Equicontinuous Geodesic Flows

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Abstract

This article is about the interplay between topological dynamics and differential geometry. One could ask how many informations of the geometry are carried in the dynamic of the geodesic flow. M. Paternain proved in [6] that an expansive geodesic flow on a surface implies that there are no conjugate points. Instead of regarding notions that describe chaotic behavior (for example expansiveness) we regard a notion that describes the stability of orbits in dynamical systems, namely equicontinuity and distality. In this paper we give a new sufficient and necessary condition for a compact Riemannian surface to have all geodesics closed (P-manifold): (M, g) is a P-manifold iff the geodesic flow $SM \times \mathbb{R} \to SM$ is equicontinuous. We also prove a weaker theorem for flows on manifolds of dimension 3. At the end we discuss some properties of equicontinuous geodesic flows on noncompact surfaces and higher dimensional manifolds.

1 Introduction

In the complete paper all geodesics are parametrized by arc length and the geodesic flow is complete. The manifolds and the Riemannian metrics are C^∞. $\pi : TM \to M$ denotes the canonical projection. At first we summarize some facts about recurrent maps and we state theorem [2.7] (due to Boris Kolev and Marie-Christine Pérouème) about the set of fixed points of recurrent maps on surfaces. In section 3 we study the geodesic return map. In section 4 we prove that equicontinuous geodesic flows are periodic. In section 5 we prove that if a flow without singularities on a 3-dimensional manifold admits a global Poincaré section and has enough periodic orbits, then the flow is pointwise periodic. In the last section we show that the existence of an equicontinuous geodesic flow on a compact manifold M implies that the fundamental group is finite. Moreover, we discuss equicontinuous geodesic flows on noncompact manifolds.

2 Recurrent Behavior

Definition 2.1. *A dynamical system* (X, T) *is called distal if* $\inf\{d(xt, yt)|t \in T\} = 0$ *implies* $x = y$.
A system (X, T) *is called equicontinuous (regular) if for all* $\epsilon > 0$ *there exists an* $\delta(\epsilon) > 0$ *such that for all* x, y *with* $d(x, y) < \delta(\epsilon)$ *we have* $d(xt, yt) < \epsilon$ *for all* $t \in T$.

Here is a well-known fact about equicontinuous systems on compact metric spaces:

Theorem 2.2. *An equicontinuous flow* $\Phi : X \times \mathbb{R} \to X$ *on a compact metric space is uniformly almost periodic, i.e. for every* $\epsilon > 0$ *there exists an* $\tau > 0$ *such that in every intervall* I *of length* τ *there exists an* $t \in I$ *such that for all* x *we have* $d(\Phi(t, x), x) < \epsilon$.

Proof: See theorem 2.2 in [1].

A weaker form of uniform almost periodicity for maps is the following:

Definition 2.3. *A continuous map* f *on a metric space* (X, d) *is recurrent if there exists a sequence* $n_k \to \infty$ *such that for* $\sup_{x\in X} d(f^{n_k}(x), x) \to 0$ $k \to \infty$.

Definition 2.4. *A continuous map* $f : X \to X$ *on a metric space* (X, d) *is called paracompact-recurrent on* $Y \subset X$ *if there exists a sequence* $n_k \to \infty$ *such that for* $k \to \infty$ *we have* $\sup_{x\in C} d(f^{n_k}(x), x) \to 0$, *where* $C \subset Y$ *is a compact subset of* X.

Note that in the definition of paracompact-recurrence the sequence n_k is fixed and does not depend on $C \subset Y$ and that paracompact-recurrence and recurrence are independent from the metric which defines the topology if the space X is compact.

Lemma 2.5. *If* f *is recurrent, then* f^m *is recurrent.*

Set $s_k := \sup_{x\in X} d(f^{n_k}(x), x)$. We conclude

$$d(f^{n_k m}(x), x) \le \sum_{i=0}^{m-1} d(f^{(m-i)n_k}(x), f^{(m-i-1)n_k}(x))$$

$$\le \sum_{i=0}^{m-1} d(f^{n_k}(f^{(m-i-1)n_k}(x)), f^{(m-i-1)n_k}(x)) \le m s_k.$$

□

Lemma 2.6. *Given a continuous map on a compact surface* S. *Let* F *be a finite non empty subset of* $Fix(f)$ *(the fixed point set). If* f *is paracompact-recurrent on* $S - F$, *then* f *is recurrent.*

W.l.o.g we can suppose that our metric is induced from a Riemannian metric. Let $\{x_1, \cdots, x_m\}$ denote F and choose a tiny $\epsilon > 0$ and define

$$C := S - (\bigcup_i B(x_i, \epsilon)).$$

$c_i = \partial B(x_i, \epsilon)$ defines a simple closed curve. Choose a $K := K(\epsilon)$ such that for all $k > K$ we have:

$$\sup_{x \in C} d(f^{n_k}(x), x) < 4\epsilon \quad \text{and} \quad f^{n_k}(c_i) \subset B(c_i, \frac{\epsilon}{2}).$$

This is possible, since there is a compact subset in $S - F$ that contains c_i and C.
Hence $f^{n_k}(B(x_i, \epsilon)) \subset B(x_i, 2\epsilon)$ thus $d(f^{n_k}(x), x) < 4\epsilon$ for all $x \in C^c$ and $k > K$. Consequently $d(f^{n_k}(x), x) < 4\epsilon$ for all $x \in X$ and $k > K$, i.e. f is recurrent. □

The following nice theorem can be found in [4] as theorem 1.1.

Theorem 2.7. *A non trivial orientation-preserving and recurrent homeomorphism of the sphere S^2 has exactly two fixed points.*

3 The Geodesic Return Map

A well-known technic to study geodesic flows on a surface is to study the geodesic return map. Let A denote the open annulus.
Suppose that we have an equicontinuous geodesic flow Φ on the unit tangent bundle SM of an orientable Riemannian surface M. Let γ be a simple closed geodesic. Let denote $W = SM|\gamma - T\gamma$ (the set of all unit tangent vectors based on γ which are not element of $T\gamma$). Note that W is homeomorphic to the union of two open annulus A_0 and A_1. We can identify $v \in A_0$ with (x, θ) where $\pi(v) = x$ and $\theta \in (0, 1)$ is the angle of v and $\dot{\gamma}(t)$ divided by π.
Since every orbit is recurrent, we can define for our flow Φ the map $F : A_0 \to A_0$ by

$$F(x, \theta) = (x_0, \theta_0),$$

where $x_0 = \pi(\Phi_{t_0}(x, \theta))$ is the next intersection point of $\{\pi(\Phi_t(x, \theta))|t > 0\}$ with γ such that $\Phi_{t_0}(x, \theta) = (x_0, \theta_0) \in A_0$. We just simple write $F : A \to A$. F can be extended to an homeomorphism of S^2 by two-point-compactification. In this case F has two fixed points $\{\infty\}$ and $\{-\infty\}$ as we will see. If for $v = (x, \theta)$ we have θ near zero, the geodesic $\Phi_t(v)$ stays near $\dot{\gamma}$ (by equicontinuity), hence $F^n(x, \theta)$ is near zero for all n, thus $\{\infty\}$ and $\{-\infty\}$ are fixed points. In this paper, we call the extension of F the geodesic return map and denote it by $F : S^2 \to S^2$.

Proposition 3.1. *If Φ is equicontinuous then F is recurrent on S^2.*

We only need to show that F is paracompact-recurrent on $S^2 - (\{\infty\} \cup \{-\infty\})$, since then we can apply lemma [2.6].
For $0 < \theta_0 < \theta_1 < 1$ we set

$$K(\theta_0, \theta_1) = \{(x, v) \in A|\theta_0 \leq v \leq \theta_1\}.$$

Lemma 3.2. *For $K := K(\theta_0, \theta_1)$ and large $M := M(\theta_0, \theta_1)$, the constant*

$$s(\theta_0, \theta_1, M) := \inf\{t_1 - t_0| - M < t_0 < t_1 < M, \Phi_{t_0}(v) \in A, \Phi_{t_1}(v) \in A, v \in K\}$$

is strict positive. Every geodesic γ_v intersects γ at least 2 times in the forward direction on the intervall $[0, M]$ and at least 2 times in the backward direction on the intervall $[-M, 0]$ if $v \in K$.

Proof: Use the compactness of K. □

Lemma 3.3. *There exists a constant $\nu(\theta_0, \theta_1, M, r) > 0$ for every $\frac{s(\theta_0,\theta_1,M)}{2} > r > 0$, large M and $K(\theta_0, \theta_1)$ such that the following holds: If for some $w \in SM$ we have an $v \in K(\theta_0, \theta_1)$ with $d(v, w) < \nu(\theta_0, \theta_1, r, M)$, then there is an unique t_w such that $|t_w| < r$ and $\Phi_{t_w}(w) \in A$.*

Proof: Use the compactness of K. □

Given $v \in A$ then for every integer n we define $t(n, v)$ as the following: $t(n, v)$ is the unique element of $\mathbb{R}$ such that $\Phi_{t(n,v)}(v) = F^n(v)$ and $t(0, v) = 0$. Moreover given $v \in A$ and $t \geq 0$ we define:

$$P(v, t) := \max\{n \geq 0 | t(n, v) \leq t\}.$$

Note that we can find a neighbourhood U_0 of γ looking like a strip (M is orientable) , therefore $U_0 - \gamma$ is the distinct union of two open strips S^* and S_*. We can define what lies in $U_0 - \gamma$ above and below γ. The area where the points of A are pointing in-ward is defined to be S^* (" above "). The other points of $U - \gamma$ are lying in S_* (" below ").

For $j \in \{0, 1\}$ choose sequences $(\theta_{j,i})_i$ with

$$\theta_{0,i+1} < \theta_{0,i} < \theta_{1,i} < \theta_{1,i+1}$$

and converging strictly to j. Set $K_i = K(\theta_{0,i}, \theta_{1,i})$ and fix a $v_0 \in \cap_i K_i$.

Lemma 3.4. *There exist sequences $T_i \to \infty$, $0 < \zeta_i \to 0$ and $0 < \beta_i \to 0$ such that:*

1. *$\pi(\Phi_{T_i+\zeta_i}(v)) \in S^*$ for all $v \in K_i$*
2. *$\pi(\Phi_{T_i-\zeta_i}(v)) \in S_*$ for all $v \in K_i$*
3. *$\Phi_{T_i}(v_0) \in A$*
4. *$d(\Phi_{T_i+s}(v), v) < \beta_i$ for all $v \in K_i$ and $|s| < 2\zeta_i$*
5. *For every $v \in K_i$ there exists an unique intersection point of γ and $\pi(\Phi_{T_i+s}(v))$, where s ranges over $|s| < 4\zeta_i$*

Proof: Choose sequences $0 < \delta_i \to 0$, $0 < s_i \to 0$ and $\alpha_i \to 0$ such that if $v \in K_i$ and $w \in SM$ with $d(v, w) < \delta_i$ then

i) $\pi(\Phi_{s_i}(w)) \in S^*$

ii) $\pi(\Phi_{-s_i}(w)) \in S_*$

iii) $d(\Phi_t(w), v) < \alpha_i$ for all $|t| \leq 8s_i$

iv) $\pi(\Phi_t(w))$ has an unique intersection point with γ where t ranges over $|t| \leq 8s_i$.

v) $\pi(\Phi_{[-20s_i,20s_i]}(w)) \subset U_0$

This can be done easy. First choose an increasing sequence M_i such that we can define $s(\theta_{0,i+1}, \theta_{1,i+1}, M_i)$ (and therefore $s(\theta_{0,i+1}, \theta_{1,i+1}, M_{i+1})$ is also defined) and a decreasing sequence $s_i \to 0$ such that

$$32s_i < \min\{\frac{s(\theta_{0,i+1}, \theta_{1,i+1}, M_i)}{2}, \frac{s(\theta_{0,i}, \theta_{1,i}, M_i)}{2}\}$$

and then a decreasing sequence $\delta_i > 0$

$$\delta_i < \nu(\theta_{0,i}, \theta_{1,i}, M_i, \frac{s_i}{2}).$$

Moreover δ_i is chosen so small, that $d(w, K_i) < \delta_i$ implies that $\Phi_{t_w}(w)$ lies in K_{i+1} (t_w is from definition [3.3]). Since $s_i, \delta_i \to 0$ and our flow Φ is defined on a compact space, surely α_i exists and by regarding only large i we have $\pi(\Phi_{[-20s_i,20s_i]}(w)) \subset U_0$. If $d(v, w) < \delta_i$ and $v \in K_i$ then i) ii) and v) holds. We check that iv) holds:
Note that for $d(v, w) < \delta_i$ and $v \in K_i$ we have $\Phi_{t_w}(w) \in K_{i+1}$ and

$$[-8s_i, 8s_i] \subset [-8s_i - t_w, 8s_i + t_w] \subset [-9s_i, 9s_i],$$

and therefore $\Phi_{[-8s_i,8s_i]}(w)$ lies in $\Phi_{[-9s_i,9s_i]}(\Phi_{t_w}(w))$.
Hence we conclude that iv) holds since $18s_i < s(\theta_{0,i+1}, \theta_{1,i+1}, M_i)$.
Now choose a sequence $S_i \to \infty$ such that $d(\Phi_{S_i}(x), x) < \delta_i$ for all x (apply theorem [2.2]). There exists an unique $|r_i| < s_i$ such that $\Phi_{S_i+r_i}(v_0) \in A$. Set $T_i := S_i + r_i$ and $\zeta_i := 2s_i$. Given $v \in K_i$ note that

$$T_i + \zeta_i = S_i + r_i + 2s_i > S_i + s_i$$

and $|T_i + \zeta_i - S_i| < 3s_i$. From this , i), iv) and v) we conclude $\Phi_{S_i+s_i}(v) \in S^*$ and therefore $\Phi_{T_i+\zeta_i}(v) \in S^*$. Analogously we conclude from ii), iv) and v) that $\Phi_{T_i-\zeta_i}(v) \in S_*$. Thus 1,2 and 3 is proven.
If $|t| \leq 2\zeta_i = 4s_i$, then $|T_i + t - S_i| < 5s_i$ thus the orbit segment $\Phi_{[T_i-2\zeta_i,T_i+2\zeta_i]}(v)$ lies in the orbit segment $\Phi_{[S_i-8s_i,S_i+8s_i]}(v)$, hence by iv) has an unique intersection point with γ. This proves 5. Since $2\zeta_i \to 0$ and uniformly $\Phi_{T_i}(x) \to x$ and M is compact, there exists a sequence β_i. $\square$

Proof of the proposition: Set $p_i := P(v_0, T_i + \zeta_i)$. We show that for this sequence F is paracompact-recurrent. Note that $P(v_0, T_i + \zeta_i) = P(v_0, T_i)$ by lemma [3.4] and $K_i \supset K_j$ if $j \leq i$. Given any compact set $C \subset A$ choose I_0 such that $C \subset K_{I_0}$. The functions

$$G_i : A \to \mathbb{N}$$

defined by $G_i(v) = P(v, T_i + \zeta_i)$ are constant on K_{I_0} if $i \geq I_0$.
Indeed, by construction the functions G_i are locally constant on K_i, hence constant $p_i = P(v_0, T_i + \zeta_i)$ on the connected set K_i. Therefore $G_i = p_i$ on K_{I_0} if $i \geq I_0$.
We have $|T_i - t(v, G_i(v))| < 2\zeta_i$ for $v \in K_i$ by construction, since we know from lemma [3.4, 1 and 2] that $P(v, T_i - \zeta_i) = p_i - 1$. Therefore we conclude from lemma [3.4, 4)] that for all $v \in K_{I_0}$ and $i \geq I_0$ we have $d(F^{p_i}(v), v) = d(\Phi_{t(v,p_i)}(v), v) < \beta_i$. □

4 Equicontinuous Geodesic Flows On Surfaces

Definition 4.1. *(M, g) is called a P-manifold if all geodesics are closed.*

The following lemma is easy to prove:

Lemma 4.2. *If (M, g) is a P-manifold then the geodesic flow (SM, Φ) is equicontinuous.*

Proof: It is a well-known fact that if M is a P-manifold then the flow is periodic (compare [8]) and M is compact. Let L denote the smallest period. Choose for $\epsilon > 0$ an $\delta > 0$ such that $d(v, w) < \delta$ implies

$$d(\Phi_t(v), \Phi_t(w)) < \epsilon$$

for $|t| < 2L$, hence it holds for all t. □

To prove our first theorem we need the following theorem.

Theorem 4.3 (Ballmann)**.** *Every compact Riemannian manifold (M, g) of dimension 2 has at least three simple closed geodesics.*

Proof: See [2]. □

Theorem 4.4. *Given a compact Riemannian manifold (M, g) of dimension 2 then the following conditions are equivalent:*

1. *M is a P-manifold.*
2. *(SM, Φ) is equicontinuous.*

1) implies 2) by lemma [4.2]. We show that 2) implies 1). Take a simple closed geodesic γ.

Lemma 4.5. *If Z is a compact set in $M - \gamma$, then there are $0 < \theta_{0,Z} < \theta_{1,Z} < 1$ such that for every geodesic α intersecting Z we have that $\Phi(Z \times \mathbb{R}) \cap A$ is a subset of $K(\theta_{0,Z}, \theta_{1,Z})$.*

Proof: Choose an open set V around γ such that $V \cap Z = \emptyset$. Choose $\delta > 0$ such that $d(w, T\gamma) < \delta$ implies $\gamma_w \subset V$. If $\theta_{0,Z}$ or $\theta_{1,Z}$ would not exist, we could find a geodesic $\gamma_w \subset V$, but starting in Z. □

Lemma 4.6. *Every geodesic intersects γ.*

Given a point $x \in M - \gamma$ and $w \in S_xM$. Choose a path-connected compact set C in SM such that $S_xM \subset C$ and $\pi(C) \cap \gamma = \emptyset$. Apply lemma [4.5] to $Z = \pi(C)$. Choose a curve β in C from $w \in S_xM$ to $q \in S_xM$ where γ_q intersects γ. Cover β with finitely many balls B_i (say n) such that for any two vectors $p, u \in B_i$ we have for a large M and all t

$$d(\Phi_t(u), \Phi_t(p)) < \nu(\theta_{0,Z} - \epsilon, \theta_{1,Z} + \epsilon, M, b),$$

where $b = \frac{s(\theta_{0,Z}-\epsilon, \theta_{1,Z}+\epsilon, M)}{4}$ and ϵ is small. By induction we conclude that if $\gamma_q(T_0) \in \gamma$, then $\gamma_w([T_0 - (n+1)b, T_0 + (n+1)b])$ intersects γ. $\square$

Proof of the theorem: Since the lifted geodesic flow on the orientable double cover N is equicontinuous, it suffices to show that the theorem holds for orientable surfaces, otherwise we regard the orientable double cover N and conclude that N is a P-manifold. Apply the construction of F in section 3 to (M, g) and γ. By theorem [4.3] and lemma [4.6] we have a periodic point in $y \in A$, thus for some m it follows that F^{2m} is an orientation-preserving homeomorphism with three fixed points ($\{\infty\}$,$\{-\infty\}$, y). From proposition [3.1] we know that F^{2m} is recurrent, hence trival by theorem [2.7]. Lemma [4.6] implies that every geodesic is closed. $\square$

Note that distality of the geodesic flow does not imply that the manifold is a P-manifold. Take the torus T^n with the standard flat metric. The flow is distal. Indeed, for an unit vector $v \in \mathbb{R}^n$ define the vector field $X_v(x) = v$. Note that the solutions of these vectorfields are our lifted geodesices.
If $\inf\{d(xt, yt) | t \in T\} = 0$ for $x, y \in ST^n$ then the lifted geodesics xt, yt are solutions of the same vectorfield, but the projected flows on T^n of these vectorfields are equicontinuous, since equicontinuity implies distality (see [1]) we know that $y = x$. Note also that on surfaces of higher genus the geodesic flow has positive entropy, but distal flows on compact metric spaces have always zero entropy (see [5]), thus the only compact surfaces M that can admit distal geodesic flows are S^2, $\mathbb{R}P^2$, T^2 and the Klein bottle and they do.

5 Flows On Manifolds Of Dimension 3

Definition 5.1. *A global surface of section Σ for a C^∞ flow Φ without singularities on a three dimensional manifold is a compact submanifold with the following properties:*

1. *If Σ has a boundary then its boundary components are periodic orbits.*
2. *The interior of the surface (stated with $\mathring{\Sigma}$) is transversal to Φ.*
3. *The orbit through a point not lying on the boundary of Σ hits the interior in forward and backward time.*
4. *Every orbit intersects Σ.*

There is a natural compactification of $\overset{\circ}{\Sigma}$ to a closed surface by collapsing the boundary components to a point. We call this unique compactification the compactification of $\overset{\circ}{\Sigma}$ to a closed surface. If the flow is equicontinuous we can do more:
The return map $F : \overset{\circ}{\Sigma} \to \overset{\circ}{\Sigma}$ can be extended to the compactification of $\overset{\circ}{\Sigma}$ to a closed surface by defining the collapsed boundary components to be fixed points. This is well defined and can be proven as in the beginning of section 3. We call this map in this section the extended poincaré section map.

Theorem 5.2. *Let Φ be a C^∞ equicontinuous flow without singularities on a three dimensional manifold that admits a global surface of section Σ. Let X be the compactification of $\overset{\circ}{\Sigma}$ to a closed surface, then the following holds:*

1. *If X is homeomorphic to the torus or the Klein bottle and the extended poincaré section map on X has at least one periodic point, then the flow is pointwise periodic.*

2. *If X is homeomorphic to the sphere and the extended poincaré section map on X has at least three periodic points, then the flow is pointwise periodic.*

3. *If X is homeomorphic to the projective plane and the extended poincaré section map on X has at least two periodic points, then the flow is pointwise periodic.*

4. *If X is homeomorphic to a surface of negative Euler characteristic, then the flow is pointwise periodic.*

Proof: The proof is quite analogue to theorem [4.4], and so we only repeat some ideas. Indeed, the proof of theorem [4.4] was actually of topological nature. The following theorem is important:

Theorem 5.3. *A recurrent homeomorphism of a compact surface with negative Euler characteristic is periodic. If a recurrent homeomorphism on the torus, the annulus or the Möbius strip or the Klein bottle has a periodic point, then the homeomorphism is periodic.*

Proof: See corollary [4.2] and remark [4.3] in [4]. □

Proof of the theorem: Let $F : \overset{\circ}{\Sigma} \to \overset{\circ}{\Sigma}$ denote our return map. If we show that our extension is recurrent, then by theorem [5.3] and our assumptions we know that F is periodic and every orbit of the flow is periodic, thus we only need to show that F is recurrent.
If the section Σ is a surface with boundary, then $\overset{\circ}{\Sigma}$ denotes the interior. Choose a finite number (say $j = 1, \cdots, N$) of connected, compact, orientable surfaces Σ_j that are diffeomorphic to discs and such that $\bigcup_j \Sigma_j = \Sigma$. Choose, for every j, a sequence of connected, compact, orientable surfaces $K_{j,i}$ such that $K_{j,i} \subset \overset{\circ}{K}_{j,i+1}$ and $\bigcup_i K_{j,i}$ is an open set. We construct $K_{j,i}$ such that given any compact set $C \subset \overset{\circ}{\Sigma}$ we have, for a large I, that $C \subset \bigcup_j K_{j,I}$. Moreover, we

suppose $\bigcup_{i,j} K_{j,i} = \overset{\circ}{\Sigma}$ and $\bigcup_j K_{j,i}$ is connected.
For every $K_{j,i}$ we can find a tubular neighbourhood $U_{j,i}$ diffeomorphic $K_{j,i} \times (-1,1)$ via a diffeomorphism $\tau_{i,j}$. We say a point in $\tau_{i,j}^{-1}(K_{j,i} \times (-1,0))$ lies above $K_{j,i}$ and a point in $\tau_{i,j}^{-1}(K_{j,i} \times (0,1))$ lies below $K_{j,i}$. We set $U_{*,i,j} := \tau_{i,j}(K_{j,i} \times (-1,0))$ and $U_{i,j}^* := \tau_{i,j}(K_{j,i} \times (0,1))$. We can suppose that $U_{*,i,j} \subset U_{*,i+1,j}$ and $U_{i,j}^* \subset U_{i+1,j}^*$. Since Φ is transversal to the section and without singularities, we conclude that $\bigcup_j K_{j,i}$ is an orientable surface. Therefore we can define in tubular neighbourhood of $\bigcup_j K_{j,i}$ what lies below and above.
Analogously we can define again for $K_{j,i}$ and large M the constant

$$s(j,i,M) := \inf\{|t_0 - t_1| \mid -M < t_0 < t_1 < M, \Phi_{t_0}(v) \in \Sigma, \Phi_{t_1}(v) \in \Sigma, v \in K_{j,i}\}.$$

We can also define analogously the constant $\nu(j,i,M,r) > 0$. Given $v \in \overset{\circ}{\Sigma}$, then for every integer n we define $t(n,v)$ as the following:
$t(n,v)$ is the unique element of $\mathbb{R}$ such that $\Phi_{t(n,v)}(v) = F^n(v)$ and $t(0,v) = 0$.
Moreover given $v \in \overset{\circ}{\Sigma}$ and $t \geq 0$ we define:

$$P(v,t) := \max\{n \geq 0 \quad | \quad t(n,v) \leq t\}.$$

The counterpart of lemma [3.4] will be the following:

Lemma 5.4. *Fix $v_j \in \bigcap_i K_{j,i}$. There exist sequences $T_i \to \infty$, $0 < \zeta_i \to 0$ and $0 < \beta_i \to 0$ such that $\Phi_{T_i}(v_1) \in \overset{\circ}{\Sigma}$ and for all $j \in \{1, \cdots N\}$ we have:*

1. *$\Phi_{T_i + \zeta_i}(v) \in U_{i+1,j}^*$ for all $v \in K_{j,i}$*
2. *$\pi(\Phi_{T_i - \zeta_i}(v)) \in U_{*,i+1,j}$ for all $v \in K_{j,i}$*
3. *$d(\Phi_{T_i+s}(v), v) < \beta_i$ for all $v \in K_{j,i}$ and $|s| < 2\zeta_i$*
4. *For every $v \in K_{j,i}$ there exists an unique intersection point of $\overset{\circ}{\Sigma}$ and $\Phi_{T_i+s}(v)$ where s ranges over $|s| < 4\zeta_i$*

Proof: Analogous to lemma [3.4]. □

The proof is now easy. Set $p_{j,i} := P(v_j, T_i + \zeta_i)$. The functions

$$G_i : A \to \mathbb{N}$$

defined by $G_i(v) = P(v, T_i + \zeta_i)$ are constant on K_{j,I_0} if $i \geq I_0$, but since $\bigcup_i K_{j,I_0}$ is connected, we know that $p_{j,i}$ is independent from j, and so $p_{j,i} = p_i$. We conclude $d(F^{p_i}(v), v) = d(\Phi_{t(v,p_i)}(v), v) < \beta_i$, thus F is paracompact-recurrent on $\overset{\circ}{\Sigma}$. Since the collapsed boundary components are fixed points, we conclude from lemma [2.6] that F is recurrent on X. □

Corollary 5.5. *If Φ is a C^∞ equicontinuous flow without singularities on a three dimensional manifold that admits a global surface of section Σ and has at least three distinct periodic orbits, then the flow is pointwise periodic.*

6 Noncompact Manifolds

In this section, we regard noncompact manifolds, therefore two metrics that generate the same topology may be non equivalent. We can prove some fact about pointwise equicontinuous geodesic flows on noncompact surfaces if we restrict to a canonical metric for geodesic flows. In our case this should be the Sasaki metric.
Let M denote a surface (maybe non compact). d will always denote the induced metric on M of the Riemannian metric and $\tilde{d}$ the induced metric on SM of the Sasaki metric. Note that we have by construction of the Sasaki metric $\tilde{d}(v, w) \geq d(\pi(v), \pi(w))$.

Definition 6.1. *A system (X, T) is called pointwise equicontinuous (pointwise regular) if for all $\epsilon > 0$ and all $x \in X$ there exists an $\delta(\epsilon, x) > 0$ such that for all y with $d(x, y) < \delta(\epsilon, x)$ we have $d(xt, yt) < \epsilon$ for all $t \in T$.*

On compact metric spaces, pointwise equicontinuity implies equicontinuity, but on noncompact metric spaces this is not true and pointwise equicontinuity is not independent from the metric that generates the topology. Moreover, given any compact set K of X, we can find an $\delta(\epsilon, x) > 0$ in the definition above that is independent from $x \in K$.

Lemma 6.2. *Let $(\Phi, SM, \tilde{d})$ be a geodesic flow that is pointwise equicontinuous. For all compact sets K there exists a number $C(K)$ such that for all $v, w \in K$ and $t \in \mathbb{R}$ we have $d(\gamma_{v_0}(t), \gamma_w(t)) \leq C(K)$.*

Proof: Choose first an open and bounded set O that contains $SM|K$. Choose for $\overline{O}$ a constant $\delta(\epsilon) > 0$ such that if for $v, w \in SM|K$ we have $d(v, w) < \delta(\epsilon)$ then $d(\Phi_t(v), \Phi_t(w)) < \epsilon$. Cover $SM|K$ with $N = N(\epsilon, K)$ balls of radius smaller than $\delta(\epsilon)$ such that the union of these balls is connected and lies in O. We conclude

$$d(\gamma_v(t), \gamma_w(t)) \leq N\epsilon := C(K)$$

for all $v, w \in SM|K$ and $t \in \mathbb{R}$. □

Proposition 6.3. *If M is compact and the geodesic flow (Φ, SM) is equicontinuous, then $\pi_1(M)$ is finite.*

Proof: (Φ, SM) is equicontinuous with respect to the Sasaki metric and therefore one can conclude that the lifted geodesic flow on the universal covering $\tilde{M}$ is equicontinuous with respect to the lifted Sasaki metric (denoted with $\tilde{d}$). Since M is compact, we conclude that for all $r > 0$ there exists a number $Z(r)$ such that for any point x of M the sphere S_xM can be covered by $Z(r)$ balls of radius r (with respect to the metric d). Given any point x and assume M is not compact. Choose $v \in S_xM$ such that $\gamma_v : \mathbb{R}^+ \to M$ is a ray and a sequence $t_i \to \infty$. Set $v_i = -\dot{\gamma_{v_i}}(t_i)$. We have by the proof of the lemma above and the existence of $Z(r)$ for an $\epsilon > 0$ $d(\gamma_{v_i}(t_i), \gamma_{-v_i}(t_i) = \gamma_v(2t_i)) \leq Z(\delta(\epsilon))\epsilon$, hence $2t_i = d(x = \gamma_{v_i}(t_i), \gamma_v(2t_i)) \leq Z(\delta(\epsilon))\epsilon$, but t_i grows. □

Corollary 6.4. *If M is noncompact and the geodesic flow (Φ, SM) is pointwise equicontinuous with respect to the Sasaki metric, then the following holds:*

1. *There is no minimal geodesic (also called line) in M.*
2. *$Diam(\partial B(x,r)) := \sup\{d(x,y)|x,y \in \partial B(x,r)\}$ is bounded by a constant $C(x)$ for all x. The constant $C(x)$ can be chosen uniformly on compact sets.*

Proof: If γ is minimal, then for some constant C

$$2t = d(\gamma(t), \gamma(-t)) \leq C,$$

and therefore γ is not minimal.
If $Diam(\partial B(x,r))$ is not bounded, choose sequences a_i and b_i such that $a_i, b_i \in \partial B(x, r_i)$ and $d(a_i, b_i) \to \infty$ where $r_i \to \infty$. Choose for a_i a sequence $v_i \in S_xM$ such that $\gamma_{v_i} : [0, d(x, a_i)]$ is a minimal geodesic segement that starts in x and ends in a_i. W.l.o.g. we suppose that $v_i \to v$ and, therefore γ_v is a ray and from equicontinuity we conclude that $d(\gamma_v(r_i), a_i) \to 0$. We repeat the construction for b_i to get a ray γ_w such that $d(\gamma_w(r_i), b_i) \to 0$. The flow is equicontinuous and therefore we have -by lemma [6.2]- $d(\gamma_v(r_i), \gamma_w(r_i))$ bounded and therefore $d(a_i, b_i)$ is bounded. It follows from the proof that $C(x)$ can be chosen uniformly on compact sets. □

Of course, one can conjecture that theorem [4.4] holds in higher dimensional cases, but it seems that there are no tools to prove this conjecture. Corollary [6.3] holds for P-manifolds (see theorem 7.37 in [3]) and using the Morse-Index theorem, one can see that noncompact manifolds with strictly positive sectional curvature have no line, thus there are some reasons to conjecture this.
We now discuss if there exists an equicontinuous (or even a pointwise equicontinuous) geodesic flow with respect to the Sasaki metric on a noncompact surface. Here is a subresult:

Proposition 6.5. *Let (M, g) be a Riemannian manifold of dimension 2 and suppose that the geodesic flow (Φ, SM) is pointwise equicontinuous with respect to the Sasaki metric, then M is homeomorphic to the plane.*

Proof: Our proof is based on the following nice theorem

Theorem 6.6. *Every surface is homeomorphic to a surface formed from a sphere S by first removing a closed totally disconnected set X from S, then removing the interiors of a finite or infinite sequence D_i of non overlapping closed discs in $S - X$ and finally suitable identifying the boundaries of these discs in pairs. It may be necessary to identify the boundary of one disc with itself to produce a "cross cap". The sequence D_i approaches X in the sense that, for any open set U in S containing X, all but a finite number of the D_i are contained in U.*

Proof: See [7]. □

d is the metric induced from the Riemannian metric. We first prove that M has genus zero! Assume that M is not diffeomorphic to $S^2 - X$. Choose a curve

β that is not contractible and lies on a "handle" or a "crosscap" such that we can find a compact set O that contains β and is diffeomorphic to a closed disc where a cylinder or a crosscap is glued in. Let $[\beta]$ denote the free homotopy class. If there is a sequence β_i such that $\beta_i \in [\beta]$ and $L(\beta_i) \to 0$, then, since β_i always intersects O, we conclude that β_i is contradictible for large i. Now let β_i be a sequence such that $\beta_i \in [\beta]$ and $L(\beta_i) \to \inf_{c\in[\beta]} L(c) > 0$. Again we know that β_i always intersects O. If β_i lies in a compact subset of M, then we know that a subsequence converges to a closed geodesic, but it is clear that β_i lies in a compact subset K of M, since otherwise there is a point $x_i \in \beta_i$ that tends to infinity, hence $L(\beta_i)$ is not bounded. Hence we have found a closed geodesic β. Since there exists a ray starting at a point of β, we conclude from lemma [6.2] that the ray does not tend to infinity, thus we get a contradiction.
We prove now that X is just a simple point. We endow S^2 with a metric d_0 that generates the standard topology of S^2. Let $-\infty$ and ∞ two different points of X. Choose sequences a_i and b_i such that $a_i \to -\infty$ and $b_i \to \infty$ with respect to d_0. We want to show that $d(a_i, b_i)$ tends to infinity for a subsequence. Suppose that $d(a_i, b_i)$ is bounded. Let $\delta_i : [0, d(a_i, b_i)] \to M$ be a minimal geodesic segment from a_i to b_i. Let K_i denote the image of δ_i. Define

$$\lim(K_i) := \{y \in S^2 \quad | \quad \exists x_i \in K_i : d_0(x_i, y) \to 0\}.$$

It is easy to see that $K := \lim(K_i)$ is closed and connected.
Assume a point of M lies in K. Then w.l.o.g. a tangent vector v_i of δ_i converges to a vector v of SM. Since we suppose that $d(a_i, b_i)$ is bounded by a constant C, then γ_v will be a geodesic that meets $-\infty$ and ∞ on the intervall $[-2C, 2C]$, since $\delta_i[0, C]$ converges to a segement of $\gamma_v[-2C, 2C]$ with respect to d. Hence K is a subset of X, but this means K is reduced to a point, thus $d_0(a_i, b_i) \to 0$ and therefore we have a contradiction to the fact that $d(a_i, b_i)$ is bounded. Thus given any sequence $a_i \to -\infty$ and $b_i \to \infty$ then $d_1(a_i, b_i)$ tends to infinity for a subsequence.
We now construct sequences $n_i \to -\infty$ and $m_i \to \infty$ such that $d(n_i, m_i)$ is bounded and therefore we derive a contradiction to the fact that X contains more than one point. Define

$$\omega(v) := \{y \in S^2 \quad | \quad \exists t_i \to \infty : d_0(\gamma_v(t_i), y) \to 0\}.$$

It is easy to see that $\omega(v)$ is closed and connected in S^2. If, for a ray γ_v, the set $\omega(v)$ is not subset of X, then there is a sequence $t_i \to \infty$ such that $\dot{\gamma_v}(t_i)$ converges to a vector v^* of SM and hence γ_{v^*} will be a minimal geodesic, since γ_v is a ray. This contradicts corollary [6.4] and therefore $\omega(v)$ is a subset of X, hence for a ray γ_v, the set $\omega(v)$ is reduced to a point of X (X is totally disconnected).
Take sequences $a_i \to -\infty$ and $b_i \to \infty$. For choose a_i a sequence $v_i \in S_xM$ such that $\gamma_{v_i} : [0, d(x, a_i)]$ is a minimal geodesic segement that starts in x and ends in a_i. v_i tends to a vector v. The curve γ_v will be a ray and therefore $\omega(v)$ will be a point. We show that $-\infty = \omega(v)$. Note that $d(a_i, \gamma_v(t_i)) \to 0$, since our flow is equicontinuous in x. Suppose that $d_0(\gamma_v(t_i), q) \to 0$ for a point $q \in S^2$. If $q \neq -\infty$, then choose a minimal geodesic segment $\delta_i : [0, d(a_i, \gamma_v(t_i))] \to M$

that starts in a_i and ends in $\gamma_v(t_i)$. We denote the image of δ_i with K_i. Again we conclude as above, that $\lim(K_i)$ will be a connected subset of X, and therefore $q = -\infty$.
Therefore we get a ray γ_v that starts in x and converges to $-\infty$. Repeating this constructions will generate a ray γ_w that starts in x and converges to ∞. From lemma [6.2] we know that $d(n_i := \gamma_v(i), m_i := \gamma_w(i))$ is bounded. $\square$

We end this paper by showing that at least a pointwise equicontinuous geodesic flow with respect to a metric d_1 exists. The metric d_1 is not equivalent to the induced metric of the Sasaki metric of h (h will be defined later). Let g_0 be the standard metric on $\mathbb{R}^2$, d_0 the metric induced from g_0 and identify $\mathbb{R}^2$ with $\mathbb{C}$. Choose a diffeomorphism $f : [0, \infty) \to [0, \infty)$ such that $f|[0, \frac{1}{2}) = Id$ and $f(t) = \exp(t)$ for $r \geq 1$. Let F denote a diffeomorphism from $\mathbb{R}^{>0} \times S^1$ to $\mathbb{R}^2 - \{0\}$ defined by $F(r, \phi) = (f(r), \phi)$ where (r, ϕ) are the polar coordinates. The pull-back of the metric g_0 under F defines a new metric h on $\mathbb{R}^2$ and its geodesic flow is complete. Regard the metric d_1 on $S\mathbb{R}^2$ defined by $d_1((x, v), (y, w)) = \|x - y\| + \|v - w\|$. We show that the geodesic flow of $(\mathbb{R}^2, h)$ will be pointwise equicontinuous with respect to d_1.
Consider the coordinante system $F : \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 - \{0\}$. Given a vector v at a point x, we regard the geodesic $g_{x,v}(t) = x + tv$ of the metric g_0. Let us fix a point $x = a_0 + ib_0$ and a vector $v = a_1 + ib_1$. Since our metric is invariant with respect to revolutions, we can assume that $a_1 \neq 0$. For a small $\epsilon > 0$ choose a $M > 0$ and a $\delta(\epsilon) > 0$ such that $d_0(\dot{g}_{x,v}(t), \dot{h}_{y,w}(t)) < \epsilon$ for $t \in [-M, M]$ and $d((x, v), (y, w)) < \delta(\epsilon)$. If M is large and $\delta(\epsilon) > 0$ small enough, then the geodesic $h_{y,w}(t)$ lies outside the compact set $F([0, 2], \mathbb{R})$ for $t \notin [-M, M]$.
Note that $|g(t)| = g(t) \cos \phi(t) + ig(t) \sin \phi(t)$,where
$\phi(t) = \arctan(\frac{b_0+tb_1}{a_0+ta_1})$ and $|g(t)|^2 = (a_0 + ta_1)^2 + (b_0 + tb_1)^2$. Note that

$$F^{-1} = g^*_{x,v}(t) := (f^{-1}(|g_{x,v}(t)|), \arctan(\frac{b_0 + tb_1}{a_0 + ta_1}))$$

will be a geodesic of our manifold $(\mathbb{R}^2, h)$.
We show that if $\epsilon > 0$ is small enough, the distance $d_1(\dot{g}^*_{x,v}(t), \dot{h}^*_{y,w}(t))$ remains arbitrary small for all t. A computation shows that for $t \notin [-M, M]$ we have

$$\dot{g}^*_{x,v}(t) = (\frac{a_1(a_0 + ta_1) + b_1(b_0 + tb_1)}{|g(t)|^2}, \frac{b_1(a_0 + ta_1) + a_1(b_0 + tb_1)}{(1 + (\frac{b_0+tb_1}{a_0+ta_1})^2)(a_0 + ta_1)^2}).$$

Hence for a small $\epsilon > 0$ and large M, we have $\|\dot{g}^*_{x,v}(t) - \dot{h}^*_{y,w}(t)\|$ small for $t \notin [-M, M]$. Let $h_{y,w}(t) = y + tw = n_0 + im_0 + t(n_1 + im_1)$, then one can compute that

$$\|h_{y,w}(t) - g_{x,y}(t)\| \leq \|f^{-1}(|g(t)|) - f^{-1}(|h(t)|)\|$$
$$+\|\arctan(\frac{b_0 + tb_1}{a_0 + ta_1}) - \arctan(\frac{m_0 + tm_1}{n_0 + tn_1})\|.$$

In the same way we conclude that for a small $\epsilon > 0$ and large M, we will have the second term small for $t \notin [-M, M]$. A computation shows that

$$\|f^{-1}(|g(t)|) - f^{-1}(|h(t)|)\| = \|\frac{1}{2} \ln(\frac{\frac{a_0^2+b_0^2}{t^2} + \frac{2(a_0a_1+b_0b_1)}{t} + a_1^2 + b_1^2}{\frac{n_0^2+m_0^2}{t^2} + \frac{2(n_0n_1+m_0m_1)}{t} + n_1^2 + m_1^2})\|,$$

but this term will be small for a small $\epsilon > 0$, large M and $t \notin [-M, M]$. Therefore we know that the geodesic flow is equicontinuous with respect to the metric d_1. The metric is not equivalent to the induced metric of h, since the Riemannian distance between two geodesics of different directions grows.

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