CONVEXITY PROPERTIES FOR GENERALIZED MOMENT MAPS I

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ABSTRACT. We study generalized moment maps for a Hamiltonian action on a connected compact H-twisted generalized complex manifold introduced by Lin and Tolman and prove the convexity and connectedness properties of the generalized moment maps for a Hamiltonian torus action.

1. INTRODUCTION

A notion of generalized complex structures was introduced by Hitchin [5] and developed by Gualtieri [4]. It provides a unifying framework for both complex and symplectic geometry, and a useful geometric language for understanding some recent development in string theory. Generalized Kähler geometry, the generalized complex geometry analogue of Kähler geometry, was introduced by Gualtieri, who also shows that it is essentially equivalent to that of a bihermitian structure, which was first discovered by physicists studying super-symmetric nonlinear sigma model in [2].

For a group action on manifolds, notions of Hamiltonian actions and moment maps play a very impotant role in many geometry. It is an interesting and important question if there exists natural notions of Hamiltonian actions and moment maps. In [10], Lin and Tolman introduced notions of a Hamiltonian action and a generalized moment map for generalized complex geometry. They showed in [10] a reduction theorem for Hamiltonian actions of compact Lie groups on an *H*-twisted generalized complex and Kähler manifold. As an application, they constructed explicit examples of bihermitian structures on \mathbb{CP}^n , Hirzebruch surfaces, the blow up of \mathbb{CP}^2 at arbitrarily many points, and other toric varieties, as well as complex Grassmannians. Their construction is a powerful tool for producing bihermitian structures on manifolds which can be produced as a symplectic reduction of \mathbb{C}^N . Moreover, it was shown by Kapustin and Tomasilleo in [6] that the mathematical notion of Hamiltonian actions on a generalized Kähler manifold corresponds exactly to the physiccal notion of general (2, 2) gauged sigma models with 3-form fluxes.

Convexity and connectedness properties for moment maps of Hamiltonian torus actions on a connected compact symplectic manifold was shown by Atiyah [1] and Guillemin and Sternberg [3]. In the present paper, we study Hamiltonian torus actions on a connected compact H-twisted generalized complex manifold and prove the convexity and connectedness properties of a generalized moment map for Hamiltonian torus actions. The main result is stated below. The detailed notations and definitions are in section 2 and section 3.

Theorem A. Let an m-dimensional torus T^m act on a connected compact Htwisted generalized complex manifold (M, \mathcal{J}) in a Hamiltonian way with a generalized moment map $\mu : M \longrightarrow \mathfrak{t}^*$ and a moment one form $\alpha \in \Omega^1(M; \mathfrak{t}^*)$. Then:

- (1) the levels of μ are connected;
- (2) the image of μ is convex;
- (3) the fixed points of the action form a finite union of connected submanifolds C_1, \dots, C_N :

$$\operatorname{Fix}(T^m) = \bigcup_{i=1}^N C_i.$$

On each component the generalized moment map μ is constant: $\mu(C_i) = \{a_i\}$, and the image of μ is the convex hull of the images a_1, \dots, a_N of the fixed points of the action, that is,

$$\mu(M) = \left\{ \sum_{i=1}^{N} \lambda_i a_i \mid \sum_{i=1}^{N} \lambda_i = 1, \lambda_i \ge 0 \right\}.$$

This paper is organized as follows. In section 2 we briefly review of the theory of generalized complex structures and generalized Kähler structures. In section 3 we introduce the notion of generalized moment maps for Hamiltonian actions on a generalized complex manifold and prove that the generalized moment map has a property of a Bott-Morse function. At the last section, we shall give a proof of Theorem A.

2. Generalized complex structures

First we recall the basic theory of generalized complex structures; see [4] for the details.

Given a closed 3-form H on an n-dimensional manifold M, we define the H-twisted Courant bracket of sections of the sum $T \oplus T^*$ of the tangent and cotangent bundles by

$$[X + \xi, Y + \eta]_H = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\eta(X) - \xi(Y)) + i_Y i_X H,$$

where \mathcal{L}_X denotes the Lie derivative along a vector field X. The vector bundle $T \oplus T^*$ is also endowed with a natural inner product of signature (n, n):

$$\langle X + \xi, Y + \eta \rangle = \frac{1}{2}(\eta(X) + \xi(Y)).$$

Definition 2.1. Let M be a manifold, and H be a closed 3-form on M. A generalized almost complex structure on M is a complex structure \mathcal{J} on the bundle $T \oplus T^*$ which preserves the natural inner product. If sections of the $\sqrt{-1}$ -eigenspace of \mathcal{J} is closed under the H-twisted Courant bracket, \mathcal{J} is called an H-twisted generalized complex structure. If H = 0, we call it simply a generalized complex structure.

An *H*-twisted generalized complex structure can be fully described in terms of its $\sqrt{-1}$ -eigenspace *L*, which is a maximal isotropic subspace of $(T \oplus T^*) \otimes \mathbb{C}$ satisfying $L \cap \overline{L} = \{0\}$ and to be closed under the *H*-twisted Courant bracket.

Let $\pi : (T \oplus T^*) \otimes \mathbb{C} \longrightarrow T \otimes \mathbb{C}$ be the natural projection. The type of an *H*-twisted generalized complex structure \mathcal{J} is the codimension of $\pi(L)$ in $T \otimes \mathbb{C}$, where *L* is the $\sqrt{-1}$ -eigenspace of \mathcal{J} .

Example 2.1 (Complex structures (type n)). Let J be a usual complex structure on a 2n-dimensional manifold M. Consider the endomorphism

$$\mathcal{J}_J = \left(\begin{array}{cc} J & 0\\ 0 & -J^* \end{array}\right),$$

where the matrix is written with respect to the direct sum $T \oplus T^*$. Then \mathcal{J}_J is a generalized complex structure of type n on M; the $\sqrt{-1}$ -eigenspace of \mathcal{J}_J is $L_J = T_{1,0} \oplus T^{0,1}$, where $T_{1,0}$ is the $\sqrt{-1}$ -eigenspace of J.

Example 2.2 (Symplectic structures (type 0)). Let ω be a symplectic structure on a 2*n*-dimensional manifold M, viewed as a skew-symmetric isomorphism $\omega : T \longrightarrow T^*$ via the interior product $X \mapsto i_X \omega$. Consider the endomorphism

$$\mathcal{J}_{\omega} = \left(\begin{array}{cc} 0 & -\omega^{-1} \\ \omega & 0 \end{array}\right)$$

Then \mathcal{J}_{ω} is a generalized complex structure of type 0 on M; the $\sqrt{-1}$ -eigenspace of \mathcal{J}_{ω} is

$$L_{\omega} = \{ X - \sqrt{-1}i_X \omega | \ X \in T \otimes \mathbb{C} \}.$$

Example 2.3 (B-field shift). Let (M, \mathcal{J}) be an H-twisted generalized complex manifold and $B \in \Omega^2(M)$ be a closed 2-form on M. Then the endomorphism

$$\mathcal{J}_B = \left(\begin{array}{cc} 1 & 0 \\ B & 1 \end{array}\right) \mathcal{J} \left(\begin{array}{cc} 1 & 0 \\ -B & 1 \end{array}\right)$$

is also an *H*-twisted generalized complex structure. It is called the *B*-field shift of \mathcal{J} . The type of \mathcal{J}_B coincides with that of \mathcal{J} . The $\sqrt{-1}$ eigenspace L_B of \mathcal{J}_B can be written by

$$L_B = \{X + f + i_X B \mid X + f \in L\},\$$

where L is the $\sqrt{-1}$ eigenspace of \mathcal{J} .

The type of an *H*-twisted generalized complex structure is not required to be constant along the manifold, and it may jump along loci. Gualtieri constructed a generalized complex structure on \mathbb{CP}^2 which is type 2 along a cubic curve and type 0 outside the cubic curve. The detailed construction can be seen in [4].

Next we briefly review the notion of *H*-twisted generalized Kähler structures.

Definition 2.2. Let M be a manifold, and H be a closed 3-form on M. an H-twisted generalized Kähler structure on M is a pair of commuting H-twisted generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ so that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric, that is, $\mathcal{G}^2 = \mathrm{id}$, \mathcal{G} preserves the natural inner product, and $\mathcal{G}(X + \xi, Y + \eta) := \langle \mathcal{G}(X + \xi), X + \xi \rangle > 0$ for all non-zero $X + \xi \in T \oplus T^*$.

Example 2.4.

(1) Let (M, g, J) be a Kähler manifold and $\omega = gJ$ be the Kähler form. By examples above, J and ω induce generalized complex structures \mathcal{J}_J and \mathcal{J}_{ω} , respectively. Moreover, \mathcal{J}_J commutes with \mathcal{J}_{ω} and

$$\mathcal{G} = -\mathcal{J}_J \mathcal{J}_\omega = \left(\begin{array}{cc} 0 & g^{-1} \\ g & 0 \end{array}\right)$$

is a positive definite metric on $T \oplus T^*$. Hence $(\mathcal{J}_J, \mathcal{J}_\omega)$ is a generalized Kähler structure on M.

(2) Let $(\mathcal{J}_1, \mathcal{J}_2)$ be an *H*-twisted generalized Kähler structure, and *B* be a closed 2-form on *M*. Then $((\mathcal{J}_1)_B, (\mathcal{J}_2)_B)$ is also an *H*-twisted generalized Kähler structure. It is called the *B*-field shift of $(\mathcal{J}_1, \mathcal{J}_2)$.

In [4], a characterization of H-twisted generalized Kähler pairs was given in terms of Hermitian geometry, which is represented below.

Theorem 2.1 (M. Gualtieri, [4]). For each *H*-twisted generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$, there exists unique 2-form b, Riemannian metric g, and two orthogonal complex structures J_{\pm} such that

$$\mathcal{J}_{1,2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^* \pm J_-^*) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix},$$

where $\omega_{\pm} = gJ_{\pm}$ satisfy

(1)
$$d_{-}^{c}\omega_{-} = -d_{+}^{c}\omega_{+} = H + db.$$

Conversely, any quadruple (g, b, J_{\pm}) satisfying condition (1) defines an H-twisted generalized Kähler structure.

Every *H*-twisted generalized complex manifold may not admit an *H*-twisted generalized Kähler structure. However, following lemma claims that every *H*-twisted generalized complex manifold admits a generalized almost Kähler structure. This is a generalized complex geometry analogue of the fact that a symplectic manifold admits an almost complex structure which is compatible with the symplectic structure.

Lemma 2.1. Let (M, \mathcal{J}) be an *H*-twisted generalized complex manifold. Then there exists a generalized almost complex structure \mathcal{J}' which is compatible with \mathcal{J} , that is, \mathcal{J}' is a generalized almost complex structure which commutes with \mathcal{J} and $\mathcal{G} = -\mathcal{J}\mathcal{J}'$ is a positive definite metric.

Proof. Choose a Riemannian metric g on M and put

$$\tilde{\mathcal{G}} = \left(\begin{array}{cc} 0 & g^{-1} \\ g & 0 \end{array}\right).$$

Then $\tilde{\mathcal{G}}$ is a postitive definite metric on $T \oplus T^*$. Now we define a symplectic structure \mathcal{W} on $T \oplus T^*$ by

$$\mathcal{W}(X+\xi,Y+\eta) = \langle \mathcal{J}(X+\xi),Y+\eta \rangle.$$

Since $\tilde{\mathcal{G}}$ and \mathcal{W} are non-degenerate, there exists an endomorphism \mathcal{A} on $T \oplus T^*$ which satisfies

$$\mathcal{W}(X+\xi,Y+\eta) = \mathcal{G}(\mathcal{A}(X+\xi),Y+\eta)$$

for all $X + \xi, Y + \eta \in T \oplus T^*$. The map \mathcal{A} is skew-symmetric because

$$\begin{split} \tilde{\mathcal{G}}(\mathcal{A}^*(X+\xi),Y+\eta) &= \tilde{\mathcal{G}}(X+\xi,\mathcal{A}(Y+\eta)) = \tilde{\mathcal{G}}(\mathcal{A}(Y+\eta),X+\xi) \\ &= \mathcal{W}(Y+\eta,X+\xi) = -\mathcal{W}(X+\xi,Y+\eta) \\ &= \tilde{\mathcal{G}}(-\mathcal{A}(X+\xi),Y+\eta), \end{split}$$

where \mathcal{A}^* denotes the adjoint operator of \mathcal{A} with respect to the positive difinite metric $\tilde{\mathcal{G}}$. Moreover since \mathcal{A} is invertible, $\mathcal{A}\mathcal{A}^*$ is symmetric and positive, that is, $(\mathcal{A}\mathcal{A}^*)^* = \mathcal{A}\mathcal{A}^*$ and

$$\tilde{\mathcal{G}}(\mathcal{A}\mathcal{A}^*(X+\xi), X+\xi) > 0$$

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for all non-zero $X + \xi \in T \oplus T^*$. So we can define $\sqrt{\mathcal{A}\mathcal{A}^*}$ the square root of $\mathcal{A}\mathcal{A}^*$. Of course $\sqrt{\mathcal{A}\mathcal{A}^*}$ is also symmetric and positive definite. Let \mathcal{J}' be an endomorphism on $T \oplus T^*$ defined by $\mathcal{J}' = (\sqrt{\mathcal{A}\mathcal{A}^*})^{-1}\mathcal{A}$. Since \mathcal{A} commutes with $\sqrt{\mathcal{A}\mathcal{A}^*}$, \mathcal{J}' also commutes with \mathcal{A} and $\sqrt{\mathcal{A}\mathcal{A}^*}$. Hence \mathcal{J}' is orthogonal with respect to $\tilde{\mathcal{G}}$ and $(\mathcal{J}')^2 = -\mathrm{id}$.

By the definition of \mathcal{A} we have $\mathcal{AJ} = -\mathcal{JA}^{-1}$, and $\sqrt{\mathcal{AA}^*}\mathcal{J} = \mathcal{J}(\sqrt{\mathcal{AA}^*})^{-1}$. Therefore we have

$$\begin{aligned} \mathcal{J}\mathcal{J}' &= \mathcal{J}(\sqrt{\mathcal{A}\mathcal{A}^*})^{-1}\mathcal{A} \\ &= -\sqrt{\mathcal{A}\mathcal{A}^*}\mathcal{A}^{-1}\mathcal{J} \\ &= -(\mathcal{J}')^{-1}\mathcal{J} \\ &= \mathcal{J}'\mathcal{J}, \end{aligned}$$

in particular \mathcal{J}' commutes with \mathcal{J} . Moreover, for each $X + \xi, Y + \eta \in T \oplus T^*$, we have

$$\begin{aligned} \langle \mathcal{J}'(X+\xi), \mathcal{J}'(Y+\eta) \rangle &= -\mathcal{W}(\mathcal{J}\mathcal{J}'(X+\xi), \mathcal{J}'(Y+\eta)) \\ &= -\tilde{\mathcal{G}}(\mathcal{A}\mathcal{J}\mathcal{J}'(X+\xi), \mathcal{J}'(Y+\eta)) \\ &= -\tilde{\mathcal{G}}(\mathcal{J}'\mathcal{A}\mathcal{J}(X+\xi), \mathcal{J}'(Y+\eta)) \\ &= -\tilde{\mathcal{G}}(\mathcal{A}\mathcal{J}(X+\xi), Y+\eta) \\ &= -\mathcal{W}(\mathcal{J}(X+\xi), Y+\eta) \\ &= \langle X+\xi, Y+\eta \rangle. \end{aligned}$$

Hence \mathcal{J}' is a generalized almost complex structure on M which commutes \mathcal{J} . Finally $\mathcal{G} := -\mathcal{J}\mathcal{J}'$ is a positive definite metric on $T \oplus T^*$ since $\mathcal{G} = \tilde{\mathcal{G}}\sqrt{\mathcal{A}\mathcal{A}^*}$. This completes the proof.

If \mathcal{J}' is a generalized almost complex structure which is compatible with an H-twisted generalized complex structure \mathcal{J} , then we can apply the argument of Gualtieri in [4] and construct a Riemannian metric g, a 2-form b, and two orthogonal almost complex structures J_{\pm} which satisfy the equation

(2)
$$\mathcal{J} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \begin{pmatrix} J_+ + J_- & -(\omega_+^{-1} - \omega_-^{-1}) \\ \omega_+ - \omega_- & -(J_+^* + J_-^*) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}.$$

Of course, J_+ and J_- are not integrable in general.

3. HAMILTONIAN ACTION ON GENERALIZED MANIFOLDS

In this section we introduce the definition of Hamiltonian actions on H-twisted generalized complex manifolds given in [10].

Definition 3.1 (Y. Lin and S. Tolman, [10]). Let a compact Lie group G with its Lie algebra \mathfrak{g} act on an H-twisted generalized complex manifold (M, \mathcal{J}) preserving \mathcal{J} , where $H \in \Omega^3(M)^G$ is a closed 3-form. The action of G is said to be Hamiltonian if there exists a G-equivariant smooth function $\mu : M \longrightarrow \mathfrak{g}^*$, called the generalized moment map, and a \mathfrak{g}^* -valued one form $\alpha \in \Omega^1(M, \mathfrak{g}^*)$, called the moment one form such that

- $\xi_M \sqrt{-1}(d\mu^{\xi} + \sqrt{-1}\alpha^{\xi})$ lies in L for all $\xi \in \mathfrak{g}$, where ξ_M denotes the induced vector field on M and $L \subset (T \oplus T^*) \otimes \mathbb{C}$ denotes the $\sqrt{-1}$ -eigenspace of \mathcal{J} , and
- $i_{\xi_M}H = d\alpha^{\xi}$ for all $\xi \in \mathfrak{g}$.

Example 3.1.

- (1) Let G act on a symplectic manifold (M, ω) preserving ω , and $\mu : M \longrightarrow \mathfrak{g}^*$ be an usual moment map, that is, μ is G-equivariant and $i_{\xi_M}\omega = d\mu^{\xi}$ for all $\xi \in \mathfrak{g}$. Then G also preserves \mathcal{J}_{ω} , μ is also a generalized moment map, and $\alpha = 0$ is a moment one form for this action. Hence the G-action on $(M, \mathcal{J}_{\omega})$ is Hamiltonian.
- (2) Let (M, J) be a complex manifold and G act on (M, \mathcal{J}_J) in a Hamiltonian way. Then G also preserves the original complex structure J. Since $L_J = T_{1,0} \oplus T^{0,1}$ and $\xi_M \in \pi(L_J)$, so we have $\xi_M = 0$ for all $\xi \in \mathfrak{g}$. Thus if G is connected, the G-action on M must be trivial.
- (3) Let G act on an H-twisted generalized complex manifold (M, \mathcal{J}) with a generalized moment map μ and a moment one form α . If $B \in \Omega^2(M)^G$ is closed, then G acts on M preserving the B-field shift of \mathcal{J} with generalized moment map μ and moment one form α' , where $(\alpha')^{\xi} = \alpha^{\xi} + i_{\xi_M} B$ for all $\xi \in \mathfrak{g}$.

By the definition and examples above, we can say that the notion of generalized moment maps is a generalization of the notion of moment maps in symplectic geometry. Generalized moment maps are studied by Lin and Tolman in [10]. They showed in [10] that a reduction theorem for Hamiltonian actions of compact Lie groups on an H-twisted generalized complex and Kähler manifold holds.

Now we can state Theorem A in Introduction. Before we begin a proof, we prove a notable property of generalized moment maps. At first we prove following lemmata.

Lemma 3.1. Let a compact Lie group G act on an H-twisted generalized conplex manifold (M, \mathcal{J}) preserving \mathcal{J} . Then there exists a G-invariant generalized almost complex structure which is compatible with \mathcal{J} .

Proof. Choose a G-invariant Riemannian metric g on M and put

$$\mathcal{G} = \left(\begin{array}{cc} 0 & g^{-1} \\ g & 0 \end{array}\right).$$

Then \mathcal{G} is a *G*-invariant positive definite metric on $T \oplus T^*$. Let \mathcal{A} be an endomorphism on $T \oplus T^*$ defined by $\mathcal{A} = \mathcal{G}^{-1}\mathcal{J}$. Since \mathcal{G} and \mathcal{J} are *G*-invariant, so \mathcal{A} is also *G*-invariant. Now if we define

$$\mathcal{J}' = (\sqrt{\mathcal{A}\mathcal{A}^*})^{-1}\mathcal{A},$$

then \mathcal{J}' is a generalized almost complex structure on M which is compatible with \mathcal{J} . Moreover since \mathcal{A} is G-invariant, \mathcal{J}' is also G-invariant. This completes the proof.

Lemma 3.2. Let an m-dimensional torus T^m act on an H-twisted generalized conplex manifold (M, \mathcal{J}) in a Hamiltonian way with a generalized moment map μ and a moment one form α . Then for an arbitrary subtorus $G \subset T^m$ the fixed point set of G-action

$$Fix(G) = \{ p \in M \mid \theta \cdot p = p \ (\forall \theta \in G) \}$$

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is an even dimensional submanifold of M.

Proof. Choose a *G*-invariant generalized almost complex structure \mathcal{J}' which is compatible with \mathcal{J} . Then there exists a Riemannian metric g, a 2-form b, and two orthogonal almost complex structures J_{\pm} which satisfies the equation (2). Since \mathcal{J} and \mathcal{J}' are *G*-invariant, so g and J_{\pm} are also *G*-invariant. For each $p \in \text{Fix}(G)$ and $\theta \in G$, the differential of the action of θ at p,

$$(\theta_*)_p: T_pM \longrightarrow T_pM,$$

preserves the almost complex structures J_{\pm} . In addition, since *G*-action preserves the metric *g*, the exponential mapping $\exp_p : T_pM \longrightarrow M$ with respect to the metric *g* is equivariant, that is,

$$\exp_p((\theta_*)_p X) = \theta \cdot \exp_p X$$

for any $\theta \in G$ and $X \in T_pM$. This concludes that the fixed point of the action of θ near p correspond to the fixed point of $(\theta_*)_p$ on T_pM by the exponential mapping, that is,

$$T_p \operatorname{Fix}(G) = \bigcap_{\theta \in G} \ker(1 - (\theta_*)_p).$$

Since $(\theta_*)_p \circ J_{\pm} = J_{\pm} \circ (\theta_*)_p$, the eigenspace with eigenvalue 1 of $(\theta_*)_p$ is invariant under J_{\pm} , and is therefore an almost complex subspace. In particular, $T_p \operatorname{Fix}(G)$ is even dimensional.

We remark in the proof of Proposition 3.2 that the fixed point set Fix(G) is an almost complex submanifold with respect to J_{\pm} . In particular, we see ω_{\pm} is non-degenerate on Fix(G). Moreover, it is known that Fix(G) is a generalized complex submanifold when H = 0 (see [11] for the details).

The following proposition claims that a generalized moment map has a property of a Bott-Morse function. This fact plays a crucial role in the proof of Theorem A.

Proposition 3.1. Let an *m*-dimensional torus T^m act on a compact *H*-twisted generalized conplex manifold (M, \mathcal{J}) in a Hamiltonian way with a generalized moment map μ and a moment one form α . Then μ^{ξ} is a Bott-Morse function with even index and coindex for all $\xi \in \mathfrak{t}$.

Proof. Let $\xi \in \mathfrak{t}$ and T^{ξ} denote the subtorus of T^m which is generated by ξ . First we shall prove that the critical set

$$Crit(\mu^{\xi}) = \{ p \in M \mid (d\mu^{\xi})_p = 0 \}$$

coincides with the fixed point set of T^{ξ} -action $\operatorname{Fix}(T^{\xi})$. Choose a T^m -invariant generalized almost complex structure \mathcal{J}' which is compatible with \mathcal{J} . Then \mathcal{J} can be written by the form of the equation (2) by corresponding quadruple (g, b, J_{\pm}) . Note that the metric g and orthogonal almost complex structures J_{\pm} are all T^m invariant.

Since $\xi_M - \sqrt{-1}(d\mu^{\xi} + \sqrt{-1}\alpha^{\xi}) \in L$ by the definition of Hamiltonian actions, so $(d\mu^{\xi})_p = 0$ implies $p \in \text{Fix}(T^{\xi})$. In particular we obtain $\text{Crit}(\mu^{\xi}) \subset \text{Fix}(T^{\xi})$. On the other hand, since $\text{Fix}(T^{\xi}) = \{p \in M \mid (\xi_M)_p = 0\}$ so we can view μ^{ξ} locally as an imaginary part of a pseudoholomorphic function on an almost complex manifold $(\text{Fix}(T^{\xi}), J_{\pm})$. By applying the principle of the maximum and compactness of $\text{Fix}(T^m)$, we see that μ^{ξ} is constant on each connected component of $\text{Fix}(T^{\xi})$. Moreover the gradient of μ^{ξ} with respect to the metric g is tangent to $\text{Fix}(T^{\xi})$ on

there because g and μ^{ξ} are T^{ξ} -invariant. This shows that $\operatorname{Fix}(T^{\xi}) \subset \operatorname{Crit}(\mu^{\xi})$, and hence we obtain $\operatorname{Crit}(\mu^{\xi}) = \operatorname{Fix}(T^{\xi})$. In particular, $\operatorname{Crit}(\mu^{\xi})$ is an even dimensional submanifold of M.

To prove μ^{ξ} is a Bott-Morse function, we shall calculate the Hessian of μ^{ξ} on $\operatorname{Crit}(\mu^{\xi})$. First note that the induced vector field ξ_M can be written by

$$\xi_M = \frac{1}{2} \Big(\omega_+^{-1}(d\mu^{\xi}) - \omega_-^{-1}(d\mu^{\xi}) \Big).$$

Let ∇ be the Riemannian connection with respect to g, and $\nabla^2 \mu^{\xi}$ denotes the Hessian of μ^{ξ} . Then by an easy calculation we have

$$g(\nabla^2 \mu^{\xi}(Y), Z) = Y(Z\mu^{\xi}) - \nabla_Y Z(\mu^{\xi})$$

= $Y(g(J_{\pm}\xi_M^{\pm}, Z)) - g(J_{\pm}\xi_M^{\pm}, \nabla_Y Z)$
= $g(\nabla_Y J_{\pm}\xi_M^{\pm}, Z)$
= $g((\nabla_Y J_{\pm})\xi_M^{\pm}, Z) + g(J_{\pm}(\nabla_Y \xi_M^{\pm}), Z),$

where $\xi_M^{\pm} = \omega_{\pm}^{-1}(d\mu^{\xi}) = -J_{\pm}g^{-1}(d\mu^{\xi})$. Thus for each $p \in \operatorname{Crit}(\mu^{\xi})$ we have

$$(\nabla^2 \mu^\xi)_p = J_{\pm}(\nabla_{Y_p} \xi_M^{\pm})$$

because $(\xi_M^{\pm})_p = 0$. Let $(L_{\xi})_p$ be an endomorphism on T_pM defined by

$$(L_{\xi})_p(Y) = [\xi_M, Y]_p = -\nabla_{Y_p} \xi_M.$$

Then since $\xi_M = \frac{1}{2} \left(\xi_M^+ - \xi_M^- \right)$, $(L_{\xi})_p$ can be written by

$$(L_{\xi})_p = -\frac{1}{2}(J_+ - J_-)(\nabla^2 \mu^{\xi})_p$$

Now we prove that $T_p \operatorname{Crit}(\mu^{\xi}) = \ker(\nabla^2 \mu^{\xi})_p$. Since $\operatorname{Crit}(\mu^{\xi})$ is a submanifold of M, it is easy to see $T_p \operatorname{Crit}(\mu^{\xi}) \subset \ker(\nabla^2 \mu^{\xi})_p$. So we may only show that $\ker(\nabla^2 \mu^{\xi})_p \subset T_p \operatorname{Crit}(\mu^{\xi})$. At first we have $\ker(\nabla^2 \mu^{\xi})_p \subset \ker(L_{\xi})_p$ by the calculation above. If we identify $(L_{\xi})_p$ with a vector field on $T_p M$, the one parameter family of diffeomorphism $\{(\exp t_{\xi*})_p\}_{t\in\mathbb{R}}$ on $T_p M$ coincides with $\{\exp t(L_{\xi})_p\}_{t\in\mathbb{R}}$. So $\ker(L_{\xi})_p$ coincides with the fixed point set of $\{(\exp t_{\xi*})_p\}_{t\in\mathbb{R}}$. Hence we have

$$\ker(\nabla^2 \mu^{\xi})_p \subset \ker(L_{\xi})_p = \bigcap_{\theta \in T^{\xi}} \ker(1 - (\theta_*)_p) = T_p \operatorname{Fix}(T^{\xi}) = T_p \operatorname{Crit}(\mu^{\xi})$$

and this shows that $T_p \operatorname{Crit}(\mu^{\xi}) = \ker(\nabla^2 \mu^{\xi})_p$. In particular, μ^{ξ} is a Bott-Morse function.

Finally, by an easy calculation we see that $(\nabla^2 \mu^{\xi})_p$ commutes with $J_+ - J_$ for all $p \in \operatorname{Crit}(\mu^{\xi})$. So we can define a non-degenerate 2-form on each non-zero eigenspace of $(\nabla^2 \mu^{\xi})_p$ by $g(J_+ - J_-)$. Hence each non-zero eigenspace of $(\nabla^2 \mu^{\xi})_p$ is even dimensional, in particular the index and coindex of the critical manifold are even.

Remark 3.1. If M is noncompact, then the generalized moment map is not a Bott-Morse function in general. Indeed, if we consider a trivial circle action on a complex manifold (M, J), then the imaginary part of an arbitrary holomorphic function is a generalized moment map for this action.

4. Proof of Theorem A

We shall prove Theorem A in this section. This proof involves induction over $m = \dim T^m$. Consider the statements:

$$A_m$$
: "the level sets of μ are connected, for any T^m -action",

 B_m : "the image of μ is convex, for any T^m -action".

Then we have

(1)
$$\Leftrightarrow$$
 A_m holds for all m ,
(2) \Leftrightarrow B_m holds for all m .

At first we see that A_1 holds by using Proposition 3.1 and the fact that level sets of a Bott-Morse function on a connected compact manifold are connected if the critical manifolds all have index and coindex $\neq 1$ (see [12] for example). B_1 holds clearly because in \mathbb{R} connectedness is convexity.

Now we prove $A_{m-1} \Longrightarrow B_m$. Choose a matrix $A \in \mathbb{Z}^{m \times (m-1)}$ of maximal rank. If we identify A with a linear mapping $A : \mathbb{R}^{m-1} \longrightarrow \mathbb{R}^m$, then A induces an action of T^{m-1} -action on M by $\theta \cdot p := (A\theta) \cdot p$ for each $\theta \in T^{m-1}$ and $p \in M$. This T^{m-1} -action is a Hamiltonian action with a generalized moment map $\mu_A(p) := A^t \mu(p)$ and a moment one form $\alpha_A^{\xi} := \alpha^{A\xi}$, where A^t denotes the transpose of A.

Given any $a \in \mathfrak{t}^*$ and $p_0 \in \mu_A^{-1}(a)$,

$$p \in \mu_A^{-1}(a) \Leftrightarrow A^t \mu(p) = a = A^t \mu(p_0) \Leftrightarrow \mu(p) - \mu(p_0) \in \ker A^t$$

so that

$$\mu_A^{-1}(a) = \{ p \in M \mid \mu(p) - \mu(p_0) \in \ker A^t \}.$$

By the statement A_{m-1} , $\mu_A^{-1}(a)$ is connected. Therefore, if we connect p_0 to p_1 by a path p_t in $\mu_A^{-1}(a)$, we obtain a path $\mu(p_t) - \mu(p_0)$ in ker A^t . Since A^t is surjective, so ker A^t is 1-dimensional. Hence $\mu(p_t)$ must go through any convex combination of $\mu(p_0)$ and $\mu(p_1)$, which shows that any point on the line segment from $\mu(p_0)$ to $\mu(p_1)$ must be in $\mu(M)$.

Any $p_0, p_1 \in M$ with $\mu(p_0) \neq \mu(p_1)$ can be approximated arbitrarily closely by points p'_0 and p'_1 with $\mu(p'_1) - \mu(p'_0) \in \ker A^t$ for a matrix $A \in \mathbb{Z}^{m \times (m-1)}$ of maximal rank. By the argument above, we see that the line segment from $\mu(p'_0)$ to $\mu(p'_1)$ must be in $\mu(M)$. By taking limits $p'_0 \longrightarrow p_0$, and $p'_1 \longrightarrow p_1$ we can conclude that $\mu(M)$ is convex.

Next we prove $A_{m-1} \Longrightarrow A_m$. By identifying \mathfrak{t} with \mathbb{R}^m , we write $\mu = (\mu_1, \cdots, \mu_m)$. The generalized moment map μ is called effective if the 1-forms $d\mu_1, \cdots, d\mu_m$ are linearly independent. Note that $p \in M$ is a regular point of μ if and only if $(d\mu_1)_p, \cdots, (d\mu_m)_p$ are linearly independent.

Lemma 4.1. If μ is not effective, the action reduces to a Hamiltonian action of an (m-1)-dimensional subtorus.

Proof. If μ is not effective, there exists $0 \neq c = (c_1, \dots, c_m) \in \mathbb{R}^m$ such that $\sum_{i=1}^m c_i d\mu_i = 0$. So if we denote the canonical basis of $\mathfrak{t} \cong \mathbb{R}^m$ by ξ_1, \dots, ξ_n , then we have

$$\sum_{i=1}^{m} c_i \Big((\xi_i)_M + \alpha_i \Big) = \sum_{i=1}^{m} c_i \Big((\xi_i)_M - \sqrt{-1} (d\mu_i + \sqrt{-1}\alpha_i) \Big) \in L,$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$. Since $\sum_{i=1}^m c_i ((\xi_i)_M + \alpha_i)$ is real and $L \cap \overline{L} = \{0\}$, we obtain $\sum_{i=1}^m c_i(\xi_i)_M = 0$. Consider $\xi = \sum_{i=1}^m c_i\xi_i \in \mathfrak{t}$. Then μ^{ξ} is constant along M because $\xi_M = 0$. For the simplicity, we may assume $\xi_1, \dots, \xi_{m-1}, \xi$ are linearly independent. Then the T^{m-1} -action generated by ξ_1, \dots, ξ_{m-1} is a Hamiltonian action with a generalized moment map $(\mu_1, \dots, \mu_{m-1})$ and a moment one form $(\alpha_1, \dots, \alpha_{m-1})$.

So we may assume that μ is effective. Then for each $0 \neq \xi \in \mathfrak{t}$, μ^{ξ} is not a constant function. So the critical manifold $\operatorname{Crit}(\mu^{\xi})$ is an even dimensional proper submanifold. Now consider the union of critical manifolds $C = \bigcup_{\eta \neq 0} \operatorname{Crit}(\mu^{\eta})$.

Lemma 4.2. The union C is a countable union of even dimensional proper submanifold, that is,

$$C = \bigcup_{0 \neq \eta \in \mathbb{Z}^m} \operatorname{Crit}(\mu^\eta).$$

Proof. We may only show that $C \subset \bigcup_{0 \neq \eta \in \mathbb{Z}^m} \operatorname{Crit}(\mu^{\eta})$. For a rational vector $\xi = \sum_{i=1}^m c_i \xi_i$, $\tilde{\xi} = (\prod_{i=1}^m q_i) \xi \in \mathbb{Z}^m$, where $c_i = p_i/q_i$ and $p_i, q_i \in \mathbb{Z}$. Since

$$\operatorname{Crit}(\mu^{\xi}) = \operatorname{Fix}(T^{\xi}) = \operatorname{Fix}(T^{\xi}) = \operatorname{Crit}(\mu^{\xi}),$$

we see that $\operatorname{Crit}(\mu^{\xi}) \subset \bigcup_{0 \neq \eta \in \mathbb{Z}^m} \operatorname{Crit}(\mu^{\eta})$. If ξ is irrational, then for rational element $\eta \in \mathfrak{t}^{\xi}$ where \mathfrak{t}^{ξ} is the Lie algebra of T^{ξ} , we have $\operatorname{Fix}(T^{\xi}) \subset \operatorname{Fix}(T^{\eta})$. So we obtain

$$\operatorname{Crit}(\mu^{\xi}) = \operatorname{Fix}(T^{\xi}) \subset \operatorname{Fix}(T^{\eta}) \subset \bigcup_{0 \neq \eta \in \mathbb{Z}^m} \operatorname{Crit}(\mu^{\eta}).$$

Hence we have proved $C = \bigcup_{0 \neq \eta \in \mathbb{Z}^m} \operatorname{Crit}(\mu^{\eta})$.

In particular, $M \setminus C$ is dense subset of M. In addition, since the condition $p \in M \setminus C$ is equivalent to the condition that $(d\mu_1)_1, \cdots, (d\mu_m)_p$ are linearly independent, we obtain $M \setminus C$ is open dense subset of M.

Lemma 4.3. The set of regular values of μ in $\mu(M)$ is a dense subset of $\mu(M)$.

Proof. For each $a = \mu(p) \in \mu(M)$, there exists a sequence $\{p_i\}_{i=1}^{\infty} \subset M \setminus C$ which satisfies that $\lim_{i\to\infty} p_i = p$. Since p_i is a regular point of μ , $\mu(M)$ contains a neighborhood of $\mu(p_i)$ by implicit function theorem. Moreover there exists a regular value $a_i \in \mathfrak{t}^*$ which is sufficiently close to $\mu(p_i)$ and $\mu^{-1}(a_i) \neq \phi$ by Sard's theorem. Hence the sequence $\{a_i\}_{i=1}^{\infty}$ approximates a.

By the similar argument, the set of $a = (a_1, \dots, a_m) \in \mathfrak{t}^*$ that (a_1, \dots, a_{m-1}) is a regular value of $(\mu_1, \dots, \mu_{m-1})$ in $\mu(M)$ is also a dense subset of $\mu(M)$. Hence, by continuity, to prove that $\mu^{-1}(a)$ is connected for every $a = (a_1, \dots, a_m) \in \mathfrak{t}^*$, it suffics to prove that $\mu^{-1}(a)$ is connected whenever (a_1, \dots, a_{m-1}) is a regular value for the reduced generalized moment map $(\mu_1, \dots, \mu_{m-1})$. By the induction hypothesis, the submanifold $Q = \bigcap_{i=1}^{m-1} \mu_i^{-1}(a_i)$ is connected whenever (a_1, \dots, a_{m-1}) is a regular value for $(\mu_1, \dots, \mu_{m-1})$. To finish the proof, we need following lemma.

Lemma 4.4. If (a_1, \dots, a_{m-1}) is a regular value for $(\mu_1, \dots, \mu_{m-1})$, the function $\mu_m : Q \longrightarrow \mathbb{R}$ is a Bott-Morse function of even index and coindex.

Proof. By the hypothesis, Q is a 2n - (m - 1) dimensional connected submanifold of M. For each $p \in Q$, the subspace W of the cotangent space T_p^*M generated by $(d\mu_1)_p, \cdots, (d\mu_{m-1})_p$ is (m-1) dimensional because p is regular. So the tangent space T_pQ of Q coincises with the annihilator of W;

$$T_pQ = \{ X \in T_pM \mid f(X) = 0 \ (\forall f \in W) \}.$$

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Hence $p \in Q$ is a critical point of $\mu_m : Q \longrightarrow \mathbb{R}$ if and only if there exists real numbers c_1, \dots, c_{m-1} such that

$$\sum_{i=1}^{n-1} c_i (d\mu_i)_p + (d\mu_m)_p = 0.$$

This means that p is a critical point of the function $\mu^{\xi} : M \longrightarrow \mathbb{R}$, where $\xi = (c_1, \dots, c_{m-1}, 1) \in \mathfrak{t} \cong \mathbb{R}^m$. By Proposition 3.1, μ^{ξ} is a Bott-Morse function with even index and coindex. We shall prove the critical manifold $\operatorname{Crit}(\mu^{\xi})$ intersects Q transversally at p, that is,

$$T_p M = T_p \operatorname{Crit}(\mu^{\xi}) + T_p Q.$$

This is equivalent to $W \cap T_p^*\operatorname{Crit}(\mu^{\xi}) = \{0\}$. To see this, we need to prove $\sum_{i=1}^{m-1} c_i(d\mu_i)_p(X) = 0$ for any $X \in T_p\operatorname{Crit}(\mu^{\xi})$ and $\sum_{i=1}^{m-1} c_i(d\mu_i)_p \in W \cap T_p^*\operatorname{Crit}(\mu^{\xi})$. This means that the linear functionals $(d\mu_1)_p, \cdots, (d\mu_{m-1})_p$ remain linearly independent when restricted to the subspace $T_p\operatorname{Crit}(\mu^{\xi})$. Consider the vector fields $\xi_1^+, \cdots, \xi_{m-1}^+$ on M defined by

$$d\mu_i = \omega_+(\xi_i^+), \quad i = 1, \cdots, m-1.$$

Then $(\xi_1^+)_p, \dots, (\xi_{m-1}^+)_p$ are linearly independent on T_pM because p is regular. So they are also linearly independent on $T_p^*\operatorname{Crit}(\mu^{\xi})$. Since the 2-form ω_+ is still nondegenerate when it is restricted to $\operatorname{Crit}(\mu^{\xi})$, so $(d\mu_1)_p, \dots, (d\mu_{m-1})_p$ are linearly independent on $T_p^*\operatorname{Crit}(\mu^{\xi})$ and hense $\operatorname{Crit}(\mu^{\xi})$ is transverse to Q as claimed.

This implies that the orthogonal complement of the subspace $T_p \operatorname{Crit}(\mu^{\xi})$ is contained in $T_p Q$. Hence the Hessian of μ^{ξ} at p is non-degenerate on this space with even index and coindex. In other words, $\operatorname{Crit}(\mu^{\xi}) \cap Q$ is the critical manifold of $\mu^{\xi}|_Q$ of even index and coindex. The same holds for $\mu_m|_Q$ since it only differs from μ^{ξ} by the constant $\sum_{i=1}^{m-1} c_i a_i$. Thus we have proved that the function $\mu_m : Q \longrightarrow \mathbb{R}$ is a Bott-Morse function with even index and coindex. \Box

By applying Lemma 4.4, if (a_1, \dots, a_{m-1}) is a regular value for $(\mu_1, \dots, \mu_{m-1})$, then the level set $\mu_m^{-1}(a_m) \cap Q = \mu^{-1}(a)$ is connected. This shows that $A_{m-1} \Longrightarrow A_m$.

Finally, we shall prove the third claim, that is, the image of the generalized moment map μ is the convex hull of the images of the fixed points of the action. By Lemma 3.2, the fixed point set $\operatorname{Fix}(T^m)$ of the action decomposes into finitely many even dimensional connected submanifolds C_1, \dots, C_N of M. The generalized moment map μ is constant on each of these sets because $C_i \subset \operatorname{Crit}(\mu^{\xi})$ for $i = 1, \dots, N$ and any $\xi \in \mathfrak{t}$. Hence there exists $a_1, \dots, a_N \in \mathfrak{t}^*$ such that

$$\mu(C_i) = \{a_i\}, \quad i = 1, \cdots, N.$$

By what we have proved so far the convex hull of the points a_1, \dots, a_N is contained in $\mu(M)$. Conversely, let $a \in \mathfrak{t}^*$ be a point which is not in the convex hull of a_1, \dots, a_N . Then there exists a vector $\xi \in \mathfrak{t}$ with rationally independent components such that

$$a_i(\xi) < a(\xi), \quad i = 1, \cdots, N.$$

Since the components of ξ are rationally independent, we have $\operatorname{Crit}(\mu^{\xi}) = \operatorname{Fix}(T^m)$. Hence the function $\mu^{\xi} : M \longrightarrow \mathbb{R}$ attains its maximum on one of the sets C_1, \dots, C_N . This implies

$$\sup_{p \in M} \mu^{\xi}(p) < a(\xi),$$

and hence $a \notin \mu(M)$. This shows that $\mu(M)$ is the convex hull of the points a_1, \dots, a_N and Theorem A is proved.

Remark 4.1. If M is an orbifold and H is a closed 3-form on M, we can define notions of H-twisted generalized complex structures of M and Hamiltonian actions of compact Lie groups on an H-twisted generalized complex orbifold in usual way. By applying same arguments of our proof and Theorem 5.1 in [9], Theorem A still holds when M is a connected compact H-twisted generalized complex orbifold. Of course, C_1, \dots, C_N are connected suborbifolds in this case.

Remark 4.2. When the manifold is noncompact and the Lie group is non abelian, the convexity and connected properties still hold in the sense of Theorem 1.1 and Theorem 4.3 in [8]. In view of the works of Lerman, Meinrenken, Tolman, and Woodward in [8], we can import notions of symplectic cuts and Cross-section theorem in symplectic geometry to generalized complex geometry. These techniques tell us that the convexity and connected properties still hold in general cases. Detailed proof can be seen in our work [13].

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