TREES, LINEAR ORDERS AND GÂTEAUX SMOOTH NORMS

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ABSTRACT. We introduce a linearly ordered set Z and use it to prove a necessity condition for the existence of a Gâteaux smooth norm on $\mathscr{C}_0(\Upsilon)$, where Υ is a tree. This criterion is directly analogous to the corresponding equivalent condition for Fréchet smooth norms. In addition, we prove that if $\mathscr{C}_0(\Upsilon)$ admits a Gâteaux smooth lattice norm then it also admits a lattice norm with strictly convex dual norm.

1. INTRODUCTION AND PRELIMINARIES

Among the most well-established geometrical properties of norms are smoothness and strict convexity. A norm $||\cdot||$ on a Banach space X is called *Gâteaux smooth*, or just *Gâteaux*, if, given any $x \in X\backslash\{0\}$, there exists a functional in X^*, denoted by $||x||'$, such that

$$\lim_{\lambda\to 0} \frac{||x+\lambda h|| - ||x||}{\lambda} \;=\; ||x||'(h)$$

for all $h \in X$. In addition, if the limit above is uniform for h in the unit sphere S_X, then $||\cdot||$ is called *Fréchet smooth*, or simply *Fréchet*.

Turning now to properties of strict convexity, we say that $||\cdot||$ is *strictly convex* if, given $x, y \in X$ satisfying $||x|| = \frac{1}{2}||x+y|| = ||y||$, we have $x = y$. Of the many stronger cousins of strictly convex norms, we mention one. The norm $||\cdot||$ is *locally uniformly rotund*, or *LUR*, if, given a point $x \in S_X$ and a sequence $(x_n) \subseteq S_X$ satisfying $||x + x_n|| \to 2$, we have $||x - x_n|| \to 0$.

Renorming theory is a branch of functional analysis that seeks to determine the extent to which a given Banach space can be endowed with equivalent norms sporting certain geometrical properties, such as the ones above. In this paper, a norm on a given Banach space is always assumed to be equivalent to the canonical norm. We refer the reader to [1] for a comprehensive account of this field up to 1993, together with the more recent surveys [2] and [12].

In recent years, trees have assumed an important role in the field, both as a source of counterexamples to existing questions and as a vehicle for exploring new avenues of research; see, for example [4], [5] and [6]. We say that a partially ordered set $(\Upsilon, \preccurlyeq)$ is a *tree* if, given arbitrary $t \in \Upsilon$, the set of predecessors $\{s \in \Upsilon \mid s \preccurlyeq t\}$, denoted by the *interval* $(0, t]$, is well-ordered. The set of *immediate successors* of $t \in \Upsilon$ is denoted by t^+. In this way, trees are a natural generalisation of ordinal numbers. As well as $(0, t]$, we define the interval $(s, t] = (0, t]\backslash(0, s]$ for $s \preccurlyeq t$, the *wedge* $[t, \infty) = \{u \in \Upsilon \mid t \preccurlyeq u\}$ and finally $(t, \infty) = [t, \infty)\backslash\{t\}$. We remark that the symbols 0 and ∞ are, in this context, convenient notational devices and not themselves elements of Υ.

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The scattered locally compact *interval topology* on Υ is the coarsest topology for which all intervals $(0,t]$ are both open and closed. This topology agrees with the standard interval topology of any ordinal Ω, if we consider Ω as a tree. To ensure that this topology is also Hausdorff, we restrict our attention to trees Υ with the property that every non-empty, linearly ordered set in Υ has at most one minimal upper bound. With this topology in mind, we consider the Banach space $\mathscr{C}_0(\Upsilon)$ of continuous real-valued functions vanishing at infinity, and the dual space of measures. We remark that as Υ is scattered, the weak topology and the topology of pointwise convergence agree on norm-bounded subsets of $\mathscr{C}_0(\Upsilon)$.

Trees and linearly ordered sets enjoy close ties. For a comprehensive review of these relationships, we refer the reader to [11]. Given partial orders P and Q, we say that the map $\rho : P \longrightarrow Q$ is called *increasing* (respectively *strictly increasing*) if $\rho(s) \preccurlyeq \rho(t)$ (respectively $\rho(s) \prec \rho(t)$) whenever $s \prec t$. *Decreasing* and *strictly decreasing* functions are defined analogously. If there exists a strictly increasing map from P to a linear order Q, we say that P is *Q-embeddable*, or $P \preccurlyeq Q$. Evidently, in this context, $\preccurlyeq$ is a transitive relation on the class of partial orders. In much of what follows, P will be a tree and Q a linear order. It is well known that $\Upsilon \preccurlyeq \mathbb{Q}$ if and only if Υ is *special*, which means that Υ can be written as a countable union of antichains (cf. [11, Theorem 9.1]). Special trees tend to have very good properties; for example, the following result can be found in [9].

Theorem 1. *Given a tree Υ, the space $\mathscr{C}_0(\Upsilon)$ admits a norm with LUR dual norm if and only if Υ is special.*

We introduce a couple of combinatorial ideas used extensively in [6].

Definition 2. Given an increasing function $\rho : \Upsilon \longrightarrow \mathbb{R}$, we say that $t \in \Upsilon$ is a *bad point for ρ* if there exists a sequence of distinct points $(u_n) \subseteq t^+$, such that $\rho(u_n) \to \rho(t)$.

Bad points are so named because their presence often indicates that the given $\mathscr{C}_0(\Upsilon)$ space has negative renorming properties. An analogue of the next simple result appears at the beginning of Section 3.

Proposition 1 ((Haydon))**.** *The tree Υ is special if and only if $\Upsilon \preccurlyeq \mathbb{R}$ and there exists an increasing map $\rho : \Upsilon \longrightarrow \mathbb{R}$ that has no bad points.*

We move on to the second combinatorial property taken from [6].

Definition 3. A subset E of a tree is said to be *ever-branching* if each element of E has a pair of strict successors in E that are incomparable in the tree order.

It is easy to see that within every ever-branching subset can be found a *dyadic tree of height ω*; that is, a tree with a single minimal element, no limit elements, and with the property that each element has exactly two immediate successors.

Many types of norm on $\mathscr{C}_0(\Upsilon)$ can be characterised in terms of increasing real-valued functions on Υ, with further combinatorial properties that can be expressed in terms of bad points and ever-branching subsets. Of particular interest to us is the following result.

Theorem 4 ((Haydon [6]))**.** *Given a tree Υ, the space $\mathscr{C}_0(\Upsilon)$ admits a Fréchet norm if and only if there exists an increasing function $\rho : \Upsilon \longrightarrow \mathbb{R}$ that has no bad points and is not constant on any ever-branching subset.*

In order to exhibit a tree that does not satisfy the statement of Theorem 4, we introduce a fundamental construction, due to Kurepa. Given a linear order Σ, we define the Hausdorff tree

$$\sigma\Sigma \;=\; \{A \subseteq \Sigma \mid\; A \text{ is well-ordered}\}.$$

We remark that some authors demand the additional requirement that elements of $\sigma\Sigma$ are bounded above. One of the reasons why Kurepa's construction is so important in the theory of trees is summed up by the following theorem.

Theorem 5 ((Kurepa [7]))**.** *If Σ is a linear order then $\sigma\Sigma \not\preccurlyeq \Sigma$.*

From Theorem 5, $\sigma\mathbb{Q}$ is not special. On the other hand, if we take an enumeration (q_n) of the rationals and consider the map $A \mapsto \sum_{q_n \in A} 2^{-n}$, we see that $\sigma\mathbb{Q} \preccurlyeq \mathbb{R}$. It follows that, by Proposition 1, every increasing, real-valued function defined on $\sigma\mathbb{Q}$ has a bad point.

Corollary 1 ((Haydon))**.** *The space $\mathscr{C}_0(\sigma\mathbb{Q})$ admits no Fréchet norm.*

While many types of norm are accounted for in [6], equivalent conditions for the existence of norms on $\mathscr{C}_0(\Upsilon)$ with strictly convex dual, or Gâteaux norms, cannot be adequately expressed in terms of increasing real-valued functions. In all that follows, ω_1 denotes the first uncountable ordinal. The following linearly ordered set is introduced in [9].

Definition 6. Let Y be the set of all strictly increasing, continuous, transfinite sequences $x = (x_\xi)_{\xi\leq\beta}$ of real numbers, where $0 \leq \beta < \omega_1$. Order Y by declaring that $x < y$ if and only if either y strictly extends x, or if there is some ordinal α such that $x_\xi = y_\xi$ for $\xi < \alpha$ and $y_\alpha < x_\alpha$.

Observe that Y is not ordered in the usual lexicographic way. Compared to the real line, Y is large.

Proposition 2 ((Smith [9]))**.** *If $\beta < \omega_1$ then $Y^\beta \preccurlyeq Y$, where Y^β is ordered lexicographically.*

As $\mathbb{R} \preccurlyeq Y$, we see that $\mathbb{R}^\beta \preccurlyeq Y$ for all $\beta < \omega_1$. On the other hand, it can be shown that Y contains no well-ordered or conversely well-ordered subsets. The next theorem is the main result of [9].

Theorem 7 ((Smith [9]))**.** *Given a tree Υ, the Banach space $\mathscr{C}_0(\Upsilon)$ admits a norm with strictly convex dual norm if and only if $\Upsilon \preccurlyeq Y$.*

Theorem 7 is a direct analogue of Theorem 1. In [9], it is shown that the spaces $\mathscr{C}_0(\sigma(\mathbb{R}^\beta))$, where $\mathbb{R}^\beta$ is ordered lexicographically, admit norms with strictly convex duals provided $\beta < \omega_1$. On the other hand, by Theorem 5, $\mathscr{C}_0(\sigma Y)$ does not admit such a norm.

The order Y can also be used to give an improved sufficient condition for the existence of Gâteaux norms in the context of trees.

Theorem 8 ((Smith [8]))**.** *If there exists an increasing function $\rho : \Upsilon \longrightarrow Y$ that is not constant on any ever-branching subset then $\mathscr{C}_0(\Upsilon)$ admits a Gâteaux norm.*

We end our review of the existing literature by presenting what was hitherto the best known necessary condition for Gâteaux norms in this context. Given a tree Υ, the *forcing topology* on Υ takes as its basis the set of all wedges $[t, \infty)$,

$t \in \Upsilon$. A subset $B \subseteq \Upsilon$ is called *Baire* if it is a Baire space with respect to the induced forcing topology; that is, any countable intersection of relatively dense, open subsets of B is again dense. When referring to the Baire property, we will only consider subsets that are *perfect* with respect to the forcing topology; in other words those without isolated points or, equivalently, maximal elements. Arguably the simplest example of such an object is the ordinal ω_1, though more interesting ones that have no uncountable linearly ordered subsets can be found in [11, Lemma 9.12] (cf. [5]).

Theorems 4 and 8 applied to a constant function on ω_1 demonstrate that, by itself, the Baire property cannot destroy Gâteaux renormability. Instead, we have the following result.

Theorem 9 ((Haydon [5]))**.** *If $\mathscr{C}_0(\Upsilon)$ admits a Gâteaux norm then Υ contains no ever-branching Baire subsets.*

We turn now to the results of this paper. In order to properly express our necessary condition for Gâteaux renormability, we must introduce a second linearly ordered set.

Definition 10. Let Z be the set of all increasing, continuous sequences $x = (x_\xi)_{\xi \leq \beta}$ of real numbers, where $0 \leq \beta < \omega_1$, and such that x is strictly increasing on $[0, \beta)$. The order of Z follows that of Y; $x < y$ if and only if either y strictly extends x, or if there is some ordinal α such that $x_\xi = y_\xi$ for $\xi < \alpha$ and $y_\alpha < x_\alpha$.

The elements of Z that are not in Y are exactly those of the form $x = (x_\xi)_{\xi \leq \beta+1}$, where $(x_\xi)_{\xi \leq \beta} \in Y$ and $x_\beta = x_{\beta+1}$. This order is a partial Dedekind completion of Y. We also need a natural definition of bad points with respect to Z.

Definition 11. Given an increasing function $\rho : \Upsilon \longrightarrow Z$, we say that $t \in \Upsilon$ is *Z-bad* for ρ if there exists a sequence of distinct points $(u_n) \subseteq t^+$ such that $\rho(u_n) \to \rho(t)$ in the order topology of Z.

Using Z-bad points, we obtain a direct analogy to the necessity part of Theorem 4; the following is the main result of this paper.

Theorem 12. *If the space $\mathscr{C}_0(\Upsilon)$ admits a Gâteaux norm, then there exists an increasing function $\rho : \Upsilon \longrightarrow Z$ that has no Z-bad points and is not constant on any ever-branching subset.*

In some sense, Y is to $\mathbb{Q}$ what Z is to $\mathbb{R}$, and these relationships correspond well to those of Theorems 7, 1, 12 and 4 respectively.

The following corollary of Theorem 12 generalises a result from [3], which states that $\mathscr{C}_0([0, \omega_1))$ does not admit any Gâteaux lattice norm.

Corollary 2. *If $\mathscr{C}_0(\Upsilon)$ admits a Gâteaux lattice norm then $\Upsilon \preccurlyeq Y$ and, consequently, $\mathscr{C}_0(\Upsilon)$ admits a lattice norm with strictly convex dual.*

We end Section 2 by proving the next proposition, which shows that Theorem 9 is a corollary of Theorem 12.

Proposition 3. *If $\rho : \Upsilon \longrightarrow Z$ is an increasing function that is not constant on any ever-branching subset, then Υ does not admit any ever-branching Baire subsets.*

The final section, devoted to examples, begins with a proof that Theorem 9 is strictly implied by Theorem 12.

Proposition 4. *The tree σY is Z-embeddable, but every increasing function $\rho : \Upsilon \longrightarrow Z$ has a Z-bad point. In particular, $\mathscr{C}_0(\sigma Y)$ does not admit a Gâteaux norm.*

Proposition 4 is analogous to Corollary 1. Section 3 ends with Example 15, which shows that there is a gap between the conditions of Theorems 8 and 12. This, together with the analogies presented above and the author's bias, prompts the following problem.

Problem 1. *If there exists an increasing function $\rho : \Upsilon \longrightarrow Z$ that has no Z-bad points and is not constant on any ever-branching subset, does $\mathscr{C}_0(\Upsilon)$ admit a Gâteaux norm?*

Recently, the author gave a purely topological formulation of Theorem 7. Given a tree Υ, the space $\mathscr{C}_0(\Upsilon)$ admits a norm with strictly convex dual norm if and only if Υ is a so-called *Gruenhage space*, with respect to its interval topology [10].

Problem 2. *Is there an internal characterisation of trees Υ, with the property that $\mathscr{C}_0(\Upsilon)$ admits a Gâteaux norm?*

Problem 2 may be restated in terms of Fréchet norms, Kadec norms and others. This section closes with further problem, motivated by Corollary 2.

Problem 3. *If L is locally compact and $\mathscr{C}_0(L)$ admits a Gâteaux lattice norm, does $\mathscr{C}_0(L)$ admit a norm with strictly convex dual? Is this statement also true with respect to a general Banach lattice?*

2. Necessity conditions for Gâteaux renormability

To help familiarise the reader with Z and Z-bad points, we begin by briefly describing some forms of sequential convergence in Z. First observe that if $x \in Y$, $y \in Z$ and $y > x$ is sufficiently close to x in the order topology of Z, then y must be a strict extension of x. On the other hand, if $x \in Z\backslash Y$ then x has no strict extensions in Z. The proof of the next lemma is a simple exercise in elementary analysis and is omitted.

Lemma 1. *Let $x \in Z$ and suppose $(z^n) \subseteq Z$ is a sequence satisfying $x < z^n$. We have the following rules for the convergence of (z^n) to x:*

1. *if $x = (x_\xi)_{\xi \leq \beta} \in Y$ then $z^n \to x$ if and only if z^n strictly extends x for large enough n, and $z^n_{\beta+1} \to \infty$.*

If $x = (x_\xi)_{\xi \leq \beta+1} \in Z\backslash Y$ then since x has no strict extensions, there exists $\alpha_n \leq \beta$ such that $z^n_\xi = x_\xi$ for $\xi < \alpha_n$ and $z^n_{\alpha_n} < x_{\alpha_n}$. In this case, we have:

2. *if $\beta = 0$ or $\beta = \alpha + 1$ for some α, then $z^n \to x$ if and only if $\alpha_n = \beta$ for large enough n, and $z^n_\beta \to x_\beta$;*
3. *if β is a limit ordinal, then $z^n \to x$ if and only if $\alpha_n \to \beta$.*

We present a simple application of Lemma 1. If $\pi : \Upsilon \longrightarrow Y$ is a strictly increasing map then it could have Z-bad points. However, if we fix an order isomorphism $\theta : \mathbb{R} \longrightarrow (0, 1)$ and define, for $x = (x_\xi)_{\xi \leq \beta} \in Y$, $\Theta(x)_\xi = \theta(x_\xi)$ whenever $\xi \leq \beta$, then by Lemma 1 part (1), the strictly increasing Y-valued map $\Theta \circ \pi$ has no Z-bad points. Thus, some Z-bad points are easily removed by making simple adjustments. More details of how Z operates can be found in Section 3.

Now, for the rest of this section, we fix a norm $||\cdot||$ on $\mathscr{C}_0(\Upsilon)$. We continue by introducing a concept that features in both [5] and [6]. Given $t \in \Upsilon$, let C_t be the set of all $f \in \mathscr{C}_0(\Upsilon)$ such that f vanishes outside $(0, t]$ and increasing on $(0, t]$.

Definition 13. If $f \in C_t$ and $\delta \geq 0$, the increasing function $\mu(f,\delta,\cdot)$ is defined on the wedge $[t,\infty)$ by

$$\mu(f,\delta,\cdot) \ = \ \inf\{||f + (f(t)+\delta)\mathbf{1}_{(t,u]} + \varphi|| \mid \varphi \in \mathscr{C}_0(\Upsilon) \text{ and } \operatorname{supp}\varphi \subseteq (u,\infty)\}$$

where $\mathbf{1}_A$ denotes the indicator function of the set A and $\operatorname{supp}\varphi$ is the support of φ. We also define the abbreviation $\mu(f,\cdot)$ by $\mu(f,u) = \mu(f,0,u)$ and the associated function μ, given by $\mu(t) = \inf\{||\mathbf{1}_{(0,t]} + \varphi|| \mid \varphi \in \mathscr{C}_0(\Upsilon) \text{ and } \operatorname{supp}\varphi \subseteq (t,\infty)\}$.

Attainment of the infimum in the definition of these so-called μ-functions has important consequences for the renormability of $\mathscr{C}_0(\Upsilon)$, and bad points and ever-branching subsets come into play. The first consequence of the following lemma is trivial, and the second and third are immediate generalisations of [6, Lemma 3.1] and [6, Proposition 3.4] respectively.

Lemma 2 ((Haydon [6]))**.** *Suppose $t \in \Upsilon$, $f \in C_t$ and $\delta \geq 0$. Then:*

(1) *if $||\cdot||$ is a lattice norm then $||f + (f(t)+\delta)\mathbf{1}_{(t,u]}|| = \mu(f,\delta,u)$ for all $u \succcurlyeq t$;*
(2) *if $u \succcurlyeq t$ is a bad point for $\mu(f,\delta,\cdot)$ then $||f + (f(t)+\delta)\mathbf{1}_{(t,u]}|| = \mu(f,\delta,u)$;*
(3) *if $\mu(f,\delta,\cdot)$ is constant on some ever-branching subset $E \subseteq (u,\infty)$, where $u \succcurlyeq t$, then there exists $\varphi \in \mathscr{C}_0(\Upsilon)$ with*

$$\operatorname{supp}\varphi \ \subseteq \ \{v \in (u,\infty) \mid v \preccurlyeq w \text{ for some } w \in E\}$$

and $\mu(f,\delta,u) = ||f + (f(t)+\delta)(\mathbf{1}_{(t,u]} + \varphi)||$.

We continue with an idea from [9].

Definition 14. A subset $V \subseteq \Upsilon$ is called a *plateau* if V has a least element 0_V and $V = \bigcup_{t\in V}[0_V, t]$. A partition $\mathscr{P}$ of Υ consisting solely of plateaux is called a *plateau partition*.

Observe that if V is a plateau then $V\backslash\{0_V\}$ is open. It follows that if we have a plateau partition $\mathscr{P}$ and define the *set of least elements* $H = \{0_V \mid V \in \mathscr{P}\}$, then H is closed in Υ. Of course, H may be regarded as a tree in its own right, with its own interval topology. Plateaux are stable under taking arbitrary intersections.

Proposition 5 ((Smith [9, Proposition 10]))**.** *Let Υ be a tree and $\mathfrak{F}$ a family of plateaux of Υ with non-empty intersection W. Then W is a plateau and $0_W = \sup_{V\in\mathfrak{F}} 0_V$.*

The connection between increasing functions and plateaux is given by the next proposition.

Proposition 6 ((Smith [9, Proposition 9]))**.** *Let $\rho : \Upsilon \longrightarrow \Sigma$ be an increasing function into a linear order Σ. Then the equivalence relation $\sim$, given by $s \sim t$ if and only if there exists $r \preccurlyeq s,t$ such that $\rho(s) = \rho(r) = \rho(t)$, defines the plateau partition of Υ, with respect to ρ. Moreover, the restriction of ρ to the set of least elements $H = \{0_V \mid V \in \mathscr{P}\}$ is strictly increasing.*

Proposition 6 applies equally well to decreasing functions. As the μ-functions from Definition 13 are increasing on their respective domains, they may be analysed using plateaux. Elements of the following technical lemma appear implicitly in the proof of [6, Theorem 8.1].

Lemma 3. *Let $||\cdot||$ be Gâteaux smooth and suppose that $\varepsilon||\cdot||_\infty \leq ||\cdot|| \leq ||\cdot||_\infty$ for some $\varepsilon \in (0,1)$. Moreover, suppose V is a plateau, $f \in C_{0_V}$ and $\mu(f,\cdot)$ is constant on V. We define a function λ on $V\backslash\{0_V\}$ by setting*

$$\lambda(t) \ = \ \sup\{\delta \geq 0 \mid \mu(f,\delta,t) \leq \mu(f,0_V) + \tfrac{1}{2}\varepsilon\delta\}.$$

We check that λ is well-defined and satisfies the following properties:

1. *λ is decreasing on $V\backslash\{0_V\}$;*
2. *if λ takes constant value ν on the plateau $W \subseteq V\backslash\{0_V\}$ then $\mu(f,\nu,\cdot)$ takes constant value $\mu(f,0_V) + \frac{1}{2}\varepsilon\nu$ on W;*
3. *if $\mathscr{P}$ is the plateau partition of $V\backslash\{0_V\}$ with respect to λ, supplied by Proposition 6, $W \in \mathscr{P}$, and $f_W \in C_{0_W}$ is defined by*
$$f_W \ = \ f + (f(0_V) + \lambda(0_W))\mathbf{1}_{(0_V,0_W]}$$
then $\mu(f_W,\cdot)$ takes constant value $\mu(f,0_V) + \frac{1}{2}\varepsilon\lambda(0_W)$ on W;
4. *if the infimum in the definition of $\mu(f,t)$ is attained then $\lambda(t) > 0$.*

Proof. Fix $t \in V\backslash\{0_V\}$ and, for $\delta \geq 0$, define $F(\delta) = \mu(f,\delta,t) - \mu(f,0_V) - \frac{1}{2}\varepsilon\delta$. Observe that F is continuous and $F(0) = 0$. Moreover, if $\operatorname{supp}\varphi$ is a subset of (t,∞), we estimate that $||f + (f(t)+\delta)\mathbf{1}_{(0_V,t]} + \varphi|| \geq \varepsilon\delta - ||f + f(t)\mathbf{1}_{(0_V,t]}||$, whence $F(\delta)$ tends to ∞ as δ does. As a result, $\lambda(t)$ is well-defined.

Now we can check the properties of λ. We see that $\mu(f,\lambda(t),t) = \mu(f,0_V) + \frac{1}{2}\varepsilon\lambda(t)$ for any $t \in V\backslash\{0_V\}$. Therefore, if $t \preccurlyeq u$ then, as $\mu(f,\lambda(u),\cdot)$ is increasing, we have

$$\mu(f,\lambda(u),t) \ \leq \ \mu(f,\lambda(u),u) \ = \ \mu(f,0_V) + \tfrac{1}{2}\varepsilon\lambda(u)$$

which shows that $\lambda(t) \geq \lambda(u)$, giving us property (1).

The second property follows immediately and the third follows from the second. To prove property (4), we let $g = f + f(t)\mathbf{1}_{(0_V,t]} + \varphi$ with $\operatorname{supp}\varphi \subseteq (t,\infty)$, such that $||g|| = \mu(f,t) = \mu(f,0_V)$. Observe that as the infimum $\mu(f,0_V)$ is attained, we have

$$||g||'(\mathbf{1}_{(0_V,t]}) \ = \ \lim_{\delta\to 0_+} \frac{||g + \delta\mathbf{1}_{(0_V,t]}|| - ||g||}{\delta} \ \geq \ 0$$

and similarly for $-\mathbf{1}_{(0_V,t]}$, whence $||g||'(\mathbf{1}_{(0_V,t]}) = 0$. Now it is evident that there exists $\delta > 0$ satisfying

$$\mu(f,\delta,t) \ \leq \ ||g + \delta\mathbf{1}_{(0_V,t]}|| \ \leq \ ||g|| + \tfrac{1}{2}\varepsilon\delta \ = \ \mu(f,0_V) + \tfrac{1}{2}\varepsilon\delta$$

which means that $\lambda(t) \geq \delta > 0$. □

While noting property (4) above, we stress that sometimes λ does vanish, and it is necessary to analyse what happens in this case.

Lemma 4. *Suppose V, f, $\mu(f,\cdot)$, λ and the partition $\mathscr{P}$ are as in Lemma 3. If $\lambda(t) = 0$ for some $t \in W \in \mathscr{P}$, then:*

1. *$W = [0_W,\infty) \cap V$;*
2. *W is finitely-branching, in other words, $u^+ \cap W$ is finite whenever $u \in W$;*
3. *W contains no ever-branching subsets.*

Proof. The first property follows because $\lambda \geq 0$ and is decreasing. To prove property (2), we suppose that $u \in V$ is such that $u^+ \cap V$ is infinite. Then u is a bad point for $\mu(f,\cdot)$ as $\mu(f,v) = \mu(f,u)$ for infinitely many $v \in u^+$. Consequently, the infimum in the definition of $\mu(f,u)$ is attained by part (2) of Lemma 2, and it follows from Lemma 3 part (4) that $\lambda(u) > 0$. As a result, $u \notin W$. For property (3), it is enough to show that if $u \in V$ and E is an ever-branching subset of $[u,\infty)\cap V$, then $\lambda(u) > 0$. Indeed, given such u and E, by part (3) of Lemma 2, the infimum in the definition of $\mu(f,u)$ is attained. Therefore, by part (4) of Lemma 3, $\lambda(u) > 0$. $\square$

The proof of Theorem 12 is similar to that of Theorem 7, in that it employs monotone real-valued functions to recursively define a refining sequence of plateaux partitions of the given tree. This sequence is used to define a Z-valued function or, in the case of Theorem 7 or Corollary 2, a Y-valued function. We will see that we must make use of the elements in $Z\backslash Y$ precisely when our λ-functions from Lemma 3 vanish.

of Theorem 12. Let $||\cdot||$ be Gâteaux smooth and suppose that $\varepsilon||\cdot||_\infty \leq ||\cdot|| \leq ||\cdot||_\infty$ for some $\varepsilon \in (0,1)$. We assemble, for each $\beta < \omega_1$, a plateau partition $\mathscr{P}_\beta$, and for each $V \in \mathscr{P}_\beta$, a function $f_{(\beta,V)} \in C_{0_V}$ such that:

(1) $\mu(f_{(\beta,V)},\cdot)$ takes constant value $\mu(f_{(\beta,V)},0_V)$ on V;
(2) $\mu(f_{(\beta,V)},0_V) - 1 \ \leq\ \frac{1}{2}\varepsilon(||f_{(\beta,V)}||_\infty - 1)$.

Following this, we define a function $\pi : \Upsilon \longrightarrow Z$ and prove that it possesses a number of properties. Our final function ρ will be a modification of π.

We begin by constructing $\mathscr{P}_0$. Recall the increasing function μ from Definition 13. Let $\mathscr{P}_0$ be its plateau partition, courtesy of Proposition 6, and define $f_{(0,V)} = \mathbf{1}_{(0,0_V]}$ for $V \in \mathscr{P}_0$. It follows that $\mu(f_{(0,V)},\cdot)$ takes constant value $\mu(f_{(0,V)},0_V) = \mu(0_V)$ on V, and that

$$\mu(f_{(0,V)},0_V) - 1 \ \leq\ ||\mathbf{1}_{(0,0_V]}|| - 1 \ \leq\ 0 \ =\ \tfrac{1}{2}\varepsilon(||f_{(0,V)}||_\infty - 1).$$

Now suppose $\mathscr{P}_\beta$ and the associated $f_{(\beta,V)}$ have been built. Let $V \in \mathscr{P}_\beta$. If $V = \{0_V\}$ then set $\mathscr{P}_V = \{V\}$ and $f_{(\beta+1,V)} = f_{(\beta,V)}$. Otherwise, Lemma 3, together with Proposition 6, furnishes us with the plateau partition of $V\backslash\{0_V\}$ associated with the λ-function. We augment this with the single element $\{0_V\}$ to give a plateau partition $\mathscr{P}_V$ of V. Set $\mathscr{P}_{\beta+1} = \bigcup\{\mathscr{P}_V \mid V \in \mathscr{P}_\beta\}$. If $W \in \mathscr{P}_V$ then either $W = \{0_V\}$ or $W \subseteq V\backslash\{0_V\}$. In the former case let $f_{(\beta+1,W)} = f_{(\beta,V)}$; it is easy to see that $f_{(\beta+1,W)}$ satisfies conditions (1) and (2) above. In the latter case, let $f_{(\beta+1,W)} = f_W$, where f_W is as in Lemma 3 part (3). We observe condition (1) is satisfied, again by Lemma 3 part (3). To see that condition (2) holds, note that

$$\mu(f_{(\beta+1,W)},0_W) - \mu(f_{(\beta,V)},0_V) \ =\ \tfrac{1}{2}\varepsilon\lambda(0_W) \ =\ \tfrac{1}{2}\varepsilon(||f_{(\beta+1,W)}||_\infty - ||f_{(\beta,V)}||_\infty)$$

and apply the inductive hypothesis.

We move on to the limit case. Suppose that $\beta < \omega_1$ is a limit ordinal and that all has been constructed for $\alpha < \beta$. Given $t \in \Upsilon$, we let $V^t_\alpha \in \mathscr{P}_\alpha$ be such that $t \in V^t_\alpha$. Set $\mathscr{P}_\beta = \{\bigcap_{\alpha<\beta} V^t_\alpha \mid t \in \Upsilon\}$. Fix some $V \in \mathscr{P}_\beta$. Let $t = 0_V$, $V_\alpha = V^t_\alpha$, $t_\alpha = 0_{V_\alpha}$ and $f_\alpha = f_{(\alpha,V_\alpha)}$. Then $t = \sup_{\alpha<\beta} t_\alpha$ by Proposition 5. What we would like to do is define $f_{(\beta,V)} = f \in \mathscr{C}_0(\Upsilon)$ to be the unique function supported on $(0,t]$, such that its restriction to $(0,t_\alpha]$ is f_α. This can indeed be done, provided that $(||f_\alpha||_\infty)_{\alpha<\beta}$ is bounded. Observe that if $g \in C_u$ satisfies condition (2) above then

$$\varepsilon||g||_\infty - 1 \ \leq\ \mu(g,u) - 1 \ \leq\ \tfrac{1}{2}\varepsilon(||g||_\infty - 1)$$

giving $||g||_\infty \leq \frac{2}{\varepsilon} - 1$. Therefore $(||f_\alpha||_\infty)_{\alpha<\beta}$ is bounded as required. Moreover, since each $f_\alpha \in C_{t_\alpha}$, we have $f \in C_t$. Now set $g_\alpha = f_\alpha + f_\alpha(t_\alpha)\mathbf{1}_{(t_\alpha,t]}$. Of course, as f_α is increasing on $(0,t_\alpha]$ and vanishes elsewhere, we have $||g_\alpha||_\infty = ||f_\alpha||_\infty$.

Moreover, as $\mu(f_\alpha,\cdot)$ takes constant value $\mu(f_\alpha,t_\alpha)$ on V_α by inductive hypothesis, and $\mu(g_\alpha,u)=\mu(f_\alpha,u)$ whenever $u\in V\subseteq V_\alpha$, it follows that $\mu(g_\alpha,\cdot)$ takes constant value $\mu(f_\alpha,t_\alpha)$ on V. The reader can verify that, as $(g_\alpha)_{\alpha<\beta}$ converges in norm to f, $(\mu(g_\alpha,\cdot))_{\alpha<\beta}$ converges uniformly to $\mu(f,\cdot)$ (cf. [6, Lemma 3.6]). As a result, f satisfies conditions (1) and (2) above. This ends the recursion.

Now we define π. Given $t\in\Upsilon$, let V^t_β be as above. In addition, we let λ^t_β be the λ-function associated with V^t_β and $f_{(\beta,V^t_\beta)}$, provided V^t_β is not a singleton. Set $\pi(t)_0=-\mu(t)$. If $\beta>0$, let $\pi(t)_\beta=\mu(f_{(\beta,V^t_\beta)},t)$ as long as $0_{V^t_\alpha}\prec t$ for all $\alpha<\beta$ and $\lambda^t_\alpha(t)>0$ whenever $\alpha+1<\beta$. Otherwise, we leave $\pi(t)_\beta$ undefined.

We verify that $\pi(t)$ is an element of Z. Observe that if $\pi(t)_\beta$ is defined, then so is $\pi(t)_\alpha$ whenever $\alpha<\beta$. If $0<\alpha<\beta$ then $\pi(t)_0<0<\pi(t)_\alpha$ and moreover

$$\begin{aligned}\pi(t)_{\alpha+1} &= \mu(f_{(\alpha+1,V^t_{\alpha+1})},t)\\ &= \mu(f_{(\alpha,V^t_\alpha)},t)+\tfrac{1}{2}\varepsilon\lambda^t_\alpha(0_{V^t_{\alpha+1}})\\ &= \pi(t)_\alpha+\tfrac{1}{2}\varepsilon\lambda^t_\alpha(t)\end{aligned}$$

whence $\pi(t)_{\alpha+1}\geq\pi(t)_\alpha$. In addition, if $\alpha+1<\beta$ then $\pi(t)_{\alpha+1}>\pi(t)_\alpha$ by our definition of π. Now, if β is a limit ordinal and $\pi(t)_\alpha$ is defined for all $\alpha<\beta$, so is $\pi(t)_\beta$. Moreover, by applying the uniform convergence of the μ-functions at limit stages of the partition construction, we see that $\pi(t)_\beta=\mu(f_{(\beta,V^t_\beta)},t)=\lim_{\alpha<\beta}\mu(f_{(\alpha,V^t_\alpha)},t)=\lim_{\alpha<\beta}\pi(t)_\alpha$. This is enough to prove that $\pi(t)\in Z$.

We observe our first property of π, namely that it is increasing. Let $s,t\in\Upsilon$ with $s\prec t$. We set γ to be the least ordinal such that $\pi(s)_\gamma$ and $\pi(t)_\gamma$ are not both defined and equal. If $\gamma=0$ then, as μ is increasing, it follows that $\pi(s)_0>\pi(t)_0$, whence $\pi(s)<\pi(t)$. If $\gamma>0$ then, by continuity, $\gamma=\beta+1$ for some β. By transfinite induction, $V^s_\alpha=V^s_\alpha$ for all $\alpha\leq\beta$. Indeed, $\mu(s)=-\pi(s)_0=-\pi(t)_0=\mu(t)$, so $V^s_0=V^t_0$. If $V^s_\alpha=U=V^t_\alpha$ and $\alpha<\beta$, set $\lambda^s_\alpha=\lambda=\lambda^t_\alpha$. Remembering property (2) of Lemma 3, we have

$$\tfrac{1}{2}\varepsilon\lambda(s) \;=\; \pi(s)_{\alpha+1}-\pi(s)_\alpha \;=\; \pi(t)_{\alpha+1}-\pi(t)_\alpha \;=\; \tfrac{1}{2}\varepsilon\lambda(t) \qquad (1)$$

whence $\lambda(s)=\lambda(t)$ and $V^s_{\alpha+1}=V^t_{\alpha+1}$. Limit stages of the induction follow by taking intersections.

Now let $V^s_\beta=V=V^t_\beta$, $\lambda^s_\beta=\lambda=\lambda^t_\beta$ and observe that $0_V\preccurlyeq s\prec t$. There are two cases to consider: either $\pi(t)_{\beta+1}$ is defined or it is not. First of all, we suppose that $\pi(t)_{\beta+1}$ is defined and prove that $\pi(s)<\pi(t)$ in this case. Indeed, if $\pi(s)_{\beta+1}$ is not defined then we are done, as $\pi(t)$ strictly extends $\pi(s)$. On the other hand, if $\pi(s)_{\beta+1}$ is defined then since $\pi(s)_{\beta+1}\neq\pi(t)_{\beta+1}$ and λ is decreasing, it must be that $\pi(s)_{\beta+1}>\pi(t)_{\beta+1}$. Therefore $\pi(s)<\pi(t)$.

The other option is that $\pi(t)_{\beta+1}$ is undefined. In this case, since $0_V\prec t$, it must be that $\lambda^t_\alpha(t)=0$ for some $\alpha+1<\beta+1$, by the definition of π. As $\pi(t)_\beta$ is defined then, again by the definition of π, it follows that $\alpha+1=\beta$. Let $V^s_\alpha=U=V^t_\alpha$ and $\lambda^s_\alpha=\lambda'=\lambda^t_\alpha$. Then by Eqn. 1 above, we have $\lambda'(s)=\lambda'(t)=0$, meaning $\pi(s)_{\beta+1}$ is not defined either. Consequently, $\pi(s)=\pi(t)$.

We have established that π is an increasing function. Now we show that it is not constant on any ever-branching subset and, given $t\in\Upsilon$, there are only finitely many $u\in t^+$ such that $\pi(u)=\pi(t)$. To prove this claim, consider $t\in\Upsilon$ and the plateau $W=\{u\in[t,\infty)\mid\pi(u)=\pi(t)\}$. If W is the singleton $\{t\}$ then there is nothing to prove, so we suppose that there exists some $u\in W$ with $t\prec u$. Let both $\pi(t)$ and $\pi(u)$ be defined on $[0,\beta]$ and fix $V = V^t_\beta$. In just the same way as above, we have that $V^t_\alpha = V^u_\alpha$ whenever $\alpha \leq \beta$ and, in particular, $V^u_\beta = V$. Observe that, as a consequence, $W \subseteq V$. Moreover, just as above, as $\pi(u)_{\beta+1}$ is undefined and $0_{V^u_\beta} \preccurlyeq t \prec u$, we have $\beta = \alpha + 1$ for some α. It follows that if we set $V^t_\alpha = U = V^u_\alpha$ and $\lambda^t_\alpha = \lambda' = \lambda^u_\alpha$, then $\lambda'(t) = \lambda'(u) = 0$. Now we can appeal to parts (2) and (3) of Lemma 4 applied to U, $f_{(\alpha,U)}$, $\mu(f_{(\alpha,U)},\cdot)$ and λ' to conclude that V is finitely-branching and contains no ever-branching subsets. As $W \subseteq V$, we are done.

We finish our appraisal of π by showing that it does not admit certain types of Z-bad points. First of all, if $\pi(t) \in Y$ then t cannot be Z-bad for π. Indeed, by Lemma 1 part (1) and the fact that the elements of $\operatorname{ran}\pi$ are uniformly bounded sequences, the only way that t can be Z-bad for π is if there are infinitely many $u \in t^+$ such that $\pi(u) = \pi(t)$. Now suppose that $\pi(t) = (\pi(t)_\xi)_{\xi\leq\beta+1} \in Z\backslash Y$, where β is a limit ordinal. We prove that t is not Z-bad for π. We know already that $\pi(u) = \pi(t)$ for only finitely many $u \in t^+$ so, for a contradiction, we must suppose that there is a sequence of distinct points $(u_n) \subseteq t^+$ such that $\pi(t) < \pi(u_n)$ and $\pi(u_n) \to \pi(t)$. We have that $\pi(t)_\beta = \pi(t)_{\beta+1}$. Let $V = V^t_\beta$, where V^t_β is the unique element $V \in \mathscr{P}_\beta$ containing t, and let $f = f_{(\beta,V)}$. Observe that if λ is the function from Lemma 3 associated with f and V then, necessarily, $\lambda(t) = 0$. Indeed, by the definition of π, we have $\frac{1}{2}\varepsilon\lambda(t) = \pi(t)_{\beta+1} - \pi(t)_\beta$. By Lemma 1 part (3), there exist ordinals $\alpha_n < \beta$ such that $\alpha_n \to \beta$, $\pi(u_n)_\xi = \pi(t)_\xi$ whenever $\xi < \alpha_n$ and $\pi(u_n)_{\alpha_n} < \pi(t)_{\alpha_n}$. By continuity and transfinite induction, $\alpha_n = \xi_n + 1$ for some ordinals ξ_n and $V^t_{\xi_n} = V^{u_n}_{\xi_n}$. Set $V_n = V^t_{\xi_n}$ and $f_n = f_{(\xi_n,V_n)}$. As $\alpha_n \to \beta$, it follows that $V = \bigcap_n V_n$ and the functions $f_n + f_n(0_{V_n})\mathbf{1}_{(0_{V_n},t]}$ converge in norm to $f + f(0_V)\mathbf{1}_{(0_V,t]}$. Moreover $\mu(f_n,u_n) = \pi(u_n)_{\xi_n} = \pi(t)_{\xi_n} \to \pi(t)_\beta = \mu(f,t)$. Now choose $\varphi_n \in \mathscr{C}_0(\Upsilon)$ to satisfy $\operatorname{supp}\varphi_n \subseteq (u_n,\infty)$ and $||f_n + f_n(0_{V_n})\mathbf{1}_{(0_{V_n},u_n]} + \varphi_n|| \leq \mu(f_n,u_n) + 2^{-n} = \mu(f_n,t) + 2^{-n}$. As the u_n are distinct, it follows that $(f_n + f_n(0_{V_n})\mathbf{1}_{(0_{V_n},u_n]} + \varphi_n)$ converges to $f + f(0_V)\mathbf{1}_{(0_V,t]}$ in the pointwise topology of $\mathscr{C}_0(\Upsilon)$. As Υ is scattered and this sequence is norm-bounded, it converges in the weak topology too. Therefore $||f + f(0_V)\mathbf{1}_{(0_V,t]}|| = \mu(f,t)$. However, by part (4) of Lemma 3, the attainment of the infimum forces $\lambda(t) > 0$, which is not the case. It follows that t cannot be a Z-bad point for π.

One case remains untreated. If $\pi(t) = (\pi(t)_\xi)_{\xi\leq\beta+1} \in Z\backslash Y$ and β is not a limit ordinal, it is possible that t is Z-bad for π. Fortunately, by making an adjustment to π akin to that given after Lemma 1, we can remove Z-bad points of this kind. Given $x = (x_\xi)_{\xi\leq\beta} \in Z$, define

$$\Phi(x)_\xi = \begin{cases} 2x_0 & \text{if } \xi = 0 \\ x_\xi + x_{\xi-1} + 1 & \text{if } \xi \text{ is a successor ordinal} \\ 2x_\xi + 1 & \text{otherwise} \end{cases}$$

for $\xi \leq \beta$. It is easy to establish that Φ takes values in Z and is strictly increasing. Set $\rho = \Phi \circ \pi$. As Φ is strictly increasing, ρ is increasing and, if we consider Proposition 6, partitions Υ in exactly the same way as π. In particular, ρ is not constant on any ever-branching subset of Υ. Again, as Φ is strictly increasing, if t is Z-bad for ρ then it is also Z-bad for π. Therefore, to prove that ρ has no Z-bad points, we suppose that $\pi(t) = (\pi(t)_\xi)_{\xi\leq\beta+1} \in Z\backslash Y$ and β is not a limit ordinal. We have that $\pi(t)_\beta = \pi(t)_{\beta+1}$ so, by the construction of π, there exists an ordinal α such that $\beta = \alpha + 1$. Therefore, $\pi(t)_\alpha < \pi(t)_\beta$ and thus $\rho(t)_\beta < \rho(t)_{\beta+1}$, giving $\rho(t) \in Y$. Again by appealing to Lemma 1 part (1), if t is Z-bad for ρ then $\rho(u) = \rho(t)$ for infinitely many $u \in t^+$. However, that would force $\pi(u) = \pi(t)$ for infinitely many $u \in t^+$, and we have already established that this is impossible. $\square$

of Corollary 2. If $||\cdot||$ is a lattice norm then, by part (1) of Lemma 2, the infima in the definition of the μ-functions are always attained. It follows that the λ-functions of Lemma 3 never vanish. Now, we prove that in this case, the map π defined in the proof of Theorem 12 is Y-valued and strictly increasing. Indeed, if we return to the point where we prove that $\pi(t) \in Z$, we see that, as the λ-functions never vanish, $\pi(t)_\alpha < \pi(t)_{\alpha+1}$ whenever $\alpha + 1 \leq \beta$. Consequently $\pi(t) \in Y$. To show that π is strictly increasing, we let $s \prec t$ and return to the point in the proof where π is shown to be increasing, specifically, where γ is defined. If $\gamma = 0$ then we are done. Otherwise, $\gamma = \beta + 1$ for some β. Since the λ-functions never vanish, it is impossible that $\pi(t)_{\beta+1}$ is undefined, therefore $\pi(s) < \pi(t)$. This proves that $\Upsilon \preccurlyeq Y$. The second statement of Corollary 2 holds because the strictly convex dual norm constructed in Theorem 7 is a lattice norm. $\square$

We finish the section with a proof of Proposition 3. It will help to introduce a useful game-theoretic characterisation of Baire trees [5]. Players **A** and **B** take turns to nominate elements of a tree Υ, beginning with t_0 played by **B**. In general, **A** follows t_{2n} with $t_{2n+1} \succcurlyeq t_{2n}$, and **B** responds with $t_{2n+2} \succcurlyeq t_{2n+1}$. The game is won by **B** if the sequence (t_n) has no upper bound in Υ. The tree Υ is Baire if and only if **B** has no winning strategy in this so-called Υ-*game*. Using this game, it is possible to prove the following result.

Proposition 7 ((Haydon [5, Proposition 1.4]))**.** *If Υ is Baire and $\rho : \Upsilon \longrightarrow \mathbb{R}$ is increasing, then there exists $t \in \Upsilon$ such that ρ is constant on the wedge $[t, \infty)$.*

One trivial consequence of Proposition 7 is that if the increasing map $\rho : \Upsilon \longrightarrow \mathbb{R}$ is not constant on any ever-branching subset then Υ contains no ever-branching Baire subsets. Indeed, if $E \subseteq \Upsilon$ were ever-branching and Baire then, by Proposition 7, we could find $t \in E$ such that ρ is constant on $[t, \infty) \cap E$, which is an ever-branching subset of Υ. We observe that the same holds if we replace $\mathbb{R}$ with any linear order Σ satisfying the statement of Proposition 7. Therefore, to establish Proposition 3, it is enough to prove the following result.

Proposition 8. *If Υ is Baire and $\rho : \Upsilon \longrightarrow Z$ is increasing, then there exists $t \in \Upsilon$ such that ρ is constant on $[t, \infty)$.*

Proof. The following order will be used in this and a subsequent proof. Define

$$Z_0 \;=\; \{x = (x_\alpha)_{\alpha \leq \beta} \in Z \mid x \subseteq [0, 1],\, x_0 = 0 \text{ and } \beta \text{ is a limit whenever } x_\beta = 1\}.$$

By considering the map Θ, introduced after Lemma 1, we observe that $Z \preccurlyeq Z_0$ and, accordingly, we can assume that our increasing function ρ takes values in Z_0.

We show that ρ is constant on some wedge of Υ by playing the Υ-game with a particular strategy for **B**. Given $u \in \Upsilon$ and an ordinal α, we call (α, u) a *fixed pair* if $\rho(v)_\xi$ is defined and equal to $\rho(u)_\xi$ whenever $v \in [u, \infty)$ and $\xi \leq \alpha$. If (α, u) is fixed, $v \in [u, \infty)$ and $\xi \leq \alpha$, then (ξ, v) is also fixed. Let **B** play arbitrary t_0 as the first move and put $\alpha_0 = 0$. Note that $(0, t_0)$ is fixed. Now suppose that $n \geq 1$ and that moves $t_0 \preccurlyeq t_1 \preccurlyeq \ldots \preccurlyeq t_{2n-1}$ have been played alternately by **B** and **A**. We choose the next move t_{2n} played by **B**, together with α_n, in the following manner.

Let

$$r_n \;=\; \sup\{\rho(u)_\alpha \mid u \succcurlyeq t_{2n-1} \text{ and } (\alpha, u) \text{ is a fixed pair}\}.$$

Let **B** choose fixed (α_n, t_{2n}) such that $t_{2n} \succcurlyeq t_{2n-1}$ and $\rho(t_{2n})_{\alpha_n} > r_n - 2^{-n}$. This strategy does not guarantee a win for **B**, so there exist moves (t_{2n+1}) of **A** such that (t_n) has an upper bound $u \in \Upsilon$. If $\alpha = \sup \alpha_n$, we see that (α, u) is fixed. This follows by continuity and the fact that (α_n, u) is fixed for all n.

If $\rho(v)_{\alpha+1}$ is not defined for any $v \succcurlyeq u$ then ρ takes constant value $\rho(u)$ on $[u, \infty)$, and we are done. Suppose instead that $\rho(v)_{\alpha+1}$ exists for some $v \succcurlyeq u$. Because (α, v) is fixed and ρ is increasing, the real-valued map $\rho(\cdot)_{\alpha+1}$ must be decreasing on $[v, \infty)$. As the forcing-open set $[v, \infty)$ is Baire, by Proposition 7, there exists $w \succcurlyeq v$ such that $\rho(\cdot)_{\alpha+1}$ is constant on $[w, \infty)$, and it follows that $(\alpha + 1, w)$ is a fixed pair. We note that the inequalities

$$r_n - 2^{-n} \;<\; \rho(t_{2n})_{\alpha_n} \;=\; \rho(w)_{\alpha_n} \;\le\; \rho(w)_\alpha \;\le\; \rho(w)_{\alpha+1} \;\le\; r_n$$

hold for all n, and conclude that $\rho(w)_{\alpha+1} = \rho(w)_\alpha$. Consequently, by the definition of elements of Z, ρ takes constant value $\rho(w)$ on $[w, \infty)$. □

3. Examples

In this section, we prove Proposition 4 and present Example 15. Before giving the proof of Proposition 4, we make an observation about embeddability and Z-bad points that is analogous to Proposition 1.

Given a tree Υ, let $\Upsilon \preccurlyeq Z$ and suppose that there is an increasing function $\rho : \Upsilon \longrightarrow Z$ with no Z-bad points. We claim that if this is the case then $\Upsilon \preccurlyeq Y$. In order to prove this claim, we introduce the following algebraic operation on Z. Recall the order isomorphism $\theta : \mathbb{R} \longrightarrow (0, 1)$, fixed after Lemma 1. For $x = (x_\xi)_{\xi \le \alpha}$ and $y = (y_\xi)_{\xi \le \beta}$ of Z, define $x \cdot y$ for $\xi \le \max\{\alpha, \beta\}$ by

$$(x \cdot y)_\xi \;=\; \begin{cases} \theta^{-1}(\theta(x_\xi)\theta(y_\xi)) & \text{if } \xi \le \min\{\alpha, \beta\} \\ x_\xi & \text{if } \alpha < \xi \le \beta \\ y_\xi & \text{if } \beta < \xi \le \alpha \end{cases}$$

where $\theta(x_\xi)\theta(y_\xi)$ is an ordinary real product. We leave the reader with the simple task of verifying that $\cdot$ is a semigroup operation on Z that respects the order; in other words, if $x \le y$ and $u \le v$ then $x \cdot u \le y \cdot v$ and, moreover, the third inequality is strict if either of the first two are. Now, let the increasing function $\nu : \Upsilon \longrightarrow Z$ have no Z-bad points and suppose $\tau : \Upsilon \longrightarrow Z$ is strictly increasing. As $\cdot$ respects order, it follows that the pointwise product $\pi = \nu \cdot \tau$ is strictly increasing and has no Z-bad points. By Lemma 1, any element of Z can be approached from above by a strictly decreasing sequence. Therefore, as $t \in \Upsilon$ is not a Z-bad point for π, there exists $\pi^*(t) \in Z$ such that $\pi(t) < \pi^*(t) \le \pi(u)$ whenever $u \in t^+$. Finally, since Y is dense in Z, we can pick $\rho(t) \in Y$ between $\pi(t)$ and $\pi^*(t)$; the resulting function ρ is strictly increasing.

of Proposition 4. In the light of Theorem 5 and our observation above, all we need to do is prove that $\sigma Y \preccurlyeq Z$. Recall the order Z_0 from the proof of Proposition 8. As $Z \preccurlyeq Z_0$, elements of σY can and are considered as subsets of Z_0. Our proof that $\sigma Y \preccurlyeq Z$ rests on the claim that Z_0 is Dedekind complete; that is, each subset of A of Z_0 has a least upper bound, denoted by $\sup A$.

For now, we assume that this claim holds and define a strictly increasing map $\rho : \sigma Y \longrightarrow Z$. Given $A \in \sigma Y$, treated as a subset of Z_0, let $\rho(A) = \sup A$ if $\sup A \in$

$Z_0 \backslash Y$ or if A has no greatest element, and let $\rho(A) = (\sup A, 2)$ otherwise. Here, $(x, 2)$ denotes the sequence obtained by extending $x \in Z_0 \cap Y$ by a single element, namely 2. Observe that if $x \in Z_0 \cap Y$, $y \in Z_0$ and $x < y$ then $(x, 2) < y$ because every element of y is strictly less than 2. Let $A, B \in \sigma Y$ satisfy $A \prec B$. If $\sup A < \sup B$ then $\rho(A) < \sup B \leq \rho(B)$. Alternatively, if $\sup A = \sup B$ then $B = A \cup \{\sup A\}$; indeed, if $x \in B \backslash A$ then $\sup A \leq x \leq \sup B = \sup A$. In particular, B has greatest element $\sup A \in Y$, whereas A has no greatest element. Therefore $\rho(A) = \sup A < (\sup A, 2) = \rho(B)$. This proves that ρ is strictly increasing.

To finish, we define $\sup A$ for $A \subseteq Z_0$. If A is empty then its least upper bound is the one-element sequence (0). From now on, we assume that A is non-empty and has no greatest element. Taking our cue from the proof of Proposition 8, given an ordinal α and $x \in A$, we will call (α, x) a *fixed pair* if x_ξ and y_ξ are both defined and equal whenever $y \in A$, $x \leq y$ and $\xi \leq \alpha$. If (α, x) is fixed, $y \in A$, $x \leq y$ and $\xi \leq \alpha$, then (ξ, y) is also fixed. Now let β be minimal, subject to the condition that there is no fixed pair (β, x). As A is non-empty and $(0, x)$ is fixed whenever $x \in A$, it follows that $\beta > 0$. We define a sequence $z = (z_\alpha)_{\alpha \leq \beta}$. If $\alpha < \beta$, let $z_\alpha = x_\alpha$, where (α, x) is some fixed pair. By the nature of fixed pairs, this is well-defined. If β is a limit, let $z_\beta = \sup_{\alpha<\beta} z_\alpha$. Instead, if $\beta = \alpha + 1$ for some α then, as A has no greatest element, there exists a fixed pair (α, x), such that x_β is defined. Let z_β be the infimum of all such x_β. It is easy to verify that $z \in Z_0$; it can be that $z_\beta = 1$, but only if β is a limit ordinal. We omit the pedestrian task of proving that z is the least upper bound of A. $\square$

Our last task is to show that there is a tree Ψ satisfying the condition of Theorem 12 but not that of Theorem 8. Before doing so, we must make some remarks. Recall the plateau partitions of Proposition 6 and note the following slightly reworded version of a result from [8].

Proposition 9 ((Smith [8, Corollary 3]))**.** *Suppose that Υ is a tree, Σ a linear order, and $\rho : \Upsilon \longrightarrow \Sigma$ an increasing function that is not constant on any ever-branching subset of Υ. Then there exists an increasing function $\pi : \Upsilon \longrightarrow \Sigma \times \omega$, such that the plateau partition $\mathscr{P}$ of Υ with respect to π consists solely of linearly ordered subsets.*

Let Υ, Σ, π and $\mathscr{P}$ be as in Proposition 9 and, moreover, let us suppose that Υ admits no uncountable linearly ordered subsets. In this case, each $V \in \mathscr{P}$ identifies with a finite or countable ordinal and, therefore, there exists a strictly increasing function $\pi_V : V \longrightarrow \mathbb{Q}$. It is apparent that the function $\tau : \Upsilon \longrightarrow \Sigma \times \omega \times \mathbb{Q}$, defined by $\tau(t) = (\pi(t), \pi_{V_t}(t))$, where V_t is the unique element of $\mathscr{P}$ containing t, is strictly increasing. As $\omega \times \mathbb{Q} \preccurlyeq \mathbb{Q}$, it follows that $\Upsilon \preccurlyeq \Sigma \times \mathbb{Q}$.

Example 15. Observe that Y has cardinality continuum $\mathfrak{c}$. If $A \in \sigma Y$ then A^+ identifies with the set $u(A)$ of all upper bounds of A and, thus, has cardinality $\mathfrak{c}$ if $u(A)$ is non-empty. Fix a well-order $\sqsubseteq$ of Y, and let $\Psi = \sigma Y \times \mathfrak{c}$. We order Ψ by declaring that $(A, \alpha) \preccurlyeq (B, \beta)$ if and only if either $A = B$ and $\alpha \leq \beta$, or if $A \prec B$ and α is no greater than the order type of $\{x \in u(A) \mid x \sqsubset \min(B \backslash A, \leq)\}$, with respect to $\sqsubset$.

With respect to this order, each element of Ψ has between one and two immediate successors. Indeed, if $(A, \alpha) \in \Psi$ then $(A, \alpha + 1)$ is always an immediate successor. If $u(A)$ is non-empty then $(A \cup \{y\}, 0)$ is also such a successor, where $y \in u(A)$ and

$\{x \in u(A) \mid x \sqsubset y\}$ has order type α. The set $\sigma Y \times \{0\}$ is a natural copy of σY inside Ψ that is closed with respect to the interval topology.

Now, by Proposition 4, there exists a strictly increasing map $\pi : \sigma Y \longrightarrow Z$. Define $\rho : \Psi \longrightarrow Z$ by $\rho(A, \alpha) = \pi(A)$. By Proposition 6, the plateau partition of Ψ with respect to ρ consists exactly of the sets $\{(A, \alpha) \mid \alpha < \mathfrak{c}\}$, where $A \in \sigma Y$. Therefore, ρ is not constant on any ever-branching subset. Because the number of immediate successors of any element of Ψ is at most two, ρ has no Z-bad points either. Therefore Ψ satisfies the condition of Proposition 12.

On the other hand, there exists no increasing Y-valued function on Ψ that is not constant on any ever-branching subset. Indeed, if there were such a function, by considering its restriction to $\sigma Y \times \{0\}$, there would be a map $\tau : \sigma Y \longrightarrow Y$, also not constant on any ever-branching subset. However, by following a similar argument to that given after Proposition 7, being Z-embeddable, σY has no perfect Baire subsets. In particular, σY does not contain a copy of ω_1. Therefore, by Proposition 2 and the remarks following Proposition 9, we would have $\sigma Y \preccurlyeq Y \times \mathbb{Q} \preccurlyeq Y$ which, by Theorem 5, is impossible.

We recall Problem 1 and conjecture that $\mathscr{C}_0(\Psi)$ admits a Gâteaux norm. The Gâteaux norms presented in [8] are built by combining norms obtained from existing techniques, namely the Fréchet norms of Talagrand and Haydon, and norms with strictly convex duals. In the author's opinion, if Problem 1 is to be resolved positively, we require a method of constructing Gâteaux norms on $\mathscr{C}(K)$ spaces that unifies these techniques on a more fundamental level.

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