for $x \leq 1$, and $f(x) = 0$ for $x \geq 2$. Choose a constant N such that the following inequalities hold for all x:

$$\left| f'(x) \right| \leq N, \quad \left| f''(x) \right| \leq N.$$

Take $\delta > 0$ and $\lambda \in (0, 1)$ and put

$$\varphi(x) = f\left(\frac{x^\lambda}{\delta} \right).$$

It is clear that, for the fixed λ, one can choose δ such that the function φ would satisfy the third condition. Moreover, it is obvious that the function φ satisfies the first two conditions. We will now show that one can choose λ such that for each δ the function φ would satisfy the fourth condition. We have

$$\varphi'(x) = f'\left(\frac{x^\lambda}{\delta} \right) \delta^{-1} \lambda x^{\lambda-1},$$

$$\varphi''(x) = f''\left(\frac{x^\lambda}{\delta} \right) \delta^{-2} \lambda^2 x^{2\lambda-2} + f'\left(\frac{x^\lambda}{\delta} \right) \delta^{-1} \lambda(\lambda - 1) x^{\lambda-2}.$$

Therefore, it will suffice to satisfy the following inequalities:

$$N\delta^{-1} \lambda x^\lambda \leq \varepsilon,$$
$$N(\delta^{-2} \lambda^2 x^{2\lambda} + \delta^{-1} \lambda(1 - \lambda) x^\lambda) \leq \varepsilon.$$

But we have $\varphi(x) = 0$ for $x^\lambda > 2\delta$, so we need to check the inequalities only in the case $x^\lambda \leq 2\delta$; thus, it is enough to obtain the inequalities

$$N\delta^{-1}\lambda 2\delta \leq \varepsilon,$$
$$N(\delta^{-2}\lambda^2(2\delta)^2 + \delta^{-1}\lambda(1-\lambda)2\delta) \leq \varepsilon$$

or, equivalently,

$$2N\lambda \leq \varepsilon, \quad 2N\lambda(1+\lambda) \leq \varepsilon.$$

Since the left-hand sides of the inequalities converge to zero as $\lambda \to 0$, we can choose λ which satisfies the required conditions. $\qquad\square$

Lemma 4.2. *Define the function* $\psi : \mathbb{R}^n \to \mathbb{R}$ *as follows:* $\psi(x) = \varphi(|x|)$. *Then* $|x| \cdot |\nabla\psi(x)| \leq \varepsilon$ *and* $|x|^2 \cdot |\nabla^2\psi(x)| \leq 2\varepsilon$ *for all* x.

Proof. We have

$$\frac{\partial\psi}{\partial x_i} = \varphi'(|x|)\frac{x_i}{|x|}, \quad \frac{\partial^2\psi}{\partial x_i \partial x_j} = \varphi''(|x|)\frac{x_i x_j}{|x|^2} + \varphi'(|x|)\frac{\delta_{ij}|x|^2 - x_i x_j}{|x|^3};$$

therefore,

$$|x| \cdot \left|\frac{\partial\psi}{\partial x_i}\right| \leq \varepsilon, \quad |x|^2 \cdot \left|\frac{\partial^2\psi}{\partial x_i \partial x_j}\right| \leq 2\varepsilon.$$

$\qquad\square$

Suppose that M is a Riemannian manifold with the metric g_0, N is its compact submanifold, $m = \dim M$, and $n = \dim N$. Define the map $r : M \to \mathbb{R}$ using the following formula:

$$r(\exp_p v) = |v|. \tag{4.1}$$

Here $p \in N$ and $v \in T_p N^\perp$.

Lemma 4.3. *Assume that we have a metric* g_1 *in a neighborhood of* N *such that*

1. *g_0 and g_1 have positive sectional curvature.*

2. *The 1-jets of two metrics coincide at the points of N.*

Then, for each $\rho > 0$, there exists a smooth Riemannian metric g on M having positive sectional curvature and such that $g = g_0$ for $r > \rho$ and $g = g_1$ for r sufficiently small.

Proof. In a neighborhood of N, define the family of metrics

$$\bar{g}(q,s) = (1-s)g_0(q) + sg_1(q), \quad s \in [0,1]. \tag{4.2}$$

Choose the function φ from Lemma 4.1 for some $\varepsilon \leq \rho$. Put

$$g(q) = \bar{g}\Big(q, \varphi\big(r(q)\big)\Big). \tag{4.3}$$

It follows from the properties of the function φ that g is a Riemannian metric which coincides with g_0 for $r > \rho$ and with g_1 for a sufficiently small r. We will show that, for each point $p \in N$, there are an open neighborhood $U(p)$ and a number $\varepsilon(p) > 0$ such that the metric g has positive curvature in $U(p)$ for $\varepsilon \leq \varepsilon(p)$. Having obtained that, we use the compactness of N to conclude that for sufficiently small ε, the metric g has positive sectional curvature in some neighborhood U of the submanifold N independent of ε. We are left with taking ε so small that g would have positive curvature in U and the set $\{r \leq \varepsilon\}$ would lie inside U. Then g coincides with g_0 outside U and therefore has positive curvature outside U. Thus g is the sought metric.

Fix $p \in N$ and consider a coordinate system $(x_1, \ldots, x_m)$ in a neighborhood of p such that $x_i(p) = 0$ and $r^2 = x_1^2 + \cdots + x_{m-n}^2$. Such system can be constructed by any trivialization of the normal bundle over N. It is clear that in this case the equations from Lemma 4.2 hold for the function $\psi(x) = \varphi\big(r(x)\big)$.

We use the following formulas for the sectional curvature:

$$R(X,Y,Y,X) = R_{ijkl}X^iY^jY^kX^l, \quad R_{ijkl} = R^\alpha_{jkl}g_{\alpha i},$$

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ik}}{\partial x^j} - \frac{\partial \Gamma^l_{ij}}{\partial x^k} + \Gamma^l_{j\alpha}\Gamma^\alpha_{ik} - \Gamma^l_{k\alpha}\Gamma^\alpha_{ij},$$

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).$$

Denote by Γ and R the Christoffel symbols and the components of the curvature tensor of the metric g. Also, introduce the notations $\bar{\Gamma}^k_{ij}(s)$ and $\bar{R}^l_{ijk}(s)$ for the Christoffel symbols and the components of the curvature tensor of the metric $\bar{g}$ for some fixed value of the parameter s. Denote by dot the derivative with respect to s. Note that

$$\dot{\bar{g}}_{ij} = g_{1ij} - g_{0ij}, \quad \ddot{\bar{g}}_{ij} = 0.$$

Since the metrics g_0 and g_1 together with their first derivatives coincide at the points of N, we have the following relations at $r = 0$:

$$\dot{\bar{g}}_{ij} = 0, \quad \dot{\bar{g}}^{ij} = 0, \quad \frac{\partial \dot{\bar{g}}_{ij}}{\partial x^k} = 0, \quad \frac{\partial \dot{\bar{g}}^{ij}}{\partial x^k} = 0, \quad \dot{\bar{\Gamma}}^k_{ij} = 0, \quad \ddot{\bar{R}}_{ijkl} = 0.$$

Using the Taylor formula, we conclude that the following inequalities hold for some constant N and sufficiently small r:

$$\big|g_{ij}\big| \leq N, \qquad \big|g^{ij}\big| \leq N, \qquad \left|\frac{\partial g^{ij}}{\partial x^k}\right| \leq N,$$

$$\left|\dot{g}_{ij}\right| \le Nr^2, \quad \left|\dot{g}^{ij}\right| \le Nr^2, \quad \left|\frac{\partial \dot{g}_{ij}}{\partial x^k}\right| \le Nr,$$

$$\left|\Gamma_{ij}^k\right| \le N, \quad \left|\overline{\Gamma}_{ij}^k(s)\right| \le N, \quad \left|\dot{\Gamma}_{ij}^k\right| \le Nr.$$

We have

$$\frac{\partial g_{ij}}{\partial x^k} = \frac{\partial \overline{g}_{ij}}{\partial x^k}\big(\psi(x)\big) + \dot{g}_{ij}\frac{\partial \psi}{\partial x^k}, \quad \Gamma_{ij}^k = \overline{\Gamma}_{ij}^k\big(\psi(x)\big) + \Delta\Gamma_{ij}^k,$$

where

$$\Delta\Gamma_{ij}^k = \frac{1}{2}g^{kl}\left(\dot{g}_{il}\frac{\partial \psi}{\partial x^j} + \dot{g}_{jl}\frac{\partial \psi}{\partial x^i} - \dot{g}_{ij}\frac{\partial \psi}{\partial x^l}\right);$$

moreover,

$$\frac{\partial \Gamma_{ij}^k}{\partial x^l} = \frac{\partial \overline{\Gamma}_{ij}^k}{\partial x^l}\big(\psi(x)\big) + \dot{\overline{\Gamma}}_{ij}^k\big(\psi(x)\big)\frac{\partial \psi}{\partial x^l} + \frac{\partial \Delta\Gamma_{ij}^k}{\partial x^l},$$

$$\frac{\partial}{\partial x^\alpha}\left(g^{kl}\dot{g}_{il}\frac{\partial \psi}{\partial x^j}\right) = \frac{\partial g^{kl}}{\partial x^\alpha}\dot{g}_{il}\frac{\partial \psi}{\partial x^j} + g^{kl}\left(\frac{\partial \dot{g}_{il}}{\partial x^\alpha}\frac{\partial \psi}{\partial x^j} + \dot{g}_{il}\frac{\partial^2 \psi}{\partial x^j x^\alpha}\right),$$

$$R_{ijkl} = \overline{R}_{ijkl}\big(\psi(x)\big) + g_{\alpha i}\Delta R_{jkl}^\alpha,$$

where

$$\Delta R_{ijk}^l = \dot{\overline{\Gamma}}_{ik}^l\big(\psi(x)\big)\frac{\partial \psi}{\partial x^j} + \frac{\partial \Delta\Gamma_{ik}^l}{\partial x^j} - \dot{\overline{\Gamma}}_{ij}^l\big(\psi(x)\big)\frac{\partial \psi}{\partial x^k} - \frac{\partial \Delta\Gamma_{ij}^l}{\partial x^k}$$
$$+ \overline{\Gamma}_{j\alpha}^l\big(\psi(x)\big)\Delta\Gamma_{ik}^\alpha + \Gamma_{ik}^\alpha\Delta\Gamma_{j\alpha}^l - \overline{\Gamma}_{k\alpha}^l\big(\psi(x)\big)\Delta\Gamma_{ij}^\alpha - \Gamma_{ij}^\alpha\Delta\Gamma_{k\alpha}^l.$$

Estimate the values $\Delta\Gamma_{ij}^k$, $\frac{\partial \Delta\Gamma_{ij}^k}{\partial x^l}$, and $g_{\alpha i}\Delta R_{jkl}^\alpha$ as follows:

$$|\Delta\Gamma_{ij}^k| \le 2N^2mr^2\big|\nabla\psi(x)\big| \le 2N^2m\varepsilon r,$$

$$\left|\frac{\partial \Delta\Gamma_{ij}^k}{\partial x^l}\right| \le 2N^2mr^2\big|\nabla\psi(x)\big| + 2N^2mr\cdot\big|\nabla\psi(x)\big| + 2N^2mr^2\big|\nabla^2\psi(x)\big|$$
$$\le 2N^2m\varepsilon(3+r),$$

$$|g_{\alpha i}\Delta R_{jkl}^\alpha| \le 2N^2mr\cdot\big|\nabla\psi(x)\big| + 4N^3m^2\varepsilon(3+r) + 8N^4m^3\varepsilon r$$
$$\le 2N^2m\varepsilon\big(1 + 2Nm(3+r) + 4N^2m^2r\big).$$

As we see, with a proper choice of ε one can make the value $R_{ijkl} - \overline{R}_{ijkl}\big(\psi(x)\big)$ arbitrarily small. It remains to prove that we have the following inequality in a neighborhood of p for some constant $C > 0$, all $s \in [0,1]$ and every orthonormal system X and Y:

$$\overline{R}(s)(X,Y,Y,X) > C.$$

Since $\ddot{\overline{R}}_{ijkl} = 0$ at the points of N, the following relation holds at these points:

$$\overline{R}_{ijkl}(s) = (1-s)R_{0ijkl} + sR_{1ijkl}, \tag{4.4}$$

That is, the tensor $\overline{R}$ is a convex combination of the curvature tensors of the metrics g_0 and g_1. Since the metrics g_0 and g_1 have positive curvature, we can obtain the inequality

$$\overline{R}(s)(p)(X,Y,Y,X) > 2C$$

for some constant $C > 0$. But the expression $\overline{R}(s)(X,Y,Y,X)$ is a continuous function of the variables x, s, X, and Y, the last three taking values in some compact. Then this expression is uniformly continuous for small x; therefore, if x is sufficiently close to p, the following inequality holds:

$$\left| \overline{R}(s)(x)(X,Y,Y,X) - \overline{R}(s)(p)(X,Y,Y,X) \right| \leq C.$$

From here we obtain the sought inequality for the curvature tensor $\overline{R}$ in a small neighborhood of the point p. The proof is complete. $\qquad\square$

5 Taking Quotients by a Nonfree Isometric Action

In this section we prove the theorem stated in the introduction.

We only give the proof for the case $G = S^3$ (the case $G = S^1$ is analyzed similarly). Put $M^\circ = M \setminus N$. It follows from the second condition of the theorem that the action is free on M°. So the space M°/G has the structure of a smooth manifold, and the quotient metric $(\cdot,\cdot)_{0*}$ of the metric $(\cdot,\cdot)_0$ has positive sectional curvature on it due to the first condition of the theorem. On the other hand, some neighborhood of N in the manifold M is diffeomorphic to some neighborhood of N in the normal bundle over N; the diffeomorphism can be constructed using (2.11), namely $\Phi(p,v) = \exp_p v$, where $p \in N$ and $v \in T_p N^\perp$. So henceforth we assume that M is a neighborhood of N in the normal bundle over N.

Note that, since N is the fixed point set of a group of isometries, it is totally geodesic. Furthermore, since the orbit of G at each point of N is a singleton, it follows from the first condition of the theorem that the restriction of the metric $(\cdot,\cdot)_0$ to N has positive sectional curvature. As above, we denote this restriction by g_N.

Using (3.1), for some constant L construct a g_0-proper metric $(\cdot,\cdot)_1$ in a neighborhood of N such that its 1-jets coincide at the points of N with those of the original metric $(\cdot,\cdot)_0$. (Henceforth we use the functions g_0 and g_ε from § 1.) Since the metric $(\cdot,\cdot)_1$ is unique, it is G invariant. Now, take L so large that Lemma 3.4 guarantees the positivity of the sectional curvature of the metric $(\cdot,\cdot)_1$ at the points of N. By narrowing the manifold M to a smaller neighborhood of N in the normal bundle if necessary, we can assume that the metrics $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_1$ have positive curvature everywhere on M. Denote the quotient metric of the metric $(\cdot,\cdot)_1$ by $(\cdot,\cdot)_{1*}$.

Now we study the action of G. For each point $p \in N$ consider the action of G on the space $V_p = T_p N^\perp$. It is a representation of G on the fiber V_p, which we denote by G_p. If $g \in G$, then denote by g_p the image of g under the representation G_p. Since G acts by isometries, we have the following equality for $g \in G$:

$$g\big(\Phi(p,v)\big) = \Phi(p,g_p v).$$

Since M was assumed to be a part of the normal bundle, i.e., Φ was the identity map, the group G acts on each fiber V_p according to the representation G_p. Note that each of the representations G_p has no nonzero nonfree points due to the second condition of the theorem; therefore, it is equivalent to the representation of $Sp(1)$ in $\mathbb{H}^2$ given by the formula $g(q_1, q_2) = (g \cdot q_1, g \cdot q_2)$ (such representation will be henceforth called *standard*).

Let (U, Θ) be some trivialization of M. We may assume that Θ acts from $U \times \mathbb{H}^2$. From now on we consider only such trivializations for which the action of G on M corresponds to the standard action on $\mathbb{H}^2$, i.e., for $p \in U$, $x \in \mathbb{H}^2$, and $g \in G$ holds

$$g\Theta(p, x) = \Theta(p, g \cdot x).$$

Such trivializations still exist in a neighborhood of every point of N. The map Ψ_{12} in (2.4) now belongs to $Sp(2) \subset SO(8)$; here the elements of $Sp(2)$ are embedded as right multiplications.

Next, we study the structure of the space $\mathbb{H}^2/Sp(1)$. First consider the space $S^7/Sp(1)$, where $S^7 \subset \mathbb{H}^2$ carries the standard metric of radius one. Note that there is an isometric right action of $Sp(2)$ on S^7 which commutes with the isometric left action of $Sp(1)$; therefore, $Sp(2)$ acts isometrically on $S^7/Sp(1)$. The kernel of the latter action is $\mathbb{Z}_2 = \{\pm E\}$, so the isometry group of $S^7/Sp(1)$ has dimension at least $\dim Sp(2) = 10$. On the other hand, $\dim\left(S^7/Sp(1)\right) = 4$. As is well-known (see [K, Chapter 2, Theorem 3.1]), in this case we have $S^7/Sp(1) \simeq S^4$ endowed with the standard metric multiplied by some constant.

Consider the geodesic $c(\varphi) = (\cos\varphi, \sin\varphi) \in S^7 \subset \mathbb{H}^2$ for $\varphi \in [0, \pi]$. It is everywhere orthogonal to the orbits of $Sp(1)$, so its projection c_* to S^4 is a geodesic of the same length π. On the other hand, $c_*(0) = c_*(\pi)$, and c_* has no other self-intersections. It follows from the behavior of the geodesics on round sphere that the metric on $S^4 \simeq S^7/Sp(1)$ is the same as on the standard sphere of radius $1/2$.

Let $f : S^7 \to S^4$ be the quotient map. Extend f to a map from $\mathbb{H}^2$ to $\mathbb{R}^5$ by the rule $f(tv) = tf(v)$, where $t \geq 0$ and $v \in S^7$. Note that the map f is everywhere continuous and smooth everywhere except zero. The right action of $Sp(2)$ on $S^7/Sp(1)$ determines the group homomorphism $\gamma : Sp(2) \to SO(5)$. Let $\gamma_* : \mathfrak{sp}(2) \to \mathfrak{so}(5)$ be the corresponding homomorphism of Lie algebras. Note that

$$f(x \cdot A) = f(x) \cdot \gamma(A),$$
$$df_x(x \cdot B) = f(x) \cdot \gamma_*(B)$$

for each $A \in Sp(2)$, $B \in \mathfrak{sp}(2)$, and $x \in \mathbb{H}^2$.

Let (U, Θ) be a trivialization. It naturally engenders a homeomorphism $\Theta_* : U \times \mathbb{R}^5 \to \pi^{-1}(U)/G$ such that

$$\Theta_*\big(p, f(x)\big) = \tau\big(\Theta(p, x)\big)$$

for $p \in U$ and $x \in \mathbb{H}^2$.

Now, the smooth structure on M/G is constructed by taking as coordinate charts the maps Θ_* for various trivializations. Smoothness of τ on M° follows from smoothness of f everywhere except zero. We now show smoothness of the transition maps. Let (U, Θ_1) and (U, Θ_2) be two trivializations and Ψ_{12} the map that connects them as in (2.4). Then

$$\Theta_{2*}\big(p, f(x)\big) = \tau\big(\Theta_2(p,x)\big)$$
$$= \tau\Big(\Theta_1\big(p, x \cdot \Psi_{12}(p)\big)\Big)$$
$$= \Theta_{1*}\Big(p, f\big(x \cdot \Psi_{12}(p)\big)\Big)$$
$$= \Theta_{1*}\Big(p, f(x) \cdot \gamma\big(\Psi_{12}(p)\big)\Big)$$

for $p \in U$ and $x \in \mathbb{H}^2$.

Note that M/G is a vector bundle of rank 5 over $N/G = N$ with trivializations (U, Θ_*).

We proceed to constructing the sought positively curved metric on M/G. First consider a trivialization (U, Θ). Let $Q_p : T_pN \to \mathfrak{so}(8)$ be a map used in construction of the metric $(\cdot,\cdot)_1$ in this trivialization. Since the metric $(\cdot,\cdot)_1$ and locally horizontal fields are invariant under the action of G, the orthogonal projection of every locally horizontal field is invariant under the action of G. Thus, $Q_p(X) \in \mathfrak{sp}(2)$ for each $X \in T_pN$. In this case we can define the map $Q_{*p} : T_pN \to \mathfrak{so}(5)$ by the formula

$$Q_{*p}(X) = \gamma_*\big(Q_p(X)\big).$$

Using (3.1) and Lemma 2.3, for each $\varepsilon > 0$ construct the metric $(\cdot,\cdot)_\varepsilon$ on $\pi^{-1}(U)/G$ from the metric g_N on N, the function g_ε from Lemma 1.2, the map Q_{*p}, and the constant L. The constructed metric is the unique g_ε-proper metric on the bundle $\pi^{-1}(U)/G$ such that it coincides with $(\cdot,\cdot)_{1*}$ for $r \geq \varepsilon$ and the restriction of the map π to every radial fiber is a Riemannian submersion onto N endowed with the metric $(1 - Lr^2)g_N$.

Indeed, the equality of the metrics on the fibers follows from the fact that $g_\varepsilon = g_0/2$ for $r \geq \varepsilon$ and on the spaces orthogonal to the fibers, from the fact that π is a Riemannian submersion. It remains to prove that the orthogonal projection of each locally horizontal field to the fiberw is the same in both metrics. Let X be such a field, $p \in N$, and $x \in \mathbb{H}^2$. Then the orthogonal projection of the field X to the fibers in the metric $(\cdot,\cdot)_\varepsilon$ is

$$X'^\varepsilon_{\tau\big(\Theta(p,x)\big)} = d\Theta_{*\big(p,f(x)\big)}\big(0, f(x) \cdot Q_{*p}(X_p)\big)$$
$$= d\Theta_{*\big(p,f(x)\big)}\Big(0, df_x\big(x \cdot Q_p(X_p)\big)\Big)$$
$$= d\tau_{\Theta(p,x)}\Big(d\Theta_{(p,x)}\big(0, x \cdot Q_p(X_p)\big)\Big)$$
$$= d\tau_{\Theta(p,x)}\big(X'^1_{\Theta(p,x)}\big).$$

Here X'^1 is the orthogonal projection to the fibers with respect to the metric $(\cdot,\cdot)_1$ of the locally horizontal field X° on M produced by the same field on N as X. Note that X and X° are τ related. The expression to the right equals the orthogonal projection of the field X to the fibers in the metric $(\cdot,\cdot)_{1*}$.

The uniqueness of the metric $(\cdot,\cdot)_\varepsilon$ makes it possible to define it on the whole M/G. Note that 1-jets of the obtained metric at the points of N are independent of ε. Choose constants L_0 and ρ_0 in Lemma 3.5 such that the

metric $(\cdot,\cdot)_\varepsilon$ has positive curvature for $L \geq L_0$, $r < \rho_0/L$, and every ε. Fix $L \geq L_0$. It follows from lemma 4.3 that there exists a positively curved metric $(\cdot,\cdot)'$ on M such that $(\cdot,\cdot)' = (\cdot,\cdot)_0$ outside any neighborhood of N given a priori, and $(\cdot,\cdot)' = (\cdot,\cdot)_1$ when $r < 2\varepsilon$ for some $\varepsilon > 0$. We may assume that $2\varepsilon < \rho_0/L$. As follows from the proof of Lemma 4.3 (according to (4.3) this metric is a linear combination of two G invariant metrics with a coefficient that is constant on the orbits of G), the metric $(\cdot,\cdot)'$ is invariant under the action of G.

Let $(\cdot,\cdot)'_*$ be the quotient metric of the metric $(\cdot,\cdot)'$ on M°/G. For $\varepsilon < r < 2\varepsilon$, we have $(\cdot,\cdot)'_* = (\cdot,\cdot)_{1*} = (\cdot,\cdot)_\varepsilon$. Therefore, there is a C^2 smooth metric $(\cdot,\cdot)_*$ on M/G such that $(\cdot,\cdot)_* = (\cdot,\cdot)'_*$ for $r > \varepsilon$ and $(\cdot,\cdot)_* = (\cdot,\cdot)_\varepsilon$ for $r < 2\varepsilon$. This metric coincides with the quotient metric of the metric $(\cdot,\cdot)_0$ outside a neighborhood of N. It has positive curvature for $r > \varepsilon$ by Lemma 4.3, and for $r < 2\varepsilon$ by Lemma 3.5. Therefore $(\cdot,\cdot)_*$ is the sought metric; the theorem is proven. $\qquad\square$

REMARK

Unfortunately, the author has not succeeded in using the method described above for constructing new examples of positively curved spaces. However, we managed to obtain several already known examples; two of them are described below.

1. Let $M = \mathbb{CP}^3$; the group $G = S^1$ acts as follows:

$$g \cdot (z_0 : z_1 : z_2 : z_3) = (gz_0 : gz_1 : z_2 : z_3).$$

Here $g \in S^1 \subset \mathbb{C}$ and $z_i \in \mathbb{C}$. The fixed point space N has two connected components:

$$N_1 = \big\{ (z_0 : z_1 : 0 : 0) \;\big|\; (z_0 : z_1) \in \mathbb{CP}^1 \big\},$$
$$N_2 = \big\{ (0 : 0 : z_2 : z_3) \;\big|\; (z_2 : z_3) \in \mathbb{CP}^1 \big\}.$$

Each of these components is diffeomorphic to $\mathbb{CP}^1 \simeq S^2$. Therefore N has codimension 4. The space M/G is homeomorphic to the sphere S^5.

2. Let $M = Sp(3)/Sp(1)^3$. Here $Sp(1)^3$ is embedded in $Sp(3)$ as matrixes of the form $\mathrm{diag}(q_1, q_2, q_3)$ for $q_1, q_2, q_3 \in Sp(1)$. We use the positively curved metric on M constructed in [W]. The group $G = S^3 = Sp(1)$ acts as follows: $g \cdot [A] = \big[\mathrm{diag}(g, 1, 1) \cdot A\big]$. Here $g \in Sp(1) \subset \mathbb{H}$, and $[A] \in M$ is the equivalency class of the matrix $A \in Sp(3)$. The fixed point set N has three connected components:

$$N_1 = \left\{ \big[\mathrm{diag}(1, B)\big] \;\Big|\; B \in Sp(2) \right\},$$
$$N_2 = N_1 \cdot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad N_3 = N_1 \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Each of the components is diffeomorphic to $Sp(2)/Sp(1)^2 \simeq S^4$. Therefore N has codimension 8. The space M/G is homeomorphic to the sphere S^9.

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