BOUNDING SECTIONAL CURVATURE ALONG A KÄHLER-RICCI FLOW

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ABSTRACT. If a normalized Kähler-Ricci flow $g(t), t \in [0, \infty)$, on a compact Kähler manifold M, dim_C $M = n \geq 3$, with positive first Chern class satisfies $g(t) \in 2\pi c_1(M)$ and has curvature operator uniformly bounded in L^n -norm, the curvature operator will also be uniformly bounded along the flow. Consequently the flow will converge along a subsequence to a Kähler-Ricci soliton.

1. INTRODUCTION

On a compact Kähler manifold M, dim_C M = n, with the first Chern class $c_1(M) > 0$, the normalized Kähler-Ricci flow equation is

(1.1)
$$\partial_t g(t) = -Ric(g(t)) + g(t),$$

for a family of Kähler metrics $g(t) \in 2\pi c_1(M)$, where, for brevity, g(t) denotes either Kähler metrics or Kähler forms depending on the context. In [5], it is proved that a solution g(t) of (1.1) exists for all $t \in [0, \infty)$. Perelman (cf. [20])) has proved some important properties for the solution $g(t), t \in [0, \infty)$, of (1.1): there exist constants C > 0 and $\kappa > 0$ independent of t such that

- (1) |R(g(t))| < C, and $\operatorname{diam}_{q(t)}(M) < C$,
- (2) (M, g(t)) is κ -noncollapsed, i.e. for any r < 1, if $|R(g(t))| \le r^{-2}$ on a metric ball $B_{q(t)}(x,r)$, then

(1.2)
$$\operatorname{Vol}_{g(t)}(B_{g(t)}(x,r)) \ge \kappa r^{2n}.$$

In a recent preprint [19], Sesum has proved that, if $n \geq 3$, assuming the Ricci curvatures |Ric(g(t))| < C and the integral of curvature operators $\int_M |Rm(g(t))|^n dv_t \leq C$, for a constant C in-dependent of t, then the curvature operators are uniformly bounded. In this note, we will show that the hypothesis of bounded Ricci curvature can be removed.

Theorem 1.1. Let $g(t), t \in [0, +\infty)$, be a solution of the normalized Kähler-Ricci flow (1.1) on a compact Kähler manifold M with $c_1(M) > 0$ and initial metric $g(0) \in$ $2\pi c_1(M)$. Assume that $\dim_{\mathbb{C}} M = n \geq 3$. If the L^n -norms of curvature operators are uniformly bounded by a constant C, i.e.

$$\int_{M} |Rm(g(t))|^n dv_t \le C,$$

then there exists a constant $0 < \overline{C} < \infty$ such that

$$\sup_{M \times [0,\infty)} |Rm(g(t))| \le \bar{C}.$$

Consequently the flow will converge along a subsequence to a Kähler-Ricci soliton.

From this theorem, it is a direct consequence of Hamilton's compactness theorem (c.f. [11]) that, for any $t_k \to \infty$, a subsequence of $(M, g(t_k + t)), t \in [0, 1]$, converges smoothly to $(X, h(t)), t \in [0, 1]$, where X is a compact complex manifold, and $\{h(t)\}, t \in [0, 1]$, is a family of Kähler metrics that satisfies the Kähler-Ricci flow equation. Furthermore, from the arguments in the proof of Theorem 12 in [20], $h(t), t \in [0, 1]$, satisfies the Kähler-Ricci soliton equation, i.e. there is a holomorphic vector field v on X such that

$$Ric(h) - h = \mathcal{L}_v h.$$

In [19], Sesum conjectured that Theorem 1.1, as stated for $n \geq 3$, is also true for n = 2. By the classification theory of complex surface, the only compact Kähler surfaces with $c_1(M) > 0$ are diffeomorphic to $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$, $0 \leq l \leq 8$, and $\mathbb{CP}^1 \times \mathbb{CP}^1$. By [21], each of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$, $3 \leq l \leq 8$ or l = 0, admits a Kähler-Einstein metric. In [6] and [13], it is shown that $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ admits a non-trivial Kähler-Ricci soliton metric. Wang and Zhu [24] showed the same result later for $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$. By [23], on a compact Kähler surface M with $c_1(M) > 0$, if the initial metric g(0) is invariant under a one-parameter group obtained from a Kähler-Ricci soliton metric on M, the curvatures stay uniformly bounded along the flow. The only remaining case is when M is a complex surface diffeomorphic to $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ or $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$, with an initial metric g(0) without any symmetry. In [9], it is proved that the Kähler-Ricci flow on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ or $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$

By using the method in the proof of Theorem 1.1, we can give a different proof of the convergence of the Kähler-Ricci flow on \mathbb{CP}^2 (Theorem 3.3), which is already implied by [23]. In a very recent preprint [7], Chen and Wang claimed that the bounding of curvatures along the Kähler-Ricci flow on a toric Fano surface M (including $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$) could be proved by using the fact that M is a toric manifold.

There is also an analogy to Theorem 1.1 in the real Ricci flow case.

Theorem 1.2. Let $g(t), t \in [0, T)$, be a solution to the Ricci flow, normalized or not, on a closed odd dimensional manifold M with $T < \infty$. Suppose that

$$\int_{M} |Rm(g(t))|^{n/2} dv_t \le C, \quad \text{and} \quad |R(g(t))| \le C$$

for a constant $C < \infty$ independent of t, where $n = \dim_{\mathbb{R}}(M)$. Then there is another constant $\overline{C} < \infty$ such that

$$\sup_{M \times [0,T)} |Rm(g(t))| \le C_1$$

and so the flow can be extended over T.

The organization of the paper is as follows: In §2, we prove Theorem 1.1. In §3, we give some remarks for Kähler-Ricci on Fano surfaces. Then we prove Theorem 1.2 in §4.

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2. Proof of Theorem 1.1

Let $g(t), t \in [0, +\infty)$, be a solution of the normalized Kähler-Ricci flow (1.1) on a compact Kähler manifold M, $\dim_{\mathbb{C}} M = n$, with $c_1(M) > 0$ and initial metric $g(0) \in 2\pi c_1(M)$. Assume that $\dim_{\mathbb{C}} M = n \geq 3$, and that

$$\int_M |Rm(g(t))|^n dv_t \le C,$$

for a constant C independent of t. Perelman (cf. [20])) has proved that there exist constants C > 0, $\kappa > 0$ independent of t such that

- (1) |R(g(t))| < C, and $\operatorname{diam}_{g(t)}(M) < C$,
- (2) (M, g(t)) is κ -noncollapsed, i.e. for any r < 1, if $|R(g(t))| \le r^{-2}$ on a metric ball $B_{g(t)}(x, r)$, then

(2.1)
$$\operatorname{Vol}_{g(t)}(B_{g(t)}(x,r)) \ge \kappa r^{2n}$$

The proof of Theorem 1.1 relies on the following theorem due to Gang Tian:

Theorem 2.1 (Theorem 2 in [22]). Let (N, J, g) be a complete non-compact Ricci-flat Kähler manifold with $\dim_{\mathbb{C}} M = n \geq 3$,

$$\int_{N} |Rm(g)|^{n} dv_{g} < C < \infty, \quad \text{and}$$
$$Vol_{g}(B_{g}(x, r)) \ge \kappa r^{2n},$$

for any r > 0, where C and κ are constants. Then M is a resolution of \mathbb{C}^n/Γ where Γ is a finite group $\Gamma \subset SU(n)$, which acts on $\mathbb{C}^n \setminus \{0\}$ freely, i.e. there is a holomorphic map $\pi : N \longrightarrow \mathbb{C}^n/\Gamma$ such that $\pi : N \setminus \pi^{-1}(0) \longrightarrow \mathbb{C}^n \setminus \{0\}/\Gamma$ is bi-holomorphic.

The assumptions of Euclidean volume growth and $\dim_{\mathbb{C}} M = n \geq 3$ not mentioned explicitly in Theorem 2 in [22] seem to be necessary. Let's recall several main steps in the proof of Theorem 2.1. First, an estimate for the decrease of the sectional curvature of gis obtained by assuming the Euclidean volume growth, bounded L^n -norm of curvature operator, and the Ricci-flat metric (See Lemma 4.1 in [22]). Then, for proving Theorem 2 of [22], one needs Lemma 3.4 and Lemma 3.3 of [22], which have the hypothesis $\dim_{\mathbb{C}} M = n \geq 3$. The main tool there was Kohn's estimate for \Box_b -operators that works only for $n \geq 3$ (See [22] for details). Proof of Theorem 1.1. Suppose otherwise, there exists a sequence of times $t_k \to \infty$, and a sequence of points $x_k \in M$ such that

$$Q_k = |Rm(g(t_k))|(x_k) = \sup_{M \times [0, t_k]} |Rm(g(t))| \to \infty.$$

Consider the sequence $\tilde{g}_k(t), t \in [-Q_k t_k, 0]$, where $\tilde{g}_k(t) = Q_k g(Q_k^{-1} t + t_k))$ and satisfy

(2.2)
$$\partial_t \tilde{g}_k(t) = -Ric(\tilde{g}_k(t)) + Q_k^{-1} \tilde{g}_k(t),$$

(2.3)
$$\sup_{M \times [-Q_k t_k, 0]} |Rm(\tilde{g}_k(t))| \le 1, \quad \text{and} \quad |Rm(\tilde{g}_k(0))|(x_k) = 1.$$

By Perelman's estimate, we obtain that

(2.4)
$$|R(\widetilde{g}_k(t))| < CQ_k^{-1} \longrightarrow 0,$$

when $k \to \infty$, and, for any $r < CQ_k^{\frac{1}{2}}, x \in M$,

(2.5)
$$\operatorname{Vol}_{\widetilde{g}_k(t)}(B_{\widetilde{g}_k(t)}(x,r)) \ge \kappa r^{2n}.$$

By Hamilton's compactness theorem (c.f. Appendix E in [12]), by passing to a subsequence, $\{(M, J, \tilde{g}_k(t), x_k)\}, t \in [-1, 0]$, converges smoothly to a family of pointed complete Riemannian manifold $(N, J_{\infty}, g_{\infty}(t), x_{\infty}), t \in [-1, 0]$, where $g_{\infty}(t), t \in [-1, 0]$, satisfies the Ricci-flow equation $\partial_t g_{\infty}(t) = -Ric(g_{\infty}(t))$. Particularly, for any $k \gg 1$ and r > 0, there is an embedding $F_{k,r} : B_{g_{\infty}}(x_{\infty}, r) \longrightarrow M$ such that $F_{k,r}^* \tilde{g}_k(0)$ converges smoothly to g_{∞} , and $dF_{k,r}^{-1} J dF_{k,r}$ converges smoothly to an almost complex structure J_{∞} , where $g_{\infty} = g_{\infty}(0)$. Actually, J_{∞} is integrable, and g_{∞} is a Kähler metric of J_{∞} (c.f. [17]). From (2.3) and (2.5), we obtain that

$$(2.6) |Rm(g_{\infty})| \le |Rm(g_{\infty})|(x_{\infty}) = 1$$

and, for any r > 0 and $x \in N$,

(2.7)
$$\operatorname{Vol}_{g_{\infty}(t)}(B_{g_{\infty}(t)}(x,r)) \ge \kappa r^{2n}$$

By (2.4), $R(g_{\infty}(t)) \equiv 0$, which implies that $Ric(g_{\infty}(t)) \equiv 0$ since $g_{\infty}(t)$ is a solution of the Ricci-flow equation. From the smooth convergence,

$$\int_{N} |Rm(g_{\infty})|^{n} dv_{g_{\infty}} \leq \limsup_{k \to \infty} \int_{M} |Rm(g(t_{k}))|^{n} dv_{g(t_{k})} \leq C < \infty.$$

Thus $(N, J_{\infty}, g_{\infty})$ is a complete Ricci-flat Kähler manifold with Euclidean volume growth, and L^n -norm of curvature operator bounded. By (2.6), g_{∞} is not a flat metric. Note that $\dim_{\mathbb{C}} M = n \geq 3$. By Theorem 2.1, N is a resolution of \mathbb{C}^n/Γ where Γ is a finite group $\Gamma \subset SU(n)$, which acts on $\mathbb{C}^n \setminus \{0\}$ freely, i.e. there is a holomorphic map $\pi : N \longrightarrow \mathbb{C}^n/\Gamma$ such that $\pi : N \setminus \pi^{-1}(0) \longrightarrow \mathbb{C}^n \setminus \{0\}/\Gamma$ is bi-holomorphic. If Γ is trivial, i.e. $\Gamma = \{e\}$, then (N, g_{∞}) is isometric to \mathbb{R}^{2n} by Theorem 3.5 in [1], which contradicts (2.1.5). Thus Γ is non-trivial, and $V = \pi^{-1}(0)$ is a compact analytic subvariety of (N, J_{∞}) with $0 < \dim_{\mathbb{C}} V = m < n$ (See [10] for the definition of $\dim_{\mathbb{C}} V$). We obtain that

$$\int_{V} g_{\infty}^{m} = m! Vol_{g_{\infty}}(V) > 0.$$

Note that, for any k, $F_{k,r}(V)$ is a cycle, and defines a homology class $[F_{k,r}(V)] \in H_{2m}(M,\mathbb{Z})$. By the smooth convergence of $F_{k,r}^* \tilde{g}_k$, for any $\varepsilon > 0$, there is a $k_0 > 0$ such that, for any $k \ge k_0$,

$$\left|\int_{V} g_{\infty}^{m} - \int_{F_{k,r}(V)} \widetilde{g}_{k}^{m}\right| \leq \varepsilon$$

where $\widetilde{g}_k = \widetilde{g}_k(0)$, and $r \gg 1$ such that $V \subset B_{g_\infty}(x_\infty, r)$. As $g_k = g(t_k) \in 2\pi c_1(M)$,

$$\int_{F_{k,r}(V)} \widetilde{g}_k^m = Q_k^m \int_{F_{k,r}(V)} g_k^m = Q_k^m (2\pi)^m \int_{F_{k,r}(V)} c_1^m(M).$$

By taking $\varepsilon = \frac{1}{2} \int_V g_\infty^m$ and $k \gg k_0$, we obtain that

$$0 < \frac{1}{2}Q_k^{-m}(2\pi)^{-m}\int_V g_\infty^m \le \int_{F_{k,r}(V)} c_1^m(M) \le \frac{3}{2}Q_k^{-m}(2\pi)^{-m}\int_V g_\infty^m < 1.$$

Since $0 \neq c_1^m(M) \in H^{2m}(M, \mathbb{Z})$, and $[F_{k,r}(V)] \in H_{2m}(M, \mathbb{Z})$, we have

$$\int_{F_{k,r}(V)} c_1^m(M) \in \mathbb{Z}.$$

It is a contradiction. We obtain that

$$\sup_{M \times [0,\infty)} |Rm(g(t))| \le \bar{C},$$

for a constant $\bar{C} > 0$.

Now, by Hamilton's compactness theorem (c.f. [11]), for any $t_k \to \infty$, a subsequence of $(M, g(t_k + t)), t \in [0, 1]$, converges smoothly to a family of compact Kähler manifolds $(X, h(t)), t \in [0, 1]$, where h(t) satisfies the Kähler-Ricci flow equation. Actually, h(t), $t \in [0, 1]$, satisfies the Kähler-Ricci soliton equation from the arguments in the proof of Theorem 12 in [20].

3. Remarks for Kähler surfaces

Let g(t), $t \in [0, +\infty)$, be a solution of the normalized Kähler-Ricci flow (1.1) on a compact Kähler surface M, i.e. $\dim_{\mathbb{C}} M = 2$, with $c_1(M) > 0$ and initial metric $g(0) \in 2\pi c_1(M)$.

Lemma 3.1. The L^2 -norms of curvature operators of g(t) are bounded along the flow, *i.e.* there is a constant C > 0 independent of t such that

(3.1)
$$\int_M |Rm(g(t))|^2 dv_t \le C.$$

Proof. Since, for any $t \in [0, \infty)$, (M, g(t)) is a Kähler surface, we have $\int_M c_1^2(M) = 2\chi(M) + 3\tau(M)$, $R^2(g(t)) = 24|W^+(g(t))|^2$, and Gauss-Bonnet-Chern formula and Hirzebruch formula

$$\chi(M) = \frac{1}{8\pi^2} \int_M (\frac{R^2}{24} + |W^+|^2 + |W^-|^2 - \frac{1}{2} |Ric^{\circ}|^2) dv_t,$$

$$\tau(M) = \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) dv_t$$

(c.f. [2]), where $Ric^{o} = Ric(g(t)) - \frac{R}{4}g(t)$, $W^{\pm}(g(t))$ are the self-dual and anti-self-dual Weyl tensors of g(t), and $\chi(M)$ (respectively $\tau(M)$) is the Euler number (respectively signature) of M. Then we obtain that

(3.2)
$$\int_{M} |Ric^{\circ}|^{2} dv_{t} = \int_{M} \frac{R^{2}}{4} dv_{t} - 8\pi^{2} c_{1}^{2}(M), \text{ and}$$
$$\int_{M} |W^{-}|^{2} dv_{t} = \int_{M} \frac{R^{2}}{24} dv_{t} - 12\pi^{2} \tau(M).$$

Note that

$$Rm(g(t)) = \begin{pmatrix} W^{+} + \frac{R}{12} & Ric^{\circ} \\ Ric^{\circ} & W^{-} + \frac{R}{12} \end{pmatrix}.$$

Thus we obtain that

$$\int_{M} |Rm(g(t))|^{2} dv_{t} = \int_{M} (\frac{R^{2}}{24} + |W^{+}|^{2} + |W^{-}|^{2} + 2|Ric^{0}|^{2}) dv_{t}$$
$$= \int_{M} \frac{5R^{2}}{8} dv_{t} - 16\pi^{2}c_{1}^{2}(M) - 12\pi^{2}\tau(M).$$

Hence, by Perelman's estimate for scalar curvatures, we obtain (3.1).

Unfortunately, our arguments in the proof of Theorem 1.1 can not be generalized to this case, even for $\mathbb{CP}^2 \not\models \overline{\mathbb{CP}}^2$. The essential point in the proof of Theorem 1.1 is that, in any Asymptotically Locally Euclidean Ricci-flat Kähler manifold N of $\dim_{\mathbb{C}} N \geq 3$, we can find a non-trivial class $[V] \neq 0 \in H_{2m}(N,\mathbb{Z})$ with $m \geq 1$. However, there are ALE Ricci-flat Kähler surfaces without such homology classes. For example, there is an ALE Ricci-flat Kähler metric h on $T^*\mathbb{RP}^2$ whose Betti numbers satisfy $b_2 = b_3 = b_4 = 0$. Actually, the universal covering space of $T^*\mathbb{RP}^2$ with the pull-back metric is the Eguchi-Hanson space (c.f. [8]), which is diffeomorphic to T^*S^2 .

Proposition 3.2. Assume that M is diffeomorphic to $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$. If there is a sequence of times $t_k \to \infty$ such that

$$Q_k = |Rm(g(t_k))|(x_k) = \sup_{M \times [0, t_k]} |Rm(g(t))| \to \infty,$$

where $x_k \in M$, then a subsequence of $(M, Q_k g(t_k), x_k)$ converges smoothly to an ALE Ricci-flat Kähler surface $(N, g_{\infty}, x_{\infty})$ in the pointed Gromov-Hausdorff sense. Furthermore, the fundamental group $\pi_1(N)$ of N is a non-trivial finite group. Proof. Let $\tilde{g}_k = Q_k g(t_k)$. By the same arguments as in the proof of Theorem 1.1, by passing to a subsequence, $\{(M, J, \tilde{g}_k, x_k)\}$ converges smoothly to a complete Ricci-flat Kähler surface $(N, J_{\infty}, g_{\infty}, x_{\infty})$, i.e. for any $k \gg 1$ and r > 0, there is an embedding $F_{k,r} : B_{g_{\infty}}(x_{\infty}, r) \longrightarrow M$ such that $F_{k,r}^* \tilde{g}_k$ converges smoothly to g_{∞} , and $dF_{k,r}^{-1} J dF_{k,r}$ converges smoothly to J_{∞} . Furthermore, $(N, J_{\infty}, g_{\infty})$ satisfies that, for any r > 0 and $x \in N$,

$$\operatorname{Vol}_{g_{\infty}}(B_{g_{\infty}}(x,r)) \geq \kappa r^{4},$$
$$\int_{N} |Rm(g_{\infty})|^{2} dv_{g_{\infty}} \leq C < \infty,$$
and
$$|Rm(g_{\infty})| \leq |Rm(g_{\infty})|(x_{\infty}) = 1$$

Thus, by Theorem 1.5 in [3], $(N, J_{\infty}, g_{\infty})$ is an Asymptotically Locally Euclidean Ricciflat Kähler surface. Since g_{∞} is not flat, it is easy to see that the fundamental group $\pi_1(N)$ of N is a finite group (c.f. [1]).

If $\pi_1(N) = \{1\}, (N, J_\infty, g_\infty)$ is an ALE hyper-Kähler 4-manifold. By the classification theory of ALE hyper-Kähler 4-manifold (c.f. [14]), there is a close surface $\Sigma \subset N$ such that $[\Sigma] \in H_2(N, \mathbb{Z})$, and $[\Sigma] \cdot [\Sigma] = -2$. Then, for $k \gg 1$ and $r \gg 1$, $F_{k,r}(\Sigma)$ is a cycle in M, and defines a homology class $[F_{k,r}(\Sigma)] \in H_2(M, \mathbb{Z})$ with $[F_{k,r}(\Sigma)] \cdot [F_{k,r}(\Sigma)] = -2$. Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ such that $[F_{k,r}(\Sigma)] = aH + bE$, where H and E are the two generators of $H_2(M, \mathbb{Z})$ such that $H \cdot H = 1$, $E \cdot E = -1$ and $H \cdot E = 0$. However, the equation $a^2 - b^2 = [F_{k,r}(\Sigma)] \cdot [F_{k,r}(\Sigma)] = -2$ does not have integer solutions. It is a contradiction. Thus $\pi_1(N) \neq \{1\}$.

Actually, by the same arguments as in the proof of (2) in Theorem 5.2 of [21], we can see that $\pi_1(N)$ is a cyclic finite group, and $(N, J_{\infty}, g_{\infty})$ is asymptotic to \mathbb{C}^2/Γ , where Γ is a finite cyclic subgroup of U(2) given by Lemma 5.5 in [21]. If one wants to use the technique in the proof of Theorem 1.1 to prove the bounding of curvatures along the Kähler-Ricci flow on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$, a method must be found to prove that N is actually simply connected. In a recently preprint [7], it is claimed that this could be done by using the fact that $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ is a toric manifold. However, the full details of the arguments in [7] have not appeared yet.

Our method can be used to give a different proof of the following theorem in term of Gromov-Hausdorff convergence, which is already implied by [23] where the Monge-Ampère flow was used.

Theorem 3.3. ([23]) If M is holomorphic to \mathbb{CP}^2 , and g(t), $t \in [0, +\infty)$, is a solution of the normalized Kähler-Ricci flow (1.1), then, for any sequence of times $t_k \to \infty$, a subsequence of $(M, g(t_k))$ converges smoothly to the unique Kähler-Einstein metric on \mathbb{CP}^2 in the Cheeger-Gromov sense.

Proof. It is well known that, on \mathbb{CP}^2 , there is a unique Kähler-Einstein metric presenting $2\pi c_1$, the Fubini-Study metric (c.f. [4]). This implies that the Mabuchi's K-energy $\nu_{g_0}(g(t))$ is bounded from below (c.f. [16]). Thus, by Perelman's estimate for scalar

curvatures, (6.1) in [16] holds, i.e.

$$\int_{M} |\partial \overline{\partial} u_t|^2 dv_t \longrightarrow 0,$$

when $t \to \infty$, where u_t are functions satisfying $-Ric(g(t)) + g(t) = \sqrt{-1}\partial\overline{\partial}u_t$. Since the Hodge Laplacian satisfies $\triangle = 2(\partial^*\partial + \partial\partial^*) = 2(\overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*), \ \Delta\overline{\partial} = \overline{\partial}\Delta$, and $\triangle u_t = R(g(t)) - 4$, we obtain that

(3.3)
$$\int_M |R(g(t)) - 4|^2 dv_t = \int_M |\Delta u_t|^2 dv_t = 4 \int_M |\partial \overline{\partial} u_t|^2 dv_t \longrightarrow 0.$$

Then

$$\begin{split} \lim_{t \to \infty} \int_M R(g(t))^2 dv_t &= \lim_{t \to \infty} \int_M (8R(g(t)) - 16) dv_t \\ &= \lim_{t \to \infty} (16 \int_M Ric(g(t)) \wedge g(t) - 8 \int_M g(t) \wedge g(t)) \\ &= 32\pi^2 \int_M c_1^2(M) = 32\pi^2 (2\chi(M) + 3\tau(M)). \end{split}$$

By (3.2), we have

$$\lim_{t \to \infty} \int_M |W^-(g(t))|^2 dv_t = \frac{8}{3}\pi^2(\chi(M) - 3\tau(M)) = 0,$$

since $\chi(M) = 3$ and $\tau(M) = 1$.

If $\sup_{t\in[0,\infty)} |Rm(g(t))| = +\infty$, then there exists a sequence of times $t_k \to \infty$, and a

sequence of points $x_k \in M$ such that

$$Q_k = |Rm(g(t_k))|(x_k) = \sup_{M \times [0, t_k]} |Rm(g(t))| \to \infty.$$

Let $\tilde{g}_k = Q_k g(t_k)$. By the same arguments as in the proof of Theorem 1.1, by passing to a subsequence, $\{(M, J, \tilde{g}_k, x_k)\}$ converges smoothly to a complete Ricci-flat Kähler surface $(N, J_{\infty}, g_{\infty}, x_{\infty})$ with

$$\sup_{N} |Rm(g_{\infty})| = 1$$

Furthermore, by the smooth convergence,

$$\int_{N} |W^{-}(g_{\infty})|^{2} dv_{\infty} \leq \lim_{t \to \infty} \int_{M} |W^{-}(g(t))|^{2} dv_{t} = 0, \text{ thus } W^{-}(g_{\infty}) \equiv 0,$$

on N. From $R^2(g_{\infty}) = 24|W^+(g_{\infty})|^2 \equiv 0$, we obtain that $|Rm(g_{\infty})| \equiv 0$ on N. It is a contradiction. Hence there is a constant C > 0 independent of t such that

$$|Rm(g(t))| \le C.$$

Finally, by the same arguments as in the proof of Theorem 1.1, for any $t_k \to \infty$, a subsequence of $(M, g(t_k + t)), t \in [0, 1]$, converges smoothly to a compact Kähler-Ricci soliton $(X, h(t)), t \in [0, 1]$. By (3.3), $R(h(t)) \equiv 4$, and, thus, h(0) is a Kähler-Einstein metric. By Kodaria classification theorem, it is well known that the Fano surface diffeomorphic to \mathbb{CP}^2 is unique. Therefore, $X \cong M$.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 follows by a similar argument as in the proof of Theorem 1.1.

Proof of Theorem 1.2. Suppose not, there exist a sequence of $t_k \to T$ and points $x_k \in M$ such that

$$Q_k = |Rm(g(t_k))|(x_k) = \sup_{M \times [0, t_k]} |Rm(g(t))| \to \infty.$$

Then consider the sequence of solutions to the Ricci flow

$$(M, Q_k g(Q_k^{-1}t + t_k), x_k), t \in [-Q_k t_k, 0].$$

First we assume g(t) is a solution to the unnormalized Ricci flow. Perelman's no local collapsing theorem (cf. [15, Theorem 4.1] or [12, Remark 12.13]) applies to show that there is a $\kappa > 0$ such that $Vol(B(x, r, g(t))) \ge \kappa r^n$ for each metric ball B(x, r, g(t))in (M, g(t)) with radius $r \le \sqrt{T}$. Then using Hamilton's compactness theorem for Ricci flow solutions, the sequence will converge modulo a subsequence to another solution to the Ricci flow, say $(M_{\infty}, g_{\infty}(t), x_{\infty}), t \in (-\infty, 0]$, which has the properties that $Vol(B(r, g_{\infty}(t))) \ge \kappa r^n$, for each metric ball $B(r, g_{\infty}(t))$ in $(M_{\infty}, g_{\infty}(t))$ of radius r, and that

$$\int_{M_{\infty}} |Rm(g_{\infty}(t))|^{n/2} dv_{g_{\infty}(t)} \leq \limsup_{k \to \infty} \int_{M} |Rm(g(t))|^{n/2} dv_{t} < \infty,$$
$$|Rm(g_{\infty}(0))|(x_{\infty}) = 1 \text{ and } R(g_{\infty}(t)) \equiv 0 \text{ over } M_{\infty} \times (-\infty, 0].$$

From the evolution of the volume Vol(g(t)) of the metric g(t):

$$\frac{d}{dt}Vol(g(t)) = -\int_M R(g(t))dv_{g(t)} \ge -CVol(g(t)),$$

we conclude that $Vol(g(t)) \geq Vol(g(0))e^{-Ct} \geq Vol(g(0))e^{-CT}$ for each metric g(t). So the limits $(M_{\infty}, g_{\infty}(t))$ are non-compact Ricci flat manifolds. After a double covering, we may also assume that the manifold M is oriented and so the limit M_{∞} is also oriented. By odd dimensional assumption of M, using theorem 3.5 of [1], we conclude that $(M_{\infty}, g_{\infty}(0))$ is in fact the Euclidean space, which contradicts the fact $|Rm(g_{\infty}(0))|(x_{\infty}) = 1.$

If g(t) is a solution to the normalized Ricci flow, then the rescaling factor from g(t) to the corresponding unnormalized Ricci flow is uniformly bounded from above and below (stays bounded away from zero), since the scalar curvature of g(t) is absolutely bounded. Thus the corresponding unnormalized Ricci flow exists in finite time, and then Perelman's no local collapsing theorem uses also. So repeatedly, $Vol(B(x, r, g(t))) \ge \kappa r^n$ for each metric ball B(x, r, g(t)) in (M, g(t)) with radius $r \le \sqrt{T}$, for some universal

 $\kappa > 0$. If g(t) does not has uniformly bounded Riemannian curvature, then there is a sequence of times $t_k \to T$ and points $x_k \in M$ such that

$$Q_k = |Rm(g(t_k))|(x_k) = \sup_{M \times [0, t_k]} |Rm(g(t))| \to \infty.$$

Then consider the sequence of solutions to the Ricci flow

$$(M, Q_k g(Q_k^{-1}t + t_k), x_k), t \in [-Q_k t_k, 0],$$

which will converge along a subsequence to a Ricci flat solution on an open manifold. The limit solution is flat by a same argument and we obtain a contradiction. \Box

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